Constr. Approx. (1993) 9: 473–500

CONSTRUCTIVE APPROXIMATION

The C_ℓ Bailey Transform and Bailey Lemma

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Abstract. The C_ℓ nonterminating ${}_6\varphi_5$ summation theorem is derived by appropriately specializing Gustafson's ${}_6\psi_6$ summation theorem for bilateral basic hypergeometric series very well-poised on symplectic C_ℓ groups. From this, the terminating ${}_6\varphi_5$ and, hence, terminating ${}_4\varphi_3$ summation theorem is obtained. A suitably modified ${}_4\varphi_3$ is then used to derive the C_ℓ generalization of the Bailey transform. The transform is then interpreted as a matrix inversion result for two infinite, lower-triangular matrices. This result is used to motivate the definition of the C_ℓ Bailey pair. The C_ℓ generalization of Bailey's lemma is then proved. This result is inverted, and the concept of the bilateral Bailey chain is discussed. The C_ℓ Bailey lemma is then used to obtain a connection coefficient result for general C_ℓ little q-Jacobi polynomials. All of this work is a natural extension of the unitary A_ℓ, or equivalently $U(\ell+1)$, case. The classical case, corresponding to A_1 or equivalently $U(2)$, contains an immense amount of the theory and application of one-variable basic hypergeometric series, including elegant proofs of the Rogers–Ramanujan–Schur identities. The C_ℓ nonterminating ${}_6\varphi_5$ summation theorem is also used to recover C. Krattenthaler's multivariable summation which he utilized in deriving his refinement of the Bender–Knuth and MacMahon generating functions for certain sets of plane partitions.

1. Introduction

The purpose of this paper is to derive a higher-dimensional generalization of the Bailey transform [6] and Bailey lemma [6] in the setting of basic hypergeometric series very well-poised on symplectic [19] groups. Our work here is a shortened version of the first author's thesis [24]. The symplectic case of the Bailey transform and Bailey lemma is a natural extension of the unitary case [30], [32] corresponding to basic hypergeometric series very well-poised on unitary [26] groups. Both types of series are directly related [19], [25] to the corresponding Macdonald identities. The series in [26] were strongly motivated by certain applications of mathematical physics and the unitary groups $U(n)$ in [13], [14], [20], and [21].

Date received: June 27, 1991. Date revised: June 4, 1992. Communicated by Tom Koornwinder.
AMS classification: Primary 33D70, 05A19; Secondary 33D20.
Key words and phrases: Multiple basic hypergeometric series, Very well-poised on unitary or symplectic groups, C_ℓ nonterminating ${}_6\varphi_5$ summation theorem, C_ℓ terminating ${}_6\varphi_5$ summation theorem, C_ℓ Bailey pair, C_ℓ Bailey transform, C_ℓ Bailey lemma, C_ℓ Bailey chain, C_ℓ little q-Jacobi polynomials, Plane partition generating functions.

The unitary series use the notation A_ℓ, or equivalently $U(\ell+1)$; the symplectic case, C_ℓ. The classical Bailey transform, lemma, and very well-poised basic hypergeometric series correspond to the case A_1, or equivalently $U(2)$.

The classical Bailey transform and Bailey lemma contain an immense amount of the theory and application of one-variable basic hypergeometric series [3], [6], [10], [16], [41]. They were ultimately inspired by Rogers' [40] second proof of the Rogers–Ramanujan–Schur identities [39]. The Bailey transform was first formulated by Bailey [11], utilized by Slater in [41], and then recast by Andrews [4] as a fundamental matrix inversion result. This last version of the Bailey transform has immediate applications to conection coefficient theory and "dual" pairs of identities [4], [6], [18], and q-Lagrange inversion and quadratic transformations [17], [18].

Let q be a complex number such that $|q| < 1$. Define

$$(1.1a)\qquad (\alpha)_\infty \equiv (\alpha; q)_\infty := \prod_{k\geq 0} (1-\alpha q^k)$$

and, thus,

$$(1.1b)\qquad (\alpha)_n \equiv (\alpha; q)_n := \frac{(\alpha)_\infty}{(\alpha q^n)_\infty}.$$

We then have Andrews' matrix inversion result.

Theorem 1.2 (Classical Bailey Transform for A_1). *Let a be indeterminate and let $i, j \geq 0$ be integers. Let the matrices M and M^* be defined as in*

$$(1.3a)\qquad M(i; j; A_1) := (q)_{i-j}^{-1}(aq)_{i+j}^{-1}$$

and

$$(1.3b)\qquad M^*(i; j; A_1) := (1-aq^{2i})(aq)_{i+j-1}(q)_{i-j}^{-1}(-1)^{i-j}q^{\binom{i-j}{2}}.$$

Then M and M^* are inverse, infinite, lower-triangular matrices. That is,

$$(1.4)\qquad \delta(i,j) = \sum_{j\leq y\leq i} M(i; y; A_1)M^*(y; j; A_1),$$

where $\delta(r,s) = 1$ if $r = s$, and 0 otherwise.

Theorem 1.2 follows from the terminating ${}_4\varphi_3$ summation theorem and a termwise rewriting of the (i,j) entry in the matrix product MM^*. Bressoud [15] has deduced an elegant extension of Theorem 1.2 for matrices $M_{a,b}$, with two free parameters, from the terminating ${}_6\varphi_5$ summation theorem. He proved that $M_{a,b}$ and $M_{b,a}$ are inverse, infinite, lower-triangular matrices. This work, as well as [1] and [4], provides a natural setting for Theorem 1.2.

Equation (1.3) motivates the definition of the A_1 Bailey pair.

Definition 1.5 (A_1 Bailey Pair). Let $n \geq 0$ and $y \geq 0$ be integers and let $\alpha = \{\alpha_y\}$ and $\beta = \{\beta_y\}$ be sequences. Let M and M^* be as in (1.3). Then we say that α and

β form an A_1 Bailey pair if

$$\beta_n = \sum_{0 \le y \le n} M(n; y; A_1)\alpha_y \tag{1.6}$$

for all $n \ge 0$.

Equation (1.4) and Definition 1.5 immediately give the following result.

Corollary 1.7 (A_1 Bailey Pair Inversion). *α and β satisfy* (1.6) *if and only if*

$$\alpha_n = \sum_{0 \le y \le n} M^*(n; y; A_1)\beta_y \tag{1.8}$$

Corollary 1.7 is responsible for the dual pairs of identities in [4], [6], and [18]. For example, with α_n and β_n as in (1.13), it follows that (1.6) and (1.8) correspond to the classical terminating very well-poised ${}_6\varphi_5$ summation [10], [16], and the balanced ${}_3\varphi_2$ summation [10], [16], respectively.

The most important application of the Bailey transform is the Bailey lemma. This result was mentioned by Bailey [11, Section 4], and he described how the proof would work. However, he never wrote the result down explicitly and thus missed the full power of *iterating* it. Andrews first established the Bailey lemma explicitly in [5] and realized its numerous possible applications in terms of the iterative "Bailey chain" concept, which produces a new Bailey pair from a given, arbitrary Bailey pair. This iteration mechanism enabled him to derive many q-series identities by "reducing" them to more elementary ones.

Andrews' explicit formulation of Bailey's lemma is provided by

Theorem 1.9 (Classical Bailey Lemma for A_1). *Let the sequences $\alpha = \{\alpha_n\}$ and $\beta = \{\beta_n\}$ form an A_1 Bailey pair. If $\alpha' = \{\alpha'_n\}$ and $\beta' = \{\beta'_n\}$ are defined by*

$$\alpha'_n := \frac{(\rho)_n(\sigma)_n}{(aq/\rho)_n(aq/\sigma)_n}\left(\frac{aq}{\rho\sigma}\right)^n \alpha_n \tag{1.10a}$$

and

$$\beta'_n := \sum_{0 \le y \le n} \frac{(\rho)_y(\sigma)_y(aq/\rho\sigma)_{n-y}}{(q)_{n-y}(aq/\rho)_n(aq/\sigma)_n}\left(\frac{aq}{\rho\sigma}\right)^y \beta_y, \tag{1.10b}$$

then α' and β' also form an A_1 Bailey pair.

If we begin with a Bailey pair (α_n, β_n), then the relationships in (1.10) give us another Bailey pair (α'_n, β'_n). If we then apply (1.10) to (α'_n, β'_n), we obtain yet another Bailey pair (α''_n, β''_n). Andrews observed that if this process is continued we obtain a sequence of Bailey pairs

$$(\alpha_n, \beta_n) \to (\alpha'_n, \beta'_n) \to (\alpha''_n, \beta''_n) \to \cdots. \tag{1.11}$$

He called (1.11) an ordinary Bailey chain. Furthermore, if (α'_n, β'_n) are given then

Andrews immediately determined α_n from (1.10a), and also solved for $\beta_0, \beta_1, \ldots, \beta_n$ from the diagonal system of equations (1.10b). Thus, he extended (1.11) to the left as well as to the right to give the bilateral Bailey chain

$$\cdots \to (\alpha_n^{(-2)}, \beta_n^{(-2)}) \to (\alpha_n^{(-1)}, \beta_n^{(-1)}) \to (\alpha_n, \beta_n) \to (\alpha_n^1, \beta_n^1) \to \cdots, \tag{1.12}$$

where each pair is related to the next through instances of (1.10a) and (1.10b).

The fact that each Bailey pair in (1.12) satisfies (1.6) has a number of well-known and important specializations. The most important occurs for

$$\alpha_n := \frac{(1 - aq^{2n})(a)_n(-1)^n q^{\binom{n}{2}}}{(1 - a)(q)_n} \tag{1.13a}$$

and

$$\beta_n := \delta(n, 0) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases} \tag{1.13b}$$

To obtain (1.13a), just substitute (1.13b) and (1.3b) into (1.8).

The Rogers–Ramanujan–Schur identities are a direct consequence of the second iteration of Theorem 1.9. That is, they follow from iterating the (1.13) case of (1.10) twice, with $\rho, \sigma \to \infty$ each time; substituting the resulting Bailey pair (α_n'', β_n'') into (1.6); letting $n \to \infty$; setting a equal to 1 or q; and finally applying the Jacobi-Triple-Product identity to the very well-poised right-hand side. Bailey's lemma reduces the proof of the Rogers–Ramanujan–Schur identities to the discovery and verification of the Bailey pair (1.13). Starting with $\beta_n = \delta(n, 0)$, these cases of (1.13) are also quickly derived from (1.6) and the q-binomial theorem [6], [10], [16]. General multiple series Rogers–Ramanujan–Schur identities [5], [6], [8], [36], [37], [41] are obtained in a similar fashion from the kth iteration of Theorem 1.9, with $\rho, \sigma \to \infty$, and the same cases of (1.13), or similar Bailey pairs (α_n, β_n).

The above Bailey chain derivation of the Rogers–Ramanujan–Schur identities is a special case of the observation in [6] that Watson's q-analog of Whipple's transformation follows immediately from the (1.13) case of the second iteration of Theorem 1.9 in which we take $\rho = b_i$ and $\sigma = c_i$ at the ith step. Furthermore, continued iteration of this same case of Theorem 1.9 yields Andrews' [2] infinite family of extensions of Watson's q-Whipple transformation. Even Whipple's original work [42], [43] fits into the $q = 1$ case of this analysis. Paule [36]–[38] independently discovered important special cases of Bailey's lemma and how those cases could be iterated. Essentially all of the depth of the Rogers–Ramanujan–Schur identities and their iterations is embedded in Bailey's lemma.

The process of iterating Bailey's lemma has led to a wide range of applications in additive number theory, combinatories, special functions, and mathematical physics. For example, see [5], [6]–[9], and [12].

The Bailey transform is a consequence of the terminating ${}_4\varphi_3$ summation theorem. The Bailey lemma is derived in [1] directly from the ${}_6\varphi_5$ summation and the matrix inversion formulation [4], [17], [18] of the Bailey transform. A similar method is employed in the A_ℓ and C_ℓ cases by starting with a suitable, higher-dimensional, terminating ${}_6\varphi_5$ summation theorem extracted from [26] and [19], respectively. The A_ℓ proofs appear in [30] and [32]. We establish the C_ℓ case in this paper. We have previously announced the A_ℓ and C_ℓ results in [35]. Many other consequences of the A_ℓ generalization of Bailey's transform and lemma have been found in [30]–[34]. These include A_ℓ q-Pfaff-Saalschütz summation theorems, q-Whipple transformations more symmetrical than the one in [27]–[29], connection coefficient results, and applications of iterating the A_ℓ Bailey lemma. Analogous C_ℓ results will appear in future works.

We organize our paper as follows. In Section 2 we first truncate, change parameters, and take limits in Gustafson's [19] C_ℓ generalization of the ${}_6\psi_6$ summation to derive a suitable C_ℓ nonterminating ${}_6\varphi_5$, and hence C_ℓ terminating very well-poised ${}_6\varphi_5$ and ${}_4\varphi_3$ summations. We conclude Section 2 by showing how the C_ℓ nonterminating ${}_6\varphi_5$ summation theorem is used to recover C. Krattenthaler's [22], [23] multivariable summation which he utilized in deriving his refinement of the Bender–Knuth and MacMahon generating functions for certain sets of plane partitions. Motivated by the A_ℓ calculation [30], the sum side of the C_ℓ terminating ${}_4\varphi_3$ is transformed termwise in Section 3 to yield the C_ℓ Bailey transform, interpreted as a matrix inversion result analogous to Theorem 1.2. The C_ℓ Bailey pair relationship is defined and then inverted in Section 4. From an arbitrary C_ℓ Bailey pair, it is shown how to construct another C_ℓ Bailey pair. This is the C_ℓ generalization of the classical Bailey lemma in Theorem 1.9. As in the classical case, the concept of the C_ℓ Bailey chain is introduced. Finally, in Section 5, we obtain a connection coefficient result for the general C_ℓ little q-Jacobi polynomials. First, the general C_ℓ little q-Jacobi polynomials are defined. This definition is in such full generality that the polynomials need not even be polynomials in $(x_1, \ldots, x_\ell)$ for the connection coefficient analysis to hold. In fact, a similar analysis should work for suitable C_ℓ Askey–Wilson polynomials. An elementary but somewhat intricate manipulation of summations together with the C_ℓ terminating ${}_4\varphi_3$ is used to prove the connection coefficient theorem.

2. Specializations of Gustafson's C_ℓ ${}_6\psi_6$

We begin with Gustafson's C_ℓ ${}_6\psi_6$ summation theorem from [19]. Specializations serve to terminate this summation theorem from below and then from above, yielding the C_ℓ ${}_6\varphi_5$ summation theorem and the C_ℓ terminating ${}_6\varphi_5$ summation theorem. The terminating ${}_6\varphi_5$ has two free parameters. When these are set equal to each other, the resulting summation theorem is a C_ℓ terminating ${}_4\varphi_3$.

The starting point is Gustafson's C_ℓ ${}_6\varphi_6$.

Theorem 2.1 (Gustafson) (The C_ℓ ${}_6\psi_6$ Summation Theorem). *Let $z_1, \ldots, z_\ell$ be indeterminate. Suppose no $z_r + z_s$ nor $z_r - z_s$ is integral. If*

$$|q^{-\ell}(b_1 b_2 \cdots b_{\ell+1}/a_1 a_2 \cdots a_{\ell+1})| < 1$$

and none of the denominators vanishes, then

$$\sum_{y_1, y_2, \ldots, y_\ell = -\infty}^{\infty} \left\{ q^{-\sum_{i=1}^{\ell} (\ell + 1) - i) y_i} \prod_{i=1}^{\ell} \left[\frac{1 - q^{2(z_i + y_i)}}{1 - q^{2z_i}} \right] \right.$$

$$\times \prod_{1 \le i < j \le \ell} \left[\frac{1 - q^{z_i + y_i - z_j - y_j}}{1 - q^{z_i - z_j}} \frac{1 - q^{z_i + y_i + z_j + y_j}}{1 - q^{z_i + z_j}} \right] \tag{2.2a}$$

$$\left. \times \prod_{i=1}^{\ell+1} \prod_{k=1}^{\ell} \left[\frac{(a_i q^{z_k})_{y_k} (a_i q^{-z_k})_{-y_k}}{(b_i q^{z_k})_{y_k} (b_i q^{-z_k})_{-y_k}} \right] \right\} \tag{2.2b}$$

$$= (q)_\infty^\ell \prod_{i,j=1}^{\ell+1} \left(\frac{b_i}{a_j} \right)_\infty \left(q^{-\ell} \frac{b_1 b_2 \cdots b_{\ell+1}}{a_1 a_2 \cdots a_{\ell+1}} \right)_\infty^{-1}$$

$$\times \prod_{1 \le i < j \le \ell+1} [(q a_i^{-1} a_j^{-1})_\infty (b_i b_j q^{-1})_\infty]$$

$$\times \prod_{1 \le i < j \le \ell} [(q^{1 + z_i + z_j})_\infty (q^{1 - z_i - z_j})_\infty (q^{1 + z_i - z_j})_\infty (q^{1 - z_i + z_j})_\infty]$$

$$\times \prod_{i=1}^{\ell+1} \prod_{k=1}^{\ell} [(b_i q^{z_k})_\infty (b_i q^{-z_k})_\infty (q a_i^{-1} q^{z_k})_\infty (q a_i^{-1} q^{-z_k})_\infty]^{-1}$$

$$\times \prod_{i=1}^{\ell} [(q^{1+2z_i})_\infty (q^{1-2z_i})_\infty]. \tag{2.2c}$$

As in the classical case, the ${}_6\psi_6$ is terminated from below to yield a ${}_6\varphi_5$.

Theorem 2.3 (The C_ℓ ${}_6\varphi_5$ Summation Theorem). *Let $z_1, \ldots, z_\ell$ be indeterminate. Suppose no $z_r + z_s$ nor $z_r - z_s$ is integral. If $|b/(a_1 a_2 \cdots a_\ell a)| < 1$ and none of the denominators vanishes, then*

$$\sum_{\substack{y_k \ge 0 \\ k = 1, 2, \ldots, \ell}} \left\{ q^{-\sum_{i=1}^{\ell} (\ell + 1 - i) y_i} \prod_{i=1}^{\ell} \left[\frac{1 - q^{2(z_i + y_i)}}{1 - q^{2z_i}} \right] \right.$$

$$\times \prod_{1 \le i < j \le \ell} \left[\frac{1 - q^{z_i + y_i - z_j - y_j}}{1 - q^{z_i - z_j}} \frac{1 - q^{z_i + y_i + z_j + y_j}}{1 - q^{z_i + z_j}} \right]$$

$$\times \prod_{i,k=1}^{\ell} \left[\frac{(a_i q^{z_k - z_i})_{y_k} (a_i q^{-z_k - z_i})_{-y_k}}{(q^{1 + z_k - z_i})_{y_k} (q^{1 - z_k - z_i})_{-y_k}} \right] \tag{2.4a}$$

$$\left. \times \prod_{k=1}^{\ell} \left[\frac{(a q^{z_k})_{y_k} (a q^{-z_k})_{-y_k}}{(b q^{z_k})_{y_k} (b q^{-z_k})_{-y_k}} \right] \right\} \tag{2.4b}$$

$$= (q)_\infty^\ell \prod_{i,j=1}^{\ell} (a_j^{-1} q^{1 + z_j - z_i})_\infty \left(\frac{1}{a_1 a_2 \cdots a_\ell} \frac{b}{a} \right)_\infty^{-1}$$

$$\times \prod_{i=1}^{\ell} (a^{-1} q^{1 - z_i})_\infty \prod_{j=1}^{\ell} \left(\frac{b}{a_j} q^{z_j} \right)_\infty \left(\frac{b}{a} \right)_\infty$$

$$\times \prod_{1\le i<j\le \ell} [(qa_i^{-1}a_j^{-1}q^{z_i+z_j})_\infty (q^{1-z_i-z_j})_\infty]$$

$$\times \prod_{i=1}^{\ell} [(q^{1+z_i}a_i^{-1}a^{-1})_\infty (bq^{-z_i})_\infty]$$

$$\times \prod_{1\le i<j\le \ell} [(q^{1+z_i+z_j})_\infty (q^{1-z_i-z_j})_\infty (q^{1+z_i-z_j})_\infty (q^{1-z_i+z_j})_\infty]$$

$$\times \prod_{i,k=1}^{\ell} [(q^{1+z_k-z_i})_\infty (q^{1-z_k-z_i})_\infty (qa_i^{-1}q^{z_k+z_i})_\infty (qa_i^{-1}q^{-z_k+z_i})_\infty]^{-1}$$

$$\times \prod_{k=1}^{\ell} [(bq^{z_k})_\infty (bq^{-z_k})_\infty (qa^{-1}q^{z_k})_\infty (qa^{-1}q^{-z_k})_\infty]^{-1}$$

$$\times \prod_{i=1}^{\ell} [(q^{1+2z_i})_\infty (q^{1-2z_i})_\infty]. \tag{2.4c}$$

Proof. Make the following substitutions in (2.2):

$$\begin{aligned} a_i &\mapsto a_i q^{-z_i} \quad \text{for} \quad i = 1, 2, \dots, \ell, \\ a_{\ell+1} &\mapsto a, \\ b_i &\mapsto b_i q^{-z_i} \quad \text{for} \quad i = 1, 2, \dots, \ell, \\ b_{\ell+1} &\mapsto b. \end{aligned} \tag{2.5}$$

Under the substitutions in (2.5), equation (2.2) becomes

$$\sum_{y_1, y_2, \dots, y_\ell = -\infty}^{\infty} \left\{ q^{-\sum_{i=1}^{\ell}(\ell+1-i)y_i} \prod_{i=1}^{\ell} \left[\frac{1-q^{2(z_i+y_i)}}{1-q^{2z_i}} \right] \right.$$

$$\times \prod_{1\le i<j\le \ell} \left[\frac{1-q^{z_i+y_i-z_j-y_j}}{1-q^{z_i-z_j}} \frac{1-q^{z_i+y_i+z_j+y_j}}{1-q^{z_i+z_j}} \right]$$

$$\times \prod_{i,k=1}^{\ell} \left[\frac{(a_i q^{z_k-z_i})_{y_k} (a_i q^{-z_k-z_i})_{-y_k}}{(b_i q^{z_k-z_i})_{y_k} (b_i q^{-z_k-z_i})_{-y_k}} \right] \tag{2.6a}$$

$$\left. \times \prod_{k=1}^{\ell} \left[\frac{(aq^{z_k})_{y_k} (aq^{-z_k})_{-y_k}}{(bq^{z_k})_{y_k} (bq^{-z_k})_{-y_k}} \right] \right\} \tag{2.6b}$$

$$= (q)_\infty^\ell \prod_{i,j=1}^{\ell} \left(\frac{b_i}{a_j} q^{z_j-z_i} \right)_\infty \left(q^{-\ell} \frac{b_1 b_2 \cdots b_\ell}{a_1 a_2 \cdots a_\ell} \frac{b}{a} \right)_\infty^{-1}$$

$$\times \prod_{i=1}^{\ell} \left(\frac{b_i}{a} q^{-z_i} \right)_\infty \prod_{j=1}^{\ell} \left(\frac{b}{a_j} q^{z_j} \right)_\infty \left(\frac{b}{a} \right)_\infty$$

$$\times \prod_{1\le i<j\le \ell} [(qa_i^{-1}a_j^{-1}q^{z_i+z_j})_\infty (b_i b_j q^{-1} q^{-z_i-z_j})_\infty]$$

$$\times \prod_{i=1}^{\ell} [(q^{1+z_i}a_i^{-1}a^{-1})_\infty (b_i b q^{-1-z_i})_\infty]$$

$$\times \prod_{1\le i<j\le \ell} [(q^{1+z_i+z_j})_\infty (q^{1-z_i-z_j})_\infty (q^{1+z_i-z_j})_\infty (q^{1-z_i+z_j})_\infty]$$

$$\times \prod_{i,k=1}^{\ell} [(b_i q^{z_k-z_i})_\infty (b_i q^{-z_k-z_i})_\infty (qa_i^{-1} q^{z_k+z_i})_\infty (qa_i^{-1} q^{-z_k+z_i})_\infty]^{-1}$$

$$\times \prod_{k=1}^{\ell} [(bq^{z_k})_\infty (bq^{-z_k})_\infty (qa^{-1}q^{z_k})_\infty (qa^{-1}q^{-z_k})_\infty]^{-1}$$

$$\times \prod_{i=1}^{\ell} [(q^{1+2z_i})_\infty (q^{1-2z_i})_\infty]. \tag{2.6c}$$

Now set $b_1 = b_2 = \cdots = b_\ell = q$ in (2.6). Under this substitution, the diagonal terms in (2.6a) will each contain a factor of the form $(q)_{y_k}^{-1}$. However, $(q)_{y_k}^{-1} = 0$ if $y_k < 0$ since $(q)_{y_k}^{-1} = (q^{1+y_k})_{-y_k}$, which contains the factor $(1 - q^{1+y_k}q^{-y_k-1})$. Therefore, the only nonzero terms occur when each $y_k \ge 0$ for $k = 1, 2, \ldots, \ell$. ■

An appropriate substitution permits us to terminate this sum from above.

Lemma 2.7 *Let $z_1, \ldots, z_\ell$ be indeterminate. Suppose no $z_r + z_s$ nor $z_r - z_s$ is integral. If N_i are nonnegative integers for $i = 1, 2, \ldots, \ell$ and none of the denominators vanishes, then*

$$\sum_{\substack{0\le y_k\le N_k\\ k=1,2,\ldots,\ell}} \left\{ q^{-\sum_{i=1}^{\ell}(\ell+1-i)y_i} \prod_{i=1}^{\ell} \left[\frac{1-q^{2(z_i+y_i)}}{1-q^{2z_i}}\right] \right.$$

$$\times \prod_{1\le i<j\le \ell} \left[\frac{1-q^{z_i+y_i-z_j-y_j}}{1-q^{z_i-z_j}} \frac{1-q^{z_i+y_i+z_j+y_j}}{1-q^{z_i+z_j}}\right]$$

$$\times \prod_{i,k=1}^{\ell} \left[\frac{(q^{z_k-z_i-N_i})_{y_k} (q^{-z_k-z_i-N_i})_{-y_k}}{(q^{1+z_k-z_i})_{y_k} (q^{1-z_k-z_i})_{-y_k}}\right] \tag{2.8a}$$

$$\left. \times \prod_{k=1}^{\ell} \left[\frac{(aq^{z_k})_{y_k} (aq^{-z_k})_{-y_k}}{(bq^{z_k})_{y_k} (bq^{-z_k})_{-y_k}}\right] \right\} \tag{2.8b}$$

$$= (q)_\infty^\ell \prod_{i,j=1}^{\ell} (q^{1+z_j-z_i+N_j})_\infty \left(q^{N_1+N_2+\cdots+N_\ell}\frac{b}{a}\right)_\infty^{-1}$$

$$\times \prod_{i=1}^{\ell} (a^{-1}q^{1-z_i})_\infty \prod_{j=1}^{\ell} (bq^{z_j+N_j})_\infty \left(\frac{b}{a}\right)_\infty$$

$$\times \prod_{1\le i<j\le \ell} [(q^{1+N_i+N_j}q^{z_i+z_j})_\infty (q^{1-z_i-z_j})_\infty]$$

$$\times \prod_{i=1}^{\ell} [(q^{1+z_i+N_i}a^{-1})_\infty (bq^{-z_i})_\infty]$$

$$\times \prod_{1\le i<j\le \ell} [(q^{1+z_i+z_j})_\infty (q^{1-z_i-z_j})_\infty (q^{1+z_i-z_j})_\infty (q^{1-z_i+z_j})_\infty]$$

$$\times \prod_{i,k=1}^{\ell} [(q^{1+z_k-z_i})_\infty (q^{1-z_k-z_i})_\infty (q^{1+N_i} q^{z_k+z_i})_\infty (q^{1+N_i} q^{-z_k+z_i})_\infty]^{-1}$$

$$\times \prod_{k=1}^{\ell} [(bq^{z_k})_\infty (bq^{-z_k})_\infty (qa^{-1}q^{z_k})_\infty (qa^{-1}q^{-z_k})_\infty]^{-1}$$

$$\times \prod_{i=1}^{\ell} [(q^{1+2z_i})_\infty (q^{1-2z_i})_\infty]. \tag{2.8c}$$

Proof. In (2.4) set $a_i = q^{-N_i}$ for $i = 1, 2, \ldots, \ell$, where each $N_i \geq 0$. The diagonal terms in (2.8a) each contain a factor of the form $(q^{-N_k})_{y_k}$. However, $(q^{-N_k})_{y_k} = 0$ if $y_k > N_k$ since it contains the factor $(1 - q^{-N_k} q^{N_k})$. Therefore, the only nonzero terms occur when each $y_k \leq N_k$ for $k = 1, 2, \ldots, \ell$. ■

Remark. Before we simplify (2.8), make the substitution $x_k = q^{z_k}$ for $k = 1, 2, \ldots, \ell$. This substitution will give the summand a form which is consistent with that in Milne [30]–[34]. Equation (2.8) becomes

$$\sum_{\substack{0 \leq y_k \leq N_k \\ k=1,2,\ldots,\ell}} \Bigg\{ q^{-\sum_{i=1}^{\ell}(\ell+1-i)y_i} \prod_{k=1}^{\ell} \left[\frac{1 - x_k^2 q^{2y_k}}{1 - x_k^2}\right]$$

$$\times \prod_{1 \leq r < s \leq \ell} \left[\frac{1 - (x_r/x_s) q^{y_r - y_s}}{1 - x_r/x_s} \frac{1 - x_r x_s q^{y_r + y_s}}{1 - x_r x_s}\right] \tag{2.9a}$$

$$\times \prod_{r,s=1}^{\ell} \left[\frac{((x_r/x_s) q^{-N_s})_{y_r} ((1/x_r x_s) q^{-N_s})_{-y_r}}{(q(x_r/x_s))_{y_r} (q(1/x_r x_s))_{-y_r}}\right] \tag{2.9b}$$

$$\times \prod_{k=1}^{\ell} \left[\frac{(ax_k)_{y_k} (a/x_k)_{-y_k}}{(bx_k)_{y_k} (b/x_k)_{-y_k}}\right]\Bigg\} \tag{2.9c}$$

$$= (q)_\infty^\ell \prod_{i,j=1}^{\ell} \left(q^{1+N_j} \frac{x_j}{x_i}\right)_\infty \left(q^{N_1+N_2+\cdots+N_\ell} \frac{b}{a}\right)_\infty^{-1}$$

$$\times \prod_{i=1}^{\ell} (a^{-1} q x_i^{-1})_\infty \prod_{j=1}^{\ell} (b x_j q^{N_j})_\infty \left(\frac{b}{a}\right)_\infty$$

$$\times \prod_{1 \leq i < j \leq \ell} \left[(q^{1+N_i+N_j} x_i x_j)_\infty \left(q \frac{1}{x_i x_j}\right)_\infty\right]$$

$$\times \prod_{i=1}^{\ell} [(x_i q^{1+N_i} a^{-1})_\infty (b x_i^{-1})_\infty]$$

$$\times \prod_{1 \leq i < j \leq \ell} \left[(q x_i x_j)_\infty \left(q \frac{1}{x_i x_j}\right)_\infty \left(q \frac{x_i}{x_j}\right)_\infty \left(q \frac{x_j}{x_i}\right)_\infty\right]$$

$$\times \prod_{i,j=1}^{\ell} \left[\left(q \frac{x_j}{x_i}\right)_\infty \left(q \frac{1}{x_i x_j}\right)_\infty (q^{1+N_i} x_i x_j)_\infty \left(q^{1+N_i} \frac{x_i}{x_j}\right)_\infty\right]^{-1}$$

$$\times \prod_{k=1}^{\ell} [(bx_k)_\infty (bx_k^{-1})_\infty (qa^{-1}x_k)_\infty (qa^{-1}x_k^{-1})_\infty]^{-1}$$

$$\times \prod_{i=1}^{\ell} [(qx_i^2)_\infty (qx_i^{-2})_\infty]. \tag{2.9d}$$

The substitutions that have been made thus far in Gustafson's $C_\ell\ {}_6\psi_6$ are

$$\begin{aligned} &a_i \mapsto a_i q^{-z_i} \mapsto q^{-N_i} q^{-z_i} \mapsto q^{-N_i} x_i^{-1} \qquad \text{for} \quad i = 1, 2, \ldots, \ell, \\ &a_{\ell+1} \mapsto a, \\ &b_i \mapsto b_i q^{-z_i} \mapsto q^{1-z_i} \mapsto qx_i^{-1} \qquad \text{for} \quad i = 1, 2, \ldots, \ell, \\ &b_{\ell+1} \mapsto b, \\ &q^{z_i} \mapsto x_i. \end{aligned} \tag{2.10}$$

Equation (2.9) motivates the following theorem:

Theorem 2.11 (C_ℓ Terminating ${}_6\varphi_5$ Summation Theorem). *Let $x_1, \ldots, x_\ell$, be indeterminate, let N_i be nonnegative integers for $i = 1, 2, \ldots, \ell$, and suppose that none of the denominators in* (2.12a) *vanishes. Then*

$$\sum_{\substack{0 \le y_k \le N_k \\ k=1,2,\ldots,\ell}} \left\{ \prod_{k=1}^{\ell} \left[\frac{1 - x_k^2 q^{2y_k}}{1 - x_k^2}\right] \prod_{1 \le r < s \le \ell} \left[\frac{1 - (x_r/x_s)q^{y_r - y_s}}{1 - (x_r/x_s)} \frac{1 - x_r x_s q^{y_r + y_s}}{1 - x_r x_s}\right] \right.$$

$$\times \prod_{r,s=1}^{\ell} \left[\frac{((x_r/x_s)q^{-N_s})_{y_r} (x_r x_s)_{y_r}}{(q(x_r/x_s))_{y_r} (qx_r x_s q^{N_s})_{y_r}}\right] \prod_{k=1}^{\ell} \left[\frac{(ax_k)_{y_k} (qx_k b^{-1})_{y_k}}{(bx_k)_{y_k} (qx_k a^{-1})_{y_k}}\right]$$

$$\left. \times\, q^{(N_1 + \cdots + N_\ell)(y_1 + \cdots + y_\ell)} q^{y_2 + 2y_3 + \cdots + (\ell-1)y_\ell} \left(\frac{b}{a}\right)^{(y_1 + \cdots + y_\ell)} \right\} \tag{2.12a}$$

$$= \prod_{k=1}^{\ell} \left[\frac{(qx_k^2)_{N_k}}{(bx_k)_{N_k} (qa^{-1}x_k)_{N_k}}\right] \prod_{1 \le r < s \le \ell} \left[\frac{(qx_r x_s)_{N_r}}{(qx_r x_s q^{N_s})_{N_r}}\right]$$

$$\times \left(\frac{b}{a}\right)_{(N_1 + \cdots + N_\ell)}. \tag{2.12b}$$

Proof. Use $(a)_n = (a)_\infty/(aq^n)_\infty$ to simplify (2.9d), the product side. Notice that in (2.9) all of the y_i's are nonnegative. Simplify (2.9) using the identity $(a)_{-n} = (-q/a)^n q^{\binom{n}{2}} (q/a)_n^{-1}$. The products in (2.9b) become

$$\prod_{r,s=1}^{\ell} \frac{((1/x_r x_s)q^{-N_s})_{-y_r}}{(q(1/x_r x_s))_{-y_r}} = \prod_{r,s=1}^{\ell} \left[q^{(1+N_s)y_r} \frac{(x_r x_s)_{y_r}}{(qx_r x_s q^{N_s})_{y_r}}\right]. \tag{2.13}$$

Those in (2.9c) become

$$\prod_{k=1}^{\ell} \frac{(ax_k^{-1})_{-y_k}}{(bx_k^{-1})_{-y_k}} = \prod_{k=1}^{\ell} \left[\frac{(qx_k b^{-1})_{y_k}}{(qx_k a^{-1})_{y_k}} \left(\frac{b}{a}\right)^{y_k}\right]. \tag{2.14}$$

We also use the following simplifications:

(2.15a) $$\prod_{r,s=1}^{\ell} q^{(1+N_s)y_r} = q^{(\ell+N_1+\cdots+N_\ell)(y_1+\cdots+y_\ell)},$$

(2.15b) $$\prod_{k=1}^{\ell}\left(\frac{b}{a}\right)^{y_k} = \left(\frac{b}{a}\right)^{(y_1+\cdots+y_\ell)},$$

(2.15c) $$q^{-\sum_{k=1}^{\ell}(\ell+1-k)y_k} = q^{-\ell(y_1+\cdots+y_\ell)}q^{y_2+2y_3+\cdots+(\ell-1)y_\ell}.$$

Combine (2.13)–(2.15) to simplify the sum side of (2.9). ■

Remark. The $\ell = 1$ case of (2.12) is the classical terminating ${}_6\varphi_5$ summation in equation (II.21) on p. 238 of [16] in which $a \mapsto x_1^2$, $n \mapsto N_1$, $b \mapsto ax_1$, and $c \mapsto qx_1b^{-1}$. That is, they are equivalent.

Remark. Just as in the classical case, the $b := aq^{1-(N_1+\cdots+N_\ell)}$ case of Theorem 2.11 yields a C_ℓ generalization of Bressoud's matrix inversion formula in [15]. We give the details of this derivation elsewhere. Setting $b := aq^{1-(N_1+\cdots+N_\ell)}$ and then taking $a \to 0$ or $a \to \infty$ gives another way of obtaining Theorem 2.16 from Theorem 2.11.

A specialization of Theorem 2.11 yields the following result:

Theorem 2.16 (C_ℓ Terminating ${}_4\varphi_3$ Summation Theorem). *Let $x_1, \ldots, x_\ell$ be indeterminate and let $N_1, N_2, \ldots, N_\ell$ be nonnegative integers. If no x_r/x_s nor x_rx_s is an integral power of q, then*

$$\sum_{\substack{0\le y_k\le N_k\\ k=1,2,\ldots,\ell}} \Bigg\{ \prod_{k=1}^{\ell}\left[\frac{1-x_k^2q^{2y_k}}{1-x_k^2}\right] \times \prod_{1\le r<s\le \ell}\left[\frac{1-(x_r/x_s)q^{y_r-y_s}}{1-x_r/x_s}\,\frac{1-x_rx_sq^{y_r+y_s}}{1-x_rx_s}\right] \times \prod_{r,s=1}^{\ell}\left[\frac{((x_r/x_s)q^{-N_s})_{y_r}(x_rx_s)_{y_r}}{(q(x_r/x_s))_{y_r}(qx_rx_sq^{N_s})_{y_r}}\right]$$

(2.17a) $$\times\, q^{(N_1+\cdots+N_\ell)(y_1+\cdots+y_\ell)}q^{y_2+2y_3+\cdots+(\ell-1)y_\ell}\Bigg\}$$

(2.17b) $$= \begin{cases} 1 & \text{if } N_1 = N_2 = \cdots = N_\ell = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Set $a = b$ in Theorem (2.11). Notice that $(1)_n = 0$ if $n > 0$. Also note that each $N_k \ge 0$. Therefore, the product side vanishes unless $N_1 = N_2 = \cdots = N_\ell = 0$. ■

In the rest of Section 2 we show how the C_ℓ nonterminating ${}_6\varphi_5$ summation theorem is used to recover C. Krattenhaler's [22], [23] multivariable summation.

We begin with

Theorem 2.18 (C_ℓ Nonterminating ${}_6\varphi_5$ Summation Theorem). *Let $a, b, a_1, \ldots, a_\ell$, and $x_1, \ldots, x_\ell$ be indeterminate, with $\ell \geq 1$. Take $0 < |q| < 1$ and $|b/a_1 a_2 \cdots a_\ell a| < 1$. Suppose that none of the denominators in* (2.19) *vanishes. Then*

$$\sum_{\substack{y_k \geq 0 \\ k=1,2,\ldots,\ell}} \left\{ \prod_{k=1}^{\ell} \left[\frac{1 - x_k^2 q^{2y_k}}{1 - x_k^2}\right] \prod_{1 \leq r < s \leq \ell} \left[\frac{1 - (x_r/x_s) q^{y_r - y_s}}{1 - x_r/x_s} \frac{1 - x_r x_s q^{y_r + y_s}}{1 - x_r x_s}\right]\right.$$

$$\times \prod_{r,s=1}^{\ell} \left[\frac{((x_r/x_s) a_s)_{y_r} (x_r x_s)_{y_r}}{(q(x_r/x_s))_{y_r} (q x_r x_s a_s^{-1})_{y_r}}\right] \prod_{k=1}^{\ell} \left[\frac{(a x_k)_{y_k} (q x_k b^{-1})_{y_k}}{(b x_k)_{y_k} (q x_k a^{-1})_{y_k}}\right]$$

$$\text{(2.19a)} \qquad \left. \times (a_1 \cdots a_\ell)^{-(y_1 + \cdots + y_\ell)} q^{y_2 + 2y_3 + \cdots + (\ell - 1) y_\ell} \left(\frac{b}{a}\right)^{(y_1 + \cdots + y_\ell)} \right\}$$

$$= \prod_{k=1}^{\ell} \left[\frac{(q x_k^2)_\infty (b x_k a_k^{-1})_\infty (q a^{-1} x_k a_k^{-1})_\infty}{(q x_k^2 a_k^{-1})_\infty (b x_k)_\infty (q a^{-1} x_k)_\infty}\right]$$

$$\times \prod_{1 \leq r < s \leq \ell} \left[\frac{(q x_r x_s)_\infty (q x_r x_s a_r^{-1} a_s^{-1})_\infty}{(q x_r x_s a_s^{-1})_\infty (q x_r x_s a_r^{-1})_\infty}\right]$$

$$\text{(2.19b)} \qquad \times \left(\frac{b}{a}\right)_\infty \left(\frac{1}{a_1 a_2 \cdots a_\ell} \frac{b}{a}\right)_\infty^{-1}.$$

Proof. Set $x_k = q^{z_k}$ for $k = 1, 2, \ldots, \ell$ in Theorem 2.3, apply the relation

$$\text{(2.20)} \qquad (A)_{-m} = (-A)^{-m} q^{m(m+1)/2} (q A^{-1})_m^{-1}$$

to suitable factors in the sum side of (2.4), and then simplify as in the proof of Theorem 2.11. ■

Remark. Setting $a_s = q^{-N_s}$ for $s = 1, 2, \ldots, \ell$ in Theorem 2.18 immediately gives Theorem 2.11. Furthermore, the $\ell = 1$ case of (2.19) is the classical nonterminating ${}_6\varphi_5$ summation in equation (II.20) on p. 238 of [16] in which $a \mapsto x_1^2$, $b \mapsto a x_1$, $c \mapsto q x_1 b^{-1}$, and $d \mapsto a_1$. That is, they are equivalent.

It is not hard to see that taking $a_s = q$ for $s = 1, 2, \ldots, \ell$ in Theorem 2.18 yields.

Corollary 2.21. *Let a, b, and $x_1, \ldots, x_\ell$ be indeterminate, with $\ell \geq 1$. Take $0 < |q| < 1$ and $|b/a| < |q|^\ell$. Suppose that none of the denominators in* (2.22) *vanishes. Then*

$$\sum_{\substack{y_k \geq 0 \\ k=1,2,\ldots,\ell}} \left\{ \prod_{k=1}^{\ell} \left(\frac{b}{q^k a}\right)^{y_k} \prod_{k=1}^{\ell} \left[\frac{(a x_k)_{y_k} (q b^{-1} x_k)_{y_k}}{(q a^{-1} x_k)_{y_k} (b x_k)_{y_k}}\right]\right.$$

$$\text{(2.22a)} \qquad \left. \times \prod_{1 \leq r < s \leq \ell} \left[\frac{1 - (x_s/x_r) q^{y_s - y_r}}{1 - (x_s/x_r)}\right] \prod_{1 \leq r < s \leq \ell} \left[\frac{1 - x_r x_s q^{y_r + y_s}}{1 - x_r x_s}\right]\right\}$$

$$\text{(2.22b)} \qquad = \prod_{1 \leq r < s \leq \ell} \left[\frac{1 - x_r x_s / q}{1 - x_r x_s}\right] \prod_{k=1}^{\ell} \left[\frac{(1 - b q^{-1} x_k)(1 - a^{-1} x_k)}{(1 - x_k^2)(1 - b a^{-1} q^{-k})}\right].$$

If we set $b = \sqrt{q}$ in (2.22) we obtain

Corollary 2.23 *Let a and $x_1, \ldots, x_\ell$ be indeterminate, with $\ell \geq 1$. Take $0 < |q| < 1$ and $1 < |aq^{\ell - 1/2}|$. Suppose that none of the denominators in* (2.24) *vanishes. Then*

$$\sum_{\substack{y_k \geq 0 \\ k=1,2,\ldots,\ell}} \left\{ \prod_{k=1}^{\ell} \left(\frac{\sqrt{q}}{q^k a}\right)^{y_k} \prod_{k=1}^{\ell} \left[\frac{(ax_k)_{y_k}}{(qa^{-1}x_k)_{y_k}}\right] \right.$$

$$\text{(2.24a)} \qquad \left. \times \prod_{1 \leq r < s \leq \ell} \left[\frac{1 - (x_s/x_r)q^{y_s - y_r}}{1 - x_s/x_r}\right] \prod_{1 \leq r \leq s \leq \ell} \left[\frac{1 - x_r x_s q^{y_r + y_s}}{1 - x_r x_s}\right] \right\}$$

$$\text{(2.24b)} \qquad = \prod_{1 \leq r < s \leq \ell} \left[\frac{1 - x_r x_s/q}{1 - x_r x_s}\right] \prod_{k=1}^{\ell} \left[\frac{(1 - (1/\sqrt{q})x_k)(1 - a^{-1}x_k)}{(1 - x_k^2)(1 - \sqrt{q}/aq^k)}\right].$$

Remark. We terminate (2.24) by setting

$$\text{(2.25)} \qquad ax_k = q^{-N_k} \qquad k = 1, 2, \ldots, \ell,$$

where the N_k are distinct nonnegative integers. Note that if (2.25) holds, then $x_s/x_r = q^{N_r - N_s} \neq 1$.

Relabeling the (2.25) case of Corollary 2.23 finally gives

Corollary 2.26 (C. Krattenthaler). *Let A and $m_1, \ldots, m_r$ be indeterminate, with $r \geq 1$. Suppose that none of the denominators in* (2.27) *vanishes. Then the summation formula*

$$\text{(2.27)} \qquad \sum_{k_1, \ldots, k_r \geq 0} \prod_{i=1}^{r} \left(\frac{\sqrt{q}}{q^i A}\right)^{k_i} \prod_{i=1}^{r} \frac{(m_i A)_{k_i}}{(qm_i/A)_{k_i}}$$

$$\times \prod_{1 \leq i < j \leq r} \frac{1 - (m_j/m_i)q^{k_j - k_i}}{1 - m_j/m_i} \prod_{1 \leq i \leq j \leq r} \frac{1 - m_i m_j q^{k_i + k_j}}{1 - m_i m_j}$$

$$= \prod_{1 \leq i < j \leq r} \frac{1 - m_i m_j/q}{1 - m_i m_j} \prod_{i=1}^{r} \frac{(1 - m_i/\sqrt{q})(1 - m_i/A)}{(1 - m_i^2)(1 - \sqrt{q}/q^i A)}$$

holds, provided that there exist nonnegative integers n_i with $n_1 > n_2 > \cdots > n_r$ such that $m_i A = q^{-n_i}$ for all $i = 1, 2, \ldots, r$.

Remark. Krattenhaler decided to formulate Corollary 2.26 in terms of the m_i's rather than the n_i's in order to keep the notation as short as possible. However, the reader should never forget that the m_i's in fact disguise the n_i's via $m_i = q^{-n_i}/A$.

Proof. Just utilize (2.25) to terminate (2.24), and then take $\ell \mapsto r$, $a \mapsto A$, $x_i \mapsto m_i$, and $N_i \mapsto n_i$ for $i = 1, 2, \ldots, r$. ∎

Krattenthaler had to prove Corollary 2.26 in [22] where he derived a refinement of the Bender–Knuth and MacMahon generating functions of certain sets of plane partitions. Corollary 2.26 is applied in the proof of Theorem 18 of [22], which in turn is crucial for obtaining Theorem 21. Corollary 2.26 is also used implicitly in the proof of Theorem 19. In this analysis, Corollary 2.26 is useful in the evaluation of certain determinants in closed form. It is possible that the additional parameter b in Corollary 2.21 will allow the evaluation in closed form of even more general determinants from [22].

Krattenthaler [22] found an impressive, complicated inductive proof of Corollary 2.26. He communicated Corollary 2.26 to us in [23], where he asked if it was a special case of a more general A_ℓ or C_ℓ summation or transformation formula. It turned out to be the above consequence of Theorem 2.18.

3. The Derivation of the C_ℓ Bailey Transform

In Theorem 2.16 make the substitution $x_k \mapsto x_k q^{j_k}$ for $k = 1, 2, \ldots, \ell$. The point of this substitution is to afford some room for manipulation of the terms. The sum side of the ${}_4\varphi_3$ is then modified in a manner motivated by the $U(n+1)$ calculation.

The modified ${}_4\varphi_3$ is then transformed under a number of simplifications to yield the C_ℓ generalization of Bailey's transform. It is vitally important to subsequent applications of this result that the transformation from the modified ${}_4\varphi_3$ to the C_ℓ Bailey transform be a termwise calculation. The transform is then reinterpreted as a matrix inversion result of two infinite, lower-triangular matrices. We begin by multiplying each side of the

$$x_k \mapsto x_k q^{j_k}, \qquad N_k \mapsto i_k - j_k$$

case of Theorem 2.16 by the product

$$\prod_{r,s=1}^{\ell} \left[\left(q \frac{x_r}{x_s} q^{j_r - j_s} \right)_{i_r - j_r}^{-1} (q x_r x_s q^{j_r + j_s})_{i_r - j_r}^{-1} \right]. \tag{3.1}$$

This choice of factors is motivated by the $U(n+1)$ case. Notice that the product side of (2.17) remains unchanged. This gives us the starting point for the derivation of the C_ℓ Bailey transform. We begin with

$$\sum_{\substack{0 \le y_k \le i_k - j_k \\ k = 1, 2, \ldots, \ell}} \left\{ \prod_{1 \le r < s \le \ell} \left[\frac{1 - (x_r/x_s) q^{y_r - y_s} q^{j_r - j_s}}{1 - (x_r/x_s) q^{j_r - j_s}} \frac{1 - x_r x_s q^{y_r + y_s} q^{j_r + j_s}}{1 - x_r x_s q^{j_r + j_s}} \right] \right.$$
$$\times \prod_{k=1}^{\ell} \left[\frac{1 - x_k^2 q^{2(y_k + j_k)}}{1 - x_k^2 q^{2 j_k}} \right]$$
$$\times \prod_{r,s=1}^{\ell} \left[\frac{((x_r/x_s) q^{-(i_s - j_s)} q^{j_r - j_s})_{y_r} (x_r x_s q^{j_r + j_s})_{y_r}}{(q (x_r/x_s) q^{j_r - j_s})_{y_r} (q x_r x_s q^{i_s - j_s} q^{j_r + j_s})_{y_r}} \right]$$
$$\left. \times\, q^{((i_1 + \cdots + i_\ell) - (j_1 + \cdots + j_\ell))(y_1 + \cdots + y_\ell)} q^{y_2 + 2y_3 + \cdots + (\ell - 1) y_\ell} \right\}$$
$$\times \prod_{r,s=1}^{\ell} \left[\left(q \frac{x_r}{x_s} q^{j_r - j_s} \right)_{i_r - j_r}^{-1} (q x_r x_s q^{j_r + j_s})_{i_r - j_r}^{-1} \right] \tag{3.2a}$$

$$= \prod_{k=1}^{\ell} \delta(i_k, j_k). \tag{3.2b}$$

The object is to separate the factors in (3.2a) into a function of $\mathbf{j} := \{j_1, \ldots, j_\ell\}$ and $\mathbf{y} := \{y_1, \ldots, y_\ell\}$ times a function of $\mathbf{i} := \{i_1, \ldots, i_\ell\}$ and $\mathbf{j} + \mathbf{y}$. Once the index of summation is shifted to $j_k \le y_k \le i_k$ for $k = 1, 2, \ldots, \ell$, the summand will then become the product of a function which is independent of $\mathbf{i}$ times a function that is independent of $\mathbf{j}$.

A term-by-term simplification of the sum follows three technical lemmas. The first two lemmas are exactly those which are used in the $U(n+1)$ calculation.

Lemma 3.3 (Milne). *Let $x_1, \ldots, x_\ell$ be indeterminate. Suppose that no x_r/x_s is an integral power of q. Then*

$$(3.4)\qquad \prod_{r,s=1}^{\ell} \left(q\frac{x_r}{x_s}\right)_{y_r - y_s}^{-1} = \left\{ \prod_{1 \le r < s \le \ell} \left[\frac{1 - x_r/x_s}{1 - (x_r/x_s)q^{y_r - y_s}}\right] (-1)^{-(\ell-1)(y_1 + \cdots + y_\ell)} \right.$$
$$\times \prod_{r,s=1}^{\ell} \left(\frac{x_r}{x_s}\right)^{-y_r} q^{\sigma_2(\mathbf{y})} q^{-(y_2 + 2y_3 + \cdots + (\ell-1)y_\ell)}$$
$$\left. \times\, q^{-(\ell-1)\left[\binom{y_1}{2} + \binom{y_2}{2} + \cdots + \binom{y_\ell}{2}\right]} \right\},$$

where $\sigma_2(\mathbf{y})$ is the second elementary symmetric function of $\mathbf{y} := \{y_1, \ldots, y_\ell\}$.

Proof. The first step in the proof of equation (3.4) is the $m = y_r - y_s$, $A = x_r/x_s$ case of

$$(qA)_m (qA^{-1})_{-m} = (-A)^m q^{\binom{m}{2}} \left[\frac{1 - Aq^m}{1 - A}\right].$$

The rest follows by elementary manipulation. ■

Lemma 3.5 (Milne). *Let $x_1, \ldots, x_\ell$ be indeterminate. Suppose that no x_r/x_s is an integral power of q. Then*

$$(3.6a)\qquad \frac{((x_s/x_r)q^{-i_r + j_s} q^{y_s - y_r})_{y_r}}{(q(x_r/x_s)q^{j_r - j_s} q^{y_r - y_s})_{i_r - j_r}}$$

$$(3.6b)\qquad = \frac{((x_s/x_r)q^{j_s - i_r})_{y_s} (q(x_r/x_s)q^{j_r - j_s})_{y_r - y_s}}{((x_s/x_r)q^{j_s - i_r})_{y_s - y_r} (q(x_r/x_s)q^{i_r - j_s})_{y_r - y_s} (q(x_r/x_s)q^{j_r - j_s})_{i_r - j_r}}.$$

Proof. Rewrite (3.6a) using the identity $(A)_m = (A)_\infty/(Aq^m)_\infty$. Then multiply and divide the result by

$$\left(\frac{x_s}{x_r} q^{j_s - i_r}\right)_\infty \left(q\frac{x_r}{x_s} q^{j_r - j_s}\right)_\infty \left(q\frac{x_r}{x_s} q^{i_r - j_s}\right)_\infty.$$

Rearrange the terms, and rewrite as (3.6b). ■

The following lemma is analogous to Lemma 3.5. However, instead of x_r/x_s, the parameters appear as products.

Lemma 3.7. *Let $x_1, \ldots, x_\ell$ be indeterminate. Suppose that no $x_r x_s$ is an integral power of q. Then*

$$\frac{((1/x_r x_s)q^{-(i_r-j_r)}q^{-y_s-y_r}q^{-j_r-j_s})_{y_r}}{(qx_r x_s q^{j_r+j_s}q^{y_r+y_s})_{i_r-j_r}} \tag{3.8a}$$

$$= \frac{((1/x_r x_s)q^{-j_r-j_s}q^{-(i_r-j_r)})_{-y_s}(qx_r x_s q^{j_r+j_s})_{y_r+y_s}}{((1/x_r x_s)q^{-j_r-j_s}q^{-(i_r-j_r)})_{-y_r-y_s}(qx_r x_s q^{j_r+j_s}q^{y_r+y_s})_{y_r+y_s}(qx_r x_s q^{j_r+j_s})_{i_r-j_r}}. \tag{3.8b}$$

Proof. As in Lemma 3.5, make use of the identity $(A)_m = (A)_\infty/(Aq^m)_\infty$. Multiply and divide (3.8a) by

$$\left(\frac{1}{x_r x_s}\, q^{-i_r-j_s}\right)_\infty (qx_r x_s q^{j_r+j_s})_\infty (qx_r x_s q^{i_r+j_s})_\infty.$$

Rearrange the terms, and rewrite as (3.8b). ■

We now use Lemmas 3.3, 3.5, and 3.7 along with standard facts about q-rising factorials to perform a rather lengthy series of calculations on the general term of (3.2a).

Lemma 3.9. *Let $x_1, \ldots, x_\ell$ be indeterminate. Suppose that no x_r/x_s nor $x_r x_s$ is an integral power of q. Then*

$$\begin{aligned}
&\Bigg\{\prod_{r,s=1}^{\ell}\left(q\,\frac{x_r}{x_s}\,q^{j_r-j_s}\right)^{-1}_{i_r-j_r}\prod_{1\le r<s\le\ell}\left[\frac{1-(x_r/x_s)q^{j_r-j_s}q^{y_r-y_s}}{1-(x_r/x_s)q^{j_r-j_s}}\right] \\
&\quad\times\prod_{r,s=1}^{\ell}\frac{((x_r/x_s)q^{j_r-j_s}q^{-(i_s-j_s)})_{y_r}}{(q(x_r/x_s)q^{j_r-j_s})_{y_r}} \\
&\quad\times q^{y_2+2y_3+\cdots+(\ell-1)y_\ell}q^{[(i_1+\cdots+i_\ell)-(j_1+\cdots+j_\ell)](y_1+\cdots+y_\ell)} \\
&\quad\times\prod_{r,s=1}^{\ell}(qx_r x_s q^{j_r+j_s})^{-1}_{i_r-j_r}\prod_{1\le r\le s\le\ell}\left[\frac{1-x_r x_s q^{j_r+j_s}q^{y_r+y_s}}{1-x_r x_s q^{j_r+j_s}}\right] \\
&\quad\times\prod_{r,s=1}^{\ell}\frac{(x_r x_s q^{j_r+j_s})_{y_r}}{(qx_r x_s q^{j_r+j_s}q^{i_s-j_s})_{y_r}}\Bigg\}
\end{aligned} \tag{3.10a}$$

$$\begin{aligned}
&=\Bigg\{\prod_{r,s=1}^{\ell}\left[\left(q\,\frac{x_r}{x_s}\,q^{j_r-j_s}q^{y_r-y_s}\right)^{-1}_{i_r-j_r-y_r}(qx_r x_s q^{j_r+j_s}q^{y_r+y_s})^{-1}_{i_r-j_r-y_r}\right] \\
&\quad\times(-1)^{(y_1+\cdots+y_\ell)}q^{\binom{y_1+\cdots+y_\ell}{2}}\prod_{1\le r<s\le\ell}\left[\frac{1-x_r x_s q^{j_r+j_s}}{1-x_r x_s q^{j_r+j_s}q^{y_r+y_s}}\right] \\
&\quad\times\prod_{r,s=1}^{\ell}\left[\left(q\,\frac{x_r}{x_s}\,q^{j_r-j_s}\right)^{-1}_{y_r}(x_r x_s q^{j_r+j_s}q^{y_s})^{-1}_{y_r}\right]\Bigg\}.
\end{aligned} \tag{3.10b}$$

Remark. It may be noticed that throughout the following calculation the terms involving x_r/x_s are segregated from those involving $x_r x_s$. What is actually proved is that

$$\prod_{r,s=1}^{\ell}\left(q\frac{x_r}{x_s}q^{j_r-j_s}\right)^{-1}_{i_r-j_r}\prod_{1\le r<s\le\ell}\left[\frac{1-(x_r/x_s)q^{j_r-j_s}q^{y_r-y_s}}{1-(x_r/x_s)q^{j_r-j_s}}\right]$$

$$\times\prod_{r,s=1}^{\ell}\frac{((x_r/x_s)q^{j_r-j_s}q^{-(i_s-j_s)})_{y_r}}{(q(x_r/x_s)q^{j_r-j_s})_{y_r}}\,q^{y_2+2y_3+\cdots+(\ell-1)y_\ell}$$

(3.11a) $$\times\, q^{[(i_1+\cdots+i_\ell)-(j_1+\cdots+j_\ell)](y_1+\cdots+y_\ell)}$$

$$=(-1)^{(y_1+\cdots+y_\ell)}q^{\binom{y_1+\cdots+y_\ell}{2}}$$

(3.11b) $$\times\prod_{r,s=1}^{\ell}\left[\left(q\frac{x_r}{x_s}q^{j_r-j_s}\right)^{-1}_{y_r}\left(q\frac{x_r}{x_s}q^{j_r-j_s}q^{y_r-y_s}\right)^{-1}_{i_r-j_r-y_r}\right],$$

and

$$\prod_{r,s=1}^{\ell}(qx_rx_sq^{j_r+j_s})^{-1}_{i_r-j_r}\prod_{1\le r\le s\le\ell}\left[\frac{1-x_rx_sq^{j_r+j_s}q^{y_r+y_s}}{1-x_rx_sq^{j_r+j_s}}\right]$$

(3.12a) $$\times\prod_{r,s=1}^{\ell}\frac{(x_rx_sq^{j_r+j_s})_{y_r}}{(qx_rx_sq^{j_r+j_s}q^{i_s-j_s})_{y_r}}$$

$$=\prod_{1\le r<s\le\ell}\left[\frac{1-x_rx_sq^{j_r+j_s}}{1-x_rx_sq^{j_r+j_s}q^{y_r+y_s}}\right]$$

(3.12b) $$\times\prod_{r,s=1}^{\ell}\left[(x_rx_sq^{j_r+j_s}q^{y_s})^{-1}_{y_r}(qx_rx_sq^{j_r+j_s}q^{y_r+y_s})^{-1}_{i_r-j_r-y_r}\right].$$

It may also be noticed that the proof of (3.11) actually contains a derivation proof of the bulk of the A_ℓ case.

Proof. We use Lemma 3.3 with x_k replaced by $x_kq^{j_k}$ to transform (3.10a). Then apply

$$\left(\frac{q}{A}\right)_m=(A)^{-1}_{-m}(-A)^{-m}q^{\binom{m+1}{2}}$$

with $A=(1/x_rx_s)q^{-j_r-j_s}$ and $m=y_r+y_s$ to that result. Some additional algebraic simplification together with the $A=x_rx_sq^{j_r+j_s}$ and $m=y_r+y_s$ case of

$$\left[\frac{1-Aq^m}{1-A}\right](qA^{-1})^{-1}_{-m}=(qA)_m(-A)^{-m}q^{-\binom{m}{2}}$$

allows us to transform that result. Then we switch r and s in $(qx_rx_sq^{j_r+j_s}q^{i_s-j_s})^{-1}_{y_r}$. We may do this because the product is taken over the square $1\le r,s\le\ell$. Also, apply

$$(-A)^mq^{-\binom{m+1}{2}}=(A)^{-1}_{-m}\left(\frac{q}{A}\right)^{-1}_m$$

with $A = (x_r/x_s)q^{j_s - i_r}$ and $m = y_r - y_s$. To this result, apply two more transformations. First, apply

$$\left(\frac{q}{A}\right)_m^{-1} = (A)_{-m}(-A)^m q^{-\binom{m+1}{2}}$$

with $A = (1/x_r x_s)q^{-j_r - j_s}q^{-(i_r - j_r)}$ and $m = y_s$. Also apply

$$q^{-\binom{n+1}{2}}q^{Nn}(-B)^n = (B)_{N-n}^{-1}(B)_N\left(\frac{q^{1-N}}{B}\right)_n^{-1}$$

with $B = q(x_r/x_s)q^{j_r - j_s}q^{y_r - y_s}$, $N = i_r - j_r$, and $n = y_r$. Again, we switch r and s. This time in $((x_r/x_s)q^{j_r - j_s}q^{-(i_s - j_s)})_{y_r}$. Lemma 3.5 is applied to these factors. The simplification

$$(-A)^m q^{-\binom{m+1}{2}} = (A)_{-m}^{-1}\left(\frac{q}{A}\right)_m^{-1}$$

with $A = (1/x_r x_s)q^{-j_r - j_s}q^{-(i_r - j_r)}$ and $m = y_r + y_s$ is also applied. The x_r/x_s terms are now in the desired form. They will remain unchanged. Lemma 3.7 is used to transform the $x_r x_s$ terms. We then apply the $A = qx_r x_s q^{j_r + j_s}q^{y_r + y_s}$, $N = i_r - j_r$, and $n = y_r$ case of

$$q^{Nn}(-A)^n q^{-\binom{n+1}{2}}(A)_N^{-1}\left(\frac{q^{1-N}}{A}\right)_n = (A)_{N-n}^{-1}.$$

The result is transformed using some simple algebra and the symmetry of the double product. An application of

$$(A)_n(A)_{N+n}^{-1} = (Aq^n)_N^{-1}$$

with $A = x_r x_s q^{j_r + j_s}$, $N = y_r$, and $n = y_s$ gives us the terms of (3.10b). A rearrangement of these terms yields the desired result. The explicit calculations described in this proof are contained in equations (3.39a)–(3.39m) of [24]. ■

Lemma 3.9 concerns the general term of the modified $C_\ell\ {}_4\varphi_3$. It is crucial to the work in Sections 4 and 5 that the calculations be valid termwise. We may, though, attempt to sum both sides of (3.10). The result is the C_ℓ Bailey transform.

Theorem 3.13 (The C_ℓ Bailey Transform). *Let $x_1, \ldots, x_\ell$ be indeterminate. Suppose that no x_r/x_s nor $x_r x_s$ is an integral power of q. Then*

$$\sum_{\substack{j_k \le y_k \le i_k \\ k=1,2,\ldots,\ell}} \left\{ \prod_{r,s=1}^{\ell} \left[\left(q\frac{x_r}{x_s}q^{y_r - y_s}\right)_{i_r - y_r}^{-1} (qx_r x_s q^{y_r + y_s})_{i_r - y_r}^{-1} \right] \right.$$
$$\times (-1)^{(y_1 + \cdots + y_\ell) - (j_1 + \cdots + j_\ell)} q^{\binom{(y_1 + \cdots + y_\ell) - (j_1 + \cdots + j_\ell)}{2}}$$
$$\times \prod_{1 \le r < s \le \ell} \left[\frac{1 - x_r x_s q^{j_r + j_s}}{1 - x_r x_s q^{y_r + y_s}} \right]$$

$$\text{(3.14a)} \qquad \times \prod_{r,s=1}^{\ell} \left[\left(q \frac{x_r}{x_s} q^{j_r - j_s} \right)_{y_r - j_r}^{-1} (x_r x_s q^{j_r + y_s})_{y_r - j_r}^{-1} \right] \Bigg\}$$

$$\text{(3.14b)} \qquad = \prod_{k=1}^{\ell} \delta(i_k, j_k).$$

Proof. Shift the index of summation from $j_k \le y_k \le i_k$ to $0 \le y_k \le i_k - j_k$. Apply Lemma 3.9 to the general term. Equation (3.2) then gives us the desired result. ■

Let us define the matrix $M = \{M_{(\mathbf{i},\mathbf{y})}\}$ as follows:

Definition 3.15 (The C_ℓ Matrix M).

$$\text{(3.16)} \qquad M_{(\mathbf{i};\mathbf{y};C_\ell)} \equiv M_{(\mathbf{i};\mathbf{y})} := \prod_{r,s=1}^{\ell} \left[\left(q \frac{x_r}{x_s} q^{y_r - y_s} \right)_{i_r - y_r}^{-1} (q x_r x_s q^{y_r + y_s})_{i_r - y_r}^{-1} \right].$$

Also define the matrix $M^* = \{M^*_{(\mathbf{y};\mathbf{j})}\}$.

Definition 3.17 (The C_ℓ Matrix M^*).

$$\text{(3.18)} \qquad M^*_{(\mathbf{y};\mathbf{j};C_\ell)} \equiv M^*_{(\mathbf{y};\mathbf{j})} := (-1)^{(y_1 + \cdots + y_\ell) - (j_1 + \cdots + j_\ell)} q^{\binom{(y_1 + \cdots + y_\ell) - (j_1 + \cdots + j_\ell)}{2}}$$
$$\times \prod_{1 \le r < s \le \ell} \left[\frac{1 - x_r x_s q^{j_r + j_s}}{1 - x_r x_s q^{y_r + y_s}} \right]$$
$$\times \prod_{r,s=1}^{\ell} \left[\left(q \frac{x_r}{x_s} q^{j_r - j_s} \right)_{y_r - j_r}^{-1} (x_r x_s q^{j_r + y_s})_{y_r - j_r}^{-1} \right].$$

The matrices M and M^* allow us to rewrite Theorem 3.13 in a way that will be very useful in subsequent calculations.

Theorem 3.19 (The C_ℓ Bailey Transform Matrices). *Let M and M^* be as defined in* (3.16) *and* (3.18). *If the entries of M and M^* are ordered lexicographically, then M and M^* are infinite, lower-triangular matrices which are also inverses of each other.*

Proof. First, order the entries lexicographically. In M suppose $i_1 = y_1, i_2 = y_2, \ldots,$ and $i_{k-1} = y_{k-1}$, but $i_k < y_k$. Consider the $r = s = k$ term in the product. It contains the factor $(q)_{i_k - y_k}^{-1}$, which equals zero since $i_k - y_k < 0$. In M^* suppose $y_1 = j_1, y_2 = j_2, \ldots,$ and $y_{k-1} = j_{k-1}$, but $y_k < j_k$. Again, consider the $r = s = k$ term in the product. It contains the factor $(q)_{y_k - j_k}^{-1}$, which equals zero since $y_k - j_k < 0$. Therefore, M and M^* are lower-triangular matrices under lexicographic ordering. They are inverses of each other because, by Theorem 3.13, the only nonzero entries in the product occur when $\mathbf{i} = \mathbf{j}$, and each of those entries equals one. ■

4. The C_ℓ Generalization of Bailey's Lemma

The first consequence of the C_ℓ Bailey transform is the C_ℓ generalization of Bailey's lemma. The concept of the C_ℓ Bailey pair is introduced. As a consequence of Lemma 3.9, the relationship between the elements of the C_ℓ Bailey pair is inverted. The matrix notation introduced in Theorem 3.19 is exploited to motivate the choice of a new C_ℓ Bailey pair. The proof of the suitability of the new pair relies heavily upon the termwise nature of the calculations leading to Lemma 3.9. Once we are able to create a new C_ℓ Bailey pair from an existing pair, the concept of the C_ℓ Bailey chain is introduced. As in the classical case, this definition is then extended to that of the bilateral C_ℓ Bailey chain.

We begin with the definition of a C_ℓ Bailey pair.

Definition 4.1 (The C_ℓ Bailey Pair). Let $A = \{A_{\mathrm{y}}\}$ and $B = \{B_{\mathrm{y}}\}$ be arbitrary sequences. A and B form a C_ℓ Bailey pair if and only if for every $N_i \geq 0$, $i = 1, 2, \ldots, \ell$,

$$B_N = \sum_{\substack{0 \leq y_k \leq N_k \\ k=1,2,\ldots,\ell}} \left\{ \prod_{r,s=1}^{\ell} \left[\left(q \frac{x_r}{x_s} q^{y_r - y_s} \right)_{N_r - y_r}^{-1} (q x_r x_s q^{y_r + y_s})_{N_r - y_r}^{-1} \right] A_{\mathrm{y}} \right\}. \tag{4.2}$$

As a consequence of the matrix inversion result, Theorem 3.19, we can invert the relationship in the definition of the C_ℓ Bailey pair. We have the following theorem.

Theorem 4.3 (C_ℓ Bailey Pair Inversion). *Let $A = \{A_{\mathrm{y}}\}$ and $B = \{B_{\mathrm{y}}\}$ form a C_ℓ Bailey pair. Then* (4.2) *holds if and only if the following also holds:*

$$A_N = \sum_{\substack{0 \leq y_k \leq N_k \\ k=1,2,\ldots,\ell}} \left\{ (-1)^{(N_1+\cdots+N_\ell)-(y_1+\cdots+y_\ell)} q^{\binom{(N_1+\cdots+N_\ell)-(y_1+\cdots+y_\ell)}{2}} \right.$$
$$\times \prod_{r,s=1}^{\ell} \left[\left(q \frac{x_r}{x_s} q^{y_r - y_s} \right)_{N_r - y_r}^{-1} (x_r x_s q^{y_r + N_s})_{N_r - y_r}^{-1} \right]$$
$$\left. \times \prod_{1 \leq r < s \leq \ell} \left[\frac{1 - x_r x_s q^{y_r + y_s}}{1 - x_r x_s q^{N_r + N_s}} \right] B_{\mathrm{y}} \right\}. \tag{4.4}$$

Proof. Substitute the definition of B_{y} as given in (4.2) into (4.4). Lemma 3.9 reduces the resulting inner sum to a product of delta functions. The only remaining nonzero term on the right-hand side is A_N. The converse follows from substituting (4.4) into (4.2) and again applying Lemma 3.9. ■

Let A and B form a C_ℓ Bailey pair. With $M = \{M_{(N;\,\mathrm{y})}\}$ and with $M^* =$

$\{M^*_{(N;\,\mathbf{y})}\}$ defined as in Theorem 3.19, (4.2) of Definition 4.1 may be written as

$$B_N = \sum_{\substack{0 \le y_k \le N_k \\ k=1,2,\dots,\ell}} M_{(N;\mathbf{y})} A_{\mathbf{y}}. \tag{4.5a}$$

We may also write (4.4) of Theorem 4.3 as

$$A_N = \sum_{\substack{0 \le y_k \le N_k \\ k=1,2,\dots,\ell}} M^*_{(N;\mathbf{y})} B_{\mathbf{y}}. \tag{4.5b}$$

Consider the sequence $A' = \{A'_N\}$ defined by

$$A'_N := C_N A_N, \tag{4.6}$$

where the sequence $C = \{C_{\mathbf{y}}\}$ is as of yet unchosen. We want to find a sequence $B' = \{B'_{\mathbf{y}}\}$ so that for every $N_i \ge 0$, $i = 1, 2, \dots, \ell$,

$$B'_N = \sum_{\substack{0 \le y_k \le N_k \\ k=1,2,\dots,\ell}} M_{(N;\mathbf{y})} A'_{\mathbf{y}}. \tag{4.7}$$

Assume that (4.5)–(4.7) hold. Then

$$B'_N = \sum_{\substack{0 \le y_k \le N_k \\ k=1,2,\dots,\ell}} \{M_{(N;\mathbf{y})} C_{\mathbf{y}} A_{\mathbf{y}}\} \tag{4.8a}$$

$$= \sum_{\substack{0 \le y_k \le N_k \\ k=1,2,\dots,\ell}} \left\{ M_{(N;\mathbf{y})} C_{\mathbf{y}} \sum_{\substack{0 \le m_i \le y_i \\ i=1,2,\dots,\ell}} [M^*_{(\mathbf{y};\mathbf{m})} B_{\mathbf{m}}] \right\} \tag{4.8b}$$

$$= \sum_{\substack{0 \le m_k \le N_k \\ k=1,2,\dots,\ell}} \left\{ B_m \sum_{\substack{m_i \le y_i \le N_i \\ i=1,2,\dots,\ell}} [M_{(N;\mathbf{y})} M^*_{(\mathbf{y};\mathbf{m})} C_{\mathbf{y}}] \right\} \tag{4.8c}$$

$$= \sum_{\substack{0 \le m_k \le N_k \\ k=1,2,\dots,\ell}} \left\{ B_{\mathbf{m}} \sum_{\substack{0 \le y_i \le N_i - m_i \\ i=1,2,\dots,\ell}} [M_{(N;\mathbf{y}+\mathbf{m})} M^*_{(\mathbf{y}+\mathbf{m};\mathbf{m})} C_{\mathbf{y}+\mathbf{m}}] \right\}. \tag{4.8d}$$

We want to choose $C = \{C_{\mathbf{y}}\}$ so that each $C_{\mathbf{y}+\mathbf{m}}$ can be factored into a function that is independent of $\mathbf{y}$ times a function of $\mathbf{m}$ and $\mathbf{y}$. The expression that is independent of $\mathbf{y}$ will then be pulled outside the sum. We also desire that the remaining terms combine with those in the inner sum of (4.8d) to form an easily summable expression. In effect, C allows us to pass from a $C_\ell\ {}_4\varphi_3$ to a $C_\ell\ {}_6\varphi_5$ which is summable by Theorem 2.11. Let us take

$$C_{\mathbf{m}} := \prod_{k=1}^{\ell} \left[\frac{(\alpha x_k)_{m_k} (q x_k \beta^{-1})_{m_k}}{(\beta x_k)_{m_k} (q x_k \alpha^{-1})_{m_k}} \right] \left(\frac{\beta}{\alpha} \right)^{(m_1 + \cdots + m_\ell)}. \tag{4.9}$$

Substituting the definitions of M, M^*, and C into the inner sum of (4.8d) gives us

$$\text{(4.10a)}\quad \prod_{r,s=1}^{\ell}\left[\left(q\frac{x_r}{x_s}q^{m_r-m_s}\right)_{N_r-m_r}^{-1}(qx_rx_sq^{m_r+m_s})_{N_r-y_r}^{-1}\right]$$

$$\text{(4.10b)}\quad \times\prod_{k=1}^{\ell}\left[\frac{(\alpha x_k)_{m_k}(qx_k\beta^{-1})_{m_k}}{(\beta x_k)_{m_k}(qx_k\alpha^{-1})_{m_k}}\right]\left(\frac{\beta}{\alpha}\right)^{(m_1+\cdots+m_\ell)}$$

$$\times\sum_{\substack{0\le y_k\le N_k-m_k\\ k=1,2,\ldots,\ell}}\left\{\prod_{1\le r\le s\le\ell}\left[\frac{1-x_rx_sq^{m_r+m_s}q^{y_r+y_s}}{1-x_rx_sq^{m_r+m_s}}\right]\right.$$

$$\times\prod_{r,s=1}^{\ell}\left[\frac{(x_rx_sq^{m_r+m_s})_{y_r}}{(qx_rx_sq^{m_r+m_s}q^{N_s-m_s})_{y_r}}\right]$$

$$\times\prod_{1\le r<s\le\ell}\left[\frac{1-(x_r/x_s)q^{m_r-m_s}q^{y_r-y_s}}{1-(x_r/x_s)q^{m_r-m_s}}\right]$$

$$\times\prod_{r,s=1}^{\ell}\left[\frac{((x_r/x_s)q^{m_r-m_s}q^{-(N_s-m_s)})_{y_r}}{(q(x_r/x_s)q^{m_r-m_s})_{y_r}}\right]$$

$$\times q^{y_2+2y_3+\cdots+(\ell-1)y_\ell}q^{[(N_1+\cdots+N_\ell)-(m_1+\cdots+m_\ell)](y_1+\cdots+y_\ell)}$$

$$\text{(4.10c)}\quad \left.\times\prod_{k=1}^{\ell}\left[\frac{(\alpha x_kq^{m_k})_{y_k}(qx_k\beta^{-1}q^{m_k})_{y_k}}{(\beta x_kq^{m_k})_{y_k}(qx_k\alpha^{-1}q^{m_k})_{y_k}}\right]\left(\frac{\beta}{\alpha}\right)^{(y_1+\cdots+y_\ell)}\right\}.$$

Utilizing Theorem 2.11 to sum (4.10c) and then simplifying, we find that (4.10) becomes

$$\text{(4.11)}\quad \left(\frac{\beta}{\alpha}\right)_{(N_1+\cdots+N_\ell)-(m_1+\cdots+m_\ell)}$$

$$\times\prod_{k=1}^{\ell}\left[\frac{(\alpha x_k)_{m_k}(qx_k\beta^{-1})_{k_k}}{(\beta x_k)_{N_k}(qx_k\alpha^{-1})_{N_k}}\right]\left(\frac{\beta}{\alpha}\right)^{(m_1+\cdots+m_\ell)}$$

$$\times\prod_{1\le r<s\le\ell}[(qx_rx_sq^{m_r+m_s})_{N_s-m_s}^{-1}(qx_rx_sq^{N_s-m_s})_{N_r-m_r}^{-1}]$$

$$\times\prod_{r,s=1}^{\ell}\left(q\frac{x_r}{x_s}q^{m_r-m_s}\right)_{N_r-m_r}^{-1}.$$

We now have the C_ℓ generalization of Bailey's lemma.

Theorem 4.12 (The C_ℓ Generalization of Bailey's Lemma). *Let $A=\{A_{\mathbf{y}}\}$ and $B=\{B_{\mathbf{y}}\}$ be sequences that satisfy*

$$\text{(4.13)}\quad B_N=\sum_{\substack{0\le y_k\le N_k\\ k=1,2,\ldots,\ell}}\left\{\prod_{r,s=1}^{\ell}\left[\left(q\frac{x_r}{x_s}q^{y_r-y_s}\right)_{N_r-y_r}^{-1}(qx_rx_sq^{y_r+y_s})_{N_r-y_r}^{-1}\right]A_{\mathbf{y}}\right\}$$

for every $N_i \geq 0$, $i = 1, 2, \ldots, \ell$. (*A and B form a* C_ℓ *Bailey pair.*) *If we define*

$$(4.14)\qquad A'_N := \left\{ \prod_{k=1}^{\ell} \left[\frac{(\alpha x_k)_{N_k} (q x_k \beta^{-1})_{N_k}}{(\beta x_k)_{N_k} (q x_k \alpha^{-1})_{N_k}} \right] \left(\frac{\beta}{\alpha} \right)^{(N_1 + \cdots + N_\ell)} A_N \right\},$$

and if we also define

$$(4.15)\quad B'_N := \sum_{\substack{0 \leq m_k \leq N_k \\ k = 1, 2, \ldots, \ell}} \left\{ \left(\frac{\beta}{\alpha} \right)_{(N_1 + \cdots + N_\ell) - (m_1 + \cdots + m_\ell)} \times \prod_{k=1}^{\ell} \left[\frac{(\alpha x_k)_{m_k} (q x_k \beta^{-1})_{m_k}}{(\beta x_k)_{N_k} (q x_k \alpha^{-1})_{N_k}} \right] \left(\frac{\beta}{\alpha} \right)^{(m_1 + \cdots + m_\ell)} \times \prod_{1 \leq r < s \leq \ell} \left[(q x_r x_s q^{m_r + m_s})^{-1}_{N_s - m_s} (q x_r x_s q^{N_s - m_s})^{-1}_{N_r - m_r} \right] \times \prod_{r,s=1}^{\ell} \left(q \frac{x_r}{x_s} q^{m_r - m_s} \right)^{-1}_{N_r - m_r} B_{\mathbf{m}} \right\},$$

then $A' = \{A'_{\mathbf{y}}\}$ *and* $B' = \{B'_{\mathbf{y}}\}$ *also satisfy* (4.13), *that is they also form a* C_ℓ *Bailey pair.*

Proof. In the definition of B'_N, rewrite the product

$$\prod_{1 \leq r < s \leq \ell} \left[(q x_r x_s q^{m_r + m_s})_{N_s - m_s} (q x_r x_s q^{N_s - m_s})_{N_r - m_r} \right]^{-1}$$

so that the

$$N_k \mapsto N_k - m_k, \qquad x_k \mapsto x_k q^{m_k}, \qquad b \mapsto \beta, \qquad a \mapsto \alpha$$

case of the product side of the C_ℓ ${}_6\varphi_5$ summation theorem is obtained. Replace these products by the sum side of the C_ℓ ${}_6\varphi_5$. At this point, the terms of the inner sum are the product of the extra terms which were multiplied with the C_ℓ ${}_4\varphi_3$, (3.1), times $C_{\mathbf{m}}$ and the above case of the sum side of the C_ℓ ${}_6\varphi_5$. Rearrange these factors, after pulling all of the factors inside the inner sum, to yield the modified C_ℓ ${}_4\varphi_3$ from (3.2) times $C_{\mathbf{m}+\mathbf{y}}$. Use the termwise nature of Lemma 3.9 to rewrite this inner sum as $M_{(N;\mathbf{y}+\mathbf{m})} M^*_{(\mathbf{y}+\mathbf{m};\mathbf{m})} C_{\mathbf{y}+\mathbf{m}}$. Then use the calculations in (4.8) along with the definition of $A'_{\mathbf{y}}$ to obtain the desired result. ∎

Corollary 4.16. *With* $A' = \{A'_{\mathbf{y}}\}$ *and* $B' = \{B'_{\mathbf{y}}\}$ *defined as in Theorem* 4.12, A' *and* B' *satisfy* (4.4).

Notice that we may apply the C_ℓ Bailey lemma to the new C_ℓ Bailey pair A' and B'. Call the resulting C_ℓ Bailey pair (A'', B''). We may continue applying the C_ℓ Bailey lemma and create a sequence of C_ℓ Bailey pairs:

$$(A, B) \to (A', B') \to (A'', B'') \to \cdots.$$

We call this sequence the "C_ℓ Bailey chain." This definition is motivated by Andrews [6].

We may also move from (A', B') back to (A, B). Given a C_ℓ Bailey pair (A', B'), we may determine A from (4.14) and then B from (4.2). Thus, we can move from right to left in the C_ℓ Bailey chain. This gives us the "bilateral C_ℓ Bailey chain":

$$\cdots \to (A^{(-2)}, B^{(-2)}) \to (A^{(-1)}, B^{(-1)}) \to (A, B) \to (A', B') \to (A'', B'') \to \cdots.$$

In the classical case, special cases of the bilateral Bailey chain give the Rogers–Ramanujan–Schur identities and also Watson's q-analog of Whipple's transformation. Similar results should be obtainable in the C_ℓ case.

5. A Connection Coefficient Result

As a direct consequence of the C_ℓ Bailey pair inversion in Theorem 4.3, we may obtain a connection coefficient result for the general C_ℓ little q-Jacobi polynomials. This connection coefficient result is done in full generality—it is not even necessary to define the C_ℓ little q-Jacobi polynomials fully. This observation was made in the $U(n+1)$ case by Milne in [33]. Gessel and Stanton [17, Section 8] have observed that the same argument as used for the classical little q-Jacobi polynomials works for the Askey–Wilson polynomials. In fact, the one-variable case of (5.7) below is even more general: for appropriate choices of the $\Theta_{\mathbf{m}}$ we arrive at the Askey–Wilson polynomials or little q-Jacobi polynomials. The same situation should hold for the C_ℓ case.

The definition of the general C_ℓ little q-Jacobi polynomials is motivated by the matrix M^* from the C_ℓ Bailey transform. The same change of summation lemma that was used in the $U(n+1)$ case is used in the C_ℓ case. An appropriate choice of factors in the definition of the C_ℓ little q-Jacobi polynomials allows us to use this summation lemma together with the C_ℓ terminating ${}_4\varphi_3$ to reduce the triple multiple sum to a single multiple sum. The result is the connection coefficient theorem for the general C_ℓ little q-Jacobi polynomials.

We begin with a change of summation lemma.

Lemma 5.1. *Let* $\mathbf{j} = (j_1, j_2, \ldots, j_\ell)$, $\mathbf{k} = (k_1, k_2, \ldots, k_\ell)$, $\mathbf{m} = (m_1, m_2, \ldots, m_\ell)$, *and* $\mathbf{n} = (n_1, n_1, \ldots, n_\ell)$. *Then*

$$\text{(5.2a)} \qquad \sum_{\substack{0 \le k_i \le n_i \\ i=1,2,\ldots,\ell}} \; \sum_{\substack{k_i \le j_i \le n_i \\ i=1,2,\ldots,\ell}} \; \sum_{\substack{0 \le m_i \le k_i \\ i=1,2,\ldots,\ell}} F(\mathbf{j}, \mathbf{k}, \mathbf{m}, \mathbf{n})$$

$$\text{(5.2b)} \qquad = \sum_{\substack{0 \le m_i \le n_i \\ i=1,2,\ldots,\ell}} \; \sum_{\substack{m_i \le j_i \le n_i \\ i=1,2,\ldots,\ell}} \; \sum_{\substack{m_i \le k_i \le j_i \\ i=1,2,\ldots,\ell}} F(\mathbf{j}, \mathbf{k}, \mathbf{m}, \mathbf{n}).$$

Proof. We repeatedly apply the one-dimensional result

$$\text{(5.3)} \qquad \sum_{0 \le \alpha \le \beta} \; \sum_{0 \le \gamma \le \alpha} f(\alpha, \beta, \gamma) = \sum_{0 \le \gamma \le \beta} \; \sum_{\gamma \le \alpha \le \beta} f(\alpha, \beta, \gamma).$$

At each stage, apply (5.3) to a pair of multiple sums for each $i = 1, 2, \ldots, \ell$. Apply (5.3) to the first two multiple sums in (5.2a). Then, apply (5.3) to the inner two multiple sums. Finally, apply (5.3) to the first two multiple sums. The result is (5.2b). ■

When we apply Lemma 5.1, we will want the innermost sum to be easily summable. Moreover, if it is the C_ℓ terminating ${}_4\varphi_3$ the two inner sums will collapse to a single term. It is this observation that motivates the definition of the C_ℓ little q-Jacobi polynomials. We begin with a technical lemma.

Lemma 5.4. *Let $x_1, x_2, \ldots, x_\ell$ be indeterminate. Suppose that no $x_r x_s$ nor x_r/x_s is an integral power of q. Then*

$$\left\{\prod_{1\le r<s\le \ell}\left[\frac{1-x_rx_sq^{m_r+m_s}}{1-x_rx_s}\,\frac{1-(x_r/x_s)q^{m_r-m_s}}{1-x_r/x_s}\right]\right.$$
$$\times\prod_{r,s=1}^{\ell}\left[\left(q^{-k_s}\frac{x_r}{x_s}\right)_{m_r}(x_rx_sq^{k_s})_{m_r}\right]$$
$$\left.\times\, q^{(m_1+\cdots+m_\ell)}q^{m_2+2m_3+\cdots+(\ell-1)m_\ell}\right\} \tag{5.5a}$$
$$=\left\{(-1)^{(k_1+\cdots+k_\ell)-(m_1+\cdots+m_\ell)}q^{\binom{(k_1+\cdots+k_\ell)}{2}-\binom{(m_1+\cdots+m_\ell)}{2}}\right.$$
$$\times\prod_{1\le r<s\le \ell}\left[\frac{1-x_rx_sq^{m_r+m_s}}{1-x_rx_sq^{k_r+k_s}}\right]$$
$$\left.\times\prod_{r,s=1}^{\ell}\left[\left(q\frac{x_r}{x_s}q^{m_r-m_s}\right)^{-1}_{k_r-m_r}(x_rx_sq^{m_r+k_s})^{-1}_{k_r-m_r}\right]\right\} \tag{5.5b}$$
$$\times\left\{(-1)^{-(k_1+\cdots+k_\ell)}q^{-\binom{(k_1+\cdots+k_\ell)}{2}}\right.$$
$$\left.\times\prod_{1\le r<s\le \ell}\left[\frac{1-x_rx_sq^{k_r+k_s}}{1-x_rx_s}\right]\prod_{r,s=1}^{\ell}\left[\left(q\frac{x_r}{x_s}\right)_{k_r}(x_rx_sq^{k_s})_{k_r}\right]\right\}. \tag{5.5c}$$

Proof. First, notice that (5.5b) is $M^*_{(\mathbf{k};\mathbf{m})}$ where the matrix M^* is as defined in (3.18). Observe that

$$\left(q\frac{x_r}{x_s}q^{m_r-m_s}\right)^{-1}_{k_r-m_r}\left(q\frac{x_r}{x_s}\right)_{k_r}=\left(q\frac{x_r}{x_s}\right)_{m_r-m_s}\left(q\frac{x_r}{x_s}q^{k_r-m_s}\right)_{m_s}.$$

Apply Lemma 3.3 to $(q(x_r/x_s))_{m_r-m_s}$. Then apply $(A)_m=(-1)^m A^m q^{\binom{m}{2}}(A^{-1}q^{1-m})_m$ to the factor $(q(x_r/x_s)q^{k_r-m_s})_{m_s}$. Notice that $(x_rx_sq^{k_s})_{k_r}(x_rx_sq^{m_r+k_s})^{-1}_{k_r-m_r}=(x_rx_sq^{k_s})_{m_r}$. Combine the resulting terms with the resulting factors in (5.5b) and (5.5c). The desired result follows after some elementary manipulation. ■

Lemmas 5.1 and 5.4 motivate the definition of the general C_ℓ little q-Jacobi polynomials.

Definition 5.6 (The Generalized C_ℓ Little q-Jacobi Polynomials). Let $\mathbf{x}:=\{x_1, x_2, \ldots, x_\ell\}$ be indeterminate. Suppose that no x_rx_s nor x_r/x_s is an integral power of q. Let $\Theta=\{\Theta_{\mathbf{k}}\}$ be arbitrary. Define the general C_ℓ little q-Jacobi polynomials, $P_{\mathbf{k}}(\mathbf{x};\Theta;C_\ell)$, by

$$(5.7)\quad P_{\mathbf{k}}(\mathbf{x};\Theta;C_\ell):=\sum_{\substack{0\le m_i\le k_i\\ i=1,2,\dots,\ell}}\left\{\prod_{1\le r<s\le\ell}\left[\frac{1-x_rx_sq^{m_r+m_s}}{1-x_rx_s}\,\frac{1-(x_r/x_s)q^{m_r-m_s}}{1-x_r/x_s}\right]\right.$$
$$\times\prod_{r,s=1}^{\ell}\left[\left(q^{-k_s}\frac{x_r}{x_s}\right)_{m_r}(x_rx_sq^{k_s})_{m_r}\right]$$
$$\left.\times\, q^{(m_1+\cdots+m_\ell)}q^{m_2+2m_3+\cdots+(\ell-1)m_\ell}\Theta_{\mathbf{m}}\right\}.$$

This definition, together with the preceding lemmas, yields the following theorem.

Theorem 5.8 (Connection Coefficient). *With $P_{\mathbf{k}}(\mathbf{x};\Theta;C_\ell)$ defined as in* (5.7) *and with $D=\{D_{\mathbf{j}}\}$ arbitrary,*

$$(5.9a)\quad \sum_{\substack{0\le m_i\le n_i\\ i=1,2,\dots,\ell}}\Theta_{\mathbf{m}}D_{\mathbf{m}}$$
$$=\sum_{\substack{0\le k_i\le n_i\\ i=1,2,\dots,\ell}}\ \sum_{\substack{k_i\le j_i\le n_i\\ i=1,2,\dots,\ell}}\left\{D_{\mathbf{j}}(-1)^{(k_1+\cdots+k_\ell)}q^{\binom{(k_1+\cdots+k_\ell)}{2}}\right.$$
$$(5.9b)\quad \times\prod_{1\le r<s\le\ell}\left[\frac{1-x_rx_s}{1-x_rx_sq^{k_r+k_s}}\right]$$
$$\times\prod_{r,s=1}^{\ell}\left[\left(q\frac{x_r}{x_s}\right)_{k_r}^{-1}(x_rx_sq^{k_s})_{k_r}^{-1}\right]$$
$$(5.9c)\quad \times\prod_{r,s=1}^{\ell}\left[\left(q\frac{x_r}{x_s}q^{k_r-k_s}\right)_{j_r-k_r}^{-1}(qx_rx_sq^{k_r+k_s})_{j_r-k_r}^{-1}\right]$$
$$(5.9d)\quad \left.\times\, P_{\mathbf{k}}(\mathbf{x};\Theta;C_\ell)\right\}.$$

Proof. Notice that (5.9b) is $D_{\mathbf{j}}M^*_{(\mathbf{k};0)}$. Also notice that (5.9c) is $M_{(\mathbf{j};\mathbf{k})}$. Substitute the definition of $P_{\mathbf{k}}(\mathbf{x};\Theta;C_\ell)$ into (5.9). The innermost terms are combined to yield $M_{(\mathbf{j};\mathbf{k})}M^*_{(\mathbf{k};\mathbf{m})}$. This step is the motivating step in the definition of $P_{\mathbf{k}}(\mathbf{x};\Theta;C_\ell)$. Apply the change of summation in Lemma 5.1 to the resulting sum. Rewrite the terms using Lemma 5.4. The innermost sum can be summed by Theorem 2.16 to yield $\delta(\mathbf{j},\mathbf{m})$. The double multiple sum then collapses to a simple multiple sum, which is the desired result. ∎

Specific definitions for the C_ℓ generalization of the little q-Jacobi polynomials can be motivated by C_ℓ generalizations of q-Saalschütz. C_ℓ generalizations of q-Saalschütz may be obtained from sequences of C_ℓ q-Whipple transformations. These, in turn, may be obtained from repeated applications of the C_ℓ generalization of Bailey's lemma. This program is the C_ℓ analogue of the $U(n+1)$ program, carried out by Milne in [30]–[34].

Acknowledgments. S. C. Milne was partially supported by NSF Grants DMS 86-04232, DMS 89-04455, and DMS 90-96254. G. M. Lilly was fully supported by NSA supplements to the above NSF grants and by NSA Grant MDA 904-88-H-2010.

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