Integral Equations
and Operator Theory
Vol. 9 (1986)

0378-620X/86/040588-04$1.50+0.20/0

THE KERNEL OF A TOEPLITZ OPERATOR

Eric Hayashi

Let T_u be a Toeplitz operator on H^2 of the unit disk. If the kernel of T_u is non-trivial, it equals $g(H^2 \ominus zbH^2)$ where g is an outer function and where b is inner. Moreover, $f \to gf$ is an isometry of $H^2 \ominus zbH^2$ onto Ker T_u.

Let u be a non-trivial essentially bounded measurable function on the unit circle $\mathbf{T}$, and let T_u denote the Toeplitz operator on H^2 of the unit disk defined by $T_u f = Puf$ where P denotes orthogonal projection onto H^2. If Ker $T_u \neq (0)$, it can be shown that there is an outer function h in H^2 such that Ker T_u = Ker $T_{\bar{h}/h}$ (see [2]). It is easily checked that

$$\text{Ker } T_{\bar{h}/h} = H^2 \cap (\bar{z}h/\bar{h})\bar{H}^2 .$$

It was shown in [2] that Ker $T_{\bar{h}/h}$ contains $L^2 \cap g(H^2 \ominus zbH^2)$ as a dense subspace for a naturally arising outer function g and inner function b. It will be shown that much more is true, but first some notation is needed. For each non-negative integer k, Let $M_k = (z^k H^2) \cap (h/\bar{h})\bar{H}^2$ and $M_k' = \bar{z}^k M_k$. If $M_k \neq (0)$, let g_k be a unit vector in $M_k' \ominus M_{k+1}'$. Then the following was shown in [2]:

PROPOSITION 1. *For $M_n \neq (0)$, there is an inner function b_n such that the following conditions hold:*

(i) $h/\bar{h} = z^n b_n g_n / \bar{g}_n$.

(ii) $b_n = (b_{n-1} - b_{n-1}(0))/z(1 - \bar{b}_{n-1}(0) b_{n-1})$ $(n \geq 1)$.

(iii) $g_n = r_n g_0 \prod_{k=0}^{n-1} (1 - \bar{b}_k(0) b_k)$ *for some scalar* r_n.

(iv) $b_n = 1$ *if and only if* $\dim M_n = 1$.

(v) $z^n g_n$ *spans* $M_n \ominus M_{n+1}$ *and* $b_n g_n$ *spans* $M_n' \ominus M_{n+1}'$.

Remark.

The spaces M_n arise naturally in the study of stochastic processes (see [1]). It is also useful to think of g_n as the unique element in the unit ball of M_n' which maximizes the functional $f \to \operatorname{Re} f(0)$. From this, it was shown in [2] that g_n^2 is an exposed point of the unit ball of H^1.

Now, from (b_n), construct a sequence (G_n) of functions as follows:

Let $G_1 = 1$, and for $n \geq 2$, let

$$G_n = \prod_{k=1}^{n-1} (1 - \bar{b}_k(0)b_k)/(1-|b_k(0)|^2)^{\frac{1}{2}} .$$

PROPOSITION 2. *If n 1 and $M_n \neq (0)$, then $g_n = G_n g_1$.*

Proof. A moment's thought reveals that $g_n = g_n(0)^{-1}P_n 1$ where P_n denotes orthognal projection onto M_n'. Also, by Proposition 1, it is evident that

$$\text{(1)} \qquad g_n = g_n(0)g_1(0)^{-1}[\prod_{k=1}^{n-1} (1-\bar{b}_k(0)b_k)/(1-|b_k(0)|^2)^{\frac{1}{2}}]g_1.$$

Thus,

$$\begin{aligned} g_n &= g_n(0)^{-1}P_n 1 \\ &= g_n(0)^{-1} \sum_{k\geq n} \langle 1,b_k g_k\rangle b_k g_k \\ &= g_n(0)^{-1} \sum_{k\geq n} \bar{b}_k(0)\bar{g}_k(0)b_k g_k . \end{aligned}$$

Since $g_k(0) \geq 0$ for each k, we have

$$g_n(0)^2 = \sum_{k\geq n} |b_k(0)|^2 g_k(0)^2 .$$

Hence,

$$g_n(0)^2(1-|b_n(0)|^2) = \sum_{k\geq n+1} |b_k(0)|^2 g_k(0)^2 = g_{n+1}(0)^2 ,$$

so,

$$\text{(2)} \qquad g_{n+1}(0) = g_n(0)[1-|b_n(0)|^2]^{\frac{1}{2}}$$

$$= \left[\prod_{k=1}^{n} (1-|b_k(0)|^2)^{\frac{1}{2}} \right] g_1(0) .$$

Now, (1) and (2) imply

$$(3) \qquad g_n = \left[\prod_{k=1}^{n-1} (1-\bar{b}_k(0)b_k)/(1-|b_k(0)|^2)^{\frac{1}{2}} \right] g_1$$

$$= G_n g_1$$

which proves the propostion.

Applying the above to the outer function $h_1 = 1 + b_1$, let $m_k = (z^k H^2) \cap (h_1/\bar{h}_1)\bar{H}^2$ and $m_k' = \bar{z}^k m_k$. Then $h_1/\bar{h}_1 = b_1$ and m_0 has $\{ G_1=1, zG_2, z^2G_3, \ldots \}$ as an orthonormal basis. Since $\{g_1, zG_2g_1, z^2G_3g_1, \ldots\} = \{g_1, zg_2, z^2g_3, \ldots\}$ which is an orthonormal basis of M_1', the map $f \to fg_1$ induces an isometry of m_0 onto M_1'. Thus, $\text{Ker } T_u = \text{Ker } T_{\bar{h}/h} = M_1' = g_1(m_0) = g_1(H^2 \ominus zb_1H^2)$ so we can write

THEOREM 3. *Let T_u be a Toeplitz operator on H^2 with a non-trivial kernel. Then there is an outer function g and an inner function b such that $\text{Ker } T_u = g(H^2 \ominus zbH^2)$. Moreover, the map $f \to fg$ is an isometry of $H^2 \ominus zbH^2$ onto $\text{Ker } T_u$.*

As an application, it is a simple matter to write down the reproducing kernel for Ker T_u. For w in the open unit disk D let $k_w(z) = 1/(1-\bar{w}z)$ be the reproducing kernel for H^2. Then the reproducing kernel for $H^2 \ominus zbH^2$ is easily seen to be given by

$$k_{b,w}(z) = (1 - \overline{wb(w)}zb(z))k_w(z) .$$

Thus, for f in Ker T_u, we have, by Theorem 3, that f/g is in $H^2 \ominus zbH^2$. Therefore,

$$f(w) = g(w)\langle f/g, k_{b,w}\rangle$$
$$= \langle f, \bar{g}(w)\, g\, k_{b,w}\rangle .$$

Because $gk_{b,w}$ is in Ker T_u for each w in D, the reproducing kernel for Ker T_u is given by

$$K_{u,w}(z) = \overline{g(w)}\, g(z)\, k_{b,w}(z).$$

The functions g and b which appear in Theorem 3 are obviously closely related. It would be of interest to get some quantitative information linking the local behaviour of g, b, and h. Such a study, which the author intends to pursue, might prove useful in analyzing the latitude one has in the prescription of side conditions for solutions of Wiener-Hopf equations.

REFERENCES

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Department of Mathematics
San Francisco State University
1600 Holloway Ave.
San Francisco, CA 94132

Submitted: September 24, 1985
Revised: January 6, 1986