Integr Equat Oper Th 15 (1992) 0378-620X/92/020262-39$1.50+0.20/0

Factorizations of and extensions to J-unitary rational matrix functions on the unit circle

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This paper concerns two topics: (1) minimal factorizations in the class of J-unitary rational matrix functions on the unit circle and (2) completions of contractive rational matrix functions on the unit circle to two by two block unitary rational matrix functions which do not increase the McMillan degree. The results are given in terms of a special realization which does not require any additional properties at zero and at infinity. The unitary completion result may be viewed as a generalization of Darlington synthesis.

1. Introduction

In this paper we study unitary and J-unitary rational matrix functions on the unit circle W, using the concept of realization. Usually a realization for a rational matrix function W is a representation of W in the form $W(\lambda)=D+C(\lambda I-A)^{-1}B$, which holds whenver W is analytic at ∞. The latter condition makes this type of realization less suitable for the study of J-unitary functions on the unit circle. The usual procedure is to assume first that the J-unitary function does not have a pole or a zero at infinity, use the aformentionend standard realization to derive the desired result for this particular case, and then derive the result for the general case using Möbius transform. In this way the final formulas do not appear explicitly in terms of the original data. Recently, another realization was proposed, which allows to study arbitrary regular rational matrix functions without constraints on the behaviour at infinity (see [GK], see also [BGR], Section 5.2). It is a representation of the function W in the form

$$W(\lambda)=D+(\alpha-\lambda)C(\lambda G-A)^{-1}B. \tag{1.1}$$

where A and G are $n\times n$ matrices with $\alpha G-A$ invertible, B is an $n\times m$ matrix and C is an $m\times n$ matrix and, finally, D is an $m\times m$ matrix. This realization is valid provided W is analytic at $\lambda=\alpha$. As any regular rational matrix functions has only a finite number of poles this is not a restriction; in practice we shall assume $\alpha=1$ mostly. Since in the present paper W is J-unitary on the unit circle it seems more natural to use that W is analytic at some point on the unit circle, than to require that W is analytic at infinity. Additional motivation for using a realization of the type (1.1) comes from the theory of reproducing kernel spaces (see Section 3 below).

In this paper we show how this realization can be used in two problems: the first one is the problem of factorization of J-unitary functions into J-unitary factors, the second one is the problem of generalized Darlington synthesis. More precisely, completion to a 2×2 block unitary rational matrix function of a given contractive function without increasing the McMillan degree. In the solution of these two problems we were inspired by [AG] and [GR], respectively. The first problem was solved in [AG] for rational matrix functions which are analytic at infinity using the standard realization. For functions on the real line the second problem was solved in [GR].

In Section 2 we study properties of realization (1.1) with respect to multiplication and factorization. In Section 3 the special properties for realization (1.1) are derived in case W is J-unitary on the unit circle. Factorizations of J-unitary functions into J-unitary factors are studied in Section 4. Special attention is given to the case where $J=I$, i.e., to unitary functions. In the last section we study unitary completions of a given contrative rational matrix function.

2. Realization, similarity, multiplication and factorization

2.1 Similarity. Let W be an $m\times m$ rational matrix function, which has an invertible value at the point $\alpha\in\mathbb{C}$. A representation of the form

$$W(\lambda)=D+(\alpha-\lambda)C(\lambda G-A)^{-1}B \tag{2.1}$$

where we assume $\alpha G-A$ is invertible, is called a *realization* of W. The realization (2.1) of $W(\lambda)$ is called *minimal* if the size of the matrices G and A is as small as possible among all realizations of W. In that case, if G and A are $n\times n$, say, the number n is called the *McMillan degree* of W; this number is denoted by $\delta(W)$. The realization is minimal if and only if it is controllable and observable, more precisely, if and only if the maps

$$C(\lambda G-A)^{-1}:\mathbb{C}^n\rightarrow\mathscr{R}(\sigma),\qquad B^*(\lambda G^*-A^*)^{-1}:\mathbb{C}^n\rightarrow\mathscr{R}(\sigma)$$

are one-one. Here $\mathscr{R}(\sigma)$ denotes the set of rational $n\times1$ vector functions with poles off σ, where σ is the set of zeros of $\det(\lambda G-A)$ including infinity. This is most easily seen by using Möbius transform. Indeed, put $\phi(\lambda)=(\alpha\lambda+1)\lambda^{-1}$, and define $V(\lambda)=W(\phi(\lambda))$. One easily checkes from (2.1) that

$$V(\lambda)=D-C(\alpha G-A)^{-1}(\lambda+G(\alpha G-A)^{-1})^{-1}B.$$

This realization for V is minimal if and only if the realization (2.1) for W is minimal. But for this type of realization it is well-known that minimality is equivalent to observability and controllability. It takes a little computation, which we leave to the reader, to see that the standard definitions of observability and controllability are equivalent to (for this particular realization for V)

$$C(\alpha G-A)^{-1}(\lambda+G(\alpha G-A)^{-1})^{-1}:\mathbb{C}^n\rightarrow\mathscr{R}(\hat{\sigma})$$

is one-one, as well as

$$B^*(\lambda+(\bar{\alpha}G^*-A^*)^{-1})^{-1}:\mathbb{C}^n\to\mathscr{R}(\hat{\sigma}).$$

Here $\hat{\sigma}=\phi^{-1}(\sigma)$. Now

$$C(\alpha G-A)^{-1}(\lambda+G(\alpha G-A)^{-1})^{-1}=\frac{1}{\lambda}C(\phi(\lambda)G-A)^{-1}.$$

Put $\mu=\phi(\lambda)$, then $\frac{1}{\lambda}=\mu-\alpha$. So $(\mu-\alpha)C(\mu G-A)^{-1}:\mathbb{C}^n\to\mathscr{R}(\sigma)$ is one-one. Likewise

$$B^*(\lambda+(\bar{\alpha}G^*-A^*)^{-1})^{-1}:\mathbb{C}^n\to\mathscr{R}(\sigma)$$

is one-one if and only if $B^*(\bar{\alpha}G^*-A^*)^{-1}(\lambda+(\bar{\alpha}G^*-A^*)^{-1})^{-1}:\mathbb{C}^n\to\mathscr{R}(\sigma)$ is one-one. A similar argument as above shows that this is equivalent to

$$B^*(\mu G^*-A^*)^{-1}:\mathbb{C}^n\to\mathscr{R}(\sigma)$$

being one-one.

Note that together with the realization (2.1) for $W(\lambda)$, we also have a realization for its inverse, given by

$$W(\lambda)^{-1}=D^{-1}-(\alpha-\lambda)D^{-1}C(\lambda G^\times-A^\times)^{-1}BD^{-1}, \tag{2.2}$$

where $G^\times=G-BD^{-1}C$, $A^\times=A-\alpha BD^{-1}C$.

Let $W(\lambda)=D_i+(\alpha-\lambda)C_i(\lambda G_i-A_i)^{-1}B_i$, $i=1, 2$, be two realizations for the same rational matrix function $W(\lambda)$, and assume that both these realizations are minimal. Then there exist unique invertible matrices E and F such that

$$E(\lambda G_1-A_1)F=(\lambda G_2-A_2),\quad C_1F=C_2,\quad EB_1=B_2. \tag{2.3}$$

We shall say that the two realizations are *strictly equivalent*, by abuse of expression sometimes also that they are *similar*.

2.2 Multiplication. Let

$$W_i(\lambda)=D_i+(\alpha-\lambda)C_i(\lambda G_i-A_i)^{-1}B_i,\qquad i=1, 2, \tag{2.4}$$

be two rational matrix functions in realized form. Then for the product we have the following realization: $W(\lambda)=W_1(\lambda)W_2(\lambda)=D+(\alpha-\lambda)C(\lambda G-A)^{-1}B$, where $D=D_1D_2$ and

$$C=\begin{bmatrix} C_1 & D_1C_2 \end{bmatrix},\quad B=\begin{bmatrix} B_1D_2 \\ B_2 \end{bmatrix}, \tag{2.5}$$

$$A=\begin{bmatrix} A_1 & \alpha B_1C_2 \\ 0 & A_2 \end{bmatrix},\quad G=\begin{bmatrix} G_1 & B_1C_2 \\ 0 & G_2 \end{bmatrix}.$$

Indeed, we have

$$W_1(\lambda)W_2(\lambda)=D_1D_2+(\alpha-\lambda)D_1C_2(\lambda G_2-A_2)^{-1}+ \tag{2.6}$$
$$+(\alpha-\lambda)C_1(\lambda G_1-A_1)^{-1}B_1D_2+(\alpha-\lambda)^2C_1(\lambda G_1-A_1)^{-1}B_1C_2(\lambda G_2-A_2)^{-1}.$$

Computing $C(\lambda G-A)^{-1}B$ from (2.5) we obtain

$$\begin{bmatrix} C_1 & D_1C_2 \end{bmatrix} \begin{bmatrix} \lambda G_1-A_1 & (\lambda-\alpha)B_1C_2 \\ 0 & \lambda G_2-A_2 \end{bmatrix}^{-1} \begin{bmatrix} B_1D_2 \\ B_2 \end{bmatrix} =$$

$$= \begin{bmatrix} C_1 & D_1C_2 \end{bmatrix} \begin{bmatrix} (\lambda G_1-A_1)^{-1} & (\alpha-\lambda)(\lambda G_1-A_1)^{-1}B_1C_2(\lambda G_2-A_2)^{-1} \\ 0 & (\lambda G_2-A_2)^{-1} \end{bmatrix}^{-1} \begin{bmatrix} B_1D_2 \\ B_2 \end{bmatrix}$$

Comparison with (2.6) learns that this is equal to $(\alpha-\lambda)^{-1}(W_1(\lambda)W_2(\lambda)-D_1D_2)$, thereby proving our claim.

Note that the inverse of the product, $W(\lambda)^{-1}=W_2(\lambda)^{-1}W_1(\lambda)^{-1}$ is given by $W(\lambda)^{-1}=D^{-1}-(\alpha-\lambda)D^{-1}C(\lambda G^{\times}-A^{\times})^{-1}BD^{-1}$, where

$$G^{\times}=G-BD^{-1}C=\begin{bmatrix} G_1^{\times} & 0 \\ -B_2D^{-1}C_1 & G_2^{\times} \end{bmatrix},$$

$$A^{\times}=A-\alpha BD^{-1}C=\begin{bmatrix} A_1^{\times} & 0 \\ -\alpha B_2D^{-1}C_1 & A_2^{\times} \end{bmatrix}.$$

2.3 Factorization. In this subsection we shall study minimal factorizations of rational matrix functions given by

$$W(\lambda)=D+(\alpha-\lambda)C(\lambda G-A)^{-1}B.$$

We start by giving a theorem that gives a sufficient condition for factorization; this result can be viewed as a converse of the formulas for multiplication obtained in the previous subsection.

Theorem 2.1. *Let* $\{M_1, M_2\}$ *be an invariant subspace pair for* $\lambda G-A$, *i.e.,* $GM_1\subset M_2$, $AM_1\subset M_2$, $\dim M_1=\dim M_2$, *and let* $\{M_1^{\times}, M_2^{\times}\}$ *be an invariant subspace pair for* $\lambda G^{\times}-A^{\times}$. *Suppose, moreover, that*

$$\mathbb{C}^n=M_1\oplus M_1^{\times}=M_2\oplus M_2^{\times}. \tag{2.7}$$

Let π_1, π_2 *be the projections along* M_1, M_2, *respectively, onto* $M_1^{\times}, M_2^{\times}$, *respectively. Then*

$$W(\lambda)=W_1(\lambda)W_2(\lambda) \tag{2.8}$$

where

$$W_1(\lambda)=D_1+(\alpha-\lambda)C(I-\pi_1)(\lambda G-A)^{-1}(I-\pi_2)BD_2^{-1}, \tag{2.9}$$

$$W_2(\lambda)=D_2+(\alpha-\lambda)D_1^{-1}C\pi_1(\lambda G-A)^{-1}\pi_2 B. \tag{2.10}$$

Proof. Consider $\lambda G-A$ as a mapping from $M_1 \oplus M_1^\times$ to $M_2 \oplus M_2^\times$:

$$\lambda G-A=\begin{bmatrix}\lambda G_1-A_1 & \lambda G_{12}-A_{12}\\ 0 & \lambda G_2-A_2\end{bmatrix}.$$

Further, consider B as a mapping from $\mathbb{C}^m$ to $M_2 \oplus M_2^\times$, and C as a mapping from $M_1 \oplus M_1^\times$ to $\mathbb{C}^m$:

$$B=\begin{bmatrix}B_1\\ B_2\end{bmatrix},\quad C=\begin{bmatrix}C_1 & C_2\end{bmatrix}.$$

Then, considering $\lambda G^\times-A^\times$ as a mapping from $M_1 \oplus M_1^\times$ to $M_2 \oplus M_2^\times$ we have:

$$\lambda G^\times-A^\times=$$

$$=\begin{bmatrix}\lambda(G_1-B_1D^{-1}C_1)-(A_1-\alpha B_1D^{-1}C_1) & \lambda(G_{12}-B_1D^{-1}C_2)-(A_{12}-\alpha B_1D^{-1}C_2)\\ (\alpha-\lambda)B_2D^{-1}C_1 & \lambda(G_2-B_2D^{-1}C_2)-(A_2-\alpha B_2D^{-1}C_2)\end{bmatrix}.$$

As $\{M_1^\times, M_2^\times\}$ is an invariant subspace pair for $\lambda G^\times-A^\times$, it follows that

$$G_{12}=B_1D^{-1}C_2,\quad A_{12}=\alpha B_1D^{-1}C_2.$$

Hence $W(\lambda)=W_1(\lambda)W_2(\lambda)$, where

$$W_1(\lambda)=D_1+(\alpha-\lambda)C_1(\lambda G_1-A_1)^{-1}B_1D_2^{-1}, \tag{2.11}$$

$$W_2(\lambda)=D_2+(\alpha-\lambda)D_1^{-1}C_2(\lambda G_2-A_2)^{-1}B_2, \tag{2.12}$$

But clearly

$$C_1(\lambda G_1-A_1)^{-1}B_1=C(I-\pi_1)(\lambda G-A)^{-1}(I-\pi_2)B,$$

and

$$C_2(\lambda G_2-A_2)^{-1}B_2=C\pi_1(\lambda G-A)^{-1}\pi_2 B.$$

This proves the theorem. □

Next we discuss minimality of the factorization $W(\lambda)=W_1(\lambda)W_2(\lambda)$. Such a factorization is called *minimal* if $\delta(W)=\delta(W_1)+\delta(W_2)$. Note that it follows from the multiplication result in Section 2.2 that we always have

$$\delta(W) \leqq \delta(W_1)+\delta(W_2). \tag{2.13}$$

Theorem 2.2. *Suppose the realization* (2.1) *is minimal. Then for every choice of invariant subspace pairs* $\{M_1, M_2\}$ *for* $\lambda G-A$ *and* $\{M_1^\times, M_2^\times\}$ *for* $\lambda G^\times - A^\times$ *such that (2.7) holds the factorization (2.8), where* $W_i(\lambda)$ $(i=1,\ 2)$ *are given by (2.9) and (2.10), is minimal.*

Conversely, if (2.8) is a minimal factorization, then $W_i(\lambda)$ $(i=1,\ 2)$ *are given by (2.9) and (2.10) for some unique projections* π_1 *and* π_2 *corresponding to invariant subspace pairs* $\{M_1, M_2\}$ *for* $\lambda G-A$ *and* $\{M_1^\times, M_2^\times\}$ *for* $\lambda G^\times - A^\times$*, respectively, for which (2.7) is satisfied.*

Proof. Suppose (2.7) holds for the invariant subspace pairs $\{M_1, M_2\}$ for $\lambda G-A$ and $\{M_1^\times, M_2^\times\}$ for $\lambda G^\times - A^\times$. By Theorem 2.1 we have a factorization $W(\lambda)=W_1(\lambda)W_2(\lambda)$, where W_1 and W_2 are given by (2.9) and (2.10), or alternatively by (2.11) and (2.12). From (2.11), (2.12) we see that

$$\delta(W_1) \leqq \dim M_1, \quad \delta(W_2) \leqq \dim M_1^\times .$$

So, by the minimality of the realization (2.1), and using (2.7) we have

$$\delta(W)=n= \dim M_1 + \dim M_1^\times \geqq \delta(W_1)+\delta(W_2).$$

Combined with (2.13) we obtain that the factorization is minimal.

To prove the converse, suppose (2.8) is a minimal factorization, and let

$$W_i(\lambda)=D_i+(\alpha-\lambda)C_i(\lambda G_i-A_i)^{-1}B_i$$

be a minimal realization for $W_i(\lambda)$, $(i=1,\ 2)$. Build a realization of the product $W(\lambda)$ as in Section 2.2:

$$W(\lambda)=D+(\alpha-\lambda)\tilde{C}(\lambda\tilde{G}-\tilde{A})^{-1}\tilde{B}, \tag{2.14}$$

where $\tilde{C}$, $\tilde{B}$, $\tilde{A}$, $\tilde{G}$ are given by (2.5). As the factorization is minimal, this is a minimal realization for W. Hence (2.1) and (2.14) are similar. Let E and F be invertible matrices such that

$$E(\lambda G-A)F=\lambda\tilde{G}-\tilde{A},\quad CF=\tilde{C},\quad EB=\tilde{B}. \tag{2.15}$$

Now

$$\lambda\tilde{G}-\tilde{A}=\begin{bmatrix}\lambda G_1-A_1 & (\lambda-\alpha)B_1C_2\\ 0 & \lambda G_2-A_2\end{bmatrix},$$

considered as a mapping from $\mathbb{C}^{n_1} \oplus \mathbb{C}^{n_2}$ to itself, where $n_i=\delta(W_i)$. Put

$$M_1=F(\mathbb{C}^{n_1}\oplus(0)),\quad M_2=E^{-1}(\mathbb{C}^{n_1}\oplus(0)).$$

Then $\{M_1, M_2\}$ is an invariant subspace pair for $\lambda G-A$. Further

$$E(\lambda G^\times-A^\times)F=E(\lambda G-A)F-(\lambda-\alpha)EBD^{-1}CF=$$

$$=\lambda\tilde{G}-\tilde{A}-(\lambda-\alpha)\tilde{B}D^{-1}\tilde{C}=\lambda\tilde{G}^\times-\tilde{A}^\times==\begin{bmatrix}\lambda G_1^\times-A_1^\times & 0\\ (\alpha-\lambda)B_2D^{-1}C_1 & \lambda G_2^\times-A_2^\times\end{bmatrix}.$$

Put

$$M_1^\times=F((0)\oplus\mathbb{C}^{n_2}),\quad M_2^\times=E^{-1}((0)\oplus\mathbb{C}^{n_2}).$$

Then $\{M_1^\times, M_2^\times\}$ is an invariant subspace pair for $\lambda G^\times-A^\times$. Moreover, we have

$$\mathbb{C}^n=M_1\oplus M_1^\times=M_2\oplus M_2^\times.$$

Let π_1, π_2 be the corresponding projections, i.e., π_i is the projection along M_i onto $M_i^\times$. Then

$$\pi_1=F\begin{bmatrix}0 & 0\\ 0 & I_{n_2}\end{bmatrix}F^{-1},\quad \pi_2=E^{-1}\begin{bmatrix}0 & 0\\ 0 & I_{n_2}\end{bmatrix}E. \tag{2.16}$$

Applying Theorem 2.1 we have $W(\lambda)=\tilde{W}_1(\lambda)\tilde{W}_2(\lambda)$, where

$$\tilde{W}_1(\lambda)=D_1+(\alpha-\lambda)C(I-\pi_1)(\lambda G-A)^{-1}(I-\pi_2)BD_2^{-1}, \tag{2.17}$$

$$\tilde{W}_2(\lambda)=D_2+(\alpha-\lambda)D_1^{-1}C\pi_1(\lambda G-A)^{-1}\pi_2B. \tag{2.18}$$

Using (2.16) and (2.15) in (2.17) and (2.18) we obtain

$$\tilde{W}_1(\lambda)=D_1+(\alpha-\lambda)\tilde{C}\begin{bmatrix}I & 0\\ 0 & 0\end{bmatrix}(\lambda\tilde{G}-\tilde{A})^{-1}\begin{bmatrix}I & 0\\ 0 & 0\end{bmatrix}\tilde{B}D_2^{-1}$$

$$\tilde{W}_2(\lambda)=D_2+(\alpha-\lambda)D_1^{-1}\tilde{C}\begin{bmatrix}I & 0\\ 0 & 0\end{bmatrix}(\lambda\tilde{G}-\tilde{A})^{-1}\begin{bmatrix}I & 0\\ 0 & 0\end{bmatrix}\tilde{B}.$$

As

$$\tilde{B}=\begin{bmatrix}B_1D_2\\ B_2\end{bmatrix},\quad \tilde{C}=\begin{bmatrix}C_1 & D_1C_2\end{bmatrix},$$

it follows from the above formulas that $\tilde{W}_1(\lambda)=W_1(\lambda)$, $\tilde{W}_2(\lambda)=W_2(\lambda)$. This proves that W_1 and W_2 are of the form (2.9) and (2.10), respectively.

It remains to prove the uniqueness of the projections π_1 and π_2. Suppose P_1 and P_2 are also projections such that

$$W_1(\lambda)=D_1+(\alpha-\lambda)\tilde{C}_1(\lambda\tilde{G}_1-\tilde{A}_1)^{-1}\tilde{B}_1D_2^{-1},$$

$$W_2(\lambda)=D_2+(\alpha-\lambda)D_1^{-1}\tilde{C}_2(\lambda\tilde{G}_2-\tilde{A}_2)^{-1}\tilde{B}_2,$$

where

$$\tilde{C}_1=C(I-P_1),\ \tilde{G}_1=(I-P_1)G(I-P_1),\ \tilde{A}_1=(I-P_1)A(I-P_1),\ \tilde{B}_1=(I-P_1)B,$$

and

$$\tilde{C}_2=CP_2,\ \tilde{G}_2=P_2GP_2,\ \tilde{A}_2=P_2AP_2,\ \tilde{B}_2=P_2B.$$

We then have two minimal realizations for both W_1 and W_2. So there exist E_1 and F_1 such that

$$E_1(\lambda G_1-A_1)F_1=\lambda\tilde{G}_1-\tilde{A}_1,\quad C_1F_1=\tilde{C}_1,\quad E_1B_1=\tilde{B}_1.$$

Also, there exist E_2 and F_2 such that

$$E_2(\lambda G_2-A_2)F_2=\lambda\tilde{G}_2-\tilde{A}_2,\quad C_2F_2=\tilde{C}_2,\quad E_2B_2=\tilde{B}_2.$$

Define

$$E=\begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix},\quad F=\begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}.$$

Then we have

$$E(\lambda G-A)F=\lambda G-A,\quad CF=C,\quad EB=B.$$

In particular, $C(\lambda G-A)^{-1}=C(\lambda G-A)^{-1}E$, i.e., $C(\lambda G-A)^{-1}(E-I)\equiv 0$. By observability of the realization for $W(\lambda)$ we get $E=I$. Further, in a similar way we have $(F-I)(\lambda G-A)^{-1}B\equiv 0$, and from the controllability of the realization for $W(\lambda)$ we obtain $F=I$. But from $E=F=I$ we have $P_i=\pi_i$. □

3. Realizations for J-unitary functions

3.1 Characterization. The function $W(\lambda)$ is called J-unitary if it has J-unitary values on the unit circle except at poles of W, or equivalently

$$W(\bar{\lambda}^{-1})^*JW(\lambda)=J \tag{3.1}$$

for all λ which are not poles of W. In this section we shall consider the properties of such functions, in particular with respect to their realizations.

Theorem 3.1 *Let $W(\lambda)=D+(1-\lambda)C(\lambda G-A)^{-1}B$ be a minimal realization with G-A invertible and $D=W(1)$ invertible. Then the following are equivalent:*

(i) $W(\lambda)$ is a J-unitary rational matrix matrix function,

(ii) $D^*JD=J$ *and there is an invertible matrix* F *such that*

$$GF+F^*A^*=BJB^*, \quad D^{-1}CF=JB^*, \tag{3.2}$$

(iii) $D^*JD=J$ *and there exists an invertible Hermitian matrix* H *such that*

$$G^*HG-A^*HA=-C^*JC, \tag{3.3}$$

$$D^{-1}C(A-G)^{-1}H^{-1}=JB^*.$$

Proof. First we show the equivalence of *(i)* and *(ii)*. Assume *(i)* holds. Then $W(1)=D$ is J-unitary, so $D^*JD=J$. From (3.1) we see that $W(\lambda)^{-1}=JW(\bar{\lambda}^{-1})^*J$. Computing realizations for both the left and right hand side of this equality we obtain

$$D^{-1}-(1-\lambda)D^{-1}C(\lambda G^\times-A^\times)^{-1}BD^{-1}=$$

$$=JD^*J-(1-\lambda)JB^*(-\lambda A^*+G^*)^{-1}C^*J.$$

Since these two realizations are minimal, they must be strictly equivalent, so there exist unique invertible matrices E and F such that

$$E(\lambda G^\times-A^\times)F=-\lambda A^*+G^*, \quad D^{-1}CF=JB^*, \quad EBD^{-1}=C^*J. \tag{3.4}$$

In particular

$$EG^\times F=-A^*, \quad EA^\times F=-G^*. \tag{3.5}$$

Now take adjoints in the last two equations in (3.4) and use $D^*JD=J$ to see

$$D^{-1}CE^{-*}=JB^*, \quad F^{-*}BD^{-1}=C^*J. \tag{3.6}$$

It follows that

$$F^{-*}BD^{-1}CE^{-*}=C^*JDJB^*=C^*D^{-*}B^*. \tag{3.7}$$

Taking adjoints in (3.5) we have

$$G^*-C^*D^{-*}B^*=-F^{-*}AE^{-*}, \quad A^*-C^*D^{-*}B^*=-F^{-*}GE^{-*}. \tag{3.8}$$

Using (3.7) these formulas give

$$F^{-*}G^\times E^{-*}=-A^*, \quad F^{-*}A^\times E^{-*}=-G^*. \tag{3.9}$$

Comparing (3.6), (3.9), with (3.4), and using the uniqueness of E and F we obtain

$$E=F^{-*}. \tag{3.10}$$

So

$$G^{\times}F=-F^{*}A^{*}, \tag{3.11}$$

and using (3.4) again we have $GF+F^{*}A^{*}=BD^{-1}CF=BJB^{*}$. Thus *(i)* holds.

Conversely, suppose *(ii)* holds. Put $E=F^{-*}$. Then (3.2) gives $GF+F^{*}A^{*}=BD^{-1}CF$, i.e., $G^{\times}F=-F^{*}A^{*}$. So $EG^{\times}F=-A^{*}$. Taking adjoints, and rewriting a little, also $EA^{\times}F=-G^{*}$. Thus (3.4) holds and hence $W(\lambda)$ is a J-unitary rational matrix function.

To show *(ii)* implies *(iii)* subtract the two formulas in (3.8), using (3.10) we obtain

$$(G-A)F=F^{*}(G-A)^{*}. \tag{3.12}$$

Introduce the matrix

$$H=F^{-1}(A-G)^{-1}. \tag{3.13}$$

By (3.12) H is a Hermitian matrix. To prove (3.3), taking adjoints in (3.11) we have

$$A=-F^{*}(G^{\times})^{*}F^{-1}=-F^{*}G^{*}F^{-1}+F^{*}C^{*}D^{-*}B^{*}F^{-1}=$$

$$=-F^{*}G^{*}F^{-1}+F^{*}C^{*}JC.$$

So we have $G^{*}F^{-1}+F^{-*}A=C^{*}JC$. Taking adjoints gives $A^{*}F^{-1}+F^{-*}G=C^{*}JC$. Adding these two, and using $F^{-1}=H(A-G)$ yields $2A^{*}HA-2G^{*}HG=2C^{*}JC$, which is equivalent to (3.3). The second part of *(iii)* is an easy consequence of (3.2).

For the converse, suppose (3.3) holds. Put $F=(A-G)^{-1}H^{-1}$. Then $D^{-1}CF=JB^{*}$ and it is straightforward to check that $G^{\times}F=-F^{*}A^{*}$. The result then follows easily. □

Clearly, the matrix H is uniquely determined by the realization of $W(\lambda)$. We shall call H the *associated Hermitian matrix*. As in [AG] we shall denote the number of negative eigenvalues of H by $\nu(W)$. That this number is independent of the particular choice of the minimal realization is a consequence of Theorem 3.2 below, for which we first introduce some notation.

Introduce the kernel functions

$$K_W(\lambda,\omega)=\frac{J-W(\lambda)JW(\omega)^{*}}{1-\lambda\bar{\omega}}, \tag{3.14}$$

$$K_{W^{*}}(\omega,\lambda)=\frac{J-W(\omega)^{*}JW(\lambda)}{1-\lambda\bar{\omega}}. \tag{3.15}$$

The function K_W is said to have $\varkappa$ negative squares if for each positive integer r and any points $w_1,\dots,w_r$ which are not poles of W, and any vectors $c_1,\dots,c_r$ the $r\times r$ Hermitian matrix

$$(c_j^* K(w_j, w_i) c_i)_{i,j=1}^{r}$$

has at most $\varkappa$ negative eigenvalues, and has exactly $\varkappa$ negative eigenvalues for some choice of r, $w_1, \ldots, w_r$ and $c_1, \ldots, c_r$. (See, e.g., [AG], Section 2.2.)

With this notation we have the following result, which is comparable to [BGR], Theorems 7.4.3 and 7.5.3.

Theorem 3.2. *Let $W(\lambda) = D + (1-\lambda)C(\lambda G - A)^{-1}B$ be a minimal realization of the rational J-unitary matrix function W with G-A invertible. Then we have*

$$K_W(\lambda,\omega) = C(\lambda G - A)^{-1} H^{-1} (\bar{\omega} G^* - A^*)^{-1} C^*, \tag{3.16}$$

$$K_{W^*}(\omega,\lambda) = B^* (\bar{\omega} G^* - A^*)^{-1} F^{-*} H^{-1} F^{-1} (\lambda G - A)^{-1} B. \tag{3.17}$$

Thus the number of negative eigenvalues of H is equal to the number of negative squares of each of the functions $K_W(\lambda,\omega)$ and $K_{W^*}(\omega,\lambda)$.

Further, $\delta(W) =$ dim $\mathcal{K}(W)$, where $\mathcal{K}(W)$ is the following set of functions in λ

$$\mathcal{K}(W) = \{K_W(\lambda,\omega)c \mid c \in \mathbb{C}, \det(\omega G - A) \neq 0\}. \tag{3.18}$$

Proof. We compute the kernel function $K_W(\lambda,\omega)$ We have, using $J = J^{-1}$

$$W(\lambda) J W(\omega)^* =$$

$$= (D + (1-\lambda)C(\lambda G - A)^{-1}B) J (D^* + (1-\bar{\omega})B^*(\bar{\omega}G^* - A^*)^{-1}C^*) =$$

$$J + (1-\bar{\omega}) DJB^* (\bar{\omega}G^* - A^*)^{-1} C^* + (1-\lambda) C(\lambda G - A)^{-1} BJD^* +$$

$$(1 + \lambda\bar{\omega} - \lambda - \bar{\omega}) C(\lambda G - A)^{-1} BJB^* (\bar{\omega}G^* - A^*)^{-1} C^*.$$

Using the second equation in (3.4) twice, this equals

$$J + (1-\bar{\omega}) CF(\bar{\omega}G^* - A^*)^{-1} C^* + (1-\lambda) C(\lambda G - A)^{-1} F^* C^* +$$

$$+ \{(1-\lambda) + (1-\bar{\omega}) + (\lambda\bar{\omega} - 1)\} C(\lambda G - A)^{-1} (F^* G^* + AF)(\bar{\omega}G^* - A^*)^{-1} C^*.$$

Collect together terms with $(1-\lambda)$ and terms with $(1-\bar{\omega})$, this gives

$$J + (1-\lambda) C(\lambda G - A)^{-1} \{F^* + (F^* A^* + GF)(\bar{\omega}G^* - A^*)^{-1}\} C^* +$$

$$+ (1-\bar{\omega}) C\{F + (\lambda G - A)^{-1} (F^* G^* + AF)\} (\bar{\omega}G^* - A^*)^{-1} C^* +$$

$$+ (1 - \lambda\bar{\omega}) C(\lambda G - A)^{-1} \{-F^* G^* - AF\} (\bar{\omega}G^* - A^*)^{-1} C^*.$$

It is easy to see that this is equal to

$$=J+(1-\lambda)C(\lambda G-A)^{-1}\{\bar{\omega}F^*G^*+GF)\}(\bar{\omega}G^*-A^*)^{-1}C^*+$$

$$+(1-\bar{\omega})C(\lambda G-A)^{-1}(\lambda GF+F^*G^*)\}(\bar{\omega}G^*-A^*)^{-1}C^*+$$

$$+(1-\lambda\bar{\omega})C(\lambda G-A)^{-1}\{-F^*G^*-AF\}(\bar{\omega}G^*-A^*)^{-1}C^*.$$

Now this simply rewrites as

$$J+(1-\lambda\bar{\omega})C(\lambda G-A)^{-1}(G-A)F(\bar{\omega}G^*-A^*)^{-1}C^*.$$

We have shown that

$$W(\lambda)JW(\omega)^*=J+(1-\lambda\bar{\omega})C(\lambda G-A)^{-1}(G-A)F(\bar{\omega}G^*-A^*)^{-1}C^*. \tag{3.19}$$

This proves (3.16).

In a similar way one shows that

$$W(\omega)^*JW(\lambda)=J+(1-\lambda\bar{\omega})B^*(\bar{\omega}G^*-A^*)^{-1}(G^*-A^*)F^{-1}(\lambda G-A)^{-1}B,$$

which gives (3.17) after noting that $(A^*-G^*)F^{-1}=F^{-*}H^{-1}F^{-1}$.

To see the last statement of the theorem, note that any function in $\mathcal{H}(W)$ can be written as

$$C(\lambda G-A)^{-1}x, \quad x\in\mathbb{C}^n$$

by (3.19). Hence $\dim\mathcal{H}(W)\leqq\delta(W)$. The observability of $(C,\lambda G-A)$, i.e. the injectivity of $C(\lambda G-A)^{-1}$ shows that $\dim\mathcal{H}(W)=\delta(W)$. □

Recall that a J-unitary function $W(\lambda)$ is called J-inner if it is J-contractive for $|\lambda|<1$, i.e.

$$W(\lambda)^*JW(\lambda)\leqq J, \quad |\lambda|\leqq 1.$$

Then we have the following proposition:

Proposition 3.3. *The J-unitary function W is J-inner if and only if the associated Hermitian matrix H is positive definite.*

Proof. This is an immediate consequence of formula (3.17) and the controllability of $(\lambda G-A,B)$. □

Our next result of this section describes the associated Hermitian matrix for the product of two J-unitary rational matrix functions.

Theorem 3.4. *Suppose*

$$W_i(\lambda)=D_i+(1-\lambda)C_i(\lambda G_i-A_i)^{-1}B_i \qquad i=1,2$$

are J-unitary functions with minimal realizations. Suppose the product $W=W_1W_2$ is *minimal. Let* H_1, H_2 *be the associated Hermitian matrices. Then the matrix* $\begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix}$ *is the Hermitian matrix associated with the minimal realization* $W(\lambda) = D + (1-\lambda)C(\lambda G - A)^{-1}B$ *for the product, where* $D = D_1 D_2$,

$$C = (C_1 \quad D_1 C_2), \quad B = \begin{bmatrix} B_1 D_2 \\ B_2 \end{bmatrix},$$

$$A = \begin{bmatrix} A_1 & B_1 C_2 \\ 0 & A_2 \end{bmatrix}, \quad G = \begin{bmatrix} G_1 & B_1 C_2 \\ 0 & G_2 \end{bmatrix}.$$

Proof. Let F_1, F_2 be the invertible matrices given by $F_i = (A_i - G_i)^{-1} H_i^{-1}$. With these F_i's (3.11) holds for the realizations for W_i as given. Put $F = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}$. It suffices to show that with this matrix F (3.11) holds with A,B,C,D,G as in the theorem. Indeed, if this is the case the associated Hermitian matrix with the above realization for W is given by

$$H = F^{-1}(A - G)^{-1} = \begin{bmatrix} F_1^{-1} & 0 \\ 0 & F_2^{-1} \end{bmatrix} \begin{bmatrix} (A_1 - G_1)^{-1} & 0 \\ 0 & (A_2 - G_2)^{-1} \end{bmatrix} = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix}.$$

To check (3.11) in this particular case compute

$$\begin{bmatrix} F_1^{-*} & 0 \\ 0 & F_2^{-*} \end{bmatrix} \begin{bmatrix} G_1^{\times} & 0 \\ -B_2 D^{-1} C_1 & G_2^{\times} \end{bmatrix} \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix} =$$

$$= \begin{bmatrix} F_1^{-*} G_1^{\times} F_1 & 0 \\ -F_2^{-1} B_2 D^{-1} C_1 F_1 & F_2^{-*} G_2^{\times} F_2 \end{bmatrix} = \begin{bmatrix} -A_1^* & 0 \\ C_2^* J D_2 D^{-1} D_1 J B_1^* & -A_2^* \end{bmatrix} =$$

$$= -\begin{bmatrix} A_1 & B_1 C_2 \\ 0 & A_1 \end{bmatrix}^*.$$

Further,

$$D^{-1} CF = (D^{-1} C_1 F_1 \quad D_2^{-1} C_2 F_2) = (D_2^{-1} J B_1^* \quad J B_2^*) =$$

$$= (J D_2^* B_1^* \quad J B_2^*) = JU \begin{bmatrix} B_1 D_2 \\ B_2 \end{bmatrix}^* = JB^*,$$

which proves the theorem. □

3.2 Inverse problem. Next, we turn our attention to the following problem. Given an observable pair $(C, \lambda G - A)$ with $G - A$ invertible, when does there exist B and D such that

$$W(\lambda)=D+(1-\lambda)C(\lambda G-A)^{-1}B$$

is a minimal realization for a J-unitary function on the unit circle? The following theorem provides the answer, which is based on Theorem 3.1.

Theorem 3.5. *Let $(C,\lambda G-A)$ be observable and let G-A be invertible. Then there exists a J-unitary function $W(\lambda)$ with minimal realization $W(\lambda)=D+(1-\lambda)C(\lambda G-A)^{-1}B$ with D invertible if and only if there is an invertible Hermitian solution H of (3.3)*

$$G^*HG-A^*HA=-C^*JC.$$

In this case one can take for D any invertible J-unitary matrix and put

$$B=H^{-1}(A^*-G^*)^{-1}C^*D^{-*}J$$

In this manner one obtains all possibilities for $W(\lambda)$ given an invertible Hermitian solution H of (3.3).

Proof. One direction of the theorem is contained in the statement of Theorem 3.1. For the converse we first show that $(\lambda G^\times - A^\times, B)$ is controllable, or equivalently, $(B^*, \lambda G^{\times *}-A^{\times *})$ is observable.

Indeed, suppose this is not the case. So, suppose that

$$B^*(\lambda G^{\times *}-A^{\times *})^{-1}x=0,$$

for some $x \neq 0$. It easily follows from (3.3) and our definition of B that

$$G^\times F=-F^*A^*, \quad A^\times F=-F^*G^*,$$

where $F=(A-G)^{-1}H^{-1}$. Noting that $B^*=JD^{-1}CF$ by definition this gives

$$0=CF(\lambda G^{\times *}-A^{\times *})^{-1}x=$$
$$=-C(\lambda A-G)^{-1}Fx=\lambda^{-1}C(\lambda^{-1}G-A)^{-1}Fx.$$

Using the observability of $(C,\lambda G-A)$, i.e. the injectivity of $C(\lambda G-A)^{-1}$ it now follows easily that $x=0$, which is a contradiction.

From Theorem 3.1 one sees that the function W given in the theorem is indeed J-unitary. To show that given an invertible Hermitian solution H of (3.3) the construction of the theorem really describes all possibilities, suppose that we have two solutions $W(\lambda)$ and $W_1(\lambda)$. From (3.19) we see that $W(\lambda)JW(\omega)^*=W_1(\lambda)JW_1(\omega)^*$. Thus $W=W_1U$ for some J-unitary constant U. This proves the theorem. □

The particular case of J-inner functions is also of importance. Because of Proposition 3.3 and the theorem above we have

Proposition 3.6 *Let $(C,\lambda G-A)$ be observable and let G-A be invertible. Then there* *exists a J-inner function $W(\lambda)$ with minimal realization $W(\lambda)=D+(1-\lambda)C(\lambda G-A)^{-1}B$ with D invertible if and only if there is a positive definite solution H of (3.3). In this case all possibilities for $W(\lambda)$ are given as in Theorem 3.5.*

Of course, the analogues of Theorem 3.5 and Proposition 3.6 for a controllable pair can also be formulated and proved. We omit the details.

3.3 Connection with reproducing kernel spaces. *) The results from Section 3.1 exhibit a close connection with the theory of reproducing kernel spaces. We shall explain this here in more detail. Suppose we have a linearly independent set of rational m-vector valued functions $f_1(\lambda),\ldots,f_n(\lambda)$ defined on the unit circle. Assume, moreover, an indefinite inner product $\langle .,. \rangle$ is defined on $P=$ span $\{f_1,\ldots,f_n\}$. Let H be the Gramm matrix of the basis $f_1,\ldots,f_n$, i.e., $H=(\langle f_j,f_i\rangle)_{i,j=1}^{n}$. Also, let us denote by $F(\lambda)$ the matrix with $f_i(\lambda)$ on its i-th column. Then P is a finite dimensional reproducing kernel space with reproducing kernel $K(\lambda,\omega)=F(\lambda)H^{-1}F(\omega)^*$. Let J be a matrix with $J=J^*=J^{-1}$. Then this reproducing kernel is of the form

$$K(\lambda,\omega)=\frac{J-W(\lambda)JW(\omega)^*}{1-\lambda\bar{\omega}}$$

for a function $W(\lambda)$ which is J-unitary for $|\lambda|=1$ (except for poles of W) if and only if H satisfies an equation of the type $A^*HA-G^*HG=C^*JC$ for some G, A and C. (See [Dy], Theorems 5.3 and 5.4.)

Let us specialise the result of the previous paragraph to the following case. Given C and $\lambda G-A$ such that $G-A$ is invertible and $(C, \lambda G-A)$ is controllable, consider the set of functions $f_i(\lambda)=C(\lambda G-A)^{-1}e_i$, where $e_1,\ldots,e_n$ runs over a basis of $\mathbb{C}^n$. Assume we have an invertible Hermitian solution H of the equation

$$A^*HA-G^*HG=C^*JC.$$

Let us denote by P the space of rational vector valued functions spanned by $f_1,\ldots,f_n$ with indefinite inner product given by H, i.e., $H=(\langle f_j,f_i\rangle)_{i,j=1}^{n}$. Then $F(\lambda)=C(\lambda G-A)^{-1}$, and by construction the reproducing kernel is of the form

$$K(\lambda,\omega)=\frac{J-W(\lambda)JW(\omega)^*}{1-\lambda\bar{\omega}}$$

for a function $W(\lambda)$ which is J-unitary for $|\lambda|=1$. In that case, putting $\omega=1$, we may

*) The authors are grateful to Harry Dym for attracting their attention to these connections.

solve for $W(\lambda)$ from the two representations of $K(\lambda,1)$:

$$W(\lambda)=W(1)-(1-\lambda)F(\lambda)H^{-1}F(1)^*JW(1)=$$

$$=D+(1-\lambda)C(\lambda G-A)^{-1}H^{-1}(A^*-G^*)^{-1}C^*JD,$$

where we have put $D=W(1)$. Clearly, this provides us with an alternative proof of part of Theorem 3.5. Moreover, this construction may be viewed as additional motivation for using realizations of the type (1.1) in the study of rational matrix functions which have J-unitary values on the unit circle.

4. Factorization of J-unitary functions into J-unitary factors

In this section we shall present several results on minimal factorizations into J-unitary factors of J-unitary functions. In the first subsection we study the general case, the special case of unitary matrix functions is studied in the second subsection.

4.1 The general case. As we know from the general results of Section 2, minimal factorizations are in one-one correspondence with invariant subspace pairs. For a J-unitary function $W(\lambda)$ with minimal realization

$$W(\lambda)=D+(1-\lambda)C(\lambda G-A)^{-1}B \tag{4.1}$$

it turns out that the existence of a certain particular invariant subspace pair of $\lambda G-A$ implies the existence of an invariant subspace pair for $\lambda G^\times-A^\times$ such that with respect to these two pairs the factorization of W is indeed a factorization into J-unitary factors. This is the content of the main theorem of this subsection.

Theorem 4.1. *Let W be a J-unitary function with minimal realization given by (4.1). Let F be the unique invertible matrix for which (3.11) holds. Suppose $\{M_1, M_2\}$ is an invariant subspace pair for $\lambda G-A$ and suppose that*

$$M_1\oplus FM_2^\perp=\mathbb{C}^n. \tag{4.2}$$

Then $\{FM_2^\perp, F^*M_1^\perp\}$ is an invariant subspace pair for $\lambda G^\times-A^\times$ and

$$M_2\oplus F^*M_1^\perp=\mathbb{C}^n. \tag{4.3}$$

Let π_1 be the projection along M_1 onto $FM_2^\perp$ and let π_2 be the projection along M_2 onto $F^*M_1^\perp$. Then

$$\pi_1=F\pi_2^*F^{-1}. \tag{4.4}$$

Put

$$W_1(\lambda)=D_1+(1-\lambda)C(I-\pi_1)(\lambda G-A)^{-1}(I-\pi_2)BD_2^{-1}, \tag{4.5}$$

$$W_2(\lambda)=D_2+(1-\lambda)D_2^{-1}C\pi_1(\lambda G-A)^{-1}\pi_2B, \tag{4.6}$$

where $D=D_1D_2$ and D_1,D_2 are J-unitary matrices. Then $W=W_1W_2$ is a minimal factorization of W, and $W_1(\lambda)$, $W_2(\lambda)$ are J-unitary functions.

Proof. First note that by (3.11) $G^\times FM_2^\perp=-F^*A^*M_2^\perp\subset F^*M_1^\perp$ and likewise $A^\times FM_2^\perp\subset F^*M_1^\perp$. So $\{FM_2^\perp, F^*M_1^\perp\}$ is an invariant subspace pair for $\lambda G^\times-A^\times$. To prove (4.3) just note that

$$[F^{-1}(M_1\cap FM_2^\perp)]^\perp=[F^{-1}M_1\cap M_2^\perp]^\perp=F^*M_1^\perp+M_2$$

and

$$[F^{-*}(M_2\cap FM_1^\perp)]^\perp=FM_2^\perp+M_1.$$

Next we show (4.4). Note that $F\pi_2^*F^{-1}$ is indeed a projection, and

$$\text{Ker } F\pi_2^*F^{-1}=\text{Ker } \pi_2^*F^{-1}=F\text{ Ker } \pi_2^*=F(\text{ Im } \pi_2)^\perp=F(F^*M_1^\perp)^\perp=M_1,$$

and

$$\text{Im } F\pi_2^*F^{-1}=\text{Im } F\pi_2^*=(\text{ Ker } \pi_2F^*)^\perp=F(\text{ Ker } \pi_2)^\perp=FM_2^\perp.$$

So indeed $F\pi_2^*F^{-1}=\pi_1$.

Now write

$$G:M_1\oplus FM_2^\perp\to M_2\oplus F^*M_1^\perp,\quad A:M_1\oplus FM_2^\perp\to M_2\oplus F^*M_1^\perp$$

$$B:\mathbb{C}^m\to M_2\oplus F^*M_1^\perp,\quad C:M_1\oplus FM_2^\perp\to\mathbb{C}^m$$

as

$$\begin{bmatrix}G_1 & B_1C_2\\ 0 & G_2\end{bmatrix},\quad \begin{bmatrix}A_1 & B_1C_2\\ 0 & A_2\end{bmatrix},\quad \begin{bmatrix}B_1\\ B_2\end{bmatrix},\quad \begin{bmatrix}C_1 & C_2\end{bmatrix},$$

respectively. Then the functions W_1, W_2 given by (4.5) and (4.6) can be written as

$$W_1(\lambda)=D_1+(1-\lambda)C_1(\lambda G_1-A_1)^{-1}B_1D_2^{-1}$$

$$W_2(\lambda)=D_2+(1-\lambda)D_1^{-1}C_2(\lambda G_2-A_2)^{-1}B_2.$$

Clearly, by multiplication we have $W(\lambda)=W_1(\lambda)W_2(\lambda)$.

It remains to show that $W_1(\lambda)$ and $W_2(\lambda)$ are J-unitary. First write

$$F:M_2\oplus F^*M_1^\perp\to M_1\oplus FM_2^\perp$$

as $F=\begin{bmatrix}F_{11} & F_{12}\\ F_{21} & F_{22}\end{bmatrix}$. Then F_{11} and F_{22} are invertible. Indeed for $x\in M_2$ we have

$Fx \notin FM_2^\perp$ so F_{11} is invertible, and for $x \in F^* M_1^\perp = (F^{-1}M_1)^\perp$ we have $Fx \notin M_1$, proving that F_{22} is invertible. Now

$$G_1^\times F_{11} = G_1^\times (I-\pi_1)F\,|_{M_2} = (I-\pi_2)G^\times F\,|_{M_2} =$$

$$= -(I-\pi_2)F^*A^*\,|_{M_2} = -(I-\pi_2)F^*(P_{M_1} + P_{M_1^\perp})A^*\,|_{M_2},$$

where P_{M_1}, respectively $P_{M_1^\perp}$ are the orthogonal projections on M_1 and $M_1^\perp$, respectively. Now $(I-\pi_2)F^*P_{M_1^\perp} = 0$, so

$$G_1^\times F_{11} = -(I-\pi_2)F^*P_{M_1}A^*\,|_{M_2} = -(I-\pi_2)F^*\,|_{M_1}A_1^*.$$

Putting $E_{11}^{-1} = (I-\pi_2)F^*\,|_{M_1}$ (which indeed is invertible as a map from M_1 to M_2) we have $G_1^\times F_{11} = -E_{11}^{-1}A_1^*$. Likewise $A_1^\times F_{11} = -E_{11}^{-1}G_1^*$. Next, by (4.4) we have $C_1F_{11} = C(I-\pi_1)F\,|_{M_2} = CF(I-\pi_2^*)\,|_{M_2}$. Using (3.11) this is equal to $DJB^*(I-\pi_2^*)\,|_{M_2} = DJ[(I-\pi_2)B]^*\,|_{M_2} = DJB_1^*$. So, since D_2 is J-unitary we obtain $D_1^{-1}C_1F_{11} = D_2JB_1^* = JD_2^{-*}B_1^* = J(B_1D_2^{-1})^*$. This proves (3.4) hold for the realization for W_1 given above. It then follows that $W_1(\lambda)$ is a J-unitary function. But then $W_2(\lambda)$ is J-unitary as well, since $W_2(\lambda) = W_1^{-1}(\lambda)W(\lambda)$. (Of course one can also check that (3.4) hold for the realization of W_2 given in the theorem, in a similar way as was done for W_1.)
□

We can also formulate condition (4.2) in terms of the associated Hermitian matrix. Indeed note that (4.2) implies that $\dim M_1 = \dim M_2$. So $(G-A)M_1 = M_2$. So

$$FM_2^\perp = F((G-A)M_1)^\perp = F(G-A)^{-*}M_1^\perp =$$

$$= -FHF^*M_1^\perp = FH(F^{-1}M_1)^\perp.$$

So (4.2) is equivalent to saying that $F^{-1}M_1 \oplus H(F^{-1}M_1)^\perp = \mathbb{C}^n$, i.e. to saying that $F^{-1}M_1$ is H-nondegenerate.

Our next goal is to prove a theorem about the splitting off of factors degree one.

Theorem 4.2 *Let $W(\lambda)$ be a J-unitary matrix function with minimal realization (4.1). Suppose $(\omega G - A)x = 0$, $x \neq 0$ (in case $\omega = \infty$ this is interpreted as $Gx = 0$.) Assume*

$$\text{span}\,\{x\} \cap F(\,\text{span}\,\{Ax\})^\perp = (0), \qquad \omega \neq 0 \tag{4.7}$$

$$\text{span}\,\{x\} \cap F(\,\text{span}\,\{Gx\})^\perp = (0), \qquad \omega = 0. \tag{4.8}$$

Then $W(\lambda) = W_1(\lambda)W_2(\lambda)$ with $W_1(\lambda)$ a J-unitary function of degree one with a pole at ω, where $W_1(\lambda)$ is given by

$$W_1(\lambda) = I - P + \frac{\omega - 1}{\bar{\omega} - 1}\,\frac{1 - \lambda\bar{\omega}}{\lambda - \omega}P, \qquad \omega \neq 0, \omega \neq \infty, \ |\omega| \neq 1, \tag{4.9}$$

$$W_1(\lambda)=I-P+\lambda P, \qquad \omega=\infty, \tag{4.10}$$

$$W_1(\lambda)=I-P+\frac{1}{\lambda}P, \qquad \omega=0, \tag{4.11}$$

$$W_1(\lambda)=I+\frac{(1-\lambda)\omega}{\lambda-\omega}\frac{Cxx^*C^*J}{\langle Ax,F^{-1}x\rangle}, \qquad |\omega|=1. \tag{4.12}$$

Here P is the one-dimensional projection given by

$$P=\frac{Cxx^*C^*J}{x^*C^*JCx}. \tag{4.13}$$

Proof. Suppose $(\omega G-A)x=0$, $x\neq 0$. We first consider the case $\omega\neq 0$, $\omega\neq\infty$, $|\omega|\neq 1$. Assume (4.7) holds. Then we can apply Theorem 4.1 with the special choice $D_1=I, D_2=D$, and

$$M_1=\text{ span }\{x\},\ \ M_2=\text{ span }\{Ax\}=\text{ span }\{Gx\}.$$

It follows that $I-\pi_1$ is the projection onto span $\{x\}$ along span $\{F^{-*}Ax\}^{\perp}$ and $I-\pi_2$ is the projection onto span $\{Ax\}$ along span $\{F^{-1}x\}^{\perp}$. So

$$(I-\pi_1)y=\frac{\langle F^{-1}y,Ax\rangle}{\langle F^{-1}x,Ax\rangle}x=\frac{x^*A^*F^{-1}y}{x^*A^*F^{-1}x}x,$$

$$(I-\pi_2)y=\frac{\langle y,F^{-1}x\rangle}{\langle Ax,F^{-1}x\rangle}Ax=\frac{x^*F^{-*}y}{x^*F^{-*}Ax}Ax.$$

Then we have

$$W_1(\lambda)=I+(1-\lambda)C(I-\pi_1)(\lambda G-A)^{-1}(I-\pi_2)BD^{-1}=$$

$$=I+(1-\lambda)C(I-\pi_1)(\lambda G-A)^{-1}(I-\pi_2)F^*C^*J=$$

$$=I+(1-\lambda)C(I-\pi_1)(\lambda G-A)^{-1}Ax\frac{x^*F^{-*}F^*C^*J}{\langle Ax,F^{-1}x\rangle}=$$

$$=I+(1-\lambda)C(I-\pi_1)\frac{\omega x}{\lambda-\omega}\frac{x^*C^*J}{\langle Ax,F^{-1}x\rangle}=$$

$$=I+\frac{(1-\lambda)\omega}{\lambda-\omega}\frac{Cxx^*C^*J}{\langle Ax,F^{-1}x\rangle}. \tag{4.14}$$

Next, we compute $\langle Ax,F^{-1}x\rangle$. In Section 3 we have seen that

$$C^*JC=G^*F^{-1}+F^{-*}A^*. \tag{4.15}$$

Because of this we now have

$$\langle C^*JCx, x\rangle = \langle F^{-1}x, Gx\rangle + \langle Ax, F^{-1}x\rangle =$$

$$\frac{1}{\bar\omega}\langle F^{-1}x, Ax\rangle + \langle Ax, F^{-1}x\rangle = -\frac{1}{\bar\omega}\langle G^{\times}x, F^{-1}x\rangle + \langle Ax, F^{-1}x\rangle =$$

$$-\frac{1}{\bar\omega}\langle Gx, F^{-1}x\rangle + \frac{1}{\bar\omega}\langle C^*JCx, x\rangle + \langle Ax, F^{-1}x\rangle =$$

$$\frac{1}{\bar\omega}\langle C^*JCx, x\rangle + (1 - \frac{1}{\omega\bar\omega})\langle Ax, F^{-1}x\rangle.$$

Hence

$$\langle Ax, F^{-1}x\rangle = \frac{\omega(\bar\omega - 1)}{\omega\bar\omega - 1} x^* C^* JCx.$$

Thus

$$W_1(\lambda) = I + \frac{(1-\lambda)(\omega\bar\omega - 1)}{(\lambda - \omega)(\bar\omega - 1)} \frac{Cxx^* C^* J}{x^* C^* JCx}.$$

Denoting by P the projection given by (4.13) we obtain (4.9) after a little calculation.

For the case $\omega = \infty$, $x \in \operatorname{Ker} G$ we have $(\lambda G - A)x = -Ax$, so $(\lambda G - A)^{-1}Ax = -x$, and by (4.15) we have $\langle Ax, F^{-1}x\rangle = \langle C^*JCx, x\rangle$. Starting again from (4.14) (which holds as long as $\omega \neq 0$), we obtain with essentially the same computation as above formula (4.10).

Note that in the case $|\omega| = 1$ formula (4.12) is just (4.14).

It remains to consider the case $\omega = 0$, $x \in \operatorname{Ker} A$. In this case we have

$$(I - \pi_1)y = \frac{\langle F^{-1}y, Gx\rangle}{\langle F^{-1}x, Gx\rangle} x = \frac{x^* G^* F^{-1} y}{x^* G^* F^{-1} x} x,$$

$$(I - \pi_2)y = \frac{\langle y, F^{-1}x\rangle}{\langle Gx, F^{-1}x\rangle} Gx = \frac{x^* F^{-*} y}{x^* F^{-*} Gx} Gx.$$

It then follows that

$$W_1(\lambda) = I + (1-\lambda)C(I - \pi_1)(\lambda G - A)^{-1} Gx \frac{x^* C^* J}{\langle Gx, F^{-1}x\rangle} =$$

$$I + (1-\lambda)\frac{1}{\lambda}\frac{Cxx^* C^* J}{\langle Gx, F^{-1}x\rangle}.$$

Again by (4.15) this is equal to (4.11). □

In the previous theorem factors are split off from the left using the poles of W. A similar result holds concerning the zeros of W. Indeed, suppose $(\omega G^{\times} - A^{\times})x = 0, x \neq 0$. Applying Theorem 4.1 with $D_2 = I$, $D_1 = D$ and

$$F^* M_1^{\perp} = \text{span}\,\{A^{\times}x\} = \text{span}\,\{G^{\times}x\}, \quad FM_2^{\perp} = \text{span}\,\{x\},$$

we obtain in case $\omega \neq 0, \omega \neq \infty$ that

$$\pi_1 y = \frac{\langle F^{-1}y, A^{\times}x\rangle}{\langle F^{-1}x, A^{\times}x\rangle}x, \quad \pi_2 y = \frac{\langle y, F^{-1}x\rangle}{\langle A^{\times}x, F^{-1}x\rangle}A^{\times}x.$$

We see that

$$W_2(\lambda)^{-1} = I - (1-\lambda)D^{-1}C\pi_1(\lambda G^{\times} - A^{\times})^{-1}\pi_2 B =$$

$$= I - \frac{(1-\lambda)\omega}{\lambda - \omega}\frac{D^{-1}Cxx^*C^*D^{-*}J^{-1}}{\langle A^{\times}x, F^{-1}x\rangle}.$$

We can now continue as in the proof of Theorem 4.2 to obtain formulas for W_2. Details are omitted.

4.2 The case of unitary functions. In this subsection we specialize the results of the previous subsection to the case of unitary functions, i.e., $J = I$. We first state and prove a result which follows from Theorem 4.2.

Theorem 4.3. *Let $W(\lambda)$ be unitary on the unit circle such that $W(1)$ is invertible. and have Mcmillan degree n . Then $W(\lambda)$ is the product of n unitary functions of McMillan degree one, which are either of the form*

$$I - P + \frac{\omega - 1}{\bar{\omega} - 1}\frac{1 - \lambda\bar{\omega}}{\lambda - \omega}P, \quad \omega \neq 0, \omega \neq \infty, \tag{4.16}$$

or of the form

$$I - P + \lambda P, \qquad \omega = \infty, \tag{4.17}$$

$$I - P + \frac{1}{\lambda}P, \qquad \omega = 0, \tag{4.18}$$

with P a projection of rank one. By the assumption that $W(1)$ is invertible we have in (4.16) that $\omega \neq 1$.

Proof. First we show that a unitary rational matrix valued function does not have poles on the unit circle. Suppose $W(\lambda)$ has a minimal realization given by (4.1), and suppose $(\omega G - A)x = 0$ for $|\omega| = 1$. Then $\omega \neq 1$, as $G - A$ is invertible. We use the same computation as in the proof of Theorem 4.2. Using (4.15) we have

$$||Cx||^2 = x^*(G^*F^{-1} + F^{-*}A)x =$$
$$= x^*G^*F^{-1}x + \omega x^*F^{-*}Gx = x^*G^*F^{-1}x - \omega x^*A^{\times *}F^{-1}x =$$
$$= x^*G^*F^{-1}x - \omega x^*A^*F^{-1}x + \omega ||Cx||^2 =$$
$$(1 - |\omega|^2)x^*G^*F^{-1}x + \omega ||Cx||^2 = \omega ||Cx||^2.$$

Hence $(1-\omega)||Cx||^2 = 0$, but then $Cx = 0$. Since $(\lambda G - A)x = (\lambda - \omega)Gx$ this implies $0 = Cx = C(\lambda G - A)^{-1}(\lambda - \omega)Gx$. Minimality then gives $Gx = 0$. But then also $Ax = 0$, and thus $x = 0$ by invertibility of $G - A$.

Next, we show that in this case (4.7), (4.8) are satisfied automatically. Indeed, suppose $\omega \neq 0$, and $x \in F(\text{ span } \{Ax\})^{\perp}$. Then $0 = \langle F^{-1}x, AX\rangle = \langle F^{-1}x, Gx\rangle$. So, again by (4.15) we have $Cx = 0$. Arguing as above we obtain $0 = Cx = (\lambda - \omega)C(\lambda G - A)^{-1}Gx$, so $Gx = 0$, and hence also $Ax = 0$, which implies $x = 0$. Hence (4.7) holds. For the case $\omega = 0$, suppose $x \in F(\text{ span } \{Gx\})^{\perp}$. Then $0 = \langle F^{-1}x, Gx\rangle$. Using also $Ax = 0$ we see from (4.15) that $Cx = 0$. This again implies $x = 0$ as above, and hence (4.8) is satisfied.

Now the theorem follows by applying Theorem 4.2 repeatedly. □

Note that the order of the factors can be chosen arbitrarely in the sense that we have freedom in the choice of which pole to factor out first. Of course, if we take another order in this, then the projections P may very well change. Part of the next theorem can also be seen from the previous result.

Theorem 4.4. *Let $W(\lambda)$ be a unitary function with minimal realization given by (4.1). Then $W(\lambda)$ can be factorized as $W(\lambda) = W_1(\lambda)W_2(\lambda)$ where W_1, W_2 are unitary and W_1 has all its poles inside the unit disc hence all its zeros outside the unit disc, and W_2 has all its poles outside the unit disc and all its zeros inside. Further*

$$\delta(W_1) = \nu(W), \quad \delta(W_2) = \delta(W) - \nu(W).$$

Formulas for W_1 and W_2 are derived as follows. Put

$$P = \frac{1}{2\pi i}\int_{\mathrm{T}} G(\lambda G - A)^{-1} d\lambda, \quad P^{\times} = \frac{1}{2\pi i}\int_{\mathrm{T}} G^{\times}(\lambda G^{\times} - A^{\times})^{-1} d\lambda, \tag{4.19}$$

$$Q = \frac{1}{2\pi i}\int_{\mathrm{T}} (\lambda G - A)^{-1} G d\lambda, \quad Q^{\times} = \frac{1}{2\pi i}\int_{\mathrm{T}} (\lambda G^{\times} - A^{\times})^{-1} G^{\times} d\lambda. \tag{4.20}$$

Then $\{\text{Im } Q, \text{Im } P\}$ is an invariant subspace pair for $\lambda G - A$ and $\{\text{Im } Q^{\times}, \text{Im } P^{\times}\}$ is an invariant subspace pair for $\lambda G^{\times} - A^{\times}$. Further

$$F^*(\text{Im } Q)^{\perp} = \text{Im } P^{\times}, \quad F(\text{Im } P)^{\perp} = \text{Im } Q^{\times}, \tag{4.21}$$

and

$$\mathbb{C}^n = \operatorname{Im} Q \oplus \operatorname{Im} Q^{\times} = \operatorname{Im} P \oplus \operatorname{Im} P^{\times}. \tag{4.22}$$

Let π_1 *be the projection along* Im Q *onto* Im $Q^{\times}$ *and* π_2 *the projection along* Im P *onto* Im $P^{\times}$. *Then*

$$W_1(\lambda) = I + (1-\lambda)C(I-\pi_1)(\lambda G - A)^{-1}(I-\pi_2)BD^{-1},$$

$$W_2(\lambda) = D + (1-\lambda)C\pi_1(\lambda G - A)^{-1}\pi_2 B.$$

Proof. First introduce $\hat{A} = (G+A)(G-A)^{-1}$. It is easy to show that P is the spectral projection of $\hat{A}$ corresponding to the left half plane. Introduce also $\hat{C} = C(G-A)^{-1}$. From (3.3) it follows that

$$\hat{A}^* H + H\hat{A} = -2\hat{C}^* \hat{C}. \tag{4.23}$$

Because of this we have that $-i\hat{A}$ is H-dissipative. Indeed

$$\frac{1}{2i}\langle\{(i\hat{A})^* H - H(i\hat{A})\}x, x\rangle = \|Cx\|^2 \geqq 0.$$

By Theorem 11.5 in [IKL] we obtain that the spectral subspace of $-i\hat{A}$ corresponding to the upper (resp. lower) halfplane is maximal H-nonnegative (resp. maximal H-nonpositive). Hence Im P is maximal H-nonnegative and Ker P is maximal H-nonpositive. We shall show that the observability of $(C, \lambda G - A)$ implies that Im P is maximal H-positive and Ker P is maximal H-negative. This in turn implies that Im P and Ker P are H-nondegenerate, and conversely, non-degeneracy implies Im P is in fact H-positive, Ker P is H-negative.

Note that the injectivity of $C(\lambda G - A)^{-1}$ implies the injectivity of $\hat{C}(\lambda - \hat{A})^{-1}$, i.e. the observability of the pair $(\hat{C}, \hat{A})$. Now suppose $x \in \operatorname{Im} P \cap (H \operatorname{Im} P)^{\perp}$. Since $\hat{A}x \in$ Im P we have $0 = \langle Hx, \hat{A}x\rangle = \langle H\hat{A}x, x\rangle$. Then (4.23) implies

$$-2\|\hat{C}x\|^2 = \langle Hx, \hat{A}x\rangle + \langle H\hat{A}x, x\rangle = 0.$$

So $x \in$ Ker $\hat{C}$. We proceed by induction. Suppose we have proved $x \in \bigcap_{i=0}^{j}$ Ker $\hat{C}\hat{A}^i$. Then for $0 \leqq i \leqq j$ we have

$$\hat{A}^* H\hat{A}^i x + H\hat{A}^{i+1} = -2\hat{C}^* \hat{C}\hat{A}^i x = 0.$$

So $\hat{A}^* H\hat{A}^i x = -H\hat{A}^{i+1}x$. Using this we have $\hat{A}^{*j} H\hat{A}^{j+1} x = (-1)^{j+1}\hat{A}^{*2j+1}Hx$. Now compute

$$-2\|\hat{C}\hat{A}^{j+1}x\|^2 = \langle\hat{A}^{*j+2}H\hat{A}^{j+1}x, x\rangle + \langle\hat{A}^{*j+1}H\hat{A}^{j+2}x, x\rangle =$$

$$=(-1)^{j+1}\langle \hat{A}^{*2j+3}Hx,x\rangle+\langle \hat{A}^{2}x,\hat{A}^{*j}H\hat{A}^{j+1}x\rangle$$

$$=(-1)^{j+1}\langle Hx,\hat{A}^{2j+3}x\rangle+(-1)^{j+1}\langle \hat{A}^{2j+3}x,Hx\rangle.$$

Since $\hat{A}^{2j+3}x\in \operatorname{Im} P$ both terms are zero, so $x\in \operatorname{Ker} \hat{C}\hat{A}^{j+1}$ as well. By induction it follows that $x\in \bigcap_{j=0}^{\infty} \operatorname{Ker} \hat{C}\hat{A}^{j}=(0)$. Thus $\operatorname{Im} P\cap(H \operatorname{Im} P)^{\perp}=(0)$. Likewise one shows

$$\operatorname{Ker} P\cap(H \operatorname{Ker} P)^{\perp}=(0).$$

So $\operatorname{Im} P$ and $\operatorname{Ker} P$ are indeed H-nondegenerate, i.e.

$$\mathbb{C}^{n}=\operatorname{Im} P\oplus(H \operatorname{Im} P)^{\perp}=\operatorname{Ker} P\oplus(H \operatorname{Ker} P)^{\perp}.$$

Now

$$(H \operatorname{Im} P)^{\perp}=H^{-1}(\operatorname{Im} P)^{\perp}=F^{*}(A-G)^{*}(\operatorname{Im} P)^{\perp}=F^{*}((A-G)^{-1}\operatorname{Im} P)^{\perp}.$$

Since $(G-A)Q=P(G-A)$, we have $(H \operatorname{Im} P)^{\perp}=F^{*}(\operatorname{Im} Q)^{\perp}=\operatorname{Ker} Q^{*}F^{-*}$. Further

$$Q^{*}F^{-*}=\frac{1}{2\pi i}\int_{\mathbb{T}} G^{*}(sG^{*}-A^{*})^{-1}F^{-*}ds=\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{1}{s}G^{*}(\frac{1}{s}G^{*}-A^{*})^{-1}F^{-*}ds$$

$$=\frac{1}{2\pi i}\int_{\mathbb{T}} -F^{-*}\frac{1}{s}A^{\times}(\frac{1}{s}G^{\times}-A^{\times})^{-1}ds$$

$$=F^{-*}\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{1}{s}-\frac{1}{s^{2}}G^{\times}(\frac{1}{s}G^{\times}-A^{\times})^{-1}ds.$$

Putting $\lambda=\frac{1}{s}$ in the last integral one sees that $Q^{*}F^{-*}=F^{-*}(I-P^{\times})$. So $(H \operatorname{Im} P)^{\perp}=\operatorname{Ker} Q^{*}F^{-*}=\operatorname{Ker}(I-P^{\times})=\operatorname{Im} P^{\times}$. Likewise $(H \operatorname{Ker} P)^{\perp}=\operatorname{Ker} P^{\times}$. Thus

$$\mathbb{C}^{n}=\operatorname{Im} P\oplus \operatorname{Im} P^{\times}=\operatorname{Ker} P\oplus \operatorname{Ker} P^{\times}. \tag{4.24}$$

Since $PG=GQ$, $PA=AQ$ we have $(G-A)^{-1}P=Q(G-A)^{-1}$ and $(G^{\times}-A^{\times})^{-1}P^{\times}=Q^{\times}(G^{\times}-A^{\times})^{-1}$. Noting that $(G^{\times}-A^{\times})=G-A$, we obtain, applying $(G-A)^{-1}$ to (4.24)

$$\mathbb{C}^{n}=\operatorname{Im} Q\oplus \operatorname{Im} Q^{\times}=\operatorname{Ker} Q\oplus \operatorname{Ker} Q^{\times}.$$

Further, as noted before $F^{*}(\operatorname{Im} Q)^{\perp}=\operatorname{Im} P^{\times}$, $F^{*}(\operatorname{Ker} Q)^{\perp}=\operatorname{Ker} P^{\times}$. In a similar way one shows

$$F(\operatorname{Im} P)^{\perp}=\operatorname{Ker} P^{*}F^{-1}=\operatorname{Ker} F^{-1}(I-Q^{\times})=\operatorname{Im} Q^{\times},$$

$$F(\operatorname{Ker} P)^{\perp}=\operatorname{Ker} Q^{\times}.$$

Then we can apply Theorem 4.1 with $M_1 = \text{Im } Q$, $M_2 = \text{Im } P$. The desired formulas for the factors then follow, and also the fact that W_1 has all its poles inside the unit disk, W_2 has all its poles outside the unit disk, as $(\lambda G - A)|_{\text{Im } Q}: \text{Im } Q \to \text{Im } P$ is invertible for $|\lambda| \geqq 1$, and $(\lambda G^\times - A^\times)|_{\text{Im } Q^\times}: \text{Im } Q^\times \to \text{Im } P^\times$ is invertible for $|\lambda| \geqq 1$. So W_2 has all its zeros inside the unit disk, hence all its poles outside the unit disk.

Finally $\delta(W_1) = \dim \text{Ker } Q = \dim \text{Ker } P$, and as $\text{Ker } P$ is maximal H-negative, $\dim \text{Ker } P = \nu(W)$. □

In particular it is a consequence of Theorem 4.5 that $\nu(W)$ is equal to the number of poles of W inside the unit disk.

In the course of the proof of the previous theorem we have also proved the following theorem.

Theorem 4.5. *Let W be as in Theorem 4.4. Then $W(\lambda) = W_3(\lambda) W_4(\lambda)$ where W_3 and W_4 are unitary for $|\lambda| = 1$, W_3 has all its poles outside the unit disk, W_4 has all its poles inside the unit disk, and $\delta(W_3) = \nu(W)$. Let π_3 be the projection along* $\text{Ker } Q$ *onto* $\text{Ker } Q^\times$ *and π_4 the projection along* $\text{Ker } P$ *onto* $\text{Ker } P^\times$. *Then*

$$W_3(\lambda) = I + (1-\lambda) C (I - \pi_3)(\lambda G - A)^{-1} (I - \pi_4) B D^{-1},$$

$$W_4(\lambda) = D + (1-\lambda) C \pi_3 (\lambda G - A)^{-1} \pi_4 B.$$

Comparing with Theorem 4.3 and [AG], Theorem 3.14 we can identify W_1, W_2, W_3 and W_4 as Blaschke-Potapov products.

We next prove an analogue to a well-known inertia theorem for the matrix equation (3.14) with $J = I$. Actually, in case either A or G is invertible the theorem below reduces to the classical case. The general case can be obtained by the use of Möbius transform, but we choose to give a slightly different proof here.

Theorem 4.6. *Let $(C, \lambda G - A)$ be observable, with G-A invertible, and let H be a Hermitian solution to the equation*

$$G^* H G - A^* H A = -C^* C. \tag{4.25}$$

Then the matrix H is invertible, $\lambda G - A$ is invertible for every λ on the unit circle, and the number of positive (resp. negative) eigenvalues of H is equal to the number of zeros of $\det(\lambda G - A)$ *outside (resp. inside) the unit circle.*

Proof. First we show the invertibility of H. As in the proof of Theorem 4.4, introduce $\hat{A} = (G + A)(G - A)^{-1}$, and $\hat{C} = C(G - A)^{-1}$. Again, note that $(\hat{C}, \hat{A})$ is observable, and that (4.25) implies that

$$\hat{A}^* H + H\hat{A} = -2\hat{C}^* \hat{C}. \tag{4.26}$$

Now suppose $Hx=0$. Then (4.26) gives $\hat{C}x=0$, and hence also $H\hat{A}x=0$. In other words, Ker H is $\hat{A}$-invariant and contained in Ker $\hat{C}$. Using the observability of $(\hat{C}, \hat{A})$ then results in Ker $H=(0)$. So H is invertible.

Next, introduce

$$W(\lambda)=I+(1-\lambda)C(\lambda G-A)^{-1}H^{-1}(A^*-G^*)^{-1}C^*$$

then $W(\lambda)$ is unitary on the unit circle by Theorem 3.4, and moreover, the above realization is minimal. As was shown in the proof of Theorem 4.3 $W(\lambda)$ has no poles on the unit circle. By Theorem 4.4 the number of poles of W inside the unit disc is equal to $\nu(W)$. Now $\nu(W)$ in turn is equal to the number of negative eigenvalues of H, while the number of poles of W inside the unit disc equals the number of zeros of det $(\lambda G-A)$ inside the unit disc. This proves the theorem. □

5. Contractions and their minimal completions on the unit circle

Given is the $p\times m$ rational matrix function $W(\lambda)$ with minimal realization

$$W(\lambda)=D+(1-\lambda)C(\lambda G-A)^{-1}B. \tag{5.1}$$

Let us assume that $W(\lambda)$ has contractive values on T , and that D is a strict contraction. In this section we shall discuss the problem of completing $W(\lambda)$ to a unitary function in the following way: we are looking for a $p+m\times p+m$ rational matrix function which is unitary on T of the form

$$\tilde{W}(\lambda)=\begin{bmatrix} W_{11}(\lambda) & W(\lambda) \\ W_{21}(\lambda) & W_{22}(\lambda)\end{bmatrix}. \tag{5.2}$$

We are particularly interested in such completions with the extra property that $\delta(\tilde{W})=\delta(W)$. Unitary completions (5.2) with this property will be called *minimal unitary completions of* $W(\lambda)$. We shall give realization formulas for the minimal unitary completions, and their associated chain scattering matrices or partial inverses, defined by

$$\Theta(\lambda)=\begin{bmatrix} W_{11}(\lambda)-W(\lambda)W_{22}(\lambda)^{-1}W_{21}(\lambda) & W(\lambda)W_{22}(\lambda)^{-1} \\ -W_{22}(\lambda)^{-1}W_{21}(\lambda) & W_{22}(\lambda)^{-1}\end{bmatrix}. \tag{5.3}$$

For the real line case, this problem was treated in [GR]. We shall follow the same approach as in [GR] for the case presently under consideration.

The problem of finding a minimal unitary completion is connected to Darlington synthesis, see [D]; in Darlington synthesis the function $W(\lambda)$ is assumed to be *stable*, i.e., all its poles are inside the unit disk. The usual solution of this problem then proceeds as follows. The functions W_{11}, W_{22}, W_{21} satisfy the following equations:

$$I-WW^{*}=W_{11}W_{11}^{*},\quad I-W^{*}W=W_{22}^{*}W_{22},\quad W_{21}=-W_{22}^{*-1}W^{*}W_{11}.$$

Hence W_{11} and W_{22} can be found as stable spectral factors, e.g., by solving a single Riccati equation; finding W_{21} is then simple. In terms of a realization of W this turns out to be rather straightforward, as from a minimal realization of W one gets for free a *minimal* realization for $I-WW^{*}$ and one for $I-W^{*}W$ (because of the stability of W). In case W is not assumed to be stable this is no longer true, i.e., the realizations one gets for $I-WW^{*}$ and $I-W^{*}W$ are not necessarily minimal, compare, however, Lemma 5.4. It is therefore surprising that the solution to the problem of finding minimal unitary completions can still be had from the solution of a single Riccati equation, see Theorem 5.6.

5.1 Characterization of contractive rational matrix functions. In this subsection we characterize contractive rational matrix functions in terms of solutions of algebraic Riccati equations. For this purpose the realization (5.1) need only be observable and stabilizable, i.e. the uncontrollable eigenvalues of $(\lambda G-A,\ B)$ are inside the unit disk; however, for some results we need minimality of (5.1). We shall make our assumptions explicit at each step. The main result of this subsection has a strong connection to [W] (cf. in particular [W], Lemma 5). It may be viewed as a form of the bounded real lemma (see, e.g., also [AV]).

Let D be a strict contraction, and introduce the following matrices:

$$\gamma=2(G^{*}-A^{*})^{-1}C^{*}(I-DD^{*})^{-1}C(G-A)^{-1}, \tag{5.4}$$

$$\alpha=(G+A)(G-A)^{-1}+2BD^{*}(I-DD^{*})^{-1}C(G-A)^{-1}, \tag{5.5}$$

$$\beta=2B(I-D^{*}D)^{-1}B^{*}. \tag{5.6}$$

The following algebraic Riccati equation plays an important role in our analysis, we shall call this equation the *state characteristic Riccati equation*

$$P\gamma P-P\alpha^{*}-\alpha P+\beta=0. \tag{5.7}$$

(Compare with [GR], where this concept was introduced, and where it plays a central role.) The importance of this equation is explained by the following theorem.

Theorem 5.1. *Let D be a strict contraction, and let (5.1) be an observable and stabilizable realization for a rational $p\times m$ matrix function $W(\lambda)$. Then the following are equivalent:*

(i) $W(\lambda)$ *has contractive values on* $\mathbb{T}$,

(ii) *the state characteristic Riccati equation has a Hermitian solution P,*

(iii) *the matrix*

$$\begin{bmatrix} -\alpha^{*} & \gamma \\ -\beta & \alpha \end{bmatrix} \tag{5.8}$$

has only even partial multiplicities at its pure imaginary eigenvalues,

(iv) *the matrix* $\begin{bmatrix} -\alpha^* & \gamma \\ -\beta & \alpha \end{bmatrix}$ *has an invariant subspace M such that* $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} M = M^{\perp}$.

The matrix (5.8) will be called the *state characteristic matrix.*

We shall prove Theorem 5.1 in several steps. The first step is to prove a lemma, which implies the equivalence of *(ii)*, *(iii)* and *(iv)*.

Lemma 5.2. *The pair* (γ, α) *is observable. In case the realization (5.1) is minimal the pair* (α, β) *is controllable.*

Proof. We first show the second part of the Lemma. Suppose $\beta^* x = 0$ and $\alpha^* x = \lambda_0 x$, for some λ_0 and some x. From (5.6) we have $\beta \geqq 0$. It follows that $B^* x = 0$, and then, using (5.5), we have $\alpha^* x = (G^* - A^*)^{-1}(G^* + A^*)x = \lambda_0 x$. So for any λ we have

$$(\lambda - \lambda_0)x = (\lambda - \alpha^*)x = (G^* - A^*)^{-1}\{\lambda(G^* - A^*) - (G^* + A^*)\}x =$$
$$= (G^* - A^*)^{-1}\{(\lambda - 1)G^* - (\lambda + 1)A^*\}x.$$

Hence for all $\lambda \neq \lambda_0$, $\lambda \neq -1$ we have

$$\frac{\lambda + 1}{\lambda - \lambda_0} x = (\frac{\lambda - 1}{\lambda + 1} G^* - A^*)^{-1}(G^* - A^*)x.$$

Since $B^* x = 0$ we obtain for all μ $B^*(\mu G^* - A^*)^{-1}(G^* - A^*)x = 0$. As $(\mu G - A, B)$ is controllable it follows that $(G^* - A^*)x = 0$. However, $G - A$ is invertible, so $x = 0$.

Next, suppose $\gamma x = 0$, $\alpha x = \lambda_0 x$. Then by (5.4) $C(G - A)^{-1}x = 0$, and by (5.5) $\alpha x = (G + A)(G - A)^{-1}x = \lambda_0 x$. Hence

$$(\lambda - \lambda_0)x = (\lambda - \alpha)x = ((\lambda - 1)G - (\lambda + 1)A)(G - A)^{-1}x.$$

So

$$\frac{\lambda + 1}{\lambda - \lambda_0}(G - A)^{-1}x = (\frac{\lambda - 1}{\lambda + 1} G - A)^{-1}x.$$

Thus for all μ we have $C(\mu G - A)^{-1}x = 0$, which gives by the observability of $(C, \mu G - A)$ that $x = 0$. □

Using this lemma we have the following proposition.

Proposition 5.3. *(i) The state characteristic Riccati equation (5.7) has a Hermitian solution P if and only if the state characteristic matrix (5.8) has only even partial multiplicities at its pure imaginary eigenvalues.*

(ii) In case the realization (5.1) is minimal every Hermitian solution P of the state characteristic Riccati equation is invertible.

Proof. *(i)* is a well-known consequence of Lemma (5.2), see e.g. [LR, S, C], also [GLR].

To prove *(ii)*, suppose $Px=0$. Multiplying (5.7) on the left with x^*, and on the right with x, we obtain $x^*\beta x=0$. As β is positive semidefinite, this implies $\beta x=0$. Then multiplying (5.7) only with x on the right we have $P\alpha^* x=0$. So $\operatorname{Ker} P$ is an α^*-invariant subspace contained in $\operatorname{Ker}\beta$. By controllability of (α, β) this gives $\operatorname{Ker} P=(0)$. □

The next step in the proof of Theorem 5.1 is to note that if $W(\lambda)$ is contractive on the unit circle, then the function $I-W(\lambda)W^*(\lambda)$ must be positive semidefinite on the unit circle, and conversely. Moreover, as D is a strict contraction this function is actually regular. Hence it must factorize as

$$I-W(\lambda)W^*(\lambda)=W_{11}(\lambda)W_{11}^*(\lambda), \tag{5.9}$$

where we can choose $W_{11}(\lambda)$ to have the same poles as $W(\lambda)$, hence, in particular $\delta(W_{11})=\delta(W)$. We shall show that if the state characteristic matrix has a Hermitian solution P, then one can construct a function $W_{11}(\lambda)$, for which (5.9) holds, thereby proving the implication *(ii)* implies *(i)*.

To achieve this we first give a realization for $I-W(\lambda)W^*(\lambda)$. From (5.1) on easily sees that

$$I-W(\lambda)W^*(\lambda)= \tag{5.10}$$

$$=I-DD^*+(1-\lambda)\begin{bmatrix}-C & DB^*\end{bmatrix}(\lambda\tilde{G}-\tilde{A})^{-1}\begin{bmatrix}BD^*\\ C^*\end{bmatrix}.$$

where

$$\tilde{G}=\begin{bmatrix}G & -BB^*\\ 0 & -A^*\end{bmatrix},\quad \tilde{A}=\begin{bmatrix}A & -BB^*\\ 0 & -G^*\end{bmatrix}.$$

For its inverse we have

$$(I-W(\lambda)W^*(\lambda))^{-1}=(I-DD^*)^{-1}-(1-\lambda)\tilde{C}(\lambda\tilde{G}^\times-\tilde{A}^\times)^{-1}\tilde{B}, \tag{5.11}$$

where

$$\tilde{C}=(I-DD^*)^{-1}\begin{bmatrix}-C & DB^*\end{bmatrix},\quad \tilde{B}=\begin{bmatrix}BD^*\\ C^*\end{bmatrix}(I-DD^*)^{-1},$$

and

$$\tilde{G}^\times=\begin{bmatrix}G+BD^*(I-DD^*)^{-1}C & -B(I-D^*D)^{-1}B^*\\ C^*(I-DD^*)^{-1}C & -A^*-C^*(I-DD^*)^{-1}DB^*\end{bmatrix},$$

$$\tilde{A}^{\times} = \begin{bmatrix} A+BD^*(I-DD^*)^{-1}C & -B(I-D^*D)^{-1}B^* \\ C^*(I-DD^*)^{-1}C & -G^*-C^*(I-DD^*)^{-1}DB^* \end{bmatrix}.$$

Proof of *(ii)* $\Rightarrow$ *(i)* **in Theorem 5.1.** Suppose $P=P^*$ solves (5.7). Put $X=(G-A)^{-1}P$, $Y=P(G^*-A^*)^{-1}$. Then $Y=X^*$, and using (5.4)-(5.6) we can rewrite (5.7) as

$$2YC^*(I-DD^*)^{-1}CX-Y(G^*+A^*)-2YC^*(I-DD^*)^{-1}DB^*+ \tag{5.12}$$

$$-(G+A)X-2BD^*(I-DD^*)^{-1}CX+2B(I-D^*D)^{-1}B=0.$$

From the definition of X and Y we have

$$Y(G^*-A^*)-(G-A)X=0. \tag{5.13}$$

Adding (5.13) and (5.12), and dividing by 2, we obtain that Y and X satisfy

$$YC^*(I-DD^*)^{-1}CX-YA^*-YC^*(I-DD^*)^{-1}DB^*= \tag{5.14}$$

$$=GX+BD^*(I-DD^*)^{-1}CX-B(I-D^*D)^{-1}B^*.$$

Taking adjoints in this equation, using $Y=X^*$ we also have

$$YC^*(I-DD^*)^{-1}CX-YG^*-YC^*(I-DD^*)^{-1}DB^*= \tag{5.15}$$

$$=AX+BD^*(I-DD^*)^{-1}CX-B(I-D^*D)^{-1}B^*.$$

Now put $M_1^{\times} = \operatorname{Im} \begin{bmatrix} X \\ I \end{bmatrix}$ and $M_2^{\times} = \operatorname{Im} \begin{bmatrix} Y \\ I \end{bmatrix}$. Then (5.14) and (5.15) are exactly telling us that

$$\tilde{G}^{\times} M_1^{\times} \subset M_2^{\times}, \quad \tilde{A}^{\times} M_1^{\times} \subset M_2^{\times},$$

respectively. In other words, the pair $\{M_1^{\times}, M_2^{\times}\}$ is an invariant subspace pair for $\lambda\tilde{G}^{\times}-\tilde{A}^{\times}$. Taking $M_1=M_2=\operatorname{Im}\begin{bmatrix} I \\ 0 \end{bmatrix}$ it is clear that the pair $\{M_1, M_2\}$ is an invariant subspace pair for $\lambda G-A$, and moreover

$$\mathbb{C}^{2n}=M_1 \oplus M_1^{\times}=M_2 \oplus M_2^{\times}.$$

It follows from Section 2 that we have the following factorization

$$I-W(\lambda)W^*(\lambda)=W_1(\lambda)W_2(\lambda),$$

where

$$W_1(\lambda)=(I-DD^*)^{1/2}+(1-\lambda)\begin{bmatrix} -C & DB^* \end{bmatrix} \times$$

$$\times\begin{bmatrix} I & -X \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \lambda G-A & (1-\lambda)BB^* \\ 0 & -\lambda A^*+G^* \end{bmatrix}\begin{bmatrix} I & -Y \\ 0 & 0 \end{bmatrix}\begin{bmatrix} BD^*-YC^* \\ 0 \end{bmatrix}(I-DD^*)^{-1/2}=$$

$$=(I-DD^*)^{1/2}+(1-\lambda)C(\lambda G-A)^{-1}(YC^*-BD^*)(I-DD^*)^{-1/2}.$$

Furthermore,

$$W_2(\lambda)=(I-DD^*)^{1/2}+(I-DD^*)^{-1/2}(1-\lambda)\begin{bmatrix} -C & DB^* \end{bmatrix}\times$$

$$\times\begin{bmatrix} 0 & X \\ 0 & I \end{bmatrix}\begin{bmatrix} \lambda G-A & (1-\lambda)BB^* \\ 0 & -\lambda A^*+G^* \end{bmatrix}\begin{bmatrix} 0 & Y \\ 0 & I \end{bmatrix}\begin{bmatrix} BD^*-YC^* \\ 0 \end{bmatrix}=$$

$$=(I-DD^*)^{1/2}+(1-\lambda)(I-DD^*)^{-1/2}(-CX+DB^*)(-\lambda A^*+G^*)^{-1}C^*.$$

Using $Y=X^*$ one sees that $W_2(\lambda)=W_1(\bar{\lambda}^{-1})^*$, so $I-W(\lambda)W^*(\lambda)=W_1(\lambda)W_1^*(\lambda)$, and hence $W(\lambda)$ is a contraction for $\lambda\in\mathbb{T}$. □

To prove the implication *(i)* ⇒ *(iii)* we need again a lemma.

Lemma 5.4. *The realization (5.10) for* $I-W(\lambda)W^*(\lambda)$, *and hence also the realization (5.11) for its inverse, are minimal for every* λ *on the unit circle.*

Proof. Take $\lambda_0\in\mathbb{T}$, and $\begin{bmatrix} x \\ y \end{bmatrix}$ such that

$$\left[\lambda_0\begin{bmatrix} G & -BB^* \\ 0 & -A^* \end{bmatrix}-\begin{bmatrix} A & -BB^* \\ 0 & -G^* \end{bmatrix}\right]\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

and $-Cx+DB^*y=0$, Then we have

$$(\lambda_0 G-A)x-(\lambda_0-1)BB^*y=0,\quad (-\lambda_0 A^*+G^*)y=0.$$

Note that $\lambda_0\neq 1$, as $\lambda_0=1$ would imply $y=0$, and hence also $x=0$, by the invertibility of $G-A$.

Now consider

$$(\lambda_0-1)\langle BB^*y,\ y\rangle=\langle(\lambda_0 G-A)x,\ y\rangle=\langle x,\ \lambda_0^{-1}(G^*-\lambda_0 A^*)y\rangle=0,$$

because $\lambda_0\in\mathbb{T}$. As $\lambda_0\neq 1$ we obtain $B^*y=0$, and from this also $Cx=0$, and $(\lambda_0 G-A)x=0$. Since $(C,\ \lambda G-A)$ is observable we obtain $x=0$, and from the stabilizability of $(\lambda G-A,\ B)$ we have $y=0$. This proves observability of the realization (5.10). Controllability is proved in essentially the same way. □

Proof of *(i)* ⇒ *(iii)* in Theorem 5.1. Suppose that $W(\lambda)$ is a contraction for $\lambda\in\mathbb{T}$. Note that

$$\begin{bmatrix} -\alpha^* & \gamma \\ -\beta & \alpha \end{bmatrix}=\begin{bmatrix} 0 & (G^*-A^*)^{-1} \\ I & 0 \end{bmatrix}(\tilde{G}^\times+\tilde{A}^\times)(\tilde{G}^\times-\tilde{A}^\times)^{-1}\begin{bmatrix} 0 & I \\ G^*-A^* & 0 \end{bmatrix}.$$

From this we obtain that

$$\begin{bmatrix} 0 & I \\ G^*-A^* & 0 \end{bmatrix} \left[\lambda - \begin{bmatrix} -\alpha^* & \gamma \\ -\beta & \alpha \end{bmatrix} \right] \begin{bmatrix} 0 & (G^*-A^*)^{-1} \\ I & 0 \end{bmatrix} =$$

$$= \frac{1}{\lambda+1}\left(\frac{\lambda-1}{\lambda+1}\tilde{G}^\times - \tilde{A}^\times\right)(\tilde{G}^\times - \tilde{A}^\times)^{-1}.$$

Now for pure imaginary λ we have that $\frac{\lambda-1}{\lambda+1}$ is on the unit circle.

By equivalence the partial multiplicities of the state characteristic matrix at $\lambda \in i\mathbb{R}$ are the same as those of the pencil $\mu\tilde{G}^\times - \tilde{A}^\times$ at $\mu = \frac{\lambda-1}{\lambda+1} \in \mathbb{T}$. Since by the previous lemma the realization (5.11) for $(I - W(\lambda)W^*(\lambda))^{-1}$ is minimal for $\lambda \in \mathbb{T}$ the partial multiplicities of $\mu\tilde{G}^\times - \tilde{A}^\times$ at unimodular eigenvalues μ_0 coincide with the partial pole multiplicities of $(I - W(\lambda)W^*(\lambda))^{-1}$ at μ_0. But $W(\lambda)$ is contractive, so $(I - W(\lambda)W^*(\lambda))^{-1}$ is positive semidefinite. Hence its partial pole multiplicities at unimodular poles are even, (as can be seen, e.g., by the unit circle version of Rellichs theorem, [R], see also [GLR], and [LRR], Proposition 3.3). Combining the above remarks proves the desired result. □

Next we consider a special case in Theorem 5.1, namely the case of a strictly contractive rational matrix function.

Theorem 5.5. *Let D be a strict contraction and let (5.1) be an observable and stabilizable realization for a rational $p \times m$ matrix function $W(\lambda)$. Then the following are equivalent:*

(i) *$W(\lambda)$ is a strict contraction for $\lambda \in \mathbb{T}$,*

(ii) *the state characteristic matrix (5.8) has no pure imaginary eigenvalues,*

(iii) *the state characteristic Riccati equation has a Hermitian solution P such that $\alpha^* - \gamma P$ is stable.*

Proof. The equivalence of *(ii)* and *(iii)* is well known, in view of Lemma 5.2.

Suppose $W(\lambda)$ is a strict contraction for $\lambda \in \mathbb{T}$. Then $I - W(\lambda)W^*(\lambda)$ is poistive defninite for unimodular λ, and so the inverse of this function is positive definite as well. Hence $I - W(\lambda)W^*(\lambda)$ has no zeros on the unit circle. Applying the same arguments as in the proof of Theorem 5.1 we have that *(ii)* holds.

Conversely, suppose that *(ii)* holds. By the argument in the last part of the proof of Theorem 5.1 we have that $\mu\tilde{G}^\times - \tilde{A}^\times$ is invertible for all $\mu \in \mathbb{T}$. But this implies, in view of (5.11) that $I - W(\lambda)W^*(\lambda)$ is invertible for all $\lambda \in \mathbb{T}$. As *(ii)* and *(iii)* are equivalent we may conclude from Theorem 5.1 that $W(\lambda)$ is a contraction, i.e., $I - W(\lambda)W^*(\lambda)$ is positive semidefinite. However, an invertible positive semidefinite matrix is positive definite.

Hence, $W(\lambda)$ is a strict contraction for $\lambda \in \mathbb{T}$. □

5.2 Description of all unitary minimal completions. Using the notation of the previous subsection we have the following theorem, which is the main result of this section.

Theorem 5.6. *Let* $W(\lambda)=D+(1-\lambda)C(\lambda G-A)^{-1}B$. *be a minimal realization for a rational matrix function which is contractive for* $\lambda \in \mathbb{T}$, *with* D *a strict contraction and* $G-A$ *invertible. Then the following hold*

(i) For every Hermitian solution P *of the state characteristic Riccati equation (5.7) the function*

$$W_P(\lambda)=\begin{bmatrix}(I-DD^*)^{1/2} & D\\ -D^* & (I-D^*D)^{1/2}\end{bmatrix}+ \tag{5.16}$$

$$+(1-\lambda)\begin{bmatrix} C \\ (I-D^*D)^{-1/2}(-D^*C+B^*X^{-1})\end{bmatrix}(\lambda G-A)^{-1}\begin{bmatrix}(YC^*-BD^*)(I-DD^*)^{-1/2} & B\end{bmatrix}$$

is a unitary minimal completion. Here $Y=P(G^*-A^*)^{-1}$, $X=(G-A)^{-1}P$.

(ii) Any unitary minimal completion of $W(\lambda)$ *has the form*

$$\tilde{W}(\lambda)=\begin{pmatrix} I & 0\\ 0 & T\end{pmatrix} W_P(\lambda)\begin{pmatrix} S & 0\\ 0 & I\end{pmatrix}$$

for some unitary matrices T *and* S *and some Hermitian solution* P *of the state characteristic Riccati equation. The matrices* S *and* T *are uniquely determined by* $\tilde{W}(1)$ *as follows*

$$S=(I-DD^*)^{-1/2}\left[\tilde{W}(1)\right]_{11},\quad T=\left[\tilde{W}(1)\right]_{22}(I-D^*D)^{-1/2}.$$

(iii) Given unitary S *and* T *the correspondence*

$$P\rightarrow \begin{pmatrix} I & 0\\ 0 & T\end{pmatrix} W_P(\lambda)\begin{pmatrix} S & 0\\ 0 & I\end{pmatrix}$$

is one to one.

Proof. To prove *(i)* let P be a Hermitian solution of (5.7). By Proposition 5.3 P is invertible. Multiplying both sides of (5.7) with P^{-1} we have

$$\gamma-\alpha^*P^{-1}-P^{-1}\alpha+P^{-1}\beta P^{-1}=0.$$

Multiplying this in turn with G^*-A^* on the left and $G-A$ on the right we obtain

$$C^*(I-DD^*)^{-1}C-G^*P^{-1}G+A^*P^{-1}A+$$

$$-C^*(I-DD^*)^{-1}DB^*P^{-1}(G-A)-(G^*-A^*)P^{-1}BD^*(I-DD^*)^{-1}C+$$

$$+(G^*-A^*)P^{-1}B(I-D^*D)^{-1}CB^*P^{-1}(G-A)=0.$$

Use $X^{-1}=P^{-1}(G-A)$ to rewrite this as

$$G^*P^{-1}G-A^*P^{-1}A= \tag{5.17}$$

$$\begin{bmatrix} C^* & (-C^*D+X^{-*}B)(I-D^*D)^{-1/2} \end{bmatrix} \begin{bmatrix} C \\ (I-D^*D)^{-1/2}(-D^*C+B^*X^{-1}) \end{bmatrix}.$$

So, with $H=-P^{-1}$ equation (3.3) is satisfied for the realization (5.16). The second equation in Theorem 3.1 *(iii)* is also satisfied for this realization as a straightforward calculation shows. So by Theorem 3.1 the function $W_P(\lambda)$ given by (5.16) is unitary. As it is clearly a minimal completion we have proved *(i)*.

To prove *(ii)* take a minimal unitary completion

$$\tilde{W}(\lambda)=\begin{bmatrix} M & D \\ N & Q \end{bmatrix}+(1-\lambda)\begin{bmatrix} C \\ Z_1 \end{bmatrix}(\lambda G-A)^{-1}\begin{bmatrix} Z_2 & B \end{bmatrix}.$$

Then $\begin{bmatrix} M & D \\ N & Q \end{bmatrix}$ is unitary, so

$$M=(I-DD^*)^{1/2}S,\quad Q=T(I-D^*D)^{1/2},\quad N=-TD^*S,$$

for some unique unitary matrices T and S. By Theorem 3.1 and the remark following it there is a unique invertible Hermitian matrix H such that

$$G^*HG-A^*HA=-C^*C-Z_1^*Z_1,$$

$$\begin{bmatrix} Z_2 & B \end{bmatrix}=H^{-1}(A^*-G^*)^{-1}\begin{bmatrix} C^* & Z_1^* \end{bmatrix}\begin{bmatrix} M & D \\ N & Q \end{bmatrix},$$

i.e.,

$$Z_2=H^{-1}(A^*-G^*)^{-1}(C^*M+Z_1^*N),$$

$$B=H^{-1}(A^*-G^*)^{-1}(C^*D+Z_1^*Q).$$

Solve the last equation for Z_1^*:

$$Z_1^*=\{(A^*-G^*)HB-C^*D\}(I-D^*D)^{-1/2}T^*. \tag{5.18}$$

Inserting this in the first equation we have after a little calculation

$$Z_2=H^{-1}(A^*-G^*)^{-1}(C^*M+Z_1^*N)=$$

$$=\{H^{-1}(A^*-G^*)^{-1}C^*-BD^*\}(I-DD^*)^{-1/2}S$$

Again using (5.18) we have

$$G^*HG-A^*HA=-C^*C-Z_1^*Z_1=$$

$$= -\{(A^* - G^*)HB - C^*D\}(I - D^*D)^{-1}\{-D^*C + B^*H(G - A)\} - C^*C,$$

which is equivalent to $P = H^{-1}$ solving (5.17), and hence to P solving the state characteristic Riccati equation. Taking all this into account we have proved *(ii)*.

Note that the matrices S, T and P appearing in the proof of *(ii)* are solved uniquely from $\tilde{W}(\lambda)$. So *(iii)* holds. □

The following corollary gives a connection between the solutions of the state characteristic equation and the number of poles of the contractive function W inside the unit disk.

Corollary 5.7. *Let $W(\lambda)$, given by the minimal realization (5.1), be contractive, with D a strict contraction. Then for any Hermitian solution P of the state characteristic Riccati equation the number of negative (resp. positive) eigenvalues of P is equal to the number of poles of $W(\lambda)$ outside (resp. inside) the unit disk.*

Proof. From the proof of the previous theorem we have that P satisfies equation (5.17)

$$G^*P^{-1}G - A^*P^{-1}A =$$

$$\begin{bmatrix} C^* & (-C^*D + X^{-*}B)(I - D^*D)^{-1/2} \end{bmatrix} \begin{bmatrix} C \\ (I - D^*D)^{-1/2}(-D^*C + B^*X^{-1}) \end{bmatrix}.$$

We can now apply Theorem 4.6, with $H = -P^{-1}$, to obtain the theorem. □

5.3 The scattering matrix. We shall now provide formulas for the partial inverse (or chain scattering matrix) of a given minimal unitary completion of $W(\lambda)$, in terms of the matrices appearing in a minimal realization of $W(\lambda)$. Recall that the partial inverse of a minimal unitary completion (5.2) is given by (5.3). Because of the fact that (5.2) is unitary we have

$$-W_{22}(\lambda)^{-1}W_{21}(\lambda) = W^*(\lambda)W_{11}^*(\lambda)^{-1} = \Theta_{21}(\lambda).$$

It follows that

$$\Theta_{11} = W_{11} - WW_{22}^{-1}W_{21} = (W_{11}W_{11}^* + WW^*)W_{11}^{*-1} = W_{11}^{*-1}.$$

So we have for the partial inverse, instead of formula (5.3),

$$\Theta(\lambda) = \begin{bmatrix} W_{11}^*(\lambda)^{-1} & W(\lambda)W_{22}(\lambda)^{-1} \\ W^*(\lambda)W_{11}^*(\lambda)^{-1} & W_{22}(\lambda)^{-1} \end{bmatrix}. \tag{5.19}$$

Using Theorem 5.6 we have the following result.

Theorem 5.8. *The partial inverse of the minimal unitary completion $W_P(\lambda)$, given by (5.16), has the following realization*

$$\Theta_P(\lambda)=\begin{bmatrix}(I-DD^*)^{-1/2} & D(I-D^*D)^{-1/2}\\ D^*(I-DD^*)^{-1/2} & (I-D^*D)^{-1/2}\end{bmatrix}+ \tag{5.20}$$

$$(1-\lambda)\begin{bmatrix}(I-DD^*)^{-1}(C-DB^*X^{-1})\\(I-D^*D)^{-1}(D^*C-B^*X^{-1})\end{bmatrix}(\lambda\hat{G}-\hat{A})^{-1}\begin{bmatrix}YC^*(I-DD^*)^{-1/2} & B(I-D^*D)^{-1/2}\end{bmatrix},$$

where

$$\hat{G}=G-B(I-D^*D)^{-1}(-D^*C+B^*X^{-1}),$$

$$\hat{A}=A-B(I-D^*D)^{-1}(-D^*C+B^*X^{-1}).$$

Here, X and Y are as in Theorem 5.6.

Moreover, any partial inverse of a minimal unitary completion is of the form

$$\tilde{\Theta}(\lambda)=\Theta_P(\lambda)\begin{bmatrix}S & 0\\ 0 & T^{-1}\end{bmatrix},$$

for some unitary matrices S and T, and a Hermitian solution P of the state characteristic Riccati equation.

Proof. As $\Theta_{22}(\lambda)=W_{22}(\lambda)^{-1}$ the formula for Θ_{22} is a direct consequence of the formula for W_{22} obtained from Theorem 5.6.*(i)*. Next, we compute $\Theta_{11}(\lambda)=W_{11}^*(\lambda)^{-1}$. From Theorem 5.6 we have

$$\Theta_{11}(\lambda)=(I-DD^*)^{-1/2}+(1-\lambda)(I-DD^*)^{-1}(CX-DB^*)\times$$

$$\times(-\lambda\{A^*-C^*(I-DD^*)^{-1}(CX-DB^*)\}+\{G^*-C^*(I-DD^*)^{-1}(CX-DB^*)\})^{-1}\times$$

$$\times C^*(I-DD^*)^{-1/2}.$$

Now using (5.14), this equals

$$\Theta_{11}(\lambda)=(I-DD^*)^{-1/2}+(1-\lambda)(I-DD^*)^{-1}(CX-DB^*)\times$$

$$\times(\lambda Y^{-1}\hat{G}X-Y^{-1}\hat{A}X)^{-1}C^*(I-DD^*)^{-1/2}=$$

$$=(I-DD^*)^{-1/2}+(1-\lambda)(I-DD^*)^{-1}(C-DB^*X^{-1})(\lambda\hat{G}-\hat{A})^{-1}YC^*(I-DD^*)^{-1/2}.$$

This is exactly the formula given in the theorem.

We now compute $\Theta_{12}(\lambda)=W(\lambda)W_{22}(\lambda)^{-1}$. By the results from Section 2 a realization for Θ_{12} is given by

$$\Theta_{12}(\lambda)=D(I-D^*D)^{-1/2}+(1-\lambda)\tilde{C}(\lambda\tilde{G}-\tilde{A})^{-1}\tilde{B},$$

where

$$\tilde{C}=\begin{bmatrix} C & D(I-D^*D)^{-1}(D^*C-B^*X^{-1}) \end{bmatrix}, \quad \tilde{B}=\begin{bmatrix} B(I-D^*D)^{-\frac{1}{2}} \\ B(I-D^*D)^{-\frac{1}{2}} \end{bmatrix},$$

$$\tilde{G}=\begin{bmatrix} G & B(I-D^*D)^{-1}(D^*C-B^*X^{-1}) \\ 0 & G+B(I-D^*D)^{-1}(D^*C-B^*X^{-1}) \end{bmatrix},$$

$$\tilde{A}=\begin{bmatrix} A & B(I-D^*D)^{-1}(D^*C-B^*X^{-1}) \\ 0 & A+B(I-D^*D)^{-1}(D^*C-B^*X^{-1}) \end{bmatrix}.$$

Put $S=\begin{bmatrix} I & I \\ 0 & I \end{bmatrix}$, then

$$S^{-1}\tilde{G}S=\begin{bmatrix} G & 0 \\ 0 & \hat{G} \end{bmatrix}, \quad S^{-1}\tilde{A}S=\begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix}, \quad S^{-1}\tilde{B}=\begin{bmatrix} 0 \\ B(I-D^*D)^{-\frac{1}{2}} \end{bmatrix},$$

and

$$\tilde{C}S=\begin{bmatrix} C & (I-DD^*)^{-1}(C-DB^*X^{-1}) \end{bmatrix}.$$

It then easily follows that $\Theta_{12}(\lambda)$ is indeed given by the formula in the theorem as the first coordinate space is uncontrollable.

Finally, we compute $\Theta_{21}(\lambda)=W^*(\lambda)W_{11}^*(\lambda)^{-1}$. A realization for this function is given by

$$\Theta_{21}(\lambda)=D^*(I-DD^*)^{-\frac{1}{2}}+(1-\lambda)C'(\lambda G'-A')^{-1}B',$$

where

$$C'=\begin{bmatrix} B^* & -D^*(I-DD^*)^{-1}(CX-DB^*) \end{bmatrix}, \quad B'=\begin{bmatrix} C^*(I-DD^*)^{-\frac{1}{2}} \\ C^*(I-DD^*)^{-\frac{1}{2}} \end{bmatrix},$$

$$G'=\begin{bmatrix} A^* & -C^*(I-DD^*)^{-1}(CX-DB^*) \\ 0 & A^*-C^*(I-DD^*)^{-1}(CX-DB^*) \end{bmatrix},$$

$$A'=\begin{bmatrix} G^* & -C^*(I-DD^*)^{-1}(CX-DB^*) \\ 0 & G^*-C^*(I-DD^*)^{-1}(CX-DB^*) \end{bmatrix}.$$

Again applying similarity with S diagonalizes A' and G', and

$$S^{-1}B'=\begin{bmatrix} 0 \\ C^*(I-DD^*)^{-\frac{1}{2}} \end{bmatrix}, \quad C'S=\begin{bmatrix} B^* & (I-D^*D)^{-1}(-D^*CX+B^*) \end{bmatrix}.$$

So we obtain

$$\Theta_{21}(\lambda)=D^*(I-DD^*)^{-\frac{1}{2}}+(1-\lambda)(I-D^*D)^{-1}(-D^*CX+B^*)\times$$

Again applying similarity with S diagonalizes A' and G', and

$$S^{-1}B' = \begin{bmatrix} 0 \\ C^*(I-DD^*)^{-\frac{1}{2}} \end{bmatrix}, \quad C'S = \begin{bmatrix} B^* & (I-D^*D)^{-1}(-D^*CX+B^*) \end{bmatrix}.$$

So we obtain

$$\Theta_{21}(\lambda) = D^*(I-DD^*)^{-\frac{1}{2}} + (1-\lambda)(I-D^*D)^{-1}(-D^*CX+B^*)\times$$
$$\times(\lambda(A^*-C^*(I-DD^*)^{-1}(CX-DB^*))-(G^*-C^*(I-DD^*)^{-1}(CX-DB^*))^{-1}\times$$
$$\times C^*(I-DD^*)^{-\frac{1}{2}},$$

and using (5.14) as before, we have that this equals the formula in the theorem. This proves the first part of the theorem.

The second part of the theorem is seen as follows. If $\tilde{W} = \begin{pmatrix} I & 0 \\ 0 & T \end{pmatrix} W_P \begin{pmatrix} S & 0 \\ 0 & I \end{pmatrix}$ then $\tilde{\Theta} = \Theta_P \begin{bmatrix} S & 0 \\ 0 & T^{-1} \end{bmatrix}$. The theorem is proved. □

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MSC 1991: 15A54, 93B15, 47A57, 47A68

Submitted: March 20, 1991
Revised: July 1, 1991