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Integral Equations
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ON THE SINGULAR BOCHNER–MARTINELLI INTEGRAL

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The Bochner-Martinelli (B.-M.) kernel inherits, for $n \geq 2$, only some of properties of the Cauchy kernel in $\mathbb{C}$. For instance it is known that the singular B.-M. operator M_n is not an involution for $n \geq 2$. M. Shapiro and N. Vasilevski found a formula for M_2^2 using methods of quaternionic analysis which are essentially complex-twodimensional. The aim of this article is to present a formula for M_n^2 for any $n \geq 2$. We use now Clifford Analysis but for $n = 2$ our formula coincides, of course, with the above-mentioned one.

1 Theorem on the square of the singular Bochner-Martinelli

1.1 We consider the m–dimensional complex space $\mathbb{C}^m$ of the variable $z = (z_1, \ldots, z_m)$. If $\{z, w\} \subset \mathbb{C}^m$ then $\langle z, w\rangle := \sum_{k=1}^{m} z_k w_k$, $|z| := \sqrt{\langle z, \overline{z}\rangle}$ where $\overline{z} := (\overline{z}_1, \ldots, \overline{z}_m)$. Topology in $\mathbb{C}^m$ is determined by the metric $dist(z, w) := |z - w|$. If $z \in \mathbb{C}^m$ then $Re\ z := (Re\ z_1, \ldots, Re\ z_m) \in \mathbb{R}^m$, $Re\ z_j = x_j$, and $Im\ z = (Im\ z_1, \ldots, Im\ z_m)$, $Im\ z_j = y_j$. Hence $\mathbb{C}^m \cong \mathbb{R}^{2m}$. Orientation in $\mathbb{C}$ is defined by the order of coordinates $(x_1, y_1, \ldots, x_m, y_m)$ which means that the differential form of volume is of the form: $dv := (-1)^{\frac{m(m-1)}{2}} dx_1 \wedge \ldots \wedge dx_m \wedge dy_1 \wedge \ldots \wedge dy_m = (-1)^{\frac{m(m-1)}{2}} dx \wedge dy = (-1)^{\frac{m(m-1)}{2}} (\frac{i}{2})^m dz \wedge d\overline{z} = (-1)^{\frac{m(m-1)}{2}} (-\frac{i}{2})^m d\overline{z} \wedge dz$ with $dz := dz_1 \wedge \ldots \wedge dz_m$. Standard notation: $\mathbb{B}(z; \varepsilon) := \{\zeta \in \mathbb{C}^m \,|\, |\zeta - z| < \varepsilon\}$.

1.2 Denote, as usually, by $\mathcal{U}(\zeta, z)$ an exterior differential form (of type $(m, m-1)$)

$$\mathcal{U}(\zeta, z) := \frac{(m-1)!}{(2\pi i)^m} \sum_{k=1}^{m} (-1)^{k-1} \frac{\overline{\zeta}_k - \overline{z}_k}{|\zeta - z|^{2m}} d\overline{\zeta}_{[k]} \wedge d\zeta,$$

where $d\overline{\zeta}_{[k]} := d\overline{\zeta}_1 \wedge \ldots \wedge d\overline{\zeta}_{k-1} \wedge d\overline{\zeta}_{k+1} \wedge \ldots \wedge d\overline{\zeta}_m$. For $m = 1$ the form $\mathcal{U}(\zeta, z)$ is nothing more than the Cauchy kernel $\frac{d\zeta}{2\pi i(\zeta - z)}$.

Let $g(\zeta,\ z)$ be a fundamental solution of the Laplace operator, i.e.

$$g(\zeta,\ z) := \begin{cases} -\frac{(m-2)!}{(2\pi)^m}\frac{1}{|\zeta-z|^{2m-2}}, & m>1, \\ \frac{1}{2\pi}ln|\zeta-z|, & m=1. \end{cases}$$

Then

$$\begin{aligned} \mathfrak{U}(\zeta,\ z) &= \sum_{k=1}^{m}(-1)^{k-1}\frac{\partial g}{\partial \zeta_k}d\overline{\zeta}_{[k]}\wedge d\zeta = \\ &= (-1)^{m-1}\partial_\zeta g\wedge\sum_{k=1}^{m}d\overline{\zeta}_{[k]}\wedge d\zeta_{[k]} \end{aligned}$$

where $\partial := \sum\limits_{k=1}^{m} d\zeta_k\frac{\partial}{\partial\zeta_k}$. $\Delta_{\mathbb{C}^m}$ will stand for the complex Laplace operator

$$\Delta_{\mathbb{C}^m} := \sum_{k=1}^{m}\frac{\partial^2}{\partial\zeta_k\partial\overline{\zeta}_k} = \frac{1}{4}\sum_{k=1}^{m}\left(\frac{\partial^2}{\partial x_k^2}+\frac{\partial^2}{\partial y_k^2}\right) = \frac{1}{4}\Delta_{\mathbb{R}^{2m}}.$$

1.3 Let Ω be a bounded domain in $\mathbb{C}^m$ with the piece–wise smooth boundary $\Gamma = \partial\Omega$. For any f holomorphic in Ω and continuous up to the boundary Γ the following Bochner–Martinelli integral representation holds:

$$f(z) = \int_{\partial\Omega} f(\zeta)\mathfrak{U}(\zeta,\ z) := F_f(z), \quad z\in\Omega. \tag{1.1}$$

For $m=1$ one has just the Cauchy integral formula but for $m>1$ the kernel in (1.1) is not holomorphic (in z and ζ), which causes many difficulties.

In particular, the Bochner–Martinelli–type integral for an integrable function f on Γ ($f\in\mathcal{L}^1(\Gamma)$):

$$F_f(z) := \int_\Gamma f(\zeta)\mathfrak{U}(\zeta,\ z), \quad z\notin\Gamma,$$

is not holomorphic in Ω unless f is a trace of a holomorphic function. If $z\in\Gamma$ then $F_f(z)$ does not, generally speaking, exist as an improper integral because the integrand has a singularity $\frac{1}{|\zeta-z|^{2m-1}}$. That is why for $z\in\Gamma$ there will be considered the Cauchy principal value of the Bochner–Martinelli integral:

$$F_f(z) := \lim_{\varepsilon\to 0}\int_{\Gamma\setminus\mathbb{B}(z;\ \varepsilon)} f(\zeta)\mathfrak{U}(\zeta,\ z), \quad z\in\Gamma.$$

We restrict ourselves to the case of a Liapunov surface Γ. It is well–known (see, for instance, [Ky]) for $f\in C^{0,\ \mu}$, the Hölder function space, $F_f(z)$ exists everywhere on Γ, and for $f\in L_p$, $p>1$, $F_f(z)$ exists a.e. on Γ and $F_f\in L_p$.

Let

$$M_m[f](z) := 2\int_\Gamma f(\zeta)\mathfrak{U}(\zeta,\ z),\quad z \in \Gamma.$$

For $m = 1$

$$M_1[f](z) = \frac{1}{\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}d\tau$$

and it is very well known that M_1 is an involution both on $C^{0,\ \mu}$ and on L_p:

$$M_1^2 = I, \tag{1.2}$$

I the identity operator. The property plays an extremly important role in many problems.

It is quite natural then to ask the question about M_m^2 for any m. In [Se] it was stated that $M_m^2 = I$ for any m, as well, but then an unremovable error was detected in the proof (see [AiYu], [Ky, §22]). The right correlation for $m = 2$ was given by M. Shapiro and N. Vasilevski, see [VaSh1], [VaSh2]:

$$M_2^2 = I + (N_2 Z)^2 \tag{1.3}$$

where the operator N_2 acts by the rule

$$N_2[f](z) := -\frac{1}{2\pi^2}\int_\Gamma \frac{(\overline{\zeta}_1 - \overline{z}_1)d\zeta_{[2]}\wedge d\overline{\zeta} + (\overline{\zeta}_2 - \overline{z}_2)d\zeta_{[1]}\wedge d\overline{\zeta}}{|\zeta - z|^4}f(\zeta),$$

and Z is the operator of complex conjugation: $Z(f) := \overline{f}$. It was shown also that

$$M_2 N_2 + N_2 M_2 Z = 0. \tag{1.4}$$

Denote by $w_2(\zeta,\ z)$ the kernel of the integral $N_2[f]$. It was proved later in [MiSh] that for any $f \in C^1(\Gamma;\ \mathbb{C})$ and $z \notin \Gamma$ there holds:

$$\int_\Gamma w_2(\zeta,\ z)f(\zeta) = \int_\Gamma g(\zeta,\ z)\cdot\frac{1}{|grad\rho(\zeta)|}\cdot\frac{\partial(f,\ \rho)}{\partial(\zeta_1,\ \zeta_2)}\cdot dS \tag{1.5}$$

where $\Gamma = \{z \in \mathbb{C}^2 | \rho(z) = 0,\ \rho \in C^1(\mathbb{C}^2;\ \mathbb{R}),\ grad\ \rho|\Gamma \notin 0\}$. Formula (1.5) explains the commutation relation (1.4).

1.4 Formula (1.3) was obtained by the methods of quaternionic analysis which, by their nature, fits the case of $\mathbb{C}^2$. In this paper we treat the general situation of $\mathbb{C}^m$ using Clifford analysis.

1.5 Introduce the following notation for $1 \le p < q \le m$:

$$\begin{aligned} N_{p,\ q}[f](z) &:= 2\frac{(m-1)!}{(2\pi i)^m}\int_\Gamma \frac{1}{|\zeta - z|^{2m}}\left((-1)^q(\overline{\zeta}_p - \overline{z}_p)f(\zeta)d\overline{\zeta}\wedge d\zeta_{[q]}\right. \\ &\quad \left. -(-1)^p(\overline{\zeta}_q - \overline{z}_q)f(\zeta)d\overline{\zeta}\wedge d\zeta_{[p]}\right) \\ M_{p,\ q} &:= N_{p,\ q}\circ Z. \end{aligned}$$

Let $w_{p,\ q}$ be a kernel of this integral, then for $m = 2$ the only possible combination is $p = 1$, $q = 2$, and hence $w_{1,\ 2}$ coincides with w_2 and $N_2 = N_{1,\ 2}$.

1.6 **Theorem** *(Square of the singular Bochner–Martinelli operator). Under the above described conditions the following equality holds on $C^{0,\ \mu}(\Gamma;\ \mathbb{C})$, $0 < \mu < 1$; and $L_p(\Gamma;\ \mathbb{C})$, $p > 1$:*

$$M_m^2 = I + \sum_{1 \leq p < q \leq m} M_{p,\ q}^2.$$

2 Prof of the Theorem

2.1 Let $Cl_{0,\ 2m}$ be a complex Clifford algebra with generators $e_1,\ e_2,\ \ldots,\ e_{2m}$. This means that

$$\begin{aligned} e_k^2 = -1 =: -e_0, &\qquad k \in \{1,\ \ldots,\ 2m\}, \\ e_k e_q + e_q e_k = 0, &\qquad k \neq q, \end{aligned}$$

any element $a \in Cl_{0,\ 2m}$ is of the form

$$a = \sum_A a_A e_A$$

where $A = (\alpha_1,\ \ldots,\ \alpha_p)$ with $1 \leq \alpha_1 < \ldots < \alpha_p \leq 2m$, $\{a_A\} \subset \mathbb{C}$, $e_A := e_{\alpha_1} \ldots e_{\alpha_p}$. All the necessary information the reader can find in [DeSoSo] and in many other sources.

Mention that for the Clifford conjugation of a we use the notation a^*:

$$a^* := \sum_A a_A e_A^*$$

with $e_A^* := e_{\alpha_p}^* \ldots e_{\alpha_1}^* := (-e_{\alpha_p}) \ldots (-e_{\alpha_1})$, and for the complex conjugation $\overline{a}$:

$$\overline{a} := \sum_A \overline{a}_A \cdot e_A$$

with $\overline{a}_A := Re\ a_A - iIm\ a_A$. Note that sometimes both conjugationes are denoted by the same symbol but we prefer to use different ones.

2.2 Let Ω be a domain in $\mathbb{R}^{2m}$ and let $f \in C^1(\Omega;\ Cl_{0,\ 2m})$. f is called hyperholomorphic, or monogenic, or regular, in Ω if

$$D[f](t) := \sum_{k=1}^{2m} e_k \cdot \frac{\partial f}{\partial t_k} = 0, \quad \forall t \in \Omega.$$

We shall denote $ker\ D =: \mathfrak{M}(\Omega)$. For $m = 1$ the definition reduces to the usual holomorphic functions of one complex variable. For an arbitrary m the theory of hyperholomorphic functions inherits many basic structural properties of one–dimensional complex analysis.

Let $t = (t_1, t_2, \ldots, t_{2m-1}, t_{2m})$, $\tau = (\tau_1, \tau_2, \ldots, \tau_{2m-1}, \tau_{2m})$ be two points in $\mathbb{R}^{2m}$. The Clifford–Cauchy kernel $\mathcal{K}(\tau - t)$ is defined as

$$\mathcal{K}(\tau - t) := \frac{1}{A_{2m}} \frac{(\tau - t)^*}{|\tau - t|^{2m}} \tag{2.6}$$

where $A_{2m} = |\mathbb{S}^{2m-1}| = \frac{2\pi^m}{\Gamma(m)} = \frac{2\pi^m}{(m-1)!}$,

$$\begin{aligned} \tau - t &= \sum_{k=1}^{2m} (\tau_k - t_k) e_k, \\ (\tau - t)^* &= \sum_{k=1}^{2m} (\tau_k - t_k) e_k^* = -\sum_{k=1}^{2m} (\tau_k - t_k) e_k. \end{aligned}$$

Let $\sigma := \sigma_\tau$ denote a differential form

$$\sigma_\tau := \sum_{j=1}^{2m} (-1)^{j-1} e_j d\tau_{[j]},$$

where $d\tau_{[j]} := d\tau_1 \wedge \ldots \wedge d\tau_{j-1} \wedge d\tau_{j+1} \wedge \ldots \wedge d\tau_{2m}$, then for any function $f \in \mathfrak{M}(\Omega;\ Cl_{0,\ 2m}) \cap C(\overline{\Omega};\ Cl_{0,\ 2m})$ the Clifford–Cauchy integral formula holds:

$$\int_\Gamma \mathcal{K}(\tau - t)\sigma_\tau f(\tau) = \begin{cases} f(t), & t \in \Omega, \\ 0, & t \in \mathbb{R}^{2m} \setminus \overline{\Omega}. \end{cases} \tag{2.7}$$

Let $f \in C^{0,\ \mu}(\Gamma;\ Cl_{0,\ 2m})$ then on Γ

$$S[f](t) := 2 \int_\Gamma \mathcal{K}(\tau - t)\sigma_\tau f(\tau) \tag{2.8}$$

exists in the sense of Cauchy's principal value; what is more on $C^{0,\ \mu}(\Gamma;\ Cl_{0,\ 2m})$

$$S^2 = I, \tag{2.9}$$

compare with (1.2) and (1.3).

If $f \in L_p(\Gamma;\ Cl_{0,\ 2m})$ then the Cauchy principal value of (2.8) exists almost everywhere on Γ and (2.9) holds on L_p.

2.3 For what follows we need some properties of idempotents in Clifford algebra $Cl_{0,\ 2m}$. Set

$$J_k := \frac{1}{2}(1 + ie_{2k-1}e_{2k})$$

for $k \in \{1, \ldots, m\}$. J_k is an idempotent: $J_k^2 = J_k$. Besides $J_j J_k = J_k J_j$. The product $J := J_1 \cdots J_m$ is a primitive idempotent: $J^2 = J$, and $J_j \cdot J = J \cdot J_j$ for any j. Very important for us is: for any j

$$\begin{aligned} e_{2j} J &= ie_{2j-1} J, \\ e_{2j-1} e_{2j} J &= -iJ. \end{aligned}$$

Both J_k and $\overline{J}_k$ are zero divisors:

$$J_k \cdot \overline{J}_k = \overline{J}_k \cdot J_k = 0.$$

2.4 Introduce notation:

$$w_j := e_{2j-1} + ie_{2j}, \quad j \in \{1, \ldots, m\},$$

then

$$\overline{w}_j := e_{2j-1} - ie_{2j}.$$

Of course

$$J_k = -\frac{1}{2} e_{2k-1} \overline{w}_k$$

which is equivalent to

$$\begin{aligned} \overline{w}_k &= 2e_{2k-1} \cdot J_k; \\ w_k &= 2e_{2k-1} \cdot \overline{J}_k. \end{aligned}$$

This gives immediately: for any j

$$w_j \cdot J_j = 0, \quad \overline{w}_j \cdot \overline{J}_j = 0,$$

and

$$w_j^2 = 0, \quad (\overline{w}_j)^2 = 0.$$

One can check up directly the following equalities:

$$\begin{aligned} w_j \cdot \overline{w}_k &= -\overline{w}_k \cdot w_j \ \text{ for } \ j \neq k; \\ \overline{w}_j \cdot \overline{w}_k &= -\overline{w}_k \cdot \overline{w}_j \ \text{ for } \ j \neq k; \\ \overline{w}_j e_{2p-1} e_{2q-1} &= e_{2p-1} e_{2q-1} \overline{w}_j \ \text{ for } \ j \neq p, \ j \neq q, \ p \neq q; \\ w_j \cdot \overline{w}_j &= -4 \cdot J_j, \\ \overline{w}_j \cdot w_j &= -4 \cdot \overline{J}_j, \\ \overline{w}_j \cdot \overline{w}_k \cdot J &= \begin{cases} 0, & j = k, \\ 4e_{2j-1} e_{2k-1} J, & j \neq k; \end{cases} \\ w_j \cdot \overline{w}_k \cdot J &= \begin{cases} -4J, & j = k, \\ 0, & j \neq k; \end{cases} \\ \overline{w}_j \overline{w}_k \overline{w}_p \overline{w}_q J &= 16 e_{2j-1} e_{2k-1} e_{2p-1} e_{2q-1} J \ \text{ for } \ j, \ k, \ p, \ q \text{ different in paires }. \end{aligned}$$

2.5 On the underlying space, $\mathbb{R}^{2m}$, of the Clifford algebra $Cl_{0,\,2m}$, we introduce now the following complex structure. If $t \in \mathbb{R}^{2m}$ then we write x_j for t_{2j-1} and y_j for t_{2j}, setting after this $z_j := x_j + iy_j$, $z := (z_1, \ldots, z_m)$. In the same fashion $\tau \in \mathbb{R}^{2m}$ gives us $\zeta_j = \xi_j + i\eta_j$, $\zeta = (\zeta_1, \ldots, \zeta_m)$. In these notations, we have:

$$\begin{aligned}(\tau - t) &= \sum_{k=1}^{2m}(\tau_k - t_k)e_k = \\ &= \sum_{k=1}^{m}((\xi_k - x_k)e_{2k-1} + (\eta_k - y_k)e_{2k}) = \\ &= \frac{1}{2}\sum_{k=1}^{m}((\zeta_k - z_k)e_{2k-1} + (\overline{\zeta}_k - \overline{z}_k)e_{2k-1} + \\ &\quad + i(\overline{\zeta}_k - \overline{z}_k)e_{2k} - i(\zeta_k - z_k)e_{2k}) = \\ &= \frac{1}{2}\sum_{k=1}^{m}((\zeta_k - z_k)\cdot \overline{w}_k + (\overline{\zeta}_k - \overline{z}_k)\cdot w_k).\end{aligned}$$

Let

$$\mathcal{K}_j(\zeta - z) := \frac{-1}{2A_{2m}}\frac{\zeta_j - z_j}{|\zeta - z|^{2m}} = -\frac{(m-1)!}{4\pi^m}\frac{\zeta_j - z_j}{|\zeta - z|^{2m}},$$

hence

$$\mathcal{K}(\zeta - z) = \sum_{j=1}^{m}(\mathcal{K}_j(\zeta - z)\cdot \overline{w}_j + \overline{\mathcal{K}}_j(\zeta - z)\cdot w_j).$$

Introduce the following notations:

$$\begin{aligned}\alpha_j &:= \frac{(-1)^{[\frac{m}{2}-\frac{1}{4}]}(-1)^j}{2^m \cdot i^m}, \\ \beta_j &:= \frac{(-1)^{[\frac{m}{2}-\frac{1}{4}]}(-1)^{m+j}}{2^m \cdot i^m}, \quad j \in \{1, \ldots, m\}.\end{aligned}$$

We have:

$$\begin{aligned}d\tau_{2j-1} &= \frac{1}{2}(d\zeta_j + d\overline{\zeta}_j), \\ d\tau_{2j} &= \frac{i}{2}(d\overline{\zeta}_j - d\zeta_j).\end{aligned}$$

Thus

$$\begin{aligned}d\tau_{[2j-1]} &= \frac{1}{2}(d\zeta_1 + d\overline{\zeta}_1) \wedge \frac{i}{2}(d\overline{\zeta}_1 - d\zeta_1) \wedge \ldots \wedge \frac{i}{2}(d\overline{\zeta}_{j-1} - d\zeta_{j-1}) \wedge \\ &\quad \wedge \frac{i}{2}(d\overline{\zeta}_j - d\zeta_j) \wedge \frac{1}{2}(d\overline{\zeta}_{j+1} + d\zeta_{j+1}) \wedge \ldots \wedge \frac{i}{2}(d\overline{\zeta}_m - d\zeta_m) \\ &= i^m \cdot \frac{1}{2^{2m-1}} \cdot (d\overline{\zeta}_j - d\zeta_j) \bigwedge_{\substack{k=1 \\ k\neq j}}^{m}(d\zeta_k + d\overline{\zeta}_k) \wedge (d\overline{\zeta}_k - d\zeta_k) = \\ &= \frac{i^m}{2^m}(d\overline{\zeta}_j - d\zeta_j) \bigwedge_{\substack{k=1 \\ k\neq j}}^{m} d\zeta_k \wedge d\overline{\zeta}_k = \\ &= \frac{i^m(-1)^{[\frac{m}{2}-\frac{1}{4}]+(m-1)}}{2^m}(d\overline{\zeta}_j - d\zeta_j) \wedge d\overline{\zeta}_{[j]} \wedge d\zeta_{[j]}\end{aligned}$$

and

$$
\begin{aligned}
d\tau_{[2j]} &= i^{m-1}\cdot\frac{1}{2^{2m-1}}\cdot(d\overline{\zeta}_j + d\zeta_j)\bigwedge_{\substack{k=1\\k\neq j}}^{m}(d\zeta_k + d\overline{\zeta}_k)\wedge(d\overline{\zeta}_k - d\zeta_k) = \\
&= \frac{i^{m-1}}{2^m}(d\overline{\zeta}_j + d\zeta_j)\bigwedge_{\substack{k=1\\k\neq j}}^{m} d\zeta_k\wedge d\overline{\zeta}_k = \\
&= \frac{(-1)^{[\frac{m}{2}-\frac{1}{4}]+(m-1)}\cdot i^{m-1}}{2^m}(d\overline{\zeta}_j + d\zeta_j)\wedge d\overline{\zeta}_{[j]}\wedge d\zeta_{[j]}.
\end{aligned}
$$

All this gives:

$$
\begin{aligned}
\sigma_\zeta &= \frac{(-1)^{[\frac{m}{2}-\frac{1}{4}]-1}}{2^m\cdot i^m}\sum_{j=1}^{m}\big(ie_{2j}(d\overline{\zeta}_j + d\zeta_j)+ \\
&\quad +e_{2j-1}(d\overline{\zeta}_j - d\zeta_j)\big)\wedge d\overline{\zeta}_{[j]}\wedge d\zeta_{[j]} = \\
&= \frac{(-1)^{[\frac{m}{2}-\frac{1}{4}]-1}}{2^m\cdot i^m}\sum_{j=1}^{m}\big(d\overline{\zeta}_j(ie_{2j} + e_{2j-1})+ \\
&\quad +d\zeta_j(ie_{2j} - e_{2j-1})\big)\wedge d\overline{\zeta}_{[j]}\wedge d\zeta_{[j]} = \\
&= \frac{(-1)^{[\frac{m}{2}-\frac{1}{4}]-1}}{2^m\cdot i^m}\sum_{j=1}^{m}\big((-1)^{j-1}(ie_{2j} + e_{2j-1})d\overline{\zeta}\wedge d\zeta_{[j]}+ \\
&\quad +(-1)^{m-j}(ie_{2j} - e_{2j-1})d\overline{\zeta}_{[j]}\wedge d\zeta\big).
\end{aligned}
$$

Thus finally

$$
\sigma_\zeta = \sum_{j=1}^{m}(\alpha_j w_j d\overline{\zeta}\wedge d\zeta_{[j]} + \beta_j\overline{w}_j d\overline{\zeta}_{[j]}\wedge d\zeta).
$$

Consider now

$$
\begin{aligned}
\mathcal{K}(\zeta - z)\cdot\sigma_\zeta &= \sum_{j,\ k=1}^{m}\big(\mathcal{K}_j(\zeta - z)\cdot\alpha_k\cdot d\overline{\zeta}\wedge d\zeta_{[k]}\cdot\overline{w}_j\cdot w_k+ \\
&\quad +\overline{\mathcal{K}}_j(\zeta - z)\cdot\alpha_k\cdot d\overline{\zeta}\wedge d\zeta_{[k]}\cdot w_j\cdot w_k + \\
&\quad +\mathcal{K}_j(\zeta - z)\cdot\beta_k\cdot d\overline{\zeta}_{[k]}\wedge d\zeta\cdot\overline{w}_j\cdot\overline{w}_k + \\
&\quad +\overline{\mathcal{K}}_j(\zeta - z)\cdot\beta_k\cdot d\overline{\zeta}_{[k]}\wedge d\zeta\cdot w_j\cdot\overline{w}_k\big).
\end{aligned}
$$

Hence:

$$
\begin{aligned}
S[f](\zeta) &:= 2\int_\Gamma \mathcal{K}(\zeta - z)\cdot\sigma_\zeta\cdot f(\zeta) \\
&= 2\sum_{j,\ k=1}^{m}\left(\int_\Gamma \mathcal{K}_j(\zeta - z)\cdot\alpha_k\cdot d\overline{\zeta}\wedge d\zeta_{[k]}\cdot\overline{w}_j w_k\cdot f(\zeta)+\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{\Gamma} \overline{\mathcal{K}}_j(\zeta - z)\cdot\alpha_k\cdot d\overline{\zeta}\wedge d\zeta_{[k]}\cdot w_j w_k\cdot f(\zeta) + \\
&+\int_{\Gamma} \mathcal{K}_j(\zeta - z)\cdot\beta_k\cdot d\overline{\zeta}_{[k]}\wedge d\zeta\cdot \overline{w}_j\overline{w}_k\cdot f(\zeta) + \\
&\left.+\int_{\Gamma} \overline{\mathcal{K}}_j(\zeta - z)\cdot\beta_k\cdot d\overline{\zeta}_{[k]}\wedge d\zeta\cdot w_j\overline{w}_k\cdot f(\zeta)\right).
\end{aligned}
$$

Denote:

$$
\begin{aligned}
S_{jk}[f](\zeta) &:= 2\int_{\Gamma}\beta_k\overline{\mathcal{K}}_j(\zeta - z)f(\zeta)d\overline{\zeta}_{[k]}\wedge d\zeta, \\
P_{jk}[f](\zeta) &:= 2\int_{\Gamma}\beta_k\mathcal{K}_j(\zeta - z)f(\zeta)d\overline{\zeta}_{[k]}\wedge d\zeta, \\
Q_{jk}[f](\zeta) &:= 2\int_{\Gamma}\alpha_k\mathcal{K}_j(\tau - z)f(\zeta)d\overline{\zeta}\wedge d\zeta_{[k]}, \\
R_{jk}[f](\zeta) &:= 2\int_{\Gamma}\alpha_k\overline{\mathcal{K}}_j(\zeta - z)f(\zeta)d\overline{\zeta}\wedge d\zeta_{[k]}.
\end{aligned}
$$

Hence

$$
S = \sum_{j,k=1}^{m}\left(Q_{jk}\overline{w}_j w_k + R_{jk}w_j w_k + S_{jk}w_j\overline{w}_k + P_{jk}\overline{w}_j\overline{w}_k\right).
$$

2.6 Consider now $S\cdot J$. Taking into account Section 2.4 we have:

$$
\begin{aligned}
S\cdot J &= \sum_{j,\ k=1}^{m}\left(S_{jk}w_j\overline{w}_k\cdot J + P_{jk}\overline{w}_j\overline{w}_k\cdot J\right) = \\
&= -4\sum_{k=1}^{m}S_{kk}J + 4\sum_{j\neq k}P_{jk}e_{2j-1}e_{2k-1}J.
\end{aligned}
$$

From here, again applying Section 2.4,

$$
\begin{aligned}
S^2\cdot J &= 16\sum_{k=1}^{m}\sum_{q=1}^{m}S_{kk}S_{qq}J - \\
&\quad -16\sum_{j\neq k}\sum_{q=1}^{m}P_{jk}S_{qq}e_{2j-1}e_{2k-1}J + \\
&\quad +\sum_{k=1}^{m}\sum_{p\neq q}Q_{kk}P_{pq}\overline{w}_k w_k\overline{w}_p\overline{w}_q J +
\end{aligned}
$$

$$+\sum_{k=1}^{m}\sum_{p\neq q} S_{kk}P_{pq}w_k\overline{w}_k\overline{w}_p\overline{w}_q J +$$
$$+\sum_{j\neq k}\sum_{p\neq q} Q_{jk}P_{pq}\overline{w}_j w_k\overline{w}_p\overline{w}_q J +$$
$$+\sum_{j\neq k}\sum_{p\neq q} R_{jk}P_{pq}w_j w_k\overline{w}_p\overline{w}_q J +$$
$$+\sum_{j\neq k}\sum_{p\neq q} S_{jk}P_{pq}w_j\overline{w}_k\overline{w}_p\overline{w}_q J +$$
$$+\sum_{j\neq k}\sum_{p\neq q} P_{jk}P_{pq}\overline{w}_j\overline{w}_k\overline{w}_p\overline{w}_q J.$$

Compute now all the terms:

1) $$\sum_{k=1}^{m}\sum_{p\neq q} Q_{kk}P_{pq}\overline{w}_k w_k\overline{w}_p\overline{w}_q J = -16\sum_{p\neq q}(Q_{pp}+Q_{qq})P_{pq}e_{2p-1}e_{2q-1}J;$$

2) $$\sum_{k=1}^{m}\sum_{p\neq q} S_{kk}P_{pq}w_k\overline{w}_k\overline{w}_p\overline{w}_q J = -16\sum_{\substack{p\neq q\\ k\neq p\\ k\neq q}} S_{kk}P_{pq}e_{2p-1}e_{2q-1}J;$$

3) $$\sum_{j\neq k}\sum_{p\neq q} Q_{jk}P_{pq}\overline{w}_j w_k\overline{w}_p\overline{w}_q J = 16\sum_{\substack{p\neq q\\ k\neq p\\ k\neq q}} Q_{pk}(P_{qk}-P_{kq})e_{2p-1}e_{2q-1}J;$$

4) $$\sum_{\substack{j\neq k\\ p\neq q}} R_{jk}P_{pq}w_j w_k\overline{w}_p\overline{w}_q J = 16\sum_{p,q}(R_{qp}-R_{pq})P_{pq}J;$$

5) $$\sum_{j\neq k}\sum_{p\neq q} S_{jk}P_{pq}w_j\overline{w}_k\overline{w}_p\overline{w}_q J = 16\sum_{\substack{p\neq q\\ k\neq p\\ k\neq q}} S_{pk}(P_{kq}-P_{qk})e_{2p-1}e_{2q-1}J;$$

6) $$\sum_{j\neq k}\sum_{p\neq q} P_{jk}P_{pq}\overline{w}_j\overline{w}_k\overline{w}_p\overline{w}_q J =$$

$$= 16 \sum_{\substack{j,\ k,\ p,\ q \\ different \\ in\ pairs}} P_{jk} P_{pq} e_{2j-1} e_{k-1} e_{2p-1} e_{2q-1} J.$$

2.7 This means that the equality

$$S^2 \cdot J = J$$

takes the form:

$$\begin{aligned} M_m^2 J - &\sum_{1 \le p < q \le m} M_{pq}^2 J - \\ -16 &\sum_{p \ne q} \sum_{k=1}^{m} P_{pq} S_{kk} e_{2p-1} e_{2q-1} J - \\ -16 &\sum_{p \ne q} (Q_{pp} + Q_{qq}) P_{pq} e_{2p-1} e_{2q-1} J - \\ -16 &\sum_{\substack{p \ne q \\ k \ne p \\ k \ne q}} S_{kk} P_{pq} e_{2p-1} e_{2q-1} J + \\ +16 &\sum_{\substack{p \ne q \\ k \ne p \\ k \ne q}} S_{kp} (P_{kq} - P_{qk}) e_{2p-1} e_{2q-1} J + \\ +16 &\sum_{\substack{p \ne q \\ k \ne p \\ k \ne q}} Q_{pk} (P_{qk} - P_{kq}) e_{2p-1} e_{2q-1} J + \\ +16 &\sum_{\substack{j,\ k,\ p,\ q \\ different \\ in\ pairs}} P_{jk} P_{pq} e_{2j-1} e_{2k-1} e_{2p-1} e_{2q-1} J = J. \end{aligned}$$

2.8 This operator equality can be applied to any function f from $C^{0,\,\mu}(\Gamma;\ Cl_{0,\,2m})$ or $L_p(\Gamma;\ Cl_{0,\,2m})$. In particular on the corresponding subsets of $\mathbb{C}$–valued functions we get:

$$\begin{aligned} M_m^2 = &\sum_{1 \le p < q \le m} M_{pq}^2 + I; \\ &((P_{pq} - P_{qp}) M_m - 2(Q_{pp} + Q_{qq})(P_{pq} - P_{qp}) - \\ &- M_m (P_{pq} - P_{qp}) + 2 \sum_{\substack{p \ne q \\ k \ne p \\ k \ne q}} ((S_{kp} - Q_{pk})(P_{kq} - P_{qk}) + \\ &+ (S_{kq} - Q_{qk})(P_{pk} - P_{kq})\ ) = 0 \end{aligned}$$

for any $1 \le p < q \le m$. The first equality gives what is written in Theorem in 1.6. Compare also the second equality with (1.4).

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