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CONSTRUCTIVE APPROXIMATION

On the Sensitivity of Radial Basis Interpolation to Minimal Data Separation Distance

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To I. J. Schoenberg—a tribute from admirers afar

Abstract. Motivated by the problem of multivariate scattered data interpolation, much interest has centered on interpolation by functions of the form

$$f(x) = \sum_{j=1}^{N} a_j g(\|x - x_j\|), \qquad x \in \mathbf{R}^s,$$

where $g: \mathbf{R}^+ \to \mathbf{R}$ is some prescribed function. For a wide range of functions g, it is known that the interpolation matrices $A = g(\|x_i - x_j\|)_{i,j=1}^N$ are invertible for given distinct data points $x_1, x_2, \ldots, x_N$. More recently, progress has been made in quantifying these interpolation methods, in the sense of estimating the (l_2) norms of the inverses of these interpolation matrices as well as their condition numbers. In particular, given a suitable function $g: \mathbf{R}^+ \to \mathbf{R}$, and data in $\mathbf{R}^s$ having minimal separation q, there exists a function $h_s: \mathbf{R}^+ \to \mathbf{R}^+$, which depends only on g and s, and a constant C_s, which depends only on s, such that the inverse of the associated interpolation matrix A satisfies the estimate $\|A^{-1}\| \le C_s h_s(q)$. The present paper seeks "converse" results to the inequality given above. That is, given a suitable function g, a spatial dimension s, and a parameter $q > 0$ (which is usually assumed to be small), it is shown that there exists a data set in $\mathbf{R}^s$ having minimal separation q, a constant $\tilde{C}_s$ depending only on s, and a function $k_s(q)$, such that the inverse of the interpolation matrix A associated with this data set satisfies $\|A^{-1}\| \ge \tilde{C}_s k_s(q)$. In some cases, it is seen that $h_s(q) = k_s(q)$, so the bounds are optimal up to constants. In certain others, $k_s(q)$ is less than $h_s(q)$, but nevertheless exhibits a behavior comparable to that of $h_s(q)$. That is, even in these cases, the bounds are close to being optimal.

1. Introduction

During the past several years, questions concerning the interpolation of scattered data in the Euclidean space $\mathbf{R}^s$ have led to various generalizations of a striking result of Schoenberg. In [S1], Schoenberg showed that if $x_1, x_2, \ldots, x_N$ are distinct points in a Hilbert space, then the matrix $A = (\|x_i - x_j\|)_{i,j=1}^N$ is invertible.

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Consequently, for given data $d_1, d_2, \ldots, d_N \in \mathbf{R}$, and distinct points $x_1, x_2, \ldots, x_N \in \mathbf{R}^s$, it is possible to find an interpolating function $f: \mathbf{R}^s \to \mathbf{R}$ of the form

$$f(x) = \sum_{j=1}^{N} a_j \|x - x_j\|.$$

Motivated by the problem of multivariate scattered data interpolation, much interest has centered on interpolation by functions of the form

$$f(x) = \sum_{j=1}^{N} a_j g(\|x - x_j\|), \qquad x \in \mathbf{R}^s,$$

where $g: \mathbf{R}^+ \to \mathbf{R}$ is some prescribed function. For a wide range of functions g (see Section 2), it was shown [M], [MN1] that the interpolation matrices $A = g(\|x_i - x_j\|)_{i,j=1}^N$ are invertible for given distinct data points $x_1, x_2, \ldots, x_N$.

More recently, progress has been made in quantifying these interpolation methods, in the sense of estimating the (l_2) norms of the inverses of these interpolation matrices as well as their condition numbers. For example, in [B], such estimates were derived for matrices associated with the function $g(r) = r$. In [NW1], a general approach, using Fourier transform techniques, was developed for obtaining such estimates for interpolation matrices arising from conditionally negative definite radial functions of order 1 (see Section 2). This approach was later adapted in [NW2] to obtain estimates for $\|A^{-1}\|$ for matrices A determined by radial functions of order $m \geq 0$ that are generated by completely monotonic functions. Moreover, in case $m = 0$ or 1, all such estimates depended only on the minimal separation distance for the data and on the dimension s of the ambient space $\mathbf{R}^s$. Thus, given a suitable function $g: \mathbf{R}^+ \to \mathbf{R}$, and data in $\mathbf{R}^s$ having minimal separation q, there exists a function $h_s: \mathbf{R}^+ \to \mathbf{R}^+$, which depends only on g and s, and a constant C_s, which depends only on s, such that the inverse of the associated interpolation matrix A satisfies the estimate $\|A^{-1}\| \leq C_s h_s(q)$.

In the present paper, we seek "converse" results to the inequality given above. In particular, given a suitable function g, a spatial dimension s, and a parameter $q > 0$ (which is usually assumed to be small), we show that there exists a data set in $\mathbf{R}^s$ having minimal separation q, a constant $\tilde{C}_s$ depending only on s, and a function $k_s(q)$, such that the inverse of the interpolation matrix A associated with this data set satisfies $\|A^{-1}\| \geq \tilde{C}_s k_s(q)$. In certain cases (see Section 4), $h_s(q) = k_s(q)$ and thus, up to constants, the results are optimal. In case $g(r)$ is the Hardy multiquadric (i.e., $g(r) = \sqrt{1 + r^2}$), it turns out that $h_s(q) > k_s(q)$. Nevertheless, our methods do show that, with s fixed and $g(r) = \sqrt{1 + r^2}$, $k_s(q)$ grows exponentially as q tends to zero. Thus, our investigations may be viewed as a systematic extension, to a broad class of functions, of the observation made in [B] (also see [Ba1]); namely, for the special case $g(r) = r$ and $s = 1$, $h_s(q) = 1/q$, $C_1 = 2$, and there exist data sets in $\mathbf{R}$ for which

$$\|A^{-1}\| \geq \frac{(2 - \varepsilon)}{q} \qquad \text{for given} \quad \varepsilon > 0.$$

So, in effect, we wish to determine the extent to which the upper bounds given in

[B], [Ba1], [NW1], and [NW2] reflect the actual value of $\|A^{-1}\|$ for matrices associated with certain data sets in $\mathbf{R}^s$.

An outline of the paper is as follows. In Section 2, we detail a suitable class of functions g to which our methods pertain. This section is necessarily technical in nature and the trustful reader may omit a significant portion of it during the first reading. Section 3 describes our main results. In particular, given a suitable function g and parameter q, a data set is constructed and the corresponding $\|A^{-1}\|$ is estimated from below. For the benefit of the casual but curious reader, we mention that the data set is a q-scaled version of a finite portion of the regular integer grid, while the estimate on $\|A^{-1}\|$ is obtained by carefully examining the quantity $\|A\lambda^{(s)}\|/\|\lambda^{(s)}\|$, where $\lambda^{(s)}$ is a specific vector which will be defined in Section 3. It transpires that $A\lambda^{(s)}$ can be expressed as a certain divided difference, and that $\|A\lambda^{(s)}\|$ can be estimated purposefully by invoking Fourier analytic methods, notably the Parseval identity and the Poisson summation formula. The final section deals with specific applications of the methods developed in Section 3.

2. Background and Preliminaries

In this section we first recall a version of the scattered data interpolation problem which forms the basis of our investigations. Second, we list pertinent definitions and notation for use throughout the remainder of the paper. Finally, we also discuss some technical results which we shall use in the subsequent sections.

Given a continuous function $F\colon \mathbf{R}^s \to \mathbf{R}$, distinct vectors $\{x_j\}_{j=1}^N$ in $\mathbf{R}^s$, and scalars $\{y_j\}_{j=1}^N$, one version of the scattered data interpolation problem [P] consists of finding scalars $\{a_j\}_{j=1}^N$ such that

$$\sum_{j=1}^N a_j F(x_k - x_j) = y_k, \qquad k = 1, 2, \ldots, N.$$

Equivalently, we wish to know when the interpolation matrix $\{F(x_k - x_j)\}_{k,j=1}^N$ is invertible.

The following class of functions has played a prominent role in the study of scattered data problems (see [GV]).

Definition 2.1. Let $F\colon \mathbf{R}^s \to \mathbf{R}$ be continuous. We say that F is conditionally negative (positive) definite of order m if for every finite set $\{x_j\}_{j=1}^N$ of distinct points in $\mathbf{R}^s$, and for every set of complex numbers $\{c_j\}_{j=1}^N$ satisfying

$$\sum_{j=1}^N c_j q(x_j) = 0$$

for every $q \in \Pi_{m-1}$ (the space of s-variate polynomials of total degree at most $m-1$), we have

$$\sum_{j,k=1}^N \bar{c}_j c_k F(x_j - x_k) \le 0 \quad (\ge 0).$$

This class of conditionally negative (positive) definite functions of order m (on $\mathbf{R}^s$) will be denoted by N_m^s (P_m^s).

Definition 2.2. We say that a continuous function g: $\mathbf{R}^+ \to \mathbf{R}$ is a conditionally negative (positive) definite *radial* function of order m if $g \circ (\|\cdot\|)$ is in N_m^s (P_m^s) for every s. (Henceforth, unless otherwise specified, $\|\cdot\|$ will denote the standard Euclidean (l_2) norm.)

The set of all conditionally negative (positive) definite radial functions of order m (on $\mathbf{R}^s$) will be denoted by RN_m^s (RP_m^s). The class RN_m^s includes those functions g which are continuous on $[0, \infty)$ and for which $(-1)^{m+1}(d^m/d\sigma^m)g(\sqrt{\sigma})$ is completely monotonic on $(0, \infty)$; i.e.,

$$(-1)^{m+1} \frac{d^m}{d\sigma^m} g(\sqrt{\sigma}) = \int_0^\infty e^{-\sigma t}\, d\mu(t),$$

where $d\mu$ is some nonnegative measure on $[0, \infty)$. This latter class of functions will be denoted by $RN_{m,c}^\infty$.

For the remainder of the paper, we will deal only with the cases $m = 0, 1$. In these cases, it is known from [S2] and [M] that $RN_{m,c}^\infty = RN_m^\infty := \bigcap_{s=1}^\infty RN_m^s$. So we will drop the subscript c. We also define the class RP_m^∞ by requiring that f belong to RP_m^∞ precisely when $-f \in RN_m^\infty$.

Now suppose that F: $\mathbf{R}^s \to \mathbf{R}$ is continuous and that it is *radially symmetric*, i.e., $F(x) = F(y)$ if $\|x\| = \|y\|$. It is manifest that F may be identified with the following function h_F: $\mathbf{R}^+ \to \mathbf{R}$ given by

$$h_F(r) = F(x) \qquad \text{where} \quad \|x\| = r.$$

Consequently, we may (and will) indulge in a slight abuse of notation and say that a function F: $\mathbf{R}^s \to \mathbf{R}$ belongs to $RN_m^\infty(RP_m^\infty)$ if F is continuous, radially symmetric, and its associated function h_F (as defined above) is in $RN_m^\infty(RP_m^\infty)$.

Next, we wish to review some relevant facts regarding divided differences. Let F: $\mathbf{R}^s \to \mathbf{R}$ be a function, n a natural number, and e_j, the jth (standard) unit vector in $\mathbf{R}^s$. We define $\nabla_j F$, $1 \le j \le s$, by

$$\nabla_j F(x) := F(x - e_j) - 2F(x) + F(x + e_j), \qquad x \in \mathbf{R}^s,$$

and

$$\nabla_j^n F(x) := \nabla_j(\nabla_j^{n-1} F(x)).$$

An induction argument shows that

$$\nabla_j^n F(x) = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} F(x - (n-k)e_j).$$

We also set

$$\nabla^n F(x) := \nabla_1^n \nabla_2^n \cdots \nabla_s^n F(x).$$

Perhaps our choice of the notation ∇^n is not altogether standard. However, in the absence of any other divided difference, no confusion should arise.

The emphasis in this paper will be on functions F (: $\mathbf{R}^s \to \mathbf{R}$) that belong to RP_0^∞ or RN_1^∞. These functions possess useful representations which we will exploit in many situations. In particular, if $F \in RP_0^\infty$, then F admits the representation [S2]

$$F(x) = \int_0^\infty e^{-\|x\|^2 t}\, d\mu(t), \qquad x \in \mathbf{R}^s, \tag{2.1}$$

where $d\mu$ is a positive measure satisfying the conditions

$$\int_0^1 d\mu(t) < \infty; \qquad \int_1^\infty e^{-t}\, d\mu(t) < \infty. \tag{2.2}$$

On the other hand, if $F \in RN_1^\infty$, then F may be realized as (see [M], [NW2], [S2], and [Su])

$$F(x) = F(0) + \int_0^\infty \frac{1 - e^{-\|x\|^2 t}}{t}\, d\mu(t), \qquad x \in \mathbf{R}^s, \tag{2.3}$$

where $d\mu$ is a positive measure such that

$$\int_0^1 d\mu(t) < \infty \quad \text{and} \quad \int_1^\infty \frac{d\mu(t)}{t} < \infty. \tag{2.4}$$

In what follows, we shall use C, C', $\tilde{C}$, and E to denote various generic constants. The dependence of the constants on certain parameters will be indicated by subscripts (e.g., C_s, $C'_{s,\alpha}$), but the actual numerical values of these constants will likely change from one occurrence to another. We will also freely employ the standard Landau symbols O and o.

Lemma 2.3. *Let F (: $\mathbf{R}^s \to \mathbf{R}$) $\in RP_0^\infty$ or RN_1^∞, $x \in \mathbf{R}^s$, and suppose that n (≥ 2) is a positive integer. Then*

(i) $|\nabla^n F(x)| \leq C_{s,n}/(1 + \|x\|)^{s+\delta}$, $\delta > 0$.
(ii) $\nabla^n F \in L^1(\mathbf{R}^s)$.

Proof. At the outset, we note that since F is continuous, so is $\nabla^n F$. So (ii) follows from (i). Second, since $\nabla^n F(x) = \nabla^{n-2}(\nabla^2 F(x))$, it is enough to prove (i) for $n = 2$. Furthermore, by continuity of $\nabla^2 F$, it is sufficient to show that

$$|\nabla^2 F(x)| \leq \frac{C_s}{\|x\|^{s+\delta}}, \qquad \|x\| \text{ large}.$$

Thus, for the duration of the proof, we set $x = (x_1, \ldots, x_s)$, $\|x\| =: r$, and assume that r is as large as necessary.

By (2.1), (2.3), and the fact that $\nabla^2(1) = 0$, we see that

$$\begin{aligned} |\nabla^2 F(x)| &\leq \int_0^\infty \frac{|\nabla^2(e^{-\|x\|^2 t})|}{t^m}\, d\mu(t) \\ &= \int_0^\infty \frac{\prod_{j=1}^s |\nabla_j^2 e^{-x_j^2 t}|}{t^m}\, d\mu(t), \end{aligned} \tag{2.5}$$

where m equals 0 if $F \in RP_0^\infty$ and equals 1 if $F \in RN_1^\infty$. Let $1 \le j \le s$ be fixed (but arbitrary), and consider $\nabla_j^2 e^{-x_j^2 t}$. It is well known that $\nabla_j^2 e^{-x_j^2 t}$ equals a constant multiple of $D_y^4(e^{-y^2 t})(\xi_j)$ for some $\xi_j \in [x_j - 2, x_j + 2]$. Since

$$D_y^4(e^{-y^2 t}) = (12t^2 - 48y^2t^3 + 16y^4t^4)e^{-y^2 t},$$

we see from (2.5) that

$$\text{(2.6)} \qquad |\nabla^2 F(x)| \le C \int_0^\infty \frac{\prod_{j=1}^s (12t^2 + 48|\xi_j|^2 t^3 + 16|\xi_j|^4 t^4) e^{-\xi_j^2 t}}{t^m} \, d\mu(t).$$

Setting $\xi := (\xi_1, \xi_2, \ldots, \xi_s)$, and recalling that $\|x\|$ is large enough, we conclude from (2.6) that

$$\text{(2.7)} \quad |\nabla^2 F(x)| \le C_s \int_0^\infty e^{-(1/2)\|x\|^2 t} t^{2s-m} \prod_{j=1}^s (12 + 48\|x\|^2 t + 16\|x\|^4 t^2) \, d\mu(t)$$
$$= C_s \int_0^\infty e^{-r^2 t/2} t^{2s-m} \left[\sum_{k=0}^{2s} a_k (r^2 t)^k \right] d\mu(t),$$

where $a_0, a_1, \ldots, a_{2s}$ are constants depending solely on s.

Let $\alpha \in (\frac{5}{3}, 2)$ be fixed and write (2.7) as a sum of three terms

$$I_1 := C_s \int_0^{2/r^\alpha} e^{-r^2 t/2} t^{2s-m} \left[\sum_{k=0}^{2s} a_k (r^2 t)^k \right] d\mu(t),$$

$$I_2 := C_s \int_{2/r^\alpha}^1 e^{-r^2 t/2} t^{2s-m} \left[\sum_{k=0}^{2s} a_k (r^2 t)^k \right] d\mu(t),$$

and

$$I_3 := C_s \int_1^\infty e^{-r^2 t/2} t^{2s-m} \left[\sum_{k=0}^{2s} a_k (r^2 t)^k \right] d\mu(t).$$

Now,

$$I_1 \le C_s \left[\frac{2}{r^\alpha} \right]^{2s-m} \left[\sum_{k=0}^{2s} a_k \left(\frac{2}{r^{\alpha-2}} \right)^k \right] \left[\int_0^1 d\mu(t) \right]$$
$$\le \frac{C_s}{r^{(2s-m)\alpha - 2s(\alpha-2)}} = \frac{C_s}{r^{s+\delta}}, \qquad \delta > 0,$$

since, for $m = 0, 1$,

$$(2s - m)\alpha + 2s(\alpha - 2) - s \ge s(4\alpha - 5) - \alpha \ge 3\alpha - 5 > 0,$$

and (2.2) and (2.4) hold.

Next,

$$I_2 \le C_s e^{-r^{2-\alpha}} r^{4s} \left[\int_0^1 d\mu(t) \right] \le \frac{C_s}{r^{s+\delta}}$$

by (2.2) and (2.4). Lastly, we turn to I_3 and note that if $m = 0$, then

$$I_3 \le C_s \int_1^\infty e^{-(r^2/2-1)t} t^{2s} r^{4s} t^{2s} e^{-t}\, d\mu(t),$$

whereas, if $m = 1$, then

$$I_3 \le C_s \int_1^\infty \frac{e^{-r^2t/2} t^{2s} r^{4s} t^{2s}}{t}\, d\mu(t).$$

For large enough r, the functions

$$t \mapsto e^{-(r^2t/2-1)} t^{4s} \qquad \text{and} \qquad t \mapsto e^{-r^2t/2} t^{4s}$$

decrease in t, so

$$I_3 \le C_s r^{4s} e^{-(r^2/2-1)} \int_1^\infty e^{-t}\, d\mu(t), \qquad m = 0,$$

and

$$I_3 \le C_s r^{4s} e^{-r^2/2} \int_1^\infty \frac{d\mu(t)}{t}, \qquad m = 1,$$

whence, by (2.2) and (2.4),

$$I_3 \le \frac{C_s}{r^{s+\delta}}.$$

This completes the proof. ■

As pointed out by M. J. D. Powell, the proof of the preceding lemma can be simplified considerably by using the elementary inequality

$$12 + 48\xi_j^2 t + 16\xi_j^4 t^2 \le 288 e^{\xi_j^2 t/3}, \qquad t \ge 0.$$

Also, F. J. Narcowich notes that the function $x \mapsto |x|^{2\alpha}$, $x \in \mathbf{R}$, $\frac{1}{2} < \alpha < 1$, serves to vitiate the lemma for $n = 1$.

Lemma 2.4. *Let F $(: \mathbf{R}^s \to \mathbf{R}) \in RP_0^\infty$ or RN_1^∞, and let $d\mu$ be its representing measure. Suppose that n (≥ 2) is a fixed positive integer, and define $G(x) := \nabla^n F(x)$. Assume that for some $\delta > 0$,*

$$(*) \qquad \int_0^\infty e^{-\|y\|^2/4t} \frac{d\mu(t)}{t^{s/2+m}} \le \frac{C_s}{\|y\|^{s+\delta}}, \qquad \|y\| \text{ large},$$

where $m = 0$ if $F \in RP_0^\infty$ while $m = 1$ if $F \in RN_1^\infty$. Then

$$|\hat{G}(y)| \le \frac{C_{s,n}}{(1 + \|y\|)^{s+\delta}} \qquad \text{for all} \quad y \in \mathbf{R}^s.$$

Proof. Since $G \in L^1(\mathbf{R}^s)$ (Lemma 2.3), $\hat{G} \in C_0(\mathbf{R}^s)$. So it suffices to demonstrate that

$$|\hat{G}(y)| \le \frac{C_{s,n}}{\|y\|^{s+\delta}}, \qquad \|y\| \text{ large.}$$

Now,

$$\hat{G}(y) = \int_{\mathbf{R}^s}\left[\int_0^\infty \frac{\nabla^n(e^{-\|x\|^2 t})}{t^m}\,d\mu(t)\right]e^{-ixy}\,dx \tag{2.8}$$

$$= \int_0^\infty\left[\int_{\mathbf{R}^s} \nabla^n(e^{-\|x\|^2 t})e^{-ixy}\,dx\right]\frac{d\mu(t)}{t^m}$$

by Fubini's theorem. We note here that for $t = 0$, $\nabla^n(e^{-\|x\|^2 t}) = \nabla^n(1) = 0$, whereas for $t > 0$, $x \mapsto e^{-\|x\|^2 t}$, and hence $x \mapsto \nabla^n(e^{-\|x\|^2 t})$, belongs to $L^1(\mathbf{R}^s)$. That is, the use of Fubini's theorem is justified.

Assume that $t \ne 0$, and $y = (y_1, y_2, \ldots, y_s)$. Since

$$\nabla^n(e^{-\|x\|^2 t}) = \prod_{j=1}^{s} \nabla_j^n(e^{-x_j^2 t}), \qquad x = (x_1, x_2, \ldots, x_s),$$

and

$$(\nabla_j^n(e^{-x_j^2 t}))^\wedge(y_j) = (-1)^n 2^{2n}\sin^{2n}\left(\frac{y_j}{2}\right)(e^{-x_j^2 t})^\wedge(y_j)$$

$$= C(-1)^n 2^{2n}\sin^{2n}\left(\frac{y_j}{2}\right)\frac{e^{-y_j^2/4t}}{\sqrt{t}},$$

(2.8) shows that

$$\hat{G}(y) = (-1)^{sn}2^{2sn}C^s\int_0^\infty\left[\prod_{j=1}^{s}\sin^{2n}\left(\frac{y_j}{2}\right)\right]e^{-\|y\|^2/4t}\,\frac{d\mu(t)}{t^{s/2+m}},$$

and the desired result follows immediately from (*). ■

Remark 2.5. It may be noted that for $y \ne 0$, the function $t \mapsto e^{-\|y\|^2/4t}/t^{s/2+m}$, $m = 0, 1$, is defined, by continuity, to be zero at $t = 0$.

Corollary 2.6. *Let F $(: \mathbf{R}^s \to \mathbf{R}) \in RP_0^\infty$ or RN_1^∞, and let $d\mu$ be its representing measure. Suppose that n $(\ge 2) \in \mathbf{N}$ and that F satisfies condition* (*) *of Lemma* 2.4. *Then, for each $y \in (0, 2\pi]^s$, the following relation holds*:

$$\sum_{k \in \mathbf{Z}^s}(\nabla^n F)(k)e^{-iky} = \sum_{k \in \mathbf{Z}^s}(\nabla^n F)^\wedge(y + 2\pi k).$$

Proof. This is a direct consequence of the Poisson summation formula [SW, Cor. 2.6, p. 252]. Its use in the present context is validated by the assertions of Lemmata 2.3 and 2.4. ■

Remark 2.7. Suppose that F $(: \mathbf{R}^s \to \mathbf{R}) \in RP_0^\infty$ or RN_1^∞, $d\mu$ is its representing measure, and $q > 0$ is a fixed number. Let $F_q(\cdot) := F(q\,\cdot)$ and

$$y = (y_1, y_2, \ldots, y_s) \in (0, 2\pi]^s.$$

If $n\,(\geq 2) \in \mathbf{N}$ and m is defined as before, then the proof of Lemma 2.4 and the fact that

$$(\nabla_j^n(e^{-q^2x_j^2t}))^\wedge(y_j) = (-1)^n 2^{2n} \sin^{2n}\left(\frac{y_j}{2}\right)(e^{-q^2x_j^2t})^\wedge(y_j)$$

$$= \sqrt{\pi}(-1)^n 2^{2n} \sin^{2n}\left(\frac{y_j}{2}\right)\frac{e^{-y_j^2/4q^2t}}{q\sqrt{t}}$$

imply that

$$(\nabla^n F_q)^\wedge(y) = \frac{\pi^{s/2}(-1)^{sn}2^{2sn}\prod_{j=1}^s \sin^{2n}(y_j/2)}{q^s}\int_0^\infty e^{-\|y\|^2/4q^2t}\,\frac{d\mu(t)}{t^{s/2+m}}.$$

In particular, if F (and hence F_q) satisfies condition (∗) of Lemma 2.4, then $(\nabla_n F_q)^\wedge(y)$ satisfies the conclusion of that lemma, and by Corollary 2.6, we have

$$\text{(2.9)}\quad \sum_{k\in\mathbf{Z}^s}(\nabla^n F_q)(k)e^{-iky} = \frac{\pi^{s/2}(-1)^{sn}2^{2sn}}{q^s}\prod_{j=1}^s \sin^{2n}\left(\frac{y_j}{2}\right)\sum_{k\in\mathbf{Z}^s}\int_0^\infty \frac{e^{-\|y+2\pi k\|^2/4q^2t}}{t^{s/2+m}}\,d\mu(t).$$

It ought to be mentioned that condition (∗) of Lemma 2.4 does not preclude any of the salient functions in RP_0^∞ or RN_1^∞ that are prevalent in the literature [D]. We now discuss examples of some such functions. The measures representing each of these functions in Examples 2.8–2.11 may be derived by using standard Laplace transform formulas (see, e.g., [EMOT]).

Example 2.8. Let $F(x) := \|x\|^{2\alpha}$, $x \in \mathbf{R}^s$, $0 < \alpha < 1$. Then $F \in RN_1^\infty$, and its representing measure $d\mu$ is given by

$$d\mu(t) = C_\alpha \frac{dt}{t^\alpha}, \qquad 0 < \alpha < 1.$$

Suppose that $\|y\| > 0$ and note that

$$\int_0^\infty \frac{e^{-\|y\|^2/4t}}{t^{s/2+1}}\,d\mu(t) = C_\alpha \int_0^\infty \frac{e^{-\|y\|^2/4t}}{t^{s/2+1+\alpha}}\,dt$$

$$= C_{\alpha,s}\int_0^\infty e^{-\|y\|^2 u}u^{s/2+\alpha-1}\,du, \qquad u = 1/4t$$

$$= \frac{C_{\alpha,s}}{\|y\|^{s+2\alpha}},$$

whence condition (∗) obtains. The last step in the preceding analysis follows from the fact that the integral in the penultimate step represents the Laplace transform, evaluated at $\|y\|^2$, of the function $u \mapsto u^{s/2+\alpha-1}$. It may be noted that the said transform exists because $\|y\| > 0$ and $s/2 + \alpha - 1 > -1$.

Example 2.9. Let $F\colon \mathbf{R}^s \to \mathbf{R}$ be given by $F(x) := \sqrt{1 + \|x\|^2}$. Then $F \in RN_1^\infty$, and its representing measure du is given by

$$d\mu(t) = C \frac{e^{-t}}{\sqrt{t}}\, dt.$$

Now, if $\|y\| > 0$, then

$$\begin{aligned}\int_0^\infty \frac{e^{-\|y\|^2/4t}}{t^{s/2+1}}\, d\mu(t) &= C \int_0^\infty \frac{e^{-\|y\|^2/4t} e^{-t}}{t^{s/2+3/2}}\, dt \\ &\le C \int_0^\infty \frac{e^{-\|y\|^2/4t}}{t^{s/2+3/2}}\, dt \\ &\le \frac{C_s}{\|y\|^{s+1}},\end{aligned}$$

as demonstrated in Example 2.8. Thus F satisfies condition $(*)$.

Example 2.10. Define $F(x) := \log(1 + \|x\|^2)$, $x \in \mathbf{R}^s$. Once again, $F \in RN_1^\infty$, and is represented by the measure

$$d\mu(t) = Ce^{-t}\, dt.$$

Suppose that $\|y\|$ is large and observe that

$$\begin{aligned}\int_0^\infty \frac{e^{-\|y\|^2/4t}}{t^{s/2+1}}\, d\mu(t) &= \int_0^\infty \frac{e^{-\|y\|^2/4t}}{t^{s/2+1}}\, e^{-t}\, dt \\ &= C_s \int_0^\infty e^{-\|y\|^2 u} e^{-1/4u} u^{s/2-1}\, du, \qquad u = 1/4t.\end{aligned}$$

Now, this last integral above is the Mellin transform, evaluated at $s/2$, of the function $u \mapsto e^{-(\|y\|^2 u + 1/4u)}$. Consequently, by [EMOT, p. 313], we conclude that

$$\int_0^\infty \frac{e^{-\|y\|^2/4t}}{t^{s/2+1}}\, d\mu(t) = \frac{C_s}{\|y\|^{s/2}} K_{s/2}(\|y\|),$$

where $K_{s/2}$ is the modified Bessel function of the third kind. The standard asymptotic expansion of $K_{s/2}(\|y\|)$ for large $\|y\|$ [AS, p. 378] guarantees that

$$K_{s/2}(\|y\|) = O(e^{-\|y\|}), \qquad \|y\| \text{ large},$$

so

$$\int_0^\infty \frac{e^{-\|y\|^2/4t}}{t^{s/2+1}}\, d\mu(t) = O(e^{-\|y\|}), \qquad \|y\| \text{ large}.$$

In particular, F satisfies condition $(*)$.

Example 2.11. Let $F\colon \mathbf{R}^s \to \mathbf{R}$ be given by $F(x) = 1/\sqrt{1 + \|x\|^2}$. This function, called the inverse multiquadric, belongs to RP_0^∞, and has for its representing measure, the measure

$$d\mu(t) = C\frac{e^{-t}}{\sqrt{t}}\,dt.$$

If $\|y\|$ is large, then

$$\begin{aligned}\int_0^\infty \frac{e^{-\|y\|^2/4t}}{t^{s/2}}\,d\mu(t) &= C\int_0^\infty \frac{e^{-\|y\|^2/4t}}{t^{s/2+1/2}}\,e^{-t}\,dt\\ &= C_s\int_0^\infty e^{-\|y\|^2u}e^{-1/4u}u^{(s-1)/2-1}\,du, \qquad u = 1/4t,\\ &= O(e^{-\|y\|}),\end{aligned}$$

exactly as in the previous example. This shows that the inverse multiquadric also satisfies condition $(*)$.

3. Main Results

This section is devoted to the development of a method which will be utilized to derive lower estimates for the norms of inverses of interpolation matrices associated with functions $F\,(:\mathbf{R}^s \to \mathbf{R}) \in RP_0^\infty$ or RN_1^∞ and certain data sets in $\mathbf{R}^s$. Our object, as discussed in the introduction, is to establish that for a given F, there exist certain data sets having (small) minimal data separation q, an appropriate function $k_s(q)$, and a constant $\tilde{C}_s$, such that the inverse of the associated interpolation matrix A satisfies $\|A^{-1}\| \geq \tilde{C}_s k_s(q)$.

In the sequel, we assume that $F\,(:\mathbf{R}^s \to \mathbf{R}) \in RP_0^\infty$ or RN_1^∞. Given a data set $S_N = \{x_j\}_{j=1}^N$ in $\mathbf{R}^s$, and a function $F \in RP_0^\infty$ or RN_1^∞, $F: \mathbf{R}^s \to \mathbf{R}$, we will call the $N \times N$ matrix, with entries $\{F(x_i - x_j)\}_{i,j=1}^N$, the *interpolation matrix associated with* S_N *and* F.

Let us also recall the following elementary, yet useful fact, that for any $N \times N$ matrix A,

$$\|A^{-1}\| = \left\{\frac{1}{\operatorname{Inf}\|A\lambda\|} : \lambda \in \mathbf{R}^N,\ \|\lambda\| = 1\right\}.$$

In particular, we have the estimate

$$\|A^{-1}\| \geq \frac{\|\lambda\|}{\|A\lambda\|} \qquad \text{for any} \quad \lambda \in \mathbf{R}^N\backslash\{0\}. \tag{3.1}$$

Crucial to most of our subsequent analysis is the introduction of the following special vector.

Definition 3.1. Let $n \in \mathbf{N}$ and let $\lambda \in \mathbf{R}^{2n+1}$ be given by

$$\lambda_j = (-1)^j\binom{2n}{j}, \qquad 0 \leq j \leq 2n.$$

Denote by $\lambda^{(s)}$, the s-fold tensor product of λ with itself. That is, $\lambda^{(s)} \in \mathbf{R}^{(2n+1)^s}$, and is given by

$$\lambda^{(s)}(j_1, j_2, \ldots, j_s) = \prod_{k=1}^{s} (-1)^{j_k} \binom{2n}{j_k}, \qquad 0 \le j_k \le 2n.$$

Lemma 3.2. *Let $\lambda^{(s)}$ be defined as above. Then there exists a constant C_s such that*

$$C_s^{-1} \frac{2^{4sn}}{(\sqrt{n})^s} \le \|\lambda^{(s)}\|^2 \le C_s \frac{2^{4sn}}{(\sqrt{n})^s}.$$

Proof. By definition of $\lambda^{(s)}$, it suffices to establish the existence of an absolute constant C such that

$$C^{-1} \frac{2^{4n}}{\sqrt{n}} \le \|\lambda\|^2 \le C \frac{2^{4n}}{\sqrt{n}}. \tag{3.2}$$

To this end, note that

$$\|\lambda\|^2 = \sum_{j=0}^{2n} \binom{2n}{j}^2 = \binom{4n}{2n} = \frac{(4n)!}{[(2n)!]^2}, \tag{3.3}$$

while the well-known Wallis' formula [AS, formula 6.1.49] warrants that

$$\operatorname*{Lim}_{m \to \infty} \left[\frac{(m!)^2 2^{2m}}{(2m)! \sqrt{m}} \right] = \sqrt{\pi}. \tag{3.4}$$

Now (3.2) follows from (3.3) and (3.4). ■

We proceed next to the description of data sets in $\mathbf{R}^s$ which we shall use to obtain appropriate norm estimates. Let $n \in \mathbf{N}$ and let $\mathscr{L}_{s,n}$ be the set of lattice points in $\mathbf{R}^s$ given by

$$\mathscr{L}_{s,n} := \{(j_1, j_2, \ldots, j_s) \colon 0 \le j_k \le 2n,\ 1 \le k \le s\}.$$

Note that the cardinality of $\mathscr{L}_{s,n}$ is $(2n+1)^s$. Suppose that $q > 0$ and that $F\,(\colon \mathbf{R}^s \to \mathbf{R}) \in RP_0^\infty$ or RN_1^∞. The interpolation matrix A on which we wish to focus is the one associated with $\mathscr{L}_{s,n}$ and F_q. (It should perhaps be pointed out that studying the interpolation matrix associated with $\mathscr{L}_{s,n}$ (which has minimal separation 1) and F_q is tantamount to studying the interpolation matrix associated with $q\mathscr{L}_{s,n} := \{ql \colon l \in \mathscr{L}_{s,n}\}$ (which has minimal separation q) and F.)

The simplest example of such an interpolation matrix is the one associated with $\mathscr{L}_{1,n}$ and F_q. In this case

$$A = \begin{bmatrix} F_q(0) & F_q(1) & \cdots & F_q(2n) \\ F_q(-1) & F_q(0) & \cdots & F_q(2n-1) \\ \vdots & \vdots & & \vdots \\ F_q(-2n) & F_q(-2n-1) & \cdots & F_q(0) \end{bmatrix}.$$

We draw the reader's attention to the fact that A is, in fact, symmetric because F is radial. The reason for displaying it as we have done will emerge momentarily.

Now we wish to consider the action of the interpolation matrix associated with $\mathscr{L}_{s,n}$ and F_q, on the vector $\lambda^{(s)}$. It turns out that the vector $A\lambda^{(s)}$ $(\in \mathbf{R}^{(2n+1)^s})$ comprises as its entries, appropriate divided differences of F_q evaluated at certain lattice points. Indeed, for $0 \le j_k \le 2n$, $1 \le k \le s$, we have $(A\lambda^{(s)})_j = \nabla^n F_q(ne - j)$ where $e = [1, 1, \ldots, 1]^{\mathrm{T}}$. To see this, we first note that

$$(A\lambda^{(s)})_j = \sum_{k_1=0}^{2n} \cdots \sum_{k_s=0}^{2n} (-1)^{k_1+\cdots+k_s} \binom{2n}{k_1} \cdots \binom{2n}{k_s} F_q(j-k).$$

However, by definition, we also have

$$\nabla^n F_q(ne-j) = \sum_{k_1=0}^{2n} \cdots \sum_{k_s=0}^{2n} (-1)^{k_1+\cdots+k_s} \binom{2n}{k_1} \cdots \binom{2n}{k_s} F_q((ne-j)-(ne-k)),$$

whence the required assertion follows because F_q is radially symmetric. Consequently, the components of $A\lambda^{(s)}$ are given by $\nabla^n F_q(j)$, $-n \le j_k \le n$, $1 \le k \le s$. (Although the purport of the preceding proof was present in the original version of our paper, it was not quite so transparent. The pithy argument given above is due to the referee whom we acknowledge gratefully at this juncture.)

Our primary objective in this section is to obtain a quantitative upper estimate for $\|A\lambda^{(s)}\|^2/\|\lambda^{(s)}\|^2$, where A is the interpolation matrix associated with $\mathscr{L}_{s,n}$ and F_q ($F \in RP_0^\infty$ or RN_1^∞). Such a result, via (3.1), will provide us with a lower bound for $\|A^{-1}\|$. Additional insight in this regard has also been offered by the referee, who points out that the sequences of Definition 3.1, appropriately normalized, form the coefficients of a suitable approximate identity centered at $[\pi, \ldots, \pi]^{\mathrm{T}}$, a fact which may be used to simplify the forthcoming estimates in Section 4. In this connection, we also refer the interested reader to [Ba2] where this very point of view is made much more explicit. (The recent articles [Ba2] and [Sc], both of which deal with issues related to those considered in the present paper, came to our attention a few months after the original version of our manuscript was submitted.)

Theorem 3.3. *Suppose that* $F\,(: \mathbf{R}^s \to \mathbf{R}) \in RP_0^\infty$ *or* RN_1^∞, *and its representing measure is* $d\mu$. *Assume that* $q > 0$, $n\,(\ge 2) \in \mathbf{N}$, *and that* A *is the interpolation matrix associated with* $\mathscr{L}_{s,n}$ *and* F_q. *If* F *satisfies condition* $(*)$ *of Lemma* 2.4, *then*

$$\frac{\|A\lambda^{(s)}\|^2}{\|\lambda^{(s)}\|^2} \le \frac{C_s(\sqrt{n})^s}{q^{2s}} \int_{(0,2\pi]^s} \prod_{j=1}^{s} \sin^{4n}\left(\frac{y_j}{2}\right) \left[\int_0^\infty \left(\sum_{k \in \mathbf{Z}^s} e^{-\|y+2\pi k\|^2/4q^2 t}\right) \frac{d\mu(t)}{t^{s/2+m}}\right]^2 dy,$$

where m *is* 0 *if* $F \in RP_0^\infty$ *and is* 1 *if* $F \in RN_1^\infty$.

Proof. From our foregoing discussion, we know that

$$\|A\lambda^{(s)}\|^2 \le \sum_{k \in \mathbf{Z}^s} |\nabla^n F_q(k)|^2.$$

By Parseval's Theorem and (2.9), we have

$$(3.5)\qquad \sum_{k\in Z^s} |\nabla^n F_q(k)|^2 = C_s \int_{(0,2\pi]^s} \left| \sum_{k\in Z^s} (\nabla^n F_q)(k) e^{-iky} \right|^2 dy$$

$$= \frac{C_s 2^{4sn}}{q^{2s}} \int_{(0,2\pi]^s} \prod_{j=1}^{s} \sin^{4n}\left(\frac{y_j}{2}\right)$$

$$\times \left[\sum_{k\in Z^s} \int_0^\infty \frac{e^{-\|y+2\pi k\|^2/4q^2t}}{t^{s/2+m}} \, d\mu(t) \right]^2 dy.$$

Using the monotone convergence theorem in (3.5), we conclude that

$$\frac{A\lambda^{(s)}\|^2}{\|\lambda^{(s)}\|^2} \le \frac{C_s 2^{4sn}}{q^{2s}\|\lambda^{(s)}\|^2} \int_{(0,2\pi]^s} \prod_{j=1}^{s} \sin^{4n}\left(\frac{y_j}{2}\right) \left[\int_0^\infty \left(\sum_{k\in Z^s} e^{-\|y+2\pi k\|^2/4q^2t} \right) \frac{d\mu(t)}{t^{s/2+m}} \right]^2 dy.$$

An appeal to Lemma 3.2 now finishes the proof. ■

The upper estimate for $\|A\lambda^{(s)}\|^2/\|\lambda^{(s)}\|^2$, as given by Theorem 3.3, involves the infinite sum $\sum_{k\in Z^s} e^{-\|y+2\pi k\|^2/4q^2t}$. It will be more convenient to replace this sum by an amenable majorant. This is done as follows. For the sake of clarity, we first detail the procedure in the bivariate case. Following that, we shall indicate how the procedure can be extended to dimensions higher than 2. The univariate case, which we do not treat explicitly, can also be dealt with similarly, if not more easily.

When $s = 2$, the sum in question is

$$(3.6)\qquad \sum_{(k,l)\in Z^2} e^{-\|y+2\pi(k,l)\|^2/4q^2t} =: \sum_{(k,l)\in Z^2} T(k,l),$$

where $y = (y_1, y_2) \in (0, 2\pi]^2$. (Assume $t \ne 0$.) The dominant terms in (3.6) are $T(0,0)$, $T(-1,0)$, $T(0,-1)$, and $T(-1,-1)$. Accordingly, we split (3.6) into four parts, each of which is assigned to one dominant term. More precisely, we set

$$S_1 := \sum_{k,l\ge 0} T(k,l), \qquad S_2 := \sum_{\substack{k\le -1\\ l\ge 0}} T(k,l),$$

$$S_3 := \sum_{\substack{k\ge 0\\ l\le -1}} T(k,l), \qquad S_4 := \sum_{\substack{k\le -1\\ l\le -1}} T(k,l).$$

Factoring the dominant term out of each of these sums, changing $k \mapsto -k$ in S_2, $l \mapsto -l$ in S_3, and $(k,l) \mapsto (-k,-l)$ in S_4, we arrive at

$$S_1 = e^{-(y_1^2+y_2^2)/4q^2t} \left[1 + \sum_{\substack{k,l\ge 0\\ (k,l)\ne(0,0)}} e^{-[(y_1+2\pi k)^2+(y_2+2\pi l)^2-y_1^2-y_2^2]/4q^2t} \right],$$

$$S_2 = e^{-[(2\pi-y_1)^2+y_2^2]/4q^2t} \left[1 + \sum_{\substack{k\ge 1,\, l\ge 0\\ (k,l)\ne(1,0)}} e^{-[(2\pi k-y_1)^2+(y_2+2\pi l)^2-(2\pi-y_1)^2-y_2^2]/4q^2t} \right],$$

$$S_3 = e^{-[y_1^2+(2\pi-y_2^2)]/4q^2t} \left[1 + \sum_{\substack{k\ge 0,\, l\ge 1\\ (k,l)\ne(0,1)}} e^{-[(y_1+2\pi k)^2+(2\pi l-y_2)^2-y_1^2-(2\pi-y_2)^2]/4q^2t} \right],$$

and

$$S_4 = e^{-[(2\pi - y_1)^2 + (2\pi - y_2)^2]/4q^2t}\left[1 + \sum_{\substack{k\geq 1, l\geq 1\\(k,l)\neq(1,1)}} e^{-[(2\pi k - y_1)^2 + (2\pi l - y_2)^2 - (2\pi - y_1)^2 - (2\pi - y_2)^2]/4q^2t}\right].$$

Let us consider S_1 and S_4. Beginning with S_1, note that if $k, l \geq 0$, then

$$(y_1 + 2\pi k)^2 + (y_2 + 2\pi l)^2 - y_1^2 - y_2^2 \geq 4\pi^2(k^2 + l^2),$$

so

$$S_1 \leq e^{-(y_1^2 + y_2^2)/4q^2t}\left[1 + \sum_{\substack{k,l\geq 0\\(k,l)\neq(0,0)}} e^{-\pi^2(k^2+l^2)/q^2t}\right]. \tag{3.7}$$

Now

$$\sum_{\substack{k,l\geq 0\\(k,l)\neq(0,0)}} e^{-\pi^2(k^2+l^2)/q^2t} = 2\sum_{k=1}^{\infty} e^{-\pi^2k^2/q^2t} + \sum_{k,l\geq 1} e^{-\pi^2(k^2+l^2)/q^2t}$$

$$=: 2S_{11} + S_{12}.$$

Observe that

$$S_{12} \leq \int_0^\infty \int_0^\infty e^{-\pi^2(u^2+v^2)/q^2t}\,du\,dv. \tag{3.8}$$

As to S_{11}, we consider two cases. If, on the one hand, $0 < q^2t \leq 1$, then

$$\sum_{k=1}^{\infty} e^{-\pi^2k^2/q^2t} \leq \sum_{k=1}^{\infty} e^{-\pi^2k^2} =: K(1). \tag{3.9}$$

On the other hand, if $q^2t > 1$, then

$$\int_0^1 e^{-\pi^2u^2/q^2t}\,du \geq e^{-\pi^2},$$

so

$$e^{\pi^2}\int_0^\infty e^{-\pi^2u^2/q^2t}\,du > e^{\pi^2}\int_0^1 e^{-\pi^2u^2/q^2t}\,du \geq 1. \tag{3.10}$$

Consequently,

$$\sum_{k=1}^{\infty} e^{-\pi^2k^2/q^2t} \leq \int_0^\infty e^{-\pi^2v^2/q^2t}\,dv < e^{\pi^2}\int_0^\infty\int_0^\infty e^{-\pi^2(u^2+v^2)/q^2t}\,du\,dv. \tag{3.11}$$

Setting $K := \max\{K(1), e^{\pi^2}\}$, we see from (3.9) and (3.11) that

$$S_{11} \leq \left[1 + \int_0^\infty\int_0^\infty e^{-\pi^2(u^2+v^2)/q^2t}\,du\,dv\right]. \tag{3.12}$$

Therefore, it follows from (3.7) that

$$(3.13)\qquad S_1 \le Ce^{-(y_1^2+y_2^2)/4q^2t}\left[1 + \int_0^\infty \int_0^\infty e^{-\pi^2(u^2+v^2)/q^2t}\,du\,dv\right]$$
$$= Ce^{-(y_1^2+y_2^2)/4q^2t}\left[1 + \frac{q^2t}{4\pi}\right],$$

where $C := 3\max\{2K, 1\}$ is a constant independent of y, q, and t. Turning to S_4, we note that for $k, l \ge 1$, $(y_1, y_2) \in (0, 2\pi]^2$,

$$(2\pi k - y_1)^2 + (2\pi l - y_2)^2 - (2\pi - y_1)^2 - (2\pi - y_2)^2 \ge 4\pi^2[(k-1)^2 + (l-1)^2],$$

so

$$S_4 \le e^{-[(2\pi-y_1)^2+(2\pi-y_2)^2]/4q^2t}\left[1 + \sum_{\substack{k,l\ge 1\\(k,l)\ne(1,1)}} e^{-\pi^2[(k-1)^2+(l-1)^2]/q^2t}\right]$$
$$\le Ce^{-[(2\pi-y_1)^2+(2\pi-y_2)^2]/4q^2t}\left[1 + \frac{q^2t}{4\pi}\right],$$

as before.

It is quite evident that S_2 and S_3 can be handled in the same fashion, thus giving

$$\sum_{i=1}^4 S_i \le C\left[1 + \frac{q^2t}{4\pi}\right]\left[e^{-(y_1^2+y_2^2)/4q^2t} + e^{-[(2\pi-y_1)^2+y_2^2]/4q^2t}\right.$$
$$\left. + e^{-[y_1^2+(2\pi-y_2)^2]/4q^2t} + e^{-[(2\pi-y_1)^2+(2\pi-y_2)^2]/4q^2t}\right]$$
$$=: \sum_{i=1}^4 f_i(y, t).$$

Consequently, Theorem 3.3 gives

$$\frac{\|A\lambda^{(2)}\|^2}{\|\lambda^{(2)}\|^2} \le \frac{C_2(\sqrt{n})^2}{q^4}\int_0^{2\pi}\int_0^{2\pi} \sin^{4n}\left(\frac{y_1}{2}\right)\sin^{4n}\left(\frac{y_2}{2}\right)\left[\sum_{i=1}^4 \int_0^\infty f_i(y,t)\frac{d\mu(t)}{t^{1+m}}\right]^2 dy_1\,dy_2$$
$$\le \frac{4C_2(\sqrt{n})^2}{q^4}\sum_{i=1}^4 \int_0^{2\pi}\int_0^{2\pi} \sin^{4n}\left(\frac{y_1}{2}\right)\sin^{4n}\left(\frac{y_2}{2}\right)\left[\int_0^\infty f_i(y,t)\frac{d\mu(t)}{t^{1+m}}\right]^2 dy_1\,dy_2,$$

by the Cauchy–Schwarz inequality. Changing $y_1 \mapsto 2\pi - y_1$ in the second integral, $y_2 \mapsto 2\pi - y_2$ in the third, and $(y_1, y_2) \mapsto (2\pi - y_1, 2\pi - y_2)$ in the fourth, we see that each of these three integrals equals the first. Thus, the following bivariate version of Theorem 3.3 results.

Theorem 3.3′. *Suppose that $s = 2$, and that F, q, n, m, and A are as in Theorem 3.3. Then*

$$\frac{\|A\lambda^{(2)}\|^2}{\|\lambda^{(2)}\|^2} \le \frac{C_2(\sqrt{n})^2}{q^4}\int_{(0,2\pi]^2} \sin^{4n}\left(\frac{y_1}{2}\right)\sin^{4n}\left(\frac{y_2}{2}\right)$$
$$\times\left[\int_0^\infty \left(1 + \frac{q^2t}{4\pi}\right)e^{-\|y\|^2/4q^2t}\frac{d\mu(t)}{t^{1+m}}\right]^2 dy. \qquad \blacksquare$$

The procedure described above may be adapted, *mutatis mutandis*, to higher dimensions. We start with the infinite sum $\sum_{k\in \mathbf{Z}^s} e^{-\|y+2\pi k\|^2/4q^2t}$ and split it into 2^s parts corresponding to the 2^s "octants" of $\mathbf{Z}^s$. Each part is dominated by an appropriate term [e.g., when $s=3$, the eight dominant terms correspond to $(0,0,0), (-1,0,0), (0,-1,0), (0,0,-1), (-1,-1,0), (-1,0,-1), (0,-1,-1)$, and $(-1,-1,-1)$, and $\mathbf{Z}^3$ is then divided suitably into eight octants]. Proceeding exactly as we did in the bivariate case, we obtain 2^s sums, each dominated by a term $f_i(y,t)$, $1\le i\le 2^s$, which consists of an appropriate exponential term multiplied by a common factor

$$C_s\left[1+\int_0^\infty\cdots\int_0^\infty e^{-\pi^2(u_1^2+u_2^2+\cdots+u_s^2)/q^2t}\,du_1\,du_2\cdots du_s\right]=C_s\left[1+\frac{q^st^{s/2}}{2^s\pi^{s/2}}\right].$$

So, once again, Theorem 3.3 and the Cauchy–Schwarz inequality give

$$\frac{\|A\lambda^{(s)}\|^2}{\|\lambda^{(s)}\|^2}\le\frac{C_s(\sqrt{n})^s}{q^{2s}}\sum_{i=1}^{2^s}\int_{(0,2\pi]^s}\prod_{j=1}^{s}\sin^{4n}\left(\frac{y_j}{2}\right)\left[\int_0^\infty f_i(y,t)\,\frac{d\mu(t)}{t^{s/2+m}}\right]^2dy.$$

Finally, we make an appropriate change of variables in each of the last 2^s-1 integrals and conclude thereby that all these integrals equal the first. This, taken in conjuction with the obvious inequality $1+q^st^{s/2}/2^s\pi^{s/2}\le 1+q^st^{s/2}$, yields

Theorem 3.4. *Suppose that F, n, q, m, and A are as in Theorem 3.3. Then*

$$\frac{\|A\lambda^{(s)}\|^2}{\|\lambda^{(s)}\|^2}\le\frac{C_s(\sqrt{n})^s}{q^{2s}}\int_{(0,2\pi]^s}\prod_{j=1}^{s}\sin^{4n}\left(\frac{y_j}{2}\right)\left[\int_0^\infty(1+q^st^{s/2})e^{-\|y\|^2/4q^2t}\,\frac{d\mu(t)}{t^{s/2+m}}\right]^2dy.$$

4. Applications

Having developed a general framework in the preceding sections, we now turn to specific applications. As in the past, our attitude towards constants will continue to be somewhat cavalier.

Theorem 4.1. *Let $n\ (\ge 2)\in\mathbf{N}$, $q>0$, and $F(x):=\|x\|^{2\alpha}$, $x\in\mathbf{R}^s$, $0<\alpha<1$. Assume that A is the interpolation matrix associated with $\mathcal{L}_{s,n}$ and F_q. Then*

$$\frac{\|A\lambda^{(s)}\|^2}{\|\lambda^{(s)}\|^2}\le C_{s,\alpha}q^{4\alpha}\left[O\left(\frac{1}{\sqrt{n}}\right)+1\right],\qquad n\to\infty.$$

Proof. Recall that $F\in RN_1^\infty$ and its representing measure is $d\mu(t)=C_\alpha(dt/t^\alpha)$. Example 2.8 shows that F satisfies condition $(*)$ of Lemma 2.4. Consequently, by Theorem 3.4,

(4.1)

$$\frac{\|A\lambda^{(s)}\|^2}{\|\lambda^{(s)}\|^2}\le\frac{C_s(\sqrt{n})^s}{q^{2s}}\int_{(0,2\pi]^s}\prod_{j=1}^{s}\sin^{4n}\left(\frac{y_j}{2}\right)\left[\int_0^\infty(1+q^st^{s/2})e^{-\|y\|^2/4q^2t}\,\frac{dt}{t^{s/2+1+\alpha}}\right]^2dy.$$

Suppose that $y \in (0, 2\pi]^s$, and consider

$$\int_0^\infty (1 + q^s t^{s/2}) \frac{e^{-\|y\|^2/4q^2t}}{t^{s/2+1+\alpha}} \, dt$$

$$= C_{s,\alpha} q^{s+2\alpha} \int_0^\infty \left(1 + \frac{1}{(4u)^{s/2}}\right) e^{-\|y\|^2 u} u^{s/2+\alpha-1} \, du, \qquad u = 1/4q^2 t,$$

$$= C_{s,\alpha} q^{s+2\alpha} \left[\int_0^\infty e^{-\|y\|^2 u} u^{s/2+\alpha-1} \, du + \int_0^\infty \frac{1}{(4u)^{s/2}} e^{-\|y\|^2 u} u^{s/2+\alpha-1} \, du \right].$$

Each of the two integrals above represents (up to constants) the Laplace transform, evaluated at $\|y\|^2$, of the functions $u \mapsto u^{s/2+\alpha-1}$ and $u \mapsto u^{\alpha-1}$, respectively. Computing these transforms leads us to conclude that

$$\text{(4.2)} \qquad \int_0^\infty (1 + q^s t^{s/2}) \frac{e^{-\|y\|^2/4q^2t}}{t^{s/2+1+\alpha}} \, dt \le q^{s+2\alpha} C_{s,\alpha} \left[\frac{1}{\|y\|^{s+2\alpha}} + \frac{1}{\|y\|^{2\alpha}} \right] \le \frac{C_{s,\alpha} q^{s+2\alpha}}{\|y\|^{s+2\alpha}}.$$

Using (4.2) in (4.1), we obtain

$$\text{(4.3)} \qquad \frac{\|A\lambda^{(s)}\|^2}{\|\lambda^{(s)}\|^2} \le C_{s,\alpha} (\sqrt{n})^s q^{4\alpha} \int_{(0,2\pi]^s} \frac{\prod_{j=1}^s \sin^{4n}(y_j/2)}{\|y\|^{2s+4\alpha}} \, dy$$

$$\le C_{s,\alpha} q^{4\alpha} \left[\sqrt{n} \int_0^{2\pi} \frac{\sin^{4n}(y_1/2)}{y_1^{2s+4\alpha}} \, dy_1 \right] (\sqrt{n})^{s-1} \prod_{j=2}^s \int_0^{2\pi} \sin^{4n}\left(\frac{y_j}{2}\right) dy_j.$$

Now, note that for each $2 \le j \le s$,

$$\int_0^{2\pi} \sin^{4n}\left(\frac{y_j}{2}\right) = 4 \int_0^{\pi/2} \sin^{4n}(y_j) \, dy_j = \frac{2\pi(4n)!}{2^{4n}(2n!)^2} = O\left(\frac{1}{\sqrt{n}}\right),$$

where the last equality follows from Wallis' formula (see (3.4)). Consequently, (4.3) implies that

$$\text{(4.4)} \qquad \frac{\|A\lambda^{(s)}\|^2}{\|\lambda^{(s)}\|^2} \le C_{s,\alpha} q^{4\alpha} \sqrt{n} \int_0^\pi \frac{\sin^{4n}(y_1)}{y_1^{2s+4\alpha}} \, dy_1.$$

Observe that

$$\text{(4.5)} \qquad \int_0^\pi \frac{\sin^{4n}(y_1)}{y_1^{2s+4\alpha}} \, dy_1 = \left(\int_0^1 + \int_1^\pi \right) \left[\frac{\sin^{4n}(y_1)}{y_1^{2s+4\alpha}} \right] dy_1$$

$$\le \int_0^1 \frac{y_1^{4n}}{y_1^{2s+4\alpha}} \, dy_1 + \int_1^\pi \sin^{4n}(y_1) \, dy_1.$$

If $4n > 2s + 4\alpha$, then the first integral in (4.5) equals $(4n - 2s - 4\alpha + 1)^{-1} = O(n^{-1})$, whereas the second integral does not exceed $\int_0^\pi \sin^{4n}(y_1) \, dy_1 = O(n^{-1/2})$, as shown before.

Thus, from (4.4),

$$\frac{\|A\lambda^{(s)}\|^2}{\|\lambda^{(s)}\|^2} \le C_{s,\alpha} q^{4\alpha}\left[O\left(\frac{1}{\sqrt{n}}\right) + 1\right], \qquad n \to \infty,$$

and the proof is complete. ∎

Corollary 4.2. *Let F, q, A be as in Theorem 4.1. Then there exists a constant $\tilde{C}_{s,\alpha}$ such that for sufficiently large n, $\|A^{-1}\| \ge \tilde{C}_{s,\alpha}/q^{2\alpha}$.*

Proof. Theorem 4.1 shows that there exists a constant $C'_{s,\alpha}$ such that for sufficiently large n, $\|A\lambda^{(s)}\|/\|\lambda^{(s)}\| \le C'_{s,\alpha} q^{2\alpha}$. So, by (3.1),

$$\|A^{-1}\| \ge \frac{\|\lambda^{(s)}\|}{\|A\lambda^{(s)}\|} \ge \frac{(C'_{s,\alpha})^{-1}}{q^{2\alpha}}.$$ ∎

Remark 4.3. (i) For the function $F(x) = \|x\|^{2\alpha}$, $x \in \mathbf{R}^s$, $0 < \alpha < 1$, the matrix A considered in Theorem 4.1 (and Corollary 4.2) exhibits "optimal" behavior with respect to the minimal separation distance q. More precisely, we recall from [NW2, Theorem 2.4] that for *any* interpolation matrix B associated with $F_q(x) = \|qx\|^{2\alpha}$, $0 < \alpha < 1$,

$$(4.6) \qquad \frac{1}{\|B^{-1}\|} \ge \frac{E_s}{q^s} \int_0^\infty \frac{e^{-\delta^2/q^2 t}}{t^{s/2+1+\alpha}}\, dt, \qquad \delta = 12\left[\frac{\pi[\Gamma((s+2)/2)]^2}{9}\right]^{1/(s+1)}$$

Setting $u = 1/q^2 t$ in (4.6) and evaluating the resulting integral, we obtain

$$(4.7) \qquad \|B^{-1}\| \le \frac{E_{s,\alpha}}{q^{2\alpha}}.$$

Assertion (4.7), taken together with Corollary 4.2, demonstrates the optimal behavior of A.

(ii) It should be noted that for $F(x) = \|x\|^{2\alpha}$, the $O(q^{-2\alpha})$ behavior of $\|A^{-1}\|$ is, in fact, a direct consequence of the homogeneity of the norm. So the additional import of Theorem 4.1, and hence of Corollary 4.2, is that the bound for $\|A^{-1}\|$ derived there is independent of n for large n.

Remark 4.4. As indicated in the introductory section, the purport of Corollary 4.2 and Remark 4.3 for the special case $s = 1$ and $\alpha = \frac{1}{2}$, was noted in [B]. A rather elegant proof of this has also been furnished recently in [Ba1].

Theorem 4.5. *Let $F(x) = \sqrt{1 + \|x\|^2}$, $x \in \mathbf{R}^s$, $n\ (\ge 2) \in \mathbf{N}$, and $0 < q < 1$. Suppose that A is the interpolation matrix associated with $\mathcal{L}_{s,n}$ and F_q. Then*

$$\frac{\|A\lambda^{(s)}\|^2}{\|\lambda^{(s)}\|^2} \le C_s q^2 [e^{-\sqrt{s}/q} + o(1)], \qquad n \to \infty.$$

Proof. Since our interest centers around small values of q, it is no loss to assume that $q \in (0, 1)$. Also, recall that $F \in RN_1^\infty$ and its representing measure is

$$d\mu(t) = C \frac{e^{-t}}{\sqrt{t}} \, dt.$$

By Example 2.9, F satisfies condition (*) of Lemma 2.4. So by Theorem 3.4,

(4.8)

$$\begin{aligned} \frac{\|A\lambda^{(s)}\|^2}{\|\lambda^{(s)}\|^2} &\le \frac{C_s(\sqrt{n})^s}{q^{2s}} \int_{(0,2\pi]^s} \prod_{j=1}^{s} \sin^{4n}\left(\frac{y_j}{2}\right) \left[\int_0^\infty (1 + q^s t^{s/2}) e^{-\|y\|^2/4q^2 t} \frac{e^{-t}}{t^{(s+3)/2}} \, dt\right]^2 dy \\ &\le C_s(\sqrt{n})^s q^2 \int_{(0,2\pi]^s} \prod_{j=1}^{s} \sin^{4n}\left(\frac{y_j}{2}\right) \\ &\quad \times \left[\int_0^\infty \left(1 + \frac{1}{u^{s/2}}\right) e^{-\|y\|^2 u} e^{-1/4q^2 u} u^{(s-1)/2} \, du\right]^2 dy, \qquad u = 1/4q^2 t. \end{aligned}$$

Suppose now that $y = (y_1, y_2, \ldots, y_s) \in (0, 2\pi]^s$, and consider

$$\begin{aligned} &\int_0^\infty \left(1 + \frac{1}{u^{s/2}}\right) e^{-\|y\|^2 u} e^{-1/4q^2 u} u^{(s-1)/2} \, du \\ &\quad = \int_0^\infty e^{-\|y\|^2 u} e^{-1/4q^2 u} u^{(s-1)/2} \, du + \int_0^\infty \frac{e^{-\|y\|^2 u}}{\sqrt{u}} e^{-1/4q^2 u} \, du \\ &\quad =: I_1 + I_2. \end{aligned}$$

From [SW, p. 6],

$$I_2 = \frac{\sqrt{\pi}}{\|y\|} e^{-\|y\|/q}. \tag{4.9}$$

Turning to I_1, we write

$$\begin{aligned} I_1 &= \left(\int_0^{1/q\|y\|} + \int_{1/q\|y\|}^\infty\right) [e^{-\|y\|^2 u} e^{-1/4q^2 u} u^{(s-1)/2}] \, du \\ &=: I_{11} + I_{12}. \end{aligned}$$

Note that the function $u \mapsto e^{-1/4q^2 u}$ increases with u, so

$$\begin{aligned} I_{11} &\le e^{-\|y\|/4q} \int_0^\infty e^{-\|y\|^2 u} u^{(s-1)/2} \, du \\ &= \frac{C_s e^{-\|y\|/4q}}{\|y\|^{s+1}}. \end{aligned} \tag{4.10}$$

Next, since $e^{-1/4q^2 u} \le 1$,

$$I_{12} \le \int_{1/q\|y\|}^\infty e^{-\|y\|^2 u} u^{(s-1)/2} \, du.$$

Now, observe that if $s = 2k + 1$, then $u^{(s-1)/2} = u^k$, whereas if $s = 2k$, then $u^{(s-1)/2} \le \sqrt{q\|y\|}u^k$, for $1/q\|y\| \le u < \infty$. Thus, in either case,

$$I_{12} \le C_s \int_{1/q\|y\|}^{\infty} e^{-\|y\|^2 u} u^k \, du, \tag{4.11}$$

where $s = 2k$ or $2k + 1$. Now

$$\begin{aligned} \int_{1/q\|y\|}^{\infty} e^{-\|y\|^2 u} u^k \, du &= \frac{1}{\|y\|^{2k+2}} e^{-\|y\|/q} \left[\left(\frac{\|y\|}{q} \right)^k + k \left(\frac{\|y\|}{q} \right)^{k-1} + \cdots + k! \right] \\ &\le \frac{C_s}{\|y\|^{3k+2}} \left(\frac{\|y\|^k}{q^k} \right) e^{-\|y\|/q} \\ &\le \frac{C_s e^{-\|y\|/2q}}{\|y\|^{3k+2}}, \end{aligned} \tag{4.12}$$

where we have used the inequalities $q < 1$ and $(x/2)^k \le (k!)e^{x/2}$, $x > 0$. Consequently,

$$I_{12} \le \frac{C_s}{\|y\|^{(3s+\gamma)/2}} e^{-\|y\|/2q}, \tag{4.13}$$

where γ equals 1 if $s = 2k + 1$, and equals 4 if $s = 2k$.

From (4.9), (4.10), and (4.13), we infer that

$$(I_1 + I_2) \le C_s \left[\frac{e^{-\|y\|/q}}{\|y\|} + \frac{e^{-\|y\|/4q}}{\|y\|^{s+1}} + \frac{e^{-\|y\|/2q}}{\|y\|^{(3s+\gamma)/2}} \right],$$

and therefore,

$$(I_1 + I_2)^2 \le \frac{C_s e^{-\|y\|/2q}}{\|y\|^{3s+\gamma}}. \tag{4.14}$$

Using (4.14) in (4.8), we get

$$\frac{\|A\lambda^{(s)}\|^2}{\|\lambda^{(s)}\|^2} \le C_s q^2 (\sqrt{n})^s \int_{(0,2\pi]^s} \frac{\prod_{j=1}^{n} \sin^{4n}(y_j/2)}{\|y\|^{3s+\gamma}} e^{-\|y\|/2q} \, dy. \tag{4.15}$$

Noting that $\|y\| \ge y_1$, and $\sqrt{s}\|y\| \ge \sum_{j=1}^{s} y_j$ by the Cauchy–Schwarz inequality, we have from (4.15),

$$\begin{aligned} \frac{\|A\lambda^{(s)}\|^2}{\|\lambda^{(s)}\|^2} &\le C_s q^2 (\sqrt{n})^s \left[\int_0^{2\pi} \frac{\sin^{4n}(y_1/2)}{y_1^{3s+\gamma}} e^{-y_1/2q\sqrt{s}} \, dy_1 \right] \\ &\quad \times \left[\prod_{j=2}^{s} \int_0^{2\pi} \sin^{4n}\left(\frac{y_j}{2} \right) e^{-y_j/2q\sqrt{s}} \, dy_j \right] \\ &\le C_s q^2 (\sqrt{n})^s \left[\int_0^{\pi} \frac{\sin^{4n}(y_1)}{y_1^{3s+\gamma}} e^{-y_1/q\sqrt{s}} \, dy_1 \right] \\ &\quad \times \left[\prod_{j=2}^{s} \int_0^{\pi} \sin^{4n}(y_j) e^{-y_j/q\sqrt{s}} \, dy_j \right]. \end{aligned} \tag{4.16}$$

Assuming that $4n > 3s + \gamma$, we note that

$$\int_0^\pi \frac{\sin^{4n}(y_1)}{y_1^{3s+\gamma}} e^{-y_1/q\sqrt{s}}\, dy_1 = \left(\int_0^1 + \int_1^\pi\right)\left[\frac{\sin^{4n}(y_1)}{y_1^{3s+\gamma}}\right] e^{-y_1/q\sqrt{s}}\, dy_1$$

$$\leq \int_0^1 y_1^{4n-3s-\gamma}\, dy_1 + e^{-1/q\sqrt{s}} \int_1^\pi \sin^{4n}(y_1)\, dy_1$$

$$= O\left(\frac{1}{n}\right) + e^{-1/q\sqrt{s}} O\left(\frac{1}{\sqrt{n}}\right),$$

as seen in the proof of Theorem 4.1. A similar argument shows that for each $2 \leq j \leq s$,

$$\int_0^\pi \sin^{4n}(y_j) e^{-y_j/q\sqrt{s}}\, dy_j = O\left(\frac{1}{n}\right) + e^{-1/q\sqrt{s}} O\left(\frac{1}{\sqrt{n}}\right).$$

Thus, by (4.16),

$$\frac{\|A\lambda^{(s)}\|^2}{\|\lambda^{(s)}\|^2} \leq C_s q^2 \left[\prod_{j=1}^{s}\left[O\left(\frac{1}{\sqrt{n}}\right) + e^{-1/q\sqrt{s}} O(1)\right]\right]$$

$$\leq C_s q^2 [e^{-\sqrt{s}/q} + o(1)], \qquad n \to \infty,$$

as desired. ■

Using (3.1) along with the preceding theorem, we obtain, as before,

Corollary 4.6. *Suppose that* $F(x) = \sqrt{1 + \|x\|^2}$, $x \in \mathbf{R}^s$. *There exists a constant* $\tilde{C}_s$ *such that, for each* $q \in (0, 1)$, *we can find an* $n_0(q) \in \mathbf{N}$ *so that for all* $n \geq n_0(q)$, *the interpolation matrix* A *associated with* $\mathcal{L}_{s,n}$ *and* F_q *satisfies*

$$\|A^{-1}\| \geq \tilde{C}_s \frac{e^{\sqrt{s}/2q}}{q}. \qquad ■$$

Remark 4.7. It is worthwhile to compare the lower estimate for $\|A^{-1}\|$ derived above with the general upper bound for the same, obtained in [NW2, Theorem 2.4]. Therein it was shown that for *any* interpolation matrix B associated with the Hardy multiquadric, and δ as in Remark 4.3, we have

$$\frac{1}{\|B^{-1}\|} \geq \frac{E_s}{q^s} \int_0^\infty e^{-\delta^2/q^2 t} \frac{e^{-t}}{t^{(s+3)/2}}\, dt$$

$$= E_s q \int_0^\infty e^{-\delta^2 u} e^{-1/q^2 u} u^{(s-1)/2}\, du, \qquad u = 1/q^2 t,$$

$$\geq E_s q \int_{1/\delta q}^\infty e^{-\delta^2 u} e^{-1/q^2 u} u^{(s-1)/2}\, du$$

$$\geq E_s q e^{-\delta/q} \int_{1/\delta q}^\infty e^{-\delta^2 u}\, du \quad (\text{as } u \mapsto e^{-1/q^2 u} \text{ increases and } q < 1)$$

$$\geq E_s q e^{-2\delta/q}.$$

Now, by Stirling's formula [AS, formula 6.1.37], $\lim_{s\to\infty} \delta/(s+2) = 6/e$; in particular, $\delta \sim s$. Thus

$$\|B^{-1}\| \le \frac{E_s^{-1} e^{2\delta/q}}{q} \sim \frac{E_s^{-1} e^{2s/q}}{q}.$$

At this time, it is not clear to the writers whether this upper estimate can in fact be sharpened to obtain $e^{\sqrt{s}/q}$ behavior.

Our next illustration concerns the inverse multiquadric.

Theorem 4.8. *Let $F\ (: \mathbf{R}^s \to \mathbf{R}) \in RP_0^\infty$ be given by $F(x) = 1/\sqrt{1 + \|x\|^2}$. Suppose that $0 < q < 1$, $n\ (\ge 2) \in \mathbf{N}$, and that A is the interpolation matrix associated with $\mathscr{L}_{s,n}$ and F_q. Then*

$$\frac{\|A\lambda^{(s)}\|^2}{\|\lambda^{(s)}\|^2} \le C_s[e^{-\sqrt{s}/q} + o(1)], \qquad n \to \infty.$$

Proof. Recall that F is represented by the measure $d\mu(t) = C(e^{-t}/\sqrt{t})\,dt$, and satisfies condition $(*)$ of Lemma 2.4. So Theorem 3.4 ensures that

(4.17)

$$\frac{\|A\lambda^{(s)}\|^2}{\|\lambda^{(s)}\|^2} \le \frac{C_s(\sqrt{n})^s}{q^{2s}} \int_{(0,2\pi]^s} \prod_{j=1}^{s} \sin^{4n}\left(\frac{y_j}{2}\right)\left[\int_0^\infty (1 + q^s t^{s/2}) e^{-\|y\|^2/4q^2 t} \frac{e^{-t}}{t^{(s+1)/2}}\,dt\right]^2 dy.$$

Let $y = (y_1, y_2, \ldots, y_s) \in (0, 2\pi]^s$, and consider

$$\begin{aligned}\int_0^\infty (1 + q^s t^{s/2}) e^{-\|y\|^2/4q^2 t} \frac{e^{-t}}{t^{(s+1)/2}}\,dt &= \int_0^\infty e^{-\|y\|^2/4q^2 t} \frac{e^{-t}}{t^{(s+1)/2}}\,dt + \int_0^\infty q^s e^{-\|y\|^2/4q^2 t} \frac{e^{-t}}{\sqrt{t}}\,dt \\ &=: I_1 + I_2.\end{aligned}$$

Again, from [SW, p. 6],

(4.18) $$I_2 = \sqrt{\pi} q^s e^{-\|y\|/q}.$$

Next,

$$\begin{aligned} I_1 &= \int_0^\infty e^{-\|y\|^2/4q^2 t} \frac{e^{-t}}{t^{(s+1)/2}}\,dt \\ &= 2^{s-1} q^{s-1} \int_0^\infty e^{-\|y\|^2 u} \frac{e^{-1/4q^2 u}}{u} u^{(s-1)/2}\,du, \qquad u = 1/4q^2 t, \\ &= 2^{s-1} q^{s-1}\left(\int_0^{1/4q\sqrt{s}} + \int_{1/4q\sqrt{s}}^\infty\right)\left[e^{-\|y\|^2 u} \frac{e^{-1/4q^2 u}}{u} u^{(s-1)/2}\right] du \\ &=: I_{11} + I_{12}.\end{aligned}$$

As the function $u \mapsto e^{-1/4q^2u}/u$ increases on $0 \le u \le 1/4q^2$, and since $1/4q\sqrt{s} \le 1/4q < 1/4q^2$, we see that

$$I_{11} \le 2^{s+1}\sqrt{s}q^s e^{-\sqrt{s}/q} \int_0^\infty e^{-\|y\|^2u} u^{(s-1)/2}\, du \le \frac{C_s q^s e^{-\sqrt{s}/q}}{\|y\|^{s+1}}. \tag{4.19}$$

Since $e^{-1/4q^2u} \le 1$,

$$I_{12} \le 2^{s+1}\sqrt{s}q^s \int_{1/4q\sqrt{s}}^\infty e^{-\|y\|^2u} u^{(s-1)/2}\, du \le C_s q^s \int_{1/4q\sqrt{s}}^\infty e^{-\|y\|^2u} u^k\, du,$$

where $s = 2k$ or $2k+1$. The rest of the proof now runs almost parallel to that of Theorem 4.5. First, we get

$$I_{12} \le \frac{C_s q^s}{\|y\|^{2s+\gamma}} e^{-\|y\|^2/8q\sqrt{s}}, \tag{4.20}$$

where γ equals 2 if $s = 2k$, and equals 0 if $s = 2k+1$. Second, from (4.18), (4.19), and (4.20), we arrive at

$$(I_1 + I_2)^2 \le C_s q^{2s}\left[e^{-2\|y\|/q} + \frac{e^{-2\sqrt{s}/q}}{\|y\|^{2s+2}} + \frac{e^{-\|y\|^2/4q\sqrt{s}}}{\|y\|^{4s+2\gamma}}\right],$$

whence, from (4.17),

$$\frac{\|A\lambda^{(s)}\|^2}{\|\lambda^{(s)}\|^2} \le C_s(\sqrt{n})^s \int_{(0,2\pi]^s} \frac{\prod_{j=1}^s \sin^{4n}(y_j/2)}{\|y\|^{4s+2\gamma}} [e^{-2\|y\|/q} + e^{-2\sqrt{s}/q} + e^{-\|y\|^2/4q\sqrt{s}}]\, dy$$
$$=: T_1 + T_2 + T_3.$$

Finally, we note that $\|y\|^2 = \sum_{j=1}^s y_j^2$, and invoke familiar techniques to conclude that for $n > s+1$,

$$T_1 \le C_s[e^{-4\sqrt{s}/q} + o(1)],$$
$$T_2 \le C_s[e^{-2\sqrt{s}/q} + o(1)],$$

and

$$T_3 \le C_s[e^{-\sqrt{s}/q} + o(1)], \qquad n \to \infty.$$

This yields the required result. ■

Corollary 4.9. *Suppose that $F(x) = 1/\sqrt{1 + \|x\|^2}$, $x \in \mathbf{R}^s$. There exists a constant $\tilde{C}_s$ such that, for each $0 < q < 1$, we can find an $n_0(q) \in \mathbf{N}$, so that for all $n \ge n_0(q)$, the interpolation matrix A associated with $\mathscr{L}_{s,n}$ and F_q satisfies*

$$\|A^{-1}\| \ge \tilde{C}_s e^{\sqrt{s}/2q}.$$

Remark 4.10. Let $0 < q < 1$ and let B be any interpolation matrix associated with $F_q(x) = 1/\sqrt{1 + \|qx\|^2}$, $x \in \mathbf{R}^s$. With δ being the same as before, we know from [NW2, Theorem 2.4] that

$$(4.21)\qquad \frac{1}{\|B^{-1}\|} \geq \frac{E_s}{q^s} \int_0^\infty e^{-\delta^2/q^2 t} \frac{e^{-t}}{t^{(s+1)/2}}\, dt$$

$$= \frac{E_s}{q} \int_0^\infty e^{-\delta^2 u} \frac{e^{-1/q^2 u}}{u} u^{(s-1)/2}\, du, \qquad u = 1/q^2 t.$$

Choose a constant a (depending only on s) such that $a/\delta < 1$ (recall that $\delta/(s+2) \to 6/e$ as $s \to \infty$). Since the function $u \mapsto e^{-1/q^2 u}/u$ increases on $[0, 1/q^2]$, and $q < 1$, we deduce from (4.21) that

$$\frac{1}{\|B^{-1}\|} \geq \frac{E_s}{q} \int_{a/2\delta q}^{a/\delta q} e^{-\delta^2 u} \frac{e^{-1/q^2 u}}{u} u^{(s-1)/2}\, du$$

$$\geq E_s e^{-2\delta/aq} \int_{a/2\delta q}^{a/\delta q} e^{-\delta^2 u}\, du$$

$$\geq E_s e^{-\delta/q(2/a + a/2)},$$

so that

$$\|B^{-1}\| \leq E_s^{-1} e^{\delta/q(2/a + a/2)} \sim E_s^{-1} e^{s/q}.$$

Remark 4.11. The techniques developed here may also be applied to $F(x) = \log(1 + \|x\|^2) \in RN_1^\infty$. We shall, however, refrain from carrying out this analysis.

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