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On the structure of varieties with equationally definable principal congruences II

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Introduction

We continue here the investigations begun in [1], but the present paper can be read independently; all the definitions and results from [1] we shall need are summarized in Section 0.

A comprehensive list of examples of varieties with equationally definable principal congruences (EDPC for short – see Section 0 for the definition) can be found in [1]. There are two principal kinds: The first are discriminator varieties. The second the varieties that arise from the algebraization of various deductive systems of classical or non-classical logic. (In fact it is shown in [2] that every variety that arises from a deductive system satisfying some reasonable version of the deduction theorem must have EDPC.) There are fundamental differences between these two kinds of examples. Discriminator varieties are both congruence-permutable and semisimple, and they comprehend all EDPC varieties with these two properties ([1] and Fried and Kiss [10]). On the other hand the varieties arising in logic may or may not be congruence-permutable, and, generally speaking, are semisimple only in the case of classical systems.

We have been motivated by two general questions: (I) Can the characterization of congruence-permutable and semisimple EDPC varieties by means of the discriminator function be extended in a useful way to classes of congruence-permutable EDPC varieties that fail to be semisimple? (II) How comprehensive is the class of EDPC varities that arise from logic? Can it be characterized by some natural algebraic conditions? If the class does in fact turn out to be extensive, then the study of EDPC varieties might be opened to the highly developed methods of algebraic logic. On the other hand, a natural algebraic characterization of the class of varieties that result from the algebraization of a deductive system, or some large subclass, would open algebraic logic to a broader study within the context of the general theory of algebras.

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Every EDPC variety has certain quaternary terms, we call them a *quaternary deductive* (or QD) *system*, that collectively have many of the characteristics of the implication connective of non-classical logic. In Sections 1–3 we investigate the extent to which this fact can be used to give an arbitrary variety with EDPC the structure of one arising from the algebraization of a deductive system. The characterization of EDPC varieties in terms of the existence of a QD system is given in Section 1. Several different characterizations of QD systems themselves are also given there. One of them (Definition 1.1) strongly suggests the definition of the normal transform operation introduced in Gould and Grätzer [11]. Another (Theorem 1.5(iii)) is a Mal'cev-style condition that generalizes McKenzie's [20] characterization of discriminator varieties; our condition is formulated in such a way however as to emphasize the connection between quaternary deduction systems and the implication connective of non-classical logic. The results of this section overlap to a considerable extent the work of Fried and Kiss [10].

In Section 3 we consider the properties of congruence-permutability and the existence of a constant term with well behaved ideals, and we study their effect when taken in conjunction with EDPC. It turns out that in combination they allow us to replace the QD system by an equivalent system of just two binary terms which correspond closely to the actual implication and conjunction connectives of non-classical logic. This results in a characterization of arbitrary congruence-permutable varieties with EDPC and a constant term with well behaved ideals as the members of a very general class of varieties arising in algebraic logic called *weak Brouwerian semilattices with filter preserving operations* (WBSO's). In preparation for this result WBSO varieties are defined in Section 2 and their basic properties are developed. A number of examples are given to illustrate how the varieties that result from the algebraization of a wide range of deductive systems that have occurred in the literature naturally assume the form of WBSO's.

The investigations of Section 4 are concerned with the first of the two questions raised above. We see that those congruence-permutable EDPC varieties whose subdirectly irreducibles have linearly ordered congruence lattices, the so-called *congruence relative Stone varieties* with permuting congruences, can be characterized by means of a natural generalization of the normal transform (Theorem 4.3). As a consequence congruence-permutable congruence relative Stone varieties share many of the special properties of discriminator varieties. Finally we prove a strong concrete representation theorem involving the congruence lattices of members of congruence-permutable congruence relative Stone varieties (Corollary 4.5). This result turns out to be very useful for constructing counterexamples to various natural conjectures about the EDPC varieties.

0. Preliminaries

By a *term function* on an algebra $\mathfrak{A}$ we mean any operation on A, the universe of $\mathfrak{A}$, of the form $t^{\mathfrak{A}}$ where t is some term in the equational language of $\mathfrak{A}$; thus term functions will be what were called polynomials in [1]. We shall not be careful to distinguish between a term and its associated term function. For example, if $s(x_0, \ldots, x_{n-1})$, $t(x_0, \ldots, x_{n-1})$ are terms in the language of $\mathfrak{A}$, and $a_0, \ldots, a_{n-1} \in A$, then we shall write $\mathfrak{A} \models (s(x_0, \ldots, x_{n-1}) = t(x_0, \ldots, x_{n-1}))(a_0, \ldots, a_{n-1})$, $\mathfrak{A} \models s(a_0, \ldots, a_{n-1}) = t(a_0, \ldots, a_{n-1})$, and $s^{\mathfrak{A}}(a_0, \ldots, a_{n-1}) = t^{\mathfrak{A}}(a_0, \ldots, a_{n-1})$ interchangeably, and, if $\mathfrak{A}$ is clear from context, we may simply write $s(a_0, \ldots, a_{n-1}) = t(a_0, \ldots, a_{n-1})$. A *constant* of a class of algebras will always mean a constant term, or the unique element in each member of the class that is the range of the associated term function.

A system $\Delta_0, \ldots, \Delta_{n-1}$ of binary terms is said to be an *equivalence system* for a variety $\mathcal{V}$ of algebras if there exist unary terms $E_0, \ldots, E_{m-1}$, $F_0, \ldots, F_{m-1}$ such that the following identities and quasi-identity hold in $\mathcal{V}$.

$$\begin{gathered} E_j(x\Delta_i x) = F_j(x\Delta_i x) \quad \text{for} \quad i<n \quad \text{and} \quad j<m, \\ \Big(\bigwedge_{i<n}\bigwedge_{j<m} E_j(x\Delta_i y) = F_j(x\Delta_i y)\Big) \quad \text{implies} \quad x = y. \end{gathered} \tag{1}$$

If the E_j all coincide with the identity and the F_j all coincide with the same constant 1, then the equivalence system is called a Gödel system (with respect to 1). Thus $\Delta_0, \ldots, \Delta_{n-1}$ is a *Gödel equivalence system for* $\mathcal{V}$ (*with respect to* 1) if

$$\begin{gathered} x\Delta_i x = 1 \quad \text{for all} \quad i<n, \\ \Big(\bigwedge_{i<n} x\Delta_i y = 1\Big) \quad \text{implies} \quad x = y \end{gathered}$$

hold in $\mathcal{V}$. If there is just one term Δ in the system we refer to it as a *Gödel equivalence term for* $\mathcal{V}$.

It is not difficult to see that $\Delta_0, \ldots, \Delta_{n-1}$ is a Gödel equivalence system for a variety $\mathcal{V}$ with respect to a constant 1 iff, for every $\mathfrak{A} \in \mathcal{V}$ and all $a, b \in A$, the following congruence condition holds:

$$\Theta(a, b) = \Theta(a\Delta_0 b, 1) \vee \cdots \vee \Theta(a\Delta_{n-1} b, 1).$$

($\Theta_{\mathfrak{A}}(a, b)$ denotes the principal congruence of $\mathfrak{A}$ generated by the pair a, b; when $\mathfrak{A}$ is clear from context we omit the subscript).

In [2] it is shown that a variety $\mathcal{V}$ is the result of algebraizing some deductive system $\mathscr{L}$ iff $\mathscr{V}$ has an equivalence system $\Delta_0, \ldots, \Delta_{n-1}$. Furthermore, the system is a Gödel system iff $\mathscr{L}$ has the property that, from any two formulas ϕ and ψ, one can deduce the equivalence of ϕ and ψ, i.e., the following so-called *Gödel rule* holds in $\mathscr{L}$:

$$\phi, \psi \vdash_{\mathscr{L}} \phi \Delta_i \psi, \quad \text{for} \quad i < n.$$

Here we identify the term Δ_i with the formula of $\mathscr{L}$ from which it naturally comes. See [2] for details. With a few notable exceptions all the algebraizable deductive systems of classical and non-classical logic occurring in the literature have the Gödel rule.

It turns out that the existence of a Gödel equivalence system can be characterized in more algebraic terms. Let e be an element of an arbitrary algebra $\mathfrak{A}$. We say that $\mathfrak{A}$ is *e-regular* if every congruence of $\mathfrak{A}$ is completely determined by its e-equivalence class; $\mathfrak{A}$ is *point-regular* if e-regular for some e. (See Henkin, Monk, and Tarski [15, p. 80] and Jónsson [16, p. 158] where the same property is described by saying that the e-ideals of $\mathfrak{A}$ are *well-behaved*.) Let $\mathscr{K}$ be a class of algebras with a constant 1. We say that $\mathscr{K}$ is a *1-regular* if each member of $\mathscr{K}$ is 1-regular in the above sense; $\mathscr{K}$ is *point-regular* if it is 1-regular for some constant 1. (In Grätzer [13] this property is called *weak regularity*.) The following result is proved in [2]; very closely related results were established earlier in Fichtner [5, 6, 7] and Grätzer [13].

THEOREM 0.1. *Let $\mathscr{V}$ be any variety. $\mathscr{V}$ is 1-regular for some constant 1 iff $\mathscr{V}$ has a Gödel equivalence system with respect to 1.*

Consequently, in view of previous remarks, a variety is point-regular iff it arises from the algebraization of some deductive system with the Gödel rule.

Let $\langle A, \cdot, 1\rangle$ be a semilattice with largest element. If it exists, the binary operation $\rightarrow$ on A defined by the condition that, for every $c \in A$,

$$c \leq a \rightarrow b \quad \text{iff} \quad a \cdot c \leq b$$

is called *relative pseudo complementation*. If $\rightarrow$ does exist $\langle A, \cdot, 1\rangle$ is said to be *relatively pseudo complemented*; the algebra $\langle A, \cdot, \rightarrow, 1\rangle$ obtained by adjoining $\rightarrow$ as a new fundamental operation is called a *Brouwerian semilattice*.

The following useful result from the general theory of algebras is easily established; see for instance [18; Lemma 2].

LEMMA 0.2. *Let $\mathfrak{A}$ be any algebra, and Φ any congruence on $\mathfrak{A}$. Then, for all*

$a, b, c, d \in A$,

$$c/\Phi \equiv d/\Phi(\Theta_{\mathfrak{A}/\Phi}(a/\Phi, b/\Phi)) \quad \textit{iff} \quad c \equiv d(\Theta_{\mathfrak{A}}(a, b) \vee \Phi).$$

A variety $\mathcal{V}$ has *equationally definable principal congruences* (EDPC) if there exists a finite system of quaternary terms $s_i(x, y, z, w)$, $t_i(x, y, z, w)$, $i < n$, such that, for every $\mathfrak{A} \in \mathcal{V}$ and all $a, b, c, d \in A$,

$$c \equiv d(\Theta_{\mathfrak{A}}(a, b)) \quad \text{iff} \quad \mathfrak{A} \models s_i(a, b, c, d) = t_i(a, b, c, d), \quad i < n.$$

(This is equivalent to the notion of restricted equationally definable principal congruences (REDPC) considered in Fried, Grätzer, and Quackenbush [9].) It is easily seen that every variety with EDPC has the congruence extension property.

In [2] it is shown that, if $\mathcal{V}$ comes from algebraizing a deductive system $\mathcal{L}$, then $\mathcal{V}$ has EDPC iff $\mathcal{L}$ satisfies the following so-called *generalized deduction theorem* (it actually generalizes both the modus ponens rule and the deduction theorem): there exists a finite system $\rightarrow_0, \ldots, \rightarrow_{m-1}$ of formulas in two propositional variables such that, for all formulas $\theta_0, \ldots, \theta_{l-1}, \phi, \psi$ of $\mathcal{L}$,

$$\theta_0, \ldots, \theta_{l-1}, \phi \vdash_{\mathcal{L}} \psi \quad \text{iff} \quad \theta_0, \ldots, \theta_{l-1} \vdash_{\mathcal{L}} \phi \rightarrow_i \psi, \quad i < m.$$

See [2] for details.

As was the case for the existence of a Gödel equivalence system, the property of having EDPC can be characterized in more algebraic terms. For any algebra $\mathfrak{A}$ the set of congruences of $\mathfrak{A}$ is denoted by $Co\mathfrak{A}$, and the lattice of congruences by $\mathfrak{Co}\mathfrak{A}$. $I_{\mathfrak{A}}$ (or just I) is the identity congruence on $\mathfrak{A}$. The set of compact (i.e., finitely generated) congruences is denoted by $Cp\mathfrak{A}$. The compact congruences form a semilattice under join with smallest element $I_{\mathfrak{A}}$. If it exists, the binary operation $*$ on $Cp\mathfrak{A}$ defined by the condition

$$\Phi * \Psi \subseteq \Theta \quad \text{iff} \quad \Psi \subseteq \Phi \vee \Theta$$

for every $\Theta \in Cp\mathfrak{A}$ is called *dual relative pseudo complementation*. The following is proved in [18].

THEOREM 0.3. *A variety* $\mathcal{V}$ *has* EDPC *iff* $\langle Cp\mathfrak{A}, \vee, I\rangle$ *is dually relatively pseudo complemented for every* $\mathfrak{A} \in \mathcal{V}$.

When $*$ exists we define $\mathfrak{Cp}\mathfrak{A} = \langle Cp\mathfrak{A}, \vee, *, I\rangle$; $\mathfrak{Cp}\mathfrak{A}$ is a *dual Brouwerian semilattice*. We shall assume the elementary theory of Brouwerian semilattices is known; see for instance Nemitz [23] or Köhler [17].

Let $h:\mathfrak{A}\to\mathfrak{B}$ be a homomorphism. For each $\Phi\in Cp\mathfrak{A}$ let $(Cp\mathfrak{A})(\Phi)$ be the congruence of $\mathfrak{B}$ generated by the set of all pairs $\langle ha, hb\rangle$ such that $a\equiv b(\Phi)$. The following result is established in [1, Lemma 4.4].

LEMMA 0.4. *Assume* $\mathfrak{A}, \mathfrak{B}$ *are members of a variety with* EDPC, *and* h *is a homomorphism from* $\mathfrak{A}$ *into* $\mathfrak{B}$. *Then* Cph *is a homomorphism from* $\mathfrak{Cp}\mathfrak{A}$ *into* $\mathfrak{Cp}\mathfrak{B}$.

In the sequel $\mathfrak{A}$ is assumed to be a member of a variety $\mathcal{V}$ with EDPC. Let $\kappa(w_0,\ldots,w_m)$ be any term in the language of $\mathfrak{Cp}\mathfrak{A}$, i.e., any term built up from the variables $w_0,\ldots,w_m$ using the constant symbol I and the binary operation symbols $\vee$ and $*$. The main result of [1, Theorem 2.2] says that there exists a conjunction $\boldsymbol{\phi}_\kappa(y_0, z_0,\ldots,y_m, z_m)$ of equations in the language of $\mathcal{V}$ such that, for all $a_0, b_0,\ldots,a_m, b_m\in A$,

$$\mathfrak{Cp}\mathfrak{A}\models\kappa(\Theta(a_0, b_0),\ldots,\Theta(a_m, b_m))=I \quad\text{iff}\quad \mathfrak{A}\models\boldsymbol{\phi}_\kappa(a_0, b_0,\ldots,a_m, b_m).$$

The conjunction of equations $\boldsymbol{\phi}_\kappa$ depends only on the Brouwerian semilattice term κ, not on $\mathfrak{A}$, and is constructed from κ by a specific recursive procedure. We can use this translation to characterize the quasi-identities satisfied by subvarieties of $\mathcal{V}$.

Consider any quasi-identity

$$\Big(\bigwedge_{i<m}\sigma_i(\bar{x})=\tau_i(\bar{x})\Big)\quad\text{implies}\quad\rho(\bar{x})=\pi(\bar{x}) \tag{2}$$

in the language of $\mathcal{V}$ where $\bar{x}$ represents the sequence of variables $x_0,\ldots,x_{n-1}$. Let $\mathcal{K}$ be an arbitrary subclass of $\mathcal{V}$, and let $H_\omega\mathcal{K}$ be the class of all algebras $\mathfrak{B}$ such that $\mathfrak{B}\cong\mathfrak{A}/\Phi$ for some $\mathfrak{A}\in\mathcal{K}$ and $\Phi\in Cp\mathfrak{A}$. It is easy to see that (2) holds in $H_\omega\mathcal{K}$ iff the congruence condition

$$\rho(\bar{a})\equiv\pi(\bar{a})\Big(\bigvee_{i<m}\Theta(\sigma_i(\bar{a}),\tau_i(\bar{a}))\Big)$$

holds for all $\mathfrak{A}\in\mathcal{K}$ and $\bar{a}\in A^n$, or, equivalently, iff

$$\mathfrak{A}\models\kappa(\Theta(\sigma_0(\bar{a}),\tau_0(\bar{a})),\ldots,\Theta(\sigma_{m-1}(\bar{a}),\tau_{m-1}(\bar{a})),\Theta(\rho(\bar{a}),\pi(\bar{a})))$$

holds for all $\mathfrak{A}\in\mathcal{K}$ and $\bar{a}\in A^n$ where

$$\kappa(w_0,\ldots,w_{m-1},w_m)=(w_0\vee\cdots\vee w_{m-1})*w_m.$$

Thus the quasi-identity (2) holds in $H_\omega\mathcal{K}$ iff the conjunction of identities

$$\Phi_\kappa(\sigma_0(\bar{x}), \tau_0(\bar{x}), \ldots, \sigma_{m-1}(\bar{x}), \tau_{m-1}(\bar{x}), \rho(\bar{x}), \pi(\bar{x}))$$

holds in $\mathcal{K}$. This gives

THEOREM 0.5. *Assume $\mathcal{V}$ is a variety with* EDPC *and $\mathcal{K} \subseteq \mathcal{V}$. Then* HSP$\mathcal{K}$ *satisfies every quasi-identity $H_\omega\mathcal{K}$ does.*

Let $P_u\mathcal{K}$ be the class of all algebras isomorphic to an ultraproduct of a system of algebras of $\mathcal{K}$.

COROLLARY 0.6. *Assume $\mathcal{V}$ and $\mathcal{K}$ are as in the theorem. Then* HSP$\mathcal{K}$ = SPP$_u H_\omega \mathcal{K}$.

Proof. Clearly SPP$_u H_\omega\mathcal{K} \subseteq$ HSP$\mathcal{K}$. By the theorem HSP$\mathcal{K}$ is included in the quasivariety generated by $H_\omega\mathcal{K}$; but this is just SPP$_u H_\omega\mathcal{K}$.

When a variety $\mathcal{V}$ has EDPC, Theorem 0.5 provides a very useful criterion for determining when $\mathcal{V}$ has an equivalence system, and hence arises from the algebraization of some deductive system. Indeed, since the conditions (1) are in the form of identities and quasi-identities, in order to show $\Delta_0, \ldots, \Delta_{n-1}$ is an equivalence system for $\mathcal{V}$, it suffices to show that it is one for $H_\omega\mathcal{K}$ where $\mathcal{K}$ is some generating subclass of $\mathcal{V}$. It turns out to be more convenient in practice to express the fact that (1) holds in $H_\omega\mathcal{K}$ in the form of a congruence condition on $\mathcal{K}$ itself. Also we shall actually formulate the result for Gödel equivalence systems since they are the ones we shall always be dealing with.

THEOREM 0.7. *Assume $\mathcal{V}$ is a variety with* EDPC *and a constant* 1, *and let $\mathcal{K} \subseteq \mathcal{V}$. Let $\Delta_0, \ldots, \Delta_{n-1}$ be a system of binary terms of $\mathcal{V}$. Then $\Delta_0, \ldots, \Delta_{n-1}$ is a Gödel equivalence system for* HSP$\mathcal{K}$ *with respect to* 1 *iff the congruence condition*

$$\Theta(a\Delta_0 b, 1) \vee \cdots \vee \Theta(a\Delta_{n-1} b, 1) = \Theta(a, b)$$

holds for all $\mathfrak{A} \in \mathcal{K}$ and all $a, b \in A$.

In [1, Theorem 4.1] the following theorem is proved. It is closely connected to 0.5 although this is not evident from comparing their statements.

THEOREM 0.8. *Let $\mathcal{U}$ be any variety of dual Brouwerian semilattices, $\mathcal{V}$ a* *variety with* EDPC, *and* $\mathscr{K} \subseteq \mathscr{V}$. *If* $\mathfrak{Cp}\mathfrak{A} \in \mathscr{U}$ *for every* $\mathfrak{A} \in \mathscr{K}$, *then* $\mathfrak{Cp}\mathfrak{A} \in \mathscr{U}$ *for every* $\mathfrak{A} \in \mathrm{HSP}\mathscr{K}$.

1. Quaternary deductive systems

We show how the property of having EDPC may be characterized in terms of the existence of a system of quaternary term functions having certain well specified properties. Most of the results of this section, with the exception of those formulated in terms of the dual relative pseudo complementation operation on compact congruences, also appear in one form or another in Fried and Kiss [10].[(1)]

DEFINITION 1.1. *Let* $q = \langle q_i(x, y, z, w) : i \leq n \rangle$ *be a finite system of quaternary terms such that* $n \geq 2$ *and* $q_0(x, y, z, w)$ *coincides with* z *and* $q_n(x, y, z, w)$ *with* w. q *is a* ***quaternary deductive*** (QD) ***system*** (***of length*** $n+1$) *for an algebra* $\mathfrak{A}$ *if* $\mathfrak{A}$ *satisfies the identities*

(i) $q_i(x, x, y, z) = q_{i+1}(x, x, y, z)$ *for even* $i < n$,

and, for all $a, b, c, d \in A$,

(ii) $q_i(a, b, c, d) = q_{i+1}(a, b, c, d)$ *for odd* $i < n$ *whenever* $c \equiv d(\Theta(a, b))$.

q *is a* QD ***system*** *for a class* $\mathscr{K}$ *of similar algebras if it is one for every member of* $\mathscr{K}$.

A system $q = \langle q_i(x, y, z, w) : i \leq n \rangle$ of quaternary terms will be called *bounded* if $n \geq 2$ and $q_0(x, y, z, w)$ and $q_n(x, y, z, w)$ coincide respectively with z and w. Note that by definition every QD system is bounded.

LEMMA 1.2. *Let* $\mathfrak{A}$ *be an algebra and* $q = \langle q_i(x, y, z, w) : i \leq n \rangle$ *a* QD *system for* $H_\omega \mathfrak{A}$. *For all* $a, b, c, d, \in A$ *let*

$$\Phi_O(a, b, c, d) = \bigvee_{\text{odd } i<n} \Theta(q_i(a, b, c, d), q_{i+1}(a, b, c, d)),$$

$$\Phi_E(a, b, c, d) = \bigvee_{\text{even } i<n} \Theta(q_i(a, b, c, d), q_{i+1}(a, b, c, d)).$$

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[1] They were originally obtained by the present authors in 1979 for congruence-permutable varieties. Subsequently, on becoming aware of the results in [10], we realized our methods could be applied with only minor technical changes to obtain the results in the general form they are presented here.

Then $\langle Cp\mathfrak{A}, \vee, I\rangle$ *is dually relatively pseudo complemented and, for all* $a, b, c, d \in A$,

(i) $\Phi_0(a, b, c, d) = \Theta(a, b) * \Theta(c, d)$,
(ii) $(\Theta(a, b) * \Theta(c, d)) * \Theta(c, d) \subseteq \Phi_E(a, b, c, d) \subseteq \Theta(a, b)$.

Proof. Write Φ_O and Φ_E respectively for $\Phi_O(a, b, c, d)$ and $\Phi_E(a, b, c, d)$. By the definition of Φ_O

$$q_i(a/\Phi_O, b/\Phi_O, c/\Phi_O, d/\Phi_O) = q_{i+1}(a/\Phi_O, b/\Phi_O, c/\Phi_O, d/\Phi_O) \quad \text{for odd} \quad i < n. \tag{1}$$

But since q is a QD polynomial for $\mathfrak{A}/\Theta(a, b)$ by hypothesis,

$$q_i(a/\Theta(a, b), \ldots, d/\Theta(a, b)) = q_{i+1}(a/\Theta(a, b), \ldots, d/\Theta(a, b)) \quad \text{for even} \quad i < n. \tag{2}$$

Combining (1) and (2) and using the premiss q is bounded, we get

$$\Theta(c, d) \subseteq \Phi_O \vee \Theta(a, b). \tag{3}$$

Let Ψ be an arbitrary compact congruence of $\mathfrak{A}$ such that

$$\Theta(c, d) \subseteq \Psi \vee \Theta(a, b). \tag{4}$$

Then by Lemma 0.2 $c/\Psi \equiv d/\Psi(\Theta(a/\Psi, b/\Psi))$, and hence, since q is also a QD system for $\mathfrak{A}/\Psi$ by hypothesis,

$$q_i(a/\Psi, \ldots, d/\Psi) = q_{i+1}(a/\Psi, \ldots, d/\Psi) \quad \text{for odd} \quad i < n.$$

This immediately gives the inclusion $\Phi_O \subseteq \Psi$ for every $\Psi \in Cp\mathfrak{A}$ satisfying (4). Together with (3) this establishes (i). In particular, the dual relative pseudo complement of any pair of principal congruences exists. From this it follows without difficulty that $\langle Cp\mathfrak{A}, \vee, I\rangle$ is dually relatively pseudo complemented; see [18] for details.

The second inclusion of (ii) follows immediately from (2). By definition of Φ_E,

$$q_i(a/\Phi_E, \ldots, d/\Phi_E) = q_{i+1}(a/\Phi_E, \ldots, d/\Phi_E) \quad \text{for even} \quad i < n.$$

Combining this with (1), and using the premiss q is bounded, we get $\Theta(c, d) \subseteq \Phi_O \vee \Phi_E$. Thus $\Phi_O * \Theta(c, d) \subseteq \Phi_E$. Hence the first inclusion of (ii) follows from (i). This completes the proof of the lemma.

A QD system for $H_\omega\mathfrak{A}$ turns out to exhibit many of the characteristics of the implication connective of non-classical logic. This idea is developed in the following definition and the subsequent lemmas. It is used to characterize in Theorem 1.5 the QD systems of $H_\omega\mathfrak{A}$ entirely in terms of the identities of $\mathfrak{A}$.

Let $q=\langle q_i(x, y, z, w): i\leq n\rangle$ be a bounded system of quaternary terms. Let $\phi=\bigwedge_{j<r}\sigma_j=\tau_j$ and $\psi=\bigwedge_{k<s}\rho_k=\pi_k$ be any two finite conjunctions of identities in the language of $\mathfrak{A}$. Using q we associate with ϕ and ψ another finite conjunction of equations that we shall denote by $\phi\Rightarrow\psi$. If ϕ is a single equation $\sigma=\tau$, then we define

$$((\sigma=\tau)\Rightarrow\psi)=\bigwedge_{k<s}\ \bigwedge_{\text{odd } i<n} q_i(\sigma, \tau, \rho_k, \pi_k)=q_{i+1}(\sigma, \tau, \rho_k, \pi_k).$$

In general

$$(\phi\Rightarrow\psi)=(\sigma_0=\tau_0)\Rightarrow((\sigma_1=\tau_1)\Rightarrow\cdots\Rightarrow((\sigma_{r-1}=\tau_{r-1})\Rightarrow\psi)\cdots).$$

LEMMA 1.3. *Let $q=\langle q_i(x, y, z, w): i\leq n\rangle$ be a bounded system of quaternary terms and let $\mathfrak{A}$ be an algebra satisfying the identities*

$$q_i(x, x, z, w)=q_{i+1}(x, x, z, w)\quad \textit{for even}\quad i<n. \tag{5}$$

Let $\phi(x_0, \ldots, x_{m-1})$, $\psi(x_0, \ldots, x_{m-1})$ be finite conjunctions of equations, and let $a_0, \ldots, a_{m-1}\in A$. If

$$\mathfrak{A}\models(\phi\Rightarrow\psi)(a_0, \ldots, a_{m-1})\quad \textit{and}\quad \mathfrak{A}\models\phi(a_0, \ldots, a_{m-1}),$$

then $\mathfrak{A}\models\psi(a_0, \ldots, a_{m-1})$.

Proof. We write $\bar{x}$ and $\bar{a}$ respectively for the sequences $x_0, \ldots, x_{m-1}$ and $a_0, \ldots, a_{m-1}$. Suppose first of all $\phi(\bar{x})$ consists of a single conjunct $\sigma(\bar{x})=\tau(\bar{x})$. Let $\psi=\bigwedge_{k<s}\rho_k(\bar{x})=\pi_k(\bar{x})$. From (5) and the premiss $\mathfrak{A}\models\phi(\bar{a})$, i.e., $\sigma(\bar{a})=\tau(\bar{a})$, we conclude that

$$q_i(\sigma(\bar{a}), \tau(\bar{a}), \rho_k(\bar{a}), \pi_k(\bar{a}))=q_{i+1}(\sigma(\bar{a}), \tau(\bar{a}), \rho_k(\bar{a}), \pi_k(\bar{a})), \tag{6}$$

for all $k<s$ and even $i<n$. But the premiss $\mathfrak{A}\models(\phi\Rightarrow\psi)(\bar{a})$ says exactly that (6) holds also for all $k<s$ and odd $i<n$. Thus, since q is bounded, $\rho_k(\bar{a})=\pi_k(\bar{a})$ for all $k<s$, i.e., $\mathfrak{A}\models\psi(\bar{a})$.

Now let $\phi=\bigwedge_{j<r}\sigma_j=\tau_j$. Then $\phi\Rightarrow\psi$ coincides with $(\sigma_0=\tau_0)\Rightarrow(\theta\Rightarrow\psi)$

where $\theta = \bigwedge_{0<j<r} \sigma_j = \tau_j$. By the first part of the proof we have $\mathfrak{A} \models (\theta \Rightarrow \psi)(\bar{a})$, and by the induction hypothesis $\mathfrak{A} \models \psi(\bar{a})$. So the lemma is proved.

For any conjunction $\phi = \bigwedge_{j<r} \sigma_j(x_0, \ldots, x_{m-1}) = \tau_j(x_0, \ldots, x_{m-1})$ of equations, any algebra $\mathfrak{A}$, and any $a_0, \ldots, a_{m-1} \in A$, let

$$\Theta(\phi, a_0, \ldots, a_{m-1}) = \bigvee_{j<r} \Theta(\sigma_j(a_0, \ldots, a_{m-1}), \tau_j(a_0, \ldots, a_{m-1})).$$

LEMMA 1.4. *Let $\mathfrak{A}$ be an algebra, q a bounded system of quaternary terms, and $\phi(x_0, \ldots, x_{m-1})$, $\psi(x_0, \ldots, x_{m-1})$ a pair of finite conjunctions of equations. If q is a* QD *system for $H_\omega \mathfrak{A}$, then, for all $a_0, \ldots, a_{m-1} \in A$,*

$$\Theta(\phi \Rightarrow \psi, a_0, \ldots, a_{m-1}) = \Theta(\phi, a_0, \ldots, a_{m-1}) * \Theta(\psi, a_0, \ldots, a_{m-1}).$$

Proof. By induction on the number of conjuncts of ϕ. Let $\psi = \bigwedge_{k<s} \rho_k(\bar{x}) = \pi_k(\bar{x})$. If ϕ is of the form $\sigma(\bar{x}) = \tau(\bar{x})$, then

$$\begin{aligned}
&\Theta(\phi \Rightarrow \psi, \bar{a}) \\
&\quad = \bigvee_{k<s} \bigvee_{\text{odd } i<n} \Theta(q_i(\sigma(\bar{a}), \tau(\bar{a}), \rho_k(\bar{a}), \pi_k(\bar{a})), q_{i+1}(\sigma(\bar{a}), \tau(\bar{a}), \rho_k(\bar{a}), \pi_k(\bar{a}))) \\
&\quad = \bigvee_{k<s} \Theta(\sigma(\bar{a}), \tau(\bar{a})) * \Theta(\rho_k(\bar{a}), \pi_k(\bar{a})) \quad \text{by 1.2(i)} \\
&\quad = \Theta(\sigma(\bar{a}), \tau(\bar{a})) * \bigvee_{k<s} \Theta(\rho_k(\bar{a}), \pi_k(\bar{a})) \quad \text{by the theory of Brouwerian semilattices} \\
&\quad = \Theta(\phi, \bar{a}) * \Theta(\psi, \bar{a}).
\end{aligned}$$

Now assume $\phi = (\sigma(\bar{x}) = \tau(\bar{x})) \wedge \theta$ so that $(\phi \Rightarrow \psi) = (\sigma(\bar{x}) = \tau(\bar{x})) \Rightarrow (\theta \Rightarrow \psi)$. Then

$$\begin{aligned}
&\Theta(\phi \Rightarrow \psi, \bar{a}) \\
&\quad = \Theta(\sigma(\bar{a}), \tau(\bar{a})) * \Theta(\theta \Rightarrow \psi, \bar{a}) \\
&\quad = \Theta(\sigma(\bar{a}), \tau(\bar{a})) * (\Theta(\theta, \bar{a}) * \Theta(\psi, \bar{a})) \quad \text{by the induction hypothesis} \\
&\quad = (\Theta(\sigma(\bar{a}), \tau(\bar{a})) \vee \Theta(\theta, \bar{a})) * \Theta(\psi, \bar{a}) \quad \text{by the theory of Brouwerian semilattices} \\
&\quad = \Theta((\sigma(\bar{x}) = \tau(\bar{x})) \wedge \theta, \bar{a}) * \Theta(\psi, \bar{a}) \\
&\quad = \Theta(\phi, \bar{a}) * \Theta(\psi, \bar{a}).
\end{aligned}$$

This completes the proof.

In connection with parts (i) and (iii) of the next theorem compare Fried and Kiss [10, Theorem 5.1].

THEOREM 1.5. *Let* $\mathfrak{A}$ *be an algebra and* $q = \langle q_i(x, y, z, w) : i \le n \rangle$ *a bounded system of quaternary terms of* $\mathfrak{A}$. *The following conditions are equivalent.*

(i) q *is a* QD *system for* $H_\omega \mathfrak{A}$.

(ii) $\mathfrak{A}$ *satisfies the following congruence conditions for all* $a, b, c, d \in A$:

$$q_i(a, b, c, d) \equiv q_{i+1}(a, b, c, d)(\Theta(a, b)) \quad \text{for even} \quad i < n; \tag{7}$$

$\Theta(a, b) * \Theta(c, d)$ *exists and*

$$q_i(a, b, c, d) \equiv q_{i+1}(a, b, c, d)(\Theta(a, b) * \Theta(c, d)) \quad \text{for odd} \quad i < n. \tag{8}$$

(iii) $\mathfrak{A}$ *satisfies the following system of identities:*

$$q_i(x, x, z, w) = q_{i+1}(x, x, z, w) \quad \text{for even} \quad i < n; \tag{9}$$

$$(x = y) \Rightarrow (x = y); \tag{10}$$

$$(x = y) \Rightarrow (z = z); \tag{11}$$

$$((x = y) \Rightarrow (z = w)) \Rightarrow ((x = y) \Rightarrow (w = z)); \tag{12}$$

$$((x = y) \Rightarrow ((z = w) \wedge (w = v))) \Rightarrow ((x = y) \Rightarrow (z = v)); \tag{13}$$

for every term $\sigma(z_0, \ldots, z_{m-1})$ *in the language of* $\mathfrak{A}$,

$$\left((x = y) \Rightarrow \bigwedge_{j<m} z_j = w_j\right) \Rightarrow ((x = y) \Rightarrow (\sigma(z_0, \ldots, z_{m-1}) = \sigma(w_0, \ldots, w_{m-1})). \tag{14}$$

Proof. That (i) implies (ii) follows trivially from Lemma 1.2. Suppose (ii) holds. (Implicit in (ii) is the requirement that $\langle Cp\mathfrak{A}, \vee, I \rangle$ is dually relatively pseudo complemented.) Consider any $\mathfrak{B} \in H_\omega \mathfrak{A}$. We identify $\mathfrak{B}$ with the quotient algebra $\mathfrak{A}/\Phi$ for some compact congruence Φ of $\mathfrak{A}$. Let $a, b, c, d \in A$ such that $c/\Phi \equiv d/\Phi(\Theta_{\mathfrak{B}}(a/\Phi, b/\Phi))$, i.e. $\Theta_{\mathfrak{B}}(c/\Phi, d/\Phi) \subseteq \Theta_{\mathfrak{B}}(a/\Phi, b/\Phi)$. Then $\Theta_{\mathfrak{A}}(c, d) \subseteq \Theta_{\mathfrak{A}}(a, b) \vee \Phi$ by 0.2, and hence $\Theta_{\mathfrak{A}}(a, b) * \Theta_{\mathfrak{A}}(c, d) \subseteq \Phi$. So by (8)

$$q_i(a, b, c, d) \equiv q_{i+1}(a, b, c, d)(\Phi) \quad \text{for odd} \quad i < n.$$

Hence in $\mathfrak{B}$,

$$q_i(a/\Phi, b/\Phi, c/\Phi, d/\Phi) = q_{i+1}(a/\Phi, b/\Phi, c/\Phi, d/\Phi) \quad \text{for odd} \quad i < n.$$

This gives 1.1(ii) for $\mathfrak{B}$, and a similar argument shows that 1.1(i) for $\mathfrak{B}$ follows from (7). So (i) and (ii) are equivalent.

Assume (i) and (ii) hold. (9)–(11) follow at once from 1.1(i) and (ii). Let $\phi(x, y, z_0, \ldots, z_{m-1}, w_0, \ldots, w_{m-1})$ be the conjunction of identities (14), and let $a, b, c_0, \ldots, c_{m-1}, d_0, \ldots, d_{m-1} \in A$. Then

$$\begin{aligned}
&\Theta(\phi, a, b, \bar{c}, \bar{d}) \\
&\quad = \Big(\Theta(a, b) * \bigvee_{j<m} \Theta(c_j, d_j)\Big) * (\Theta(a, b) * \Theta(\sigma(\bar{c}), \sigma(\bar{d}))) \quad \text{by Lemma 1.4} \\
&\quad = \Theta(a, b) * \Big(\Big(\bigvee_{j<m} \Theta(c_j, d_j)\Big) * \Theta(\sigma(\bar{c}), \sigma(\bar{d}))\Big) \\
&\qquad\qquad\qquad\qquad \text{by the theory of Brouwerian semilattices} \\
&\quad = \Theta(a, b) * I_{\mathfrak{A}} \quad \text{since} \quad \sigma(\bar{c}) \equiv \sigma(\bar{d})\Big(\bigvee_{j<m} \Theta(c_j, d_j)\Big) \\
&\quad = I_{\mathfrak{A}}.
\end{aligned}$$

Thus recalling the definition of $\Theta(\phi, a, b, \bar{c}, \bar{d})$ we immediately get $\mathfrak{A} \models \phi(a, b, \bar{c}, \bar{d})$. Hence the conjunction of identities (14) holds in $\mathfrak{A}$. The remaining identities (12) and (13) of (iii) are established in a similar way. Thus (i) implies (iii).

Now assume (iii) holds. Let $\mathfrak{B} \in H_\omega \mathfrak{A}$. Then $\mathfrak{B}$ also satisfies the identities of (iii). Let $a, b \in B$, and let

$$\Phi = \{\langle c, d\rangle \in B^2 : \mathfrak{B} \models q_i(a, b, c, d) = q_{i+1}(a, b, c, d) \text{ for odd } i < n\}.$$

Then in terms of "$\Rightarrow$" we have

$$c \equiv d(\Phi) \quad \text{iff} \quad \mathfrak{B} \models ((x = y) \Rightarrow (z = w))(a, b, c, d).$$

Assume $c_j \equiv d_j(\Phi)$ for $j < m$. Then

$$\mathfrak{B} \models \Big((x = y) \Rightarrow \bigwedge_{j<m} z_j = w_j\Big)(a, b, \bar{c}, \bar{d}).$$

Then using the premise that (14) holds identically in $\mathfrak{A}$, and applying Lemma 1.3, we conclude that

$$\mathfrak{B} \models ((x = y) \Rightarrow (\sigma(\bar{z}) = \sigma(\bar{w})))(a, b, \bar{c}, \bar{d}).$$

Hence $\sigma(\bar{c}) \equiv \sigma(\bar{d})(\Phi)$. So Φ has the substitution property with respect to all the term functions of $\mathfrak{B}$. In a similar way we use the identities (10)–(13) to show that Φ is an equivalence relation, and hence a congruence relation, and that $a \equiv b(\Phi)$. Thus $\Theta(a, b) \subseteq \Phi$. It follows at once that 1.1(ii) holds for $\mathfrak{B}$; 1.1(i) coincides with (9). Hence (ii) implies (i), and the proof of the theorem is complete.

Observe that in condition (iii) the identities (14) may be replaced by the smaller system

$$\left((x = y) \Rightarrow \bigwedge_{j<m} z_j = w_j\right) \Rightarrow ((x = y) \Rightarrow (Fz_0 \cdots z_{m-1} = Fw_0 \cdots w_{m-1}))$$

for every fundamental operation F of $\mathfrak{A}$.

COROLLARY 1.6. *Let $\mathcal{K}$ be any class of similar algebras and q a bounded system of quaternary terms of $\mathcal{K}$. Then q is a* QD *system for* HSP$\mathcal{K}$ *iff it is one for* $\mathrm{H}_\omega\mathcal{K}$

THEOREM 1.7. *Let $\mathcal{K}$ be any class of similar algebras. The following are equivalent.*

(i) HSP$\mathcal{K}$ *has* EDPC;

(ii) $H_\omega\mathcal{K}$ *has a* QD *system*;

(iii) $H_\omega\mathcal{K}$ *has a* QD *system* $q = \langle q_i(x, y, z, w) : i \leq n\rangle$ *satisfying the additional congruence condition*

$$\bigvee_{\text{even } i<n} \Theta(q_i(a, b, c, d), q_{i+1}(a, b, c, d)) = (\Theta(a, b) * \Theta(c, d)) * \Theta(c, d) \tag{15}$$

for all $\mathfrak{A} \in$ HSP$\mathcal{K}$ *and all* $a, b, c, d \in A$;

(iv) *there exists a bounded system q of quaternary terms such that the identities* (9)–(14) *of* 1.5(*iii*) *hold in* $\mathcal{K}$.

Proof. (ii) and (iv) are equivalent by 1.5, and (iii) trivially implies (ii). If $q = \langle q_i(x, y, z, w) : i \leq n\rangle$ is a QD system for $H_\omega\mathcal{K}$, then by 1.5 it is also a QD system for HSP$\mathcal{K}$. Let $\mathfrak{A} \in$ HSP$\mathcal{K}$ and $a, b, c, d \in A$. Then $c \equiv d(\Theta(a, b))$ iff $\Theta(a, b) * \Theta(c, d) = I_{\mathfrak{A}}$ iff $q_i(a, b, c, d) = q_{i+1}(a, b, c, d)$ for odd $i < n$, by 1.2(i). Hence HSP$\mathcal{K}$ has EDPC and (ii) implies (i). It remains only to show that (i) implies (iii).

Assume HSP$\mathcal{K}$ has EDPC and let $\mathfrak{F}$ be the free algebra of HSP$\mathcal{K}$ with four free generators x, y, z, w. By Theorem 0.3, $\Theta(x, y) * \Theta(z, w)$ and

$$\Theta(x, y) * \Theta(z, w)^2 = (\Theta(x, y) * \Theta(z, w)) * \Theta(z, w)$$

both exist, and by the theory of Brouwerian semilattices

$$z \equiv w((\Theta(x, y) * \Theta(z, w)) \vee (\Theta(x, y) * \Theta(z, w)^2)).$$

Thus there exists a bounded sequence $\langle q_i(x, y, z, w) : i \leq n \rangle$ of elements of $\mathfrak{F}$ (that we identify with terms) satisfying the congruence conditions

$$q_i(x, y, z, w) \equiv q_{i+1}(x, y, z, w)(\Theta(x, y) * \Theta(z, w)) \quad \text{for odd} \quad i < n, \tag{16}$$

$$q_i(x, y, z, w) \equiv q_{i+1}(x, y, z, w)(\Theta(x, y) * \Theta(z, w)^2) \quad \text{for even} \quad i < n. \tag{17}$$

Let $\mathfrak{A}$ be any algebra of HSP$\mathcal{K}$ and a, b, c, d any elements of $\mathfrak{A}$. Let h be the homomorphism from $\mathfrak{F}$ into $\mathfrak{A}$ such that $hx = a$, $hy = b$, $hz = c$, and $hw = d$. Using the fact that HSP$\mathcal{K}$ has the congruence extension property we get $(Cph)\Theta_{\mathfrak{F}}(x, y) = \Theta_{\mathfrak{A}}(a, b)$ and $(Cph)\Theta_{\mathfrak{F}}(z, w) = \Theta_{\mathfrak{A}}(c, d)$. Hence by means of Lemma 0.4 we can conclude from (16) and (17) respectively

$$q_i(a, b, c, d) \equiv q_{i+1}(a, b, c, d)(\Theta_{\mathfrak{A}}(a, b) * \Theta_{\mathfrak{A}}(c, d)) \quad \text{for odd} \quad i < n, \tag{18}$$

$$q_i(a, b, c, d) \equiv q_{i+1}(a, b, c, d)(\Theta_{\mathfrak{A}}(a, b) * \Theta_{\mathfrak{A}}(c, d)^2) \quad \text{for even} \quad i < n. \tag{19}$$

Taking $a = b$ in (19) and observing that

$$\Theta_{\mathfrak{A}}(a, a) * \Theta_{\mathfrak{A}}(c, d)^2 = (I_{\mathfrak{A}} * \Theta_{\mathfrak{A}}(c, d)) * \Theta_{\mathfrak{A}}(c, d) = \Theta_{\mathfrak{A}}(c, d) * \Theta_{\mathfrak{A}}(c, d) = I_{\mathfrak{A}}$$

we obtain the identities 1.1(i). On the other hand, if $c \equiv d(\Theta_{\mathfrak{A}}(a, b))$, then $\Theta_{\mathfrak{A}}(a, b) * \Theta_{\mathfrak{A}}(c, d) = I_{\mathfrak{A}}$, and hence from (18) we obtain condition 1.1(ii). So q is a QD system for HSP$\mathcal{K}$, and in particular for $H_\omega\mathcal{K}$. Finally, from (19) and 1.2(ii) we get (15). Hence (i) does imply (iii) and the theorem is proved.

Let q be a QD system for $H_\omega\mathcal{K}$ satisfying the additional condition (15). Let $\mathfrak{A} \in$ HSP$\mathcal{K}$ and $a, b, c \in A$. Then $\Theta(a, b) * \Theta(c, c) = \Theta(a, b) * I = I$, and similarly $(\Theta(a, b) * \Theta(c, c)) * \Theta(c, c) = I$. Thus the identity $q_i(x, y, z, z) = q_{i+1}(x, y, z, z)$ holds in HSP$\mathcal{K}$ for all $i < n$; compare Fried and Kiss [10, Theorem 4.1(b)].

Let $\mathcal{V}$ be a variety and $\mathcal{V}_{si}$ its class of subdirectly irreducibles. Assume $\mathcal{V}$ has EDPC and let q be a QD system for $\mathcal{V}$. Assume further that $\mathcal{V}$ is semisimple, i.e., every subdirectly irreducible is simple. Then for every $\mathfrak{A} \in \mathcal{V}_{si}$ and all $a, b, c, d \in A$, condition 1.1(ii) becomes

$$q_i(a, b, c, d) = q_{i+1}(a, b, c, d) \quad \text{for odd} \quad i < n \quad \text{whenever} \quad a \neq b. \tag{20}$$

Assume conversely that q is a bounded system of quaternary terms for $\mathcal{V}$ such that the identities 1.1(i) hold in $\mathcal{V}$, and (20) holds for every $\mathfrak{A} \in \mathcal{V}_{si}$ and all $a, b, c, d \in A$. The conditions 1.1(i) and (20) together imply that every member of $\mathcal{V}_{si}$ is simple. Hence $\mathcal{V}$ is semisimple, and also $H_\omega \mathcal{V}_{si} = \mathcal{V}_{si}$; it follows that (20) coincides with 1.1(ii) on $H_\omega \mathcal{V}_{si}$, and so q is a QD system for $H_\omega \mathcal{V}_{si}$. Applying 1.7(i), (ii) we conclude that $\mathcal{V}$ has EDPC.

Fried, Grätzer, and Quackenbush [9] have shown that a variety is filtral iff it is semisimple and has EDPC. Hence the above argument shows that a necessary and sufficient condition for $\mathcal{V}$ to be filtral is the existence of a bounded system q of quaternary terms satisfying the identities 1.1(i) together with the condition (20) on the subdirectly irreducible members of $\mathcal{V}$; compare Fried and Kiss [10, Theorem 4.1(a)].

2. Weak Brouwerian semilattices with filter preserving operations

In the last section, in Theorem 1.7, we saw how the property of having EDPC can be characterized in terms of the existence of an "operation" $\Rightarrow$ that exhibits, at least formally, many of the characteristics of the implication connective of non-classical logic. But $\Rightarrow$ is a metamathematical operation whose domain is the set of conjunctions of equations. In the next section we shall show that, when EDPC is combined with congruence-permutability and point-regularity (the latter property being characteristic of a very wide family of varieties coming from logic), $\Rightarrow$ can be replaced by two term functions that correspond quite closely to the implication and conjunction of non-classical logic. Moreover every EDPC variety with these two additional properties assumes a form quite close to that of the familiar varieties of algebraic logic.

In the present section we axiomatize the varieties that arise in this way and investigate some of their basic properties. Later in the section we show by examples how the familiar varieties arising in non-classical logic fit naturally into this scheme.

DEFINITION 2.1. *$\mathcal{V}$ is a variety of* ***weak Brouwerian semilattices with filter***

preserving operations (WBSO) *if there exist binary terms* $\to$, $\cdot$, *and* Δ *and a constant* 1 *such that the following identities and quasi-identities hold in* $\mathcal{V}$.

(i) $x \to x = 1$,
(ii) $x \to 1 = 1$,
(iii) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$,
(iv) $1 \cdot 1 = 1$,
(v) $(z \to x) \to ((z \to y) \to (z \to x \cdot y)) = 1$,
(vi) $(x \cdot y) \to x = 1, \quad (x \cdot y) \to y = 1$,
(vii) $x \Delta x = 1$,
(viii) $(x \Delta 1) \to x = 1, \quad x \to (x \Delta 1) = 1$
(ix) $(x \Delta y) \to (y \Delta x) = 1$,
(x) $((x \Delta y) \cdot (y \Delta z)) \to (x \Delta z) = 1$,
(xi) *for every fundamental operation* F *of* $\mathcal{V}$,

$$(\cdots((x_0 \Delta y_0) \cdot (x_1 \Delta y_1)) \cdot \cdots) \cdot (x_{m-1} \Delta y_{m-1}) \to (Fx_0 \cdots x_{m-1}) \Delta (Fy_0 \cdots y_{m-1}) = 1,$$

(xii) $1 \to x = 1$ implies $x = 1$,
(xiii) $x \Delta y = 1$ implies $x = y$.

Any quasi-identity that holds throughout a variety is equivalent to a system of identities. Consider for example the quasi-identity (xiii). The condition that it holds throughout $\mathcal{V}$ is equivalent to the congruence condition

$$a \equiv b(\Theta_{\mathfrak{A}}(a \Delta b, 1))$$

holding for all $\mathfrak{A} \in \mathcal{V}$ and all $a, b \in A$. Applying this in the case $\mathfrak{A}$ is the free algebra of $\mathcal{V}$ on two generators x, y we obtain by Mal'cev's lemma [12, Theorem 10.3] a system of ternary terms $t_0, \ldots, t_n$ such that $t_0 = x$, $t_n = y$, and, for all $i < n$,

$$t_i(\bar{g}_i(x, y), x, y) = t_{i+1}(g_i(x, y), x, y) \tag{1}$$

where $g_i(x, y)$ is either $x \Delta y$ or 1, and $\bar{g}_i(x, y)$ is 1 when $g_i(x, y)$ is $x \Delta y$ and $\bar{g}_i(x, y)$ is $x \Delta y$ when $g_i(x, y)$ is 1.

The system of identities (1) holds throughout $\mathcal{V}$ and, conversely, any variety in which they hold must satisfy (xiii). The quasi-identity (xii) can be replaced by identities in a similar way. Thus the property of being a variety of WBSO's can assume a form somewhat similar to the one characterizing varieties with EDPC given in 1.7(iv).

An algebra $\mathfrak{A}$ is a *weak Brouwerian semilattice with filter preserving operations* if it is a member of some WBSO variety, or equivalently, if it generates a WBSO variety. Clearly $\mathfrak{A}$ is a WBSO algebra if it satisfies the identities 2.1(i)–(xi), the identities (1) for some system of terms $t_0, \ldots, t_n$ and "switching functions" $g_0, \ldots, g_{n-1}$, and a similar set of identities associated with the quasi-identity 2.1(xii).

In the sequel $\mathcal{V}$ will denote a WBSO variety with special terms $\rightarrow$, $\cdot$, Δ, and 1, and $\mathfrak{A}$ will be an arbitrary member of $\mathcal{V}$.

Define a binary relation $\leqslant$ on A by the condition

$$a \leqslant b \quad \text{iff} \quad a \rightarrow b = 1.$$

By (i) $\leqslant$ is reflexive, and by (iii) and (xii) it is transitive, hence a quasiordering. Let $\approx$ be its associated equivalence relation, so that

$$a \approx b \quad \text{iff} \quad a \rightarrow b = 1 \quad \text{and} \quad b \rightarrow a = 1.$$

From (v), (vi), and the fact $\leqslant$ is a quasiordering it follows that $\approx$ is a congruence relation on $\langle A, \cdot \rangle$, and $\langle A/{\approx}, \cdot/{\approx} \rangle$ is a semilattice with $\cdot/{\approx}$ as the greatest lower bound operation for the partial ordering induced by $\leqslant$. By (xii) 1 is congruent only to itself, so we shall identify 1 with its equivalence class $\{1\}$ under $\approx$. By (ii) 1 is the largest element of the partial ordering induced by $\leqslant$.

A subset $F \subseteq A$ is a *weak filter* of $\mathfrak{A}$ if $1 \in F$, $a \cdot b \in F$ whenever $a, b \in F$, and $b \in F$ whenever $a \in F$ and $a \leqslant b$. Notice in particular that $a \in F$ and $a \approx b$ imply $b \in F$. We denote the set of weak filters of $\mathfrak{A}$ by $Wf\mathfrak{A}$. For any congruence relation Φ on $\mathfrak{A}$ let

$$\mathbf{F}\Phi = 1/\Phi,$$

the congruence class of 1. $a \equiv 1(\Phi)$ and $b \equiv 1(\Phi)$ imply (with the help of (iv)) $a \cdot b \equiv_{\Phi} 1 \cdot 1 = 1$, and $a \equiv 1(\Phi)$ and $a \leqslant b$ together imply $1 \rightarrow b \equiv_{\Phi} a \rightarrow b = 1$. Thus, by (xii) applied to the quotient $\mathfrak{A}/\Phi$, $b \equiv 1(\Phi)$. Hence $\mathbf{F}\Phi$ is a weak filter. Now suppose F is an arbitrary weak filter. Let

$$\mathbf{\Phi}F = \{\langle c, d \rangle : c \Delta d \in F\}.$$

Then (vii), (ix), and (x) imply Φ is an equivalence relation, and by (xi) $\mathbf{\Phi}F$ has the substitution property with respect to each fundamental operation of $\mathfrak{A}$. Thus $\mathbf{\Phi}F$ is a congruence relation on $\mathfrak{A}$. $a \equiv 1(\mathbf{\Phi}F)$ iff $a \Delta 1 \in F$ iff $a \in F$ by (viii). Thus $\mathbf{F\Phi}F = F$. For any congruence Φ of $\mathfrak{A}$, $a \equiv b(\mathbf{\Phi F}\Phi)$ iff $a \Delta b \in \mathbf{F}\Phi$ iff $a \Delta b \equiv 1(\Phi)$ iff $a \equiv b(\Phi)$ by (xiii) applied to the quotient algebra $\mathfrak{A}/\Phi$. Thus $\mathbf{\Phi F}\Phi = \Phi$. This immediately gives the following

LEMMA 2.2. *Let* $\mathfrak{A}$ *be a* WBSO. **F** *is an isomorphism between the congruence and weak filter lattices of* $\mathfrak{A}$; *its inverse is* $\boldsymbol{\Phi}$.

Subsets of A of the form $\mathbf{F}\Phi$ for some $\Phi \in Co\mathfrak{A}$ are just the 1-ideals of $\mathfrak{A}$. From 2.2 it follows that the 1-ideals are exactly the weak filters. Another consequence of 2.2 is that $\mathfrak{A}$ is point-regular (see the Preliminaries).

For any $a, b \in A$ we have $a \equiv b(\Phi)$ iff $a \equiv b(\boldsymbol{\Phi}\mathbf{F}\Phi)$ iff $a \Delta b \in \mathbf{F}\Phi$ iff $a \Delta b \equiv 1(\Phi)$. This gives

LEMMA 2.3. *Let* $\mathfrak{A}$ *be a* WBSO. *For all* $a, b \in A$, $\Theta(a, b) = \Theta(a \Delta b, 1)$.

Thus Δ is a Gödel equivalence term in any WBSO variety $\mathcal{V}$ (see the Preliminaries).

For every $a \in A$ define

$$[a) = \{b \in A : a \leqslant b\}.$$

$a \leqslant b \leqslant c$ implies $a \leqslant c$, and $a \leqslant b$ and $a \leqslant c$ imply $a \leqslant b \cdot c$. Thus [a) is a weak filter. Clearly [a) is the weak filter generated by a, and $[a) \vee [b) = [a \cdot b)$ where "$\vee$" denotes join in the lattice of weak filters.

LEMMA 2.4. *Let* $\mathfrak{A}$ *be a* WBSO. *For any* $a_0, b_0, \ldots, a_{n-1}, b_{n-1}, c, d \in A$, $c \equiv d(\Theta(a_0, b_0) \vee \cdots \vee \Theta(a_{n-1}, b_{n-1}))$ *iff* $(\cdots (a_0 \Delta b_0) \cdot \cdots) \cdot (a_{n-1} \Delta b_{n-1}) \leqslant c \Delta d$.

Proof. Recall for any congruence Φ of $\mathfrak{A}$ we have, by 2.3 and the definition of **F**,

$$a_i \equiv b_i(\Phi) \quad \text{iff} \quad a_i \Delta b_i \in \mathbf{F}\Phi.$$

Thus from the fact **F** is a lattice isomorphism we get

$$\begin{aligned} \mathbf{F}\Theta(a_i, b_i) &= \mathbf{F} \cap \{\Phi \in Co\mathfrak{A} : a_i \equiv b_i(\Phi)\} \\ &= \bigcap \{\mathbf{F}\Phi : a_i \equiv b_i(\Phi)\} \\ &= \bigcap \{F \in Wf\mathfrak{A} : a_i \Delta b_i \in F\} \\ &= [a_i \Delta b_i). \end{aligned}$$

Therefore,

$$\begin{aligned} &\mathbf{F}(\Theta(a_0, b_0) \vee \cdots \vee \Theta(a_{n-1}, b_{n-1})) \\ &\quad = \mathbf{F}\Theta(a_0, b_0) \vee \cdots \vee \mathbf{F}\Theta(a_{n-1}, b_{n-1}) \\ &\quad = [a_0 \Delta b_0) \vee \cdots \vee [a_{n-1} \Delta b_{n-1}) \\ &\quad = [(a_0 \Delta b_0) \cdot \cdots \cdot (a_{n-1} \Delta b_{n-1})). \end{aligned}$$

This proves the lemma.

As a particular case of the lemma we have, for all $a, b, c, d \in A$,

$$c \equiv d(\Theta(a, b)) \quad \text{iff} \quad a \Delta b \to c \Delta d = 1. \tag{2}$$

Thus $\mathcal{V}$ has EDPC, and hence by 0.3, the join-semilattice of compact congruences of $\mathfrak{A}$ is dually relatively pseudo complemented.

Observe that the principal congruences of a WBSO variety $\mathcal{V}$ are defined by a single equation. One cannot however conclude from this that $\mathcal{V}$ has a uniform congruence scheme of length 1, and is thus congruence-permutable. (See Fried, Grätzer, and Quackenbush [9, p. 178].) Indeed one of the examples of WBSO varieties considered below, Nelson algebras, fails to be congruence-permutable. Although we have not investigated the matter it seems very likely that no upper bound can be placed on the minimal length of uniform congruence schemes of WBSO varieties.

LEMMA 2.5. *Let* $\mathfrak{A}$ *be a* WBSO. *Then for all* $a, b \in A$

(i) $a \leqslant b$ *iff* $\Theta(b, 1) \subseteq \Theta(a, 1)$,
(ii) $a \approx b$ *iff* $\Theta(a, 1) = \Theta(b, 1)$,
(iii) $\Theta(\mathrm{a} \to \mathrm{b}, 1) = \Theta(a, 1) * \Theta(b, 1)$,
(iv) $\Theta(a \cdot b, 1) = \Theta(a, 1) \vee \Theta(b, 1)$.

Proof. From 2.1(viii) we have $a \Delta 1 \approx a$ and $b \Delta 1 \approx b$. Thus, by (2), $b \equiv 1(\Theta(a, 1))$ iff $a \Delta 1 \leqslant b \Delta 1$ iff $a \leqslant b$. Hence $\Theta(b, 1) \subseteq \Theta(a, 1)$ iff $a \leqslant b$. This gives (i), and (ii) follows. This also gives $b \equiv 1(\Theta(a, 1))$ iff $a \to b = 1$. Using this equivalence we get for every $\Phi \in Co\mathfrak{A}$

$$\begin{aligned} &\Theta(b, 1) \subseteq \Theta(a, 1) \vee \Phi \\ &\quad \text{iff} \quad \Theta(b/\Phi, 1/\Phi) \subseteq \Theta(a/\Phi, 1/\Phi) \quad \text{by 0.2} \\ &\quad \text{iff} \quad a/\Phi \to b/\Phi = 1/\Phi \\ &\quad \text{iff} \quad a \to b \equiv 1(\Phi) \\ &\quad \text{iff} \quad \Theta(a \to b, 1) \subseteq \Phi, \end{aligned}$$

and hence the equality (iii) holds.

From 2.1(viii) and the fact $\approx$ is a congruence relation on $\langle A, \cdot \rangle$ we get $(a \Delta 1) \cdot (b \Delta 1) \approx (a \cdot b) \Delta 1$. Thus $\mathbf{F}(\Theta(a, 1) \vee \Theta(b, 1)) = \mathbf{F}\Theta(a, 1) \vee \mathbf{F}\Theta(b, 1) = [a \Delta 1) \vee [b \Delta 1) = [(a \Delta 1) \cdot (b \Delta 1)) = [(a \cdot b) \Delta 1) = \mathbf{F}\Theta(a \cdot b, 1)$. (iv) follows since F is one-one.

Therefore the mapping $a \mapsto \Theta(a, 1)$ from A to $Co\mathfrak{A}$ is a homomorphism from $\langle A, \cdot, \rightarrow, 1\rangle$ onto $\mathfrak{Cp}\mathfrak{A}$ with $\approx$ as its relation-kernel. Thus $\langle A, \cdot, \rightarrow, 1\rangle/\approx$ is a Brouwerian semilattice in the ordinary sense since it is isomorphic to $\mathfrak{Cp}\mathfrak{A}$. Observe that $F \subseteq A$ is a weak filter of $\mathfrak{A}$ iff F is of the form $\bigcup G$ for some filter (in the usual sense) G of the Brouwerian semilattice $\langle A, \cdot, \rightarrow, 1\rangle/\approx$.

THEOREM 2.6. *$\mathcal{V}$ is a* WBSO *variety iff it has terms $\rightarrow, \cdot, \Delta$, and* 1 *such that every member $\mathfrak{A}$ of $\mathcal{V}$ satisfies the following conditions.*

(i) *$\mathfrak{A}$ is* 1*-regular and Δ is a Gödel equivalence term function with respect to* 1.

(ii) *The relation $\approx$ defined by the condition $a \approx b$ iff $a \rightarrow b = 1$ and $b \rightarrow a = 1$ is a congruence relation on $\langle A, \cdot, \rightarrow, 1\rangle$, and the quotient $\langle A, \cdot, \rightarrow, 1\rangle/\approx$ is a Brouwerian semilattice.*

(iii) *The* 1*-ideals of $\mathfrak{A}$ are exactly the subsets of A of the form $\bigcup G$ where G is a filter of $\langle A, \cdot, \rightarrow, 1\rangle/\approx$.*

Proof. We have already shown that (i)–(iii) hold in any WBSO variety. So we only need to prove that these conditions imply $\mathcal{V}$ is a WBSO variety. It is well known from the theory of Brouwerian semilattices that the mapping $a/\approx \mapsto [a/\approx)$ is a (dual) isomorphism between $\langle A, \cdot, 1\rangle/\approx$ and its join-semilattice of compact filters. By conditions (i) and (iii) the mapping $[a/\approx) \mapsto \Theta(a, 1)$ is an isomorphism from this latter semilattice onto $\langle Cp\mathfrak{A}, \vee, I\rangle$, the join-semilattice of of compact congruences of $\mathfrak{A}$. Consequently, since $\langle \mathfrak{A}, \cdot, 1\rangle/\approx$ is relatively pseudo complemented by hypothesis, so is $\langle Cp\mathfrak{A}, \vee, I\rangle$, and $a/\approx \mapsto \Theta(a, 1)$ is actually an isomorphism between the Brouwerian semilattices $\langle \mathfrak{A}, \cdot, \rightarrow, 1\rangle/\approx$ and $\mathfrak{Cp}\mathfrak{A} = \langle Cp\mathfrak{A}, \vee, *, I\rangle$. It follows almost immediately that, for all $a, b \in A$,

$$\Theta(a \rightarrow b, 1) = \Theta(a, 1) * \Theta(b, 1), \qquad \Theta(a \cdot b, 1) = \Theta(a, 1) \vee \Theta(b, 1).$$

Also, directly from (i),

$$\Theta(a \Delta b, 1) = \Theta(a, b).$$

The identities 2.1(i)–(xi) are now easily established. Consider for example 2.1(iii).

$$\begin{aligned} &\Theta((a \rightarrow b) \rightarrow ((b \rightarrow c) \rightarrow (a \rightarrow c)), 1) \\ &\quad = (\Theta(a, 1) * \Theta(b, 1)) * ((\Theta(b, 1) * \Theta(c, 1)) * (\Theta(a, 1) * \Theta(c, 1))) \\ &\quad = I \end{aligned}$$

by the theory of Brouwerian semilattices. Thus $(a \rightarrow b) \rightarrow ((b \rightarrow c) \rightarrow (a \rightarrow c)) =$

1. Consider also 2.1(xi) where, to simplify computation, we assume $m = 2$.

$$\begin{aligned}&\Theta((a_0 \Delta b_0)\cdot(a_1 \Delta b_1) \to F(a_0, a_1)\Delta F(b_0, b_1), 1)\\&\quad = (\Theta(a_0, b_0) \vee \Theta(a_1, b_1)) * \Theta(F(a_0, a_1), F(b_0, b_1))\\&\quad = I\end{aligned}$$

since $\Theta(F(a_0, a_1), F(b_0, b_1)) \subseteq \Theta(a_0, b_0) \vee \Theta(a_1, b_1)$.

The two quasi-identities 2.1(xii), (xiii) are established in the same way. For instance, if $1 \to a = 1$, then $I = \Theta(1 \to a, 1) = \Theta(1, 1) * \Theta(a, 1) = I * \Theta(a, 1) = \Theta(a, 1)$. Hence we must have $a = 1$. This completes the proof of the theorem.

Let $\mathfrak{A}$ be an arbitrary algebra (not necessarily a WBSO) and $1 \in A$. A term function $\cdot$ of $\mathfrak{A}$ is called a *weak meet* (*with respect to* 1) if, for all $a, b \in A$,

$$\Theta(a \cdot b, 1) = \Theta(a, 1) \vee \Theta(b, 1). \tag{3}$$

A binary term function $\to$ is called a *weak relative pseudo complementation* (*with respect to* 1) if $\langle Cp\mathfrak{A}, \vee, I\rangle$ is dually relatively pseudo complemented and

$$\Theta(a \to b, 1) = \Theta(a, 1) * \Theta(b, 1)$$

for all $a, b \in A$. Finally, a binary term function Δ is called a *Gödel equivalence term* (*with respect to* 1), if, for all $a, b \in A$,

$$\Theta(a, b) = \Theta(a \Delta b, 1).$$

All three of these definitions are extended to arbitrary classes of similar algebras in the obvious way; in the case of Gödel equivalence terms this agrees with the definition given in the Preliminaries for varieties.

Lemmas 2.3, 2.5(iii), (iv) and the proof of Theorem 2.6 immediately give the following useful result.

LEMMA 2.7. *$\mathcal{V}$ is a* WBSO *variety iff it has terms $\to$, $\cdot$, Δ, and* 1 *such that $\to$ is a weak relative pseudo complementation, $\cdot$ is a weak meet, and Δ is a Gödel equivalence term, all with respect to* 1.

For any WBSO $\mathfrak{A}$ the term $(x \to y)\cdot(y \to x)$ is a Gödel equivalence term for the quotient algebra $\langle A, \cdot, \to, 1\rangle/\approx$; this is simply in virtue of the fact $\langle A, \cdot, \to, 1\rangle/\approx$ is a Brouwerian semilattice. But the term is not in general a Gödel equivalence term for $\mathfrak{A}$ itself. Indeed $(a \to b) \cdot (b \to a) = 1$ iff $a \approx b$. Thus $(x \to y) \cdot (y \to x)$ is a Gödel equivalence term for $\mathfrak{A}$ iff $\approx$ coincides with the identity relation on A, or, equivalently, iff

$$\Theta(a, 1) = \Theta(b, 1) \quad \text{implies} \quad a = b$$

for all $a, b \in A$. If a variety $\mathcal{V}$ is 1-regular for some constant 1, and if the above implication holds for all $\mathfrak{A} \in \mathcal{V}$ and $a, b \in A$, then $\mathcal{V}$ is said to be *Fregean* (*with respect to* 1); see [2] for the justification of this terminology.

As suggested by the name, perhaps the most typical example of a WBSO variety is Brouwerian semilattices with the original operations of relative pseudo complementation, meet, and unit. Moreover $x \Delta y = (x \to y) \cdot (y \to x)$ is the equivalence term, so Brouwerian semilattices form a Fregean variety. For details see [17].

Boolean algebras also form a WBSO variety, but so does any discriminator variety with at least one constant term function. More precisely, take $\mathfrak{A}$ to be any member of a discriminator variety, and let 1 be an arbitrary element of $\mathfrak{A}$. Then $\mathfrak{A}$, or more exactly the algebra obtained from it by adjoining 1 if necessary as a new constant operation, is a WBSO whose weak relative pseudo complementation, weak meet, and Gödel equivalence terms are given by

$$x \to y = n(x, 1, y, 1), \qquad x \cdot y = n(x, 1, y, x), \tag{4}$$

$$x \Delta y = n(x, y, 1, n(x, 1, y, x)) \tag{5}$$

where $n(x, y, z, w)$ is the normal transform on A. See Bulman-Fleming and Werner [4, Lemma 1.3] for details; see also the remarks following 3.5, 3.8, and 4.1 below.

It is clear that the simple and hence the subdirectly irreducible members of a Fregean discriminator variety can contain only two elements. Thus (up to equational definitional equivalence) the only discriminator varieties that are Fregean are Boolean algebras and generalized Boolean algebras (i.e., distributive lattices $\langle D, +, \cdot, \to, 1\rangle$ with largest element 1 and relative-complement operation $\to$).

The paradigm for a non-Fregean variety of WBSO's is the variety of *interior algebras* (or *topological Boolean algebras* in the terminology of Rasiowa and Sikorski [27]) $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, {}^0\rangle$: these are the modal algebras associated with Lewis' system (S4), and are characterized by the condition that $\langle A, +, \cdot, -, 0, 1\rangle$ is a Boolean algebra and ^{0}, the *interior operator*, satisfies the identities $1^0 = 1$, $(x \cdot y)^0 = x^0 \cdot y^0$, $x^0 \le x$, and $x^0 \le x^{00}$. An element a of $\mathfrak{A}$ is *open* if $a^0 = a$; A^0 is the set of all open elements. McKinsey and Tarski [21] showed that every interior algebra is isomorphic to a subalgebra of the interior algebra of all subsets of a topological space X where, for each $Y \subseteq X$, Y^0 is the interior of Y in the topological sense; see also Rasiowa [26, p. 120].

A filter F of the Boolean reduct of an interior algebra $\mathfrak{A}$ is *open* if $a \in F$ implies $a^0 \in F$. The 1-ideals of $\mathfrak{A}$ are well behaved and coincide with the open filters; moreover

$$x \Delta y = (-x + y) \cdot (-y + x)$$

is a Gödel equivalence term with respect to 1 (Rasiowa [26, pp. 166 ff.]). The set A^0 of open elements of $\mathfrak{A}$ contains 0 and 1 and is closed under $+$ and $\cdot$; hence it forms a sublattice of $\mathfrak{A}$. This lattice is relatively pseudo complemented and thus forms a Heyting algebra; specifically, for any $a, b \in A^0$ the pseudo complement of b relative to a is $(-b+a)^0$ (Rasiowa [26, p. 123]). Let $\rightarrow$ be the operation defined for all $a, b \in A$ by

$$a \rightarrow b = (-a^0 + b^0)^0. \tag{6}$$

Then it is easy to check that $a \approx b$ (i.e., $a \rightarrow b = 1$ and $b \rightarrow a = 1$) iff $a^0 = b^0$. Thus $\approx$ is a congruence relation on $\langle A, \cdot, \rightarrow, 1\rangle$, and $\langle A, \cdot, \rightarrow, 1\rangle/\approx$ is isomorphic to $\langle A^0, \cdot, \rightarrow, 1\rangle$, a Brouwerian semilattice. It is clear that the weak filters of $\langle A, \cdot, \rightarrow, 1\rangle$ coincide with the open filters of $\langle A, +, \cdot, 0, 1\rangle$, and hence with the 1-ideals. Consequently, by Theorem 2.6 interior algebras form a WBSO variety.

It turns out there are several equivalent alternative choices $\ominus\!\!\!\rightarrow$ and $\odot$ for the weak relative pseudo complementation and weak meet operations of interior algebras in the sense that $\ominus\!\!\!\rightarrow$ defines the same quasiordering relation $\preccurlyeq$ as $\rightarrow$, and $\ominus\!\!\!\rightarrow$ and $\odot$ coincide with $\rightarrow$ and $\cdot$, respectively, modulo $\approx$. We can take any one of $x^0 \cdot y$, $(x \cdot y)^0$, or $x^0 \cdot y + -x^0 \cdot x$ in place of $x \cdot y$. Also either $(-x^0 + y)^0$ or $-x^0 + y$ can be taken in place of $(-x^0 + y^0)^0$ in (6). But it is not difficult to see that neither the ordinary relative complementation $-x + y$ nor its interior $(-x+y)^0$, which corresponds to the strict implication of modal logic, can be used for this purpose. The weak meet and weak relative pseudo complementation $x^0 \cdot y + -x^0 \cdot x$ and $-x^0 + y$ is a naturally occurring pair; see the remarks following Theorem 3.6 below, especially (6).

The final example we consider are *Nelson algebras*, or *quasi-pseudo Boolean algebras* in the terminology of Rasiowa [26]. They arise from the algebraization of constructive logic with strong negation; see Rasiowa [26, Chapter XII]. A *Nelson algebra* is an algebra $\mathfrak{A} = \langle A, +, \cdot, -, \rightarrow, 0, 1\rangle$ satisfying the following conditions.

(I) $\langle A, +, \cdot, -, 0, 1\rangle$ is a DeMorgan algebra with smallest element 0 and largest element 1. (II) The relation $\leqslant$ on A defined by $a \leqslant b$ iff $a \to b = 1$ is a quasi-ordering of A. (III) The relation $\approx$ on A defined by $a \approx b$ iff $a \leqslant b$ and $b \leqslant a$ is a congruence relation on $\langle A, +, \cdot, \to, 0, 1\rangle$ and the quotient $\langle A, +, \cdot, \to, 0, 1\rangle/\approx$ is a Heyting algebra. (IV) The following hold for all $a, b \in A$: $-(a \to b) \approx a \cdot -b$, $a \cdot -a \leqslant 0$, $(a \to b) \cdot (-b \to -a) = 1$ iff $a \cdot b = a$.

The definition of Nelson algebras we give here has been adapted from the one presented in Rasiowa [26]. The class of Nelson algebras turns out to be a variety (Brignole and Monteiro [3]; see also Rasiowa [26, pp. 75 ff.]). It turns out further that in any Nelson algebra $\mathfrak{A}$ the 1-ideals are well behaved and

$$x \Delta y = (x \to y) \cdot (-y \to -x) \cdot (y \to x) \cdot (-x \to -y)$$

serves as a Gödel equivalence function with respect to 1. Furthermore the 1-ideals are exactly the subsets of A of the form $\bigcup G$ where G is a filter of the Heyting algebra $\langle A, +, \cdot, \to, 0, 1\rangle/\approx$ (Monteiro [22]; see also Rasiowa [26, pp. 91–92]). Thus the variety of Nelson algebras is a WBSO variety by Theorem 2.6 with weak relative pseudo complementation $x \to y$ and weak meet $x \cdot y$; it is non-Fregean.

As WBSO's interior algebras and Nelson algebras both have a rather special character that is not shared by WBSO varieties in general, in particular by arbitrary discriminator varieties. In any interior algebra $\mathfrak{A}$ the set A^0 of open elements contains 1 and is closed under the weak relative pseudo complementation and weak meet operations $\to$ and $\cdot$. The algebra $\langle A^0, \cdot, \to, 1\rangle$ turns out to be (dually) isomorphic to $\mathfrak{Cp}\mathfrak{A}$ under the mapping that sends each open element a to $\Theta(a, 1)$. So an interior algebra actually includes, if we identify a with $\Theta(a, 1)$, $\mathfrak{Cp}\mathfrak{A}$ as a subalgebra of its term-function reduct $\langle A, \cdot, \to, 1\rangle$, and the mapping $a \mapsto \Theta(a, 1)$ is a retraction from $\langle A, \cdot, \to, 1\rangle$ onto $\mathfrak{Cp}\mathfrak{A}$; WBSO varieties with this property will be studied in detail in a sequel to the present paper.

An analogous phenomenon can be observed in a Nelson algebra $\mathfrak{A}$. The analog of the interior operator is the mapping

$$a \mapsto -(a \to 0).$$

Take $B = \{-(a \to 0) : a \in A\}$ and let $x \ominus\!\!\!\!\!\to y = -((x \to y) \to 0)$ and $x \odot y = -(x \cdot y \to 0)$. Then B is closed under these operations, contains 1, and $\langle B, \odot, \ominus\!\!\!\!\!\to, 1\rangle$ is isomorphic to $\mathfrak{Cp}\mathfrak{A}$ under the mapping $a \mapsto \Theta(a, 1)$; see Rasiowa [26, Chapter V] for details.

3. Congruence-permutability and quaternary deduction terms

In this section we examine the effect congruence-permutability has on the structure of varieties with EDPC. It allows us to replace the quaternary deductive system by a single quaternary term. If the variety is also point-regular, this quaternary term can be replaced in turn by a pair of binary terms with all the properties of weak relative pseudo complementation and weak meet. As a consequence, congruence permutable and point-regular varieties with EDPC coincide with congruence-permutable WBSO varieties (Theorem 3.6).

Let $\mathfrak{A}$ be an algebra and $q(x, y, z, w)$ a quaternary term function of $\mathfrak{A}$. It follows from Definition 1.1 that $q(x, y, z, w)$ is the middle term of a quaternary deduction system of length 3 for $\mathfrak{A}$ iff, for all $a, b, c, d \in A$,

$$q(a, b, c, d) = \begin{cases} d & \text{if} \quad c \equiv d(\Theta(a, b)) \\ c & \text{if} \quad a = b \\ \text{arbitrary otherwise.} \end{cases}$$

Any term satisfying this condition is called a *quaternary deduction* (QD) *term for* $\mathfrak{A}$. Observe that, if $\mathfrak{A}$ is simple, there is only one QD term function and it coincides with the normal transform.

The following are immediate corollaries of Lemma 1.2 and Theorem 1.5, respectively.

LEMMA 3.1. *Let* $\mathfrak{A}$ *be an algebra and* $q(x, y, z, w)$ *a* QD *term for* $H_\omega\mathfrak{A}$. *Then* $\langle Cp\mathfrak{A}, \vee, I\rangle$ *is dually relatively pseudo complemented and for all* $a, b, c, d \in A$,

(i) $\Theta(q(a, b, c, d), d) = \Theta(a, b) * \Theta(c, d)$,

(ii) $(\Theta(a, b) * \Theta(c, d)) * \Theta(c, d) \subseteq \Theta(q(a, b, c, d), c) \subseteq \Theta(a, b)$.

THEOREM 3.2. *Let* $q(x, y, z, w)$ *be a quaternary term function for an algebra* $\mathfrak{A}$. *Then the following are equivalent.*

(i) q *is a* QD *term for* $H_\omega\mathfrak{A}$;

(ii) $\mathfrak{A}$ *satisfies the following congruence conditions for all* $a, b, c, d \in A$: $\Theta(a, b) * \Theta(c, d)$ *exists and*

$$c \equiv_{\Theta(a,b)} q(a, b, c, d) \equiv_{\Theta(a,b)*\Theta(c,d)} d;$$

(iii) $\mathfrak{A}$ *satisfies the identity* $q(x, x, z, w) = z$ *together with the identities* (10)–(14) *of* 1.5(*iii*).

The next theorem is essentially a specialization of Theorem 1.7; it gives several equivalent conditions for a class $\mathcal{K}$ of algebras to generate a congruence-permutable variety with EDPC. Observe that, in view of 3.2(i), (ii), condition (ii) below can be reformulated in terms of a congruence condition on members of $\mathcal{K}$ only; in practice this proves to be the most useful of all the conditions.

THEOREM 3.3. *Let $\mathcal{K}$ be any class of similar algebras. The following are equivalent.*

(i) HSP$\mathcal{K}$ *has* EDPC *and is congruence-permutable*;

(ii) $H_\omega\mathcal{K}$ *has a* QD term;

(iii) $H_\omega\mathcal{K}$ *has a* QD *term* $q(x, y, z, w)$ *satisfying the additional congruence condition*

$$\Theta(q(a, b, c, d), c) = (\Theta(a, b) * \Theta(c, d)) * \Theta(c, d)$$

for all $\mathfrak{A} \in$ HSP$\mathcal{K}$ *and all* $a, b, c, d \in A$;

(iv) $\mathcal{K}$ *has a quaternary term satisfying the identity* $q(x, x, z, w) = z$ *together with the identities* (10)–(14) *of* 1.5(*iii*).

Proof. The equivalence of (ii)–(iv) is an immediate consequence of 1.7(ii)–(iv). An easy modification of the proof of the implication 1.7(i) to 1.7(ii) gives the implication from (i) to (ii). Suppose $q(x, y, z, w)$ is a QD polynomial for $H_\omega\mathcal{K}$, and let $m(x, y, z) = q(x, y, z, x)$. It follows easily from the two congruence conditions of 3.2(ii) that $m(x, y, z)$ is a Mal'cev polynomial for $\mathcal{K}$ and hence for HSP$\mathcal{K}$. Thus HSP$\mathcal{K}$ is congruence-permutable. It has EDPC by 1.7(i), (ii). Hence (ii) implies (i).

Observe that the single equation $q(x, y, z, w) = w$ serves to define principal congruences. Thus the existence of a uniform restricted congruence scheme of length 1 guarantees principal congruences are definable by a single equation; as we noted previously the converse fails. (See Fried, Grätzer, and Quackenbush [9].)

COROLLARY 3.4. *$\mathcal{V}$ is a discriminator variety iff $\mathcal{V}$ is semi-simple, congruence-permutable, and has* EDPC.

Proof. It is well known that every discriminator variety is semi-simple, congruence-permutable, and has EDPC. (See for instance Werner [28].) Suppose $\mathcal{V}$ has these three properties, and $q(x, y, z, w)$ is a QD term for $\mathcal{V}$. Then q coincides with the normal transform on each subdirectly irreducible of $\mathcal{V}$. So $\mathcal{V}$ must be a discriminator variety.

This result was obtained independently by Fried and Kiss [10].

Brouwerian semilattices $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1\rangle$ constitute a congruence-permutable variety with EDPC. Define

$$p(x, y, z) = (x \rightarrow y) \cdot (y \rightarrow x) \cdot z,$$
$$m(x, y, z) = ((x \rightarrow y) \rightarrow z) \cdot ((z \rightarrow y) \rightarrow x).$$

It is well known from the theory of Brouwerian semilattices that m is a Mal'cev polynomial, i.e., $m(x, x, z) = z$ and $m(x, z, z) = x$ are identities of $\mathfrak{A}$, and that, for all $a, b, c, d \in A$, $c \equiv d(\Theta(a, b))$ iff $p(a, b, c) = p(a, b, d)$ (see, for instance, [17]). Using these facts it is easy to show that

$$q(x, y, z, w) = m(p(x, y, z), p(x, y, w), w) \tag{2}$$

is a QD term for $\mathfrak{A}$.

In interior algebras

$$q(x, y, z, w) = (x\Delta y)^0 \cdot z + -(x\Delta y)^0 \cdot w \tag{3}$$

is a QD term where $x\Delta y = (-x + y) \cdot (-y + x)$. In discriminator varieties the normal transform is a QD term. Nelson algebras fail to have a QD term since they are not congruence-permutable. This can be shown by an argument very similar to the one used in [1; Section 1] to show DeMorgan algebras are not congruence-permutable.

In a sense made precise in Theorem 3.6 below Brouwerian semilattices can be thought of as the paradigm for congruence-permutable varieties with EDPC.

Throughout the remainder of the section $\mathfrak{A}$ will be, unless otherwise specified, a member of a variety with EDPC and permuting congruence relations. Moreover, q will be a term that serves as a QD term $H_\omega\mathfrak{A}$, or equivalently, satisfies condition 3.2(ii) in $\mathfrak{A}$.

Consider an arbitrary element $e \in A$. Define for $a, b \in A$

$$a \rightarrow_e b = q(a, e, b, e), \qquad a \cdot_e b = q(a, e, b, a). \tag{4}$$

THEOREM 3.5. *Assume $\mathfrak{A}$ is a member of variety with* EDPC *and permuting congruences. Let $q(x, y, z, w)$ be a* QD *term for $H_\omega\mathfrak{A}$, and let $e \in A$. Then the term functions $\rightarrow_e$ and $\cdot_e$ defined in* (4) *are respectively a weak relative pseudo complementation and weak meet for $\mathfrak{A}$ relative to e, i.e., for all $a, b \in A$*

(i) $\Theta(a \rightarrow_e b, e) = \Theta(a, e) * \Theta(b, e)$,

(ii) $\Theta(a \cdot_e b, e) = \Theta(a, e) \vee \Theta(b, e)$.

Proof. (i) is a trivial consequence of 3.1(i). From the string of equivalences

$$b \equiv_{\Theta(a,e)} q(a, e, b, a) \equiv_{\Theta(a \cdot_e b, e)} e \equiv_{\Theta(a,e)} a$$

we get

$$\Theta(b, a) \subseteq \Theta(a, e) \vee \Theta(a \cdot_e b, e), \tag{5}$$

and hence $\Theta(a, e) * \Theta(b, a) \subseteq \Theta(a \cdot_e b, e)$. Thus by 3.1(i), taking a, e, b, a in place of a, b, c, d, respectively, $a \cdot_e b \equiv a(\Theta(a \cdot_e b, e))$. Hence $\Theta(a, e) \subseteq \Theta(a \cdot_e b, e)$, and combined with (5) this also gives $\Theta(b, e) \subseteq \Theta(a \cdot_e b, e)$. Thus since $a \cdot_e b \equiv_{\Theta(a,e)} b \equiv_{\Theta(b,e)} e$ by 3.1(ii), we have (ii).

The next theorem is a natural companion to 3.3. It provides a very satisfactory characterization of varieties with EDPC that are also congruence-permutable and point-regular.

THEOREM 3.6. *Let $\mathcal{K}$ be any class of similar algebras with a constant* 1. *The following are equivalent.*

(i) HSP$\mathcal{K}$ *has* EDPC *and is congruence-permutable and* 1*-regular.*

(ii) $H_\omega \mathcal{K}$ *has a QD term q and a Gödel equivalence term Δ with respect to* 1.

(iii) $\mathcal{K}$ *has a quaternary term q and a binary term Δ such that the congruence conditions*

$$c \equiv_{\Theta(a,\, b)} q(a, b, c, d) \equiv_{\Theta(a,\, b) * \Theta(c,\, d)} d,$$
$$\Theta(a \Delta b, 1) = \Theta(a, b)$$

are satisfied by all $\mathfrak{A} \in \mathcal{K}$ and $a, b, c, d \in A$.

(iv) HSP$\mathcal{K}$ *is a congruence-permutable* WBSO *variety.*

Proof. The equivalence of (i), (ii), and (iii) follows immediately from Theorems 0.1, 0.7, 3.2(i), (ii), 3.3(i), (ii), and 3.5(ii). Clearly (iv) implies (i). (See the remarks following the proof of Lemma 2.4.) To complete the proof we show that (i) implies (iv).

Let $\mathfrak{F}$ be the free algebra of HSP$\mathcal{K}$ with two generators x and y. By hypothesis $\Theta(x, y)$ is generated by pairs of terms of the form $\langle t(x, y), 1\rangle$ where $t(x, y) \equiv 1(\Theta(x, y))$. But since $\Theta(x, y)$ is finitely generated, it can be generated by a finite number $\langle t_0(x, y), 1\rangle, \ldots, \langle t_{n-1}(x, y), 1\rangle$ of these pairs. Let

$$x \Delta y = t_0(x, y) \cdot_1 t_1(x, y) \cdot_1 \cdots \cdot_1 t_{n-1}(x, y).$$

Then by 3.5(ii) we have

$$\Theta(x, y) = \Theta(t_0(x, y), 1) \vee \cdots \vee \Theta(t_{n-1}(x, y), 1) = \Theta(x \Delta y, 1).$$

Since HSP$\mathcal{K}$ has EDPC by hypothesis, and hence the congruence extension property, the congruence condition $\Theta(a, b) = \Theta(a \Delta b, 1)$ holds for all $\mathfrak{A} \in$ HSP$\mathcal{K}$ and $a, b \in A$ (see Lemma 0.2). When this is combined with the special cases of the congruence conditions 3.5(i), (ii) obtained by taking e to be 1, we get by Lemma 2.7 that HSP$\mathcal{K}$ is a WBSO variety with weak relative pseudo complement $\rightarrow_1$ and weak meet $\cdot_1$.

It is easy to check that the operations $\rightarrow_1$ and $\cdot_1$ that one obtains from (4) by using the QD term (2) coincide with the usual relative pseudo complementation $\rightarrow$ and meet $\cdot$ of Brouwerian semilattices. With the QD term (3) for an interior algebra $\mathfrak{A}$ the definitions (4) give

$$a \rightarrow_1 b = -a^0 + b, \qquad a \cdot_1 b = a^0 \cdot b + -a^0 \cdot a \tag{6}$$

for all $a, b \in A$. Finally, applying (4) with q equal to the normal transform we obtain the same weak relative pseudo complementation and weak meet for discriminator varieties originally obtained by Bulman-Fleming and Werner [4]; see (4) of Section 2.

We mentioned in the Preliminaries that, if a deductive system that has both the deduction theorem and the Gödel rule is algebraizable, the associated variety must have EDPC and be point-regular. Thus all the congruence-permutable varieties that arise from algebraizing deductive systems of this kind take the form of WBSO's. For some insight into the metalogical significance of congruence permutability see Corollary 3.10 below.

Suppose $\mathcal{V}$ is a variety with a QD term q and a constant 1 with well behaved ideals, i.e., $\mathcal{V}$ is 1-regular. If $\mathcal{V}$ is Fregean with respect to 1 (recall the definition of this notion given in the remarks following Lemma 2.7), then $\rightarrow_1$ and $\cdot_1$ defined in (4) are actual relative pseudo complementation and meet operations since, for every $\mathfrak{A} \in \mathcal{V}$, $\langle A, \cdot_1, \rightarrow_1, 1\rangle$ is isomorphic to the Brouwerian semilattice $\mathfrak{Cp}\mathfrak{A}$. Even in the non-Fregean case these operations turn out to have some of the special properties of relative pseudo complementation and meet that are not shared by the weak relative pseudo complementation and weak meet of arbitrary WBSO varieties.

THEOREM 3.7. *Assume* $\mathfrak{A}$, $q(x, y, z, w)$, $e, \rightarrow_e, \cdot_e$ *are as in Theorem* 3.5, *and* *let*

$$x +_e y = ((x \to_e y) \to y) \cdot_e ((y \to_e x) \to_e x).$$

The following identities hold in $\mathfrak{A}$.

(i) $x \cdot_e x = x$,
(ii) $x \cdot_e e = e \cdot_e x = x$,
(iii) $e \to_e x = x$,
(iv) $x \to_e e = e$,
(v) $x +_e x = x$,
(vi) $x +_e e = e +_e x = e$.

The following quasi-identities also hold.

(vii) $x \to_e y = e$ *iff* $x \cdot_e y = x$,
(viii) $x \to_e y = e$ *iff* $x +_e y = y$.

Proof. (i)–(vi) are all easy consequences of 3.2(ii) and 3.5. Consider for example (iii). $e \to_e a = q(a, e, a, e) = a$ by 3.2(ii).

The quasi-identities (vii) and (viii) require somewhat more work. From 3.2(ii) and 3.5(i) we get the string of equivalences $a \to_e b = e$ iff $b \equiv e(\Theta(a, e))$ iff $b \equiv a(\Theta(a, e))$ iff $a \cdot_e b = q(a, e, b, a) = a$. Thus (vii) holds. For the purpose of verifying (viii) we observe first of all that

$$\Theta(b \to_e ((b \to_e a) \to_e a), e) = \Theta(b, e) * ((\Theta(b, e) * \Theta(a, e)) * \Theta(a, e)) = I$$

by 3.5 and the theory of Brouwerian semilattices. Thus

$$y \to_e ((y \to_e x) \to_e x) = e \tag{7}$$

is an identity of $\mathfrak{A}$, and in analogous manner we can show that

$$x \to_e (x +_e y) = e \tag{8}$$

is also an identity. Assume $a \to_e b = e$. Then

$$\begin{aligned} a +_e b &= (e \to_e b) \cdot_e ((b \to_e a) \to_e a) \\ &= b \cdot_e ((b \to_e a) \to_e a) \quad \text{by (iii)} \\ &= b \quad \text{by (7) and (vii).} \end{aligned}$$

Now assume $a +_e b = b$. Then $a \to_e b = e$ by (8). So (viii) holds.

The example of interior algebras (see (6)) shows that in general $\rightarrow_e$ and $\cdot_e$ fail to satisfy most of the other familiar identities of relative pseudo complementation and meet, even in the case e has well behaved ideals. For instance, $\cdot_e$ is neither associative nor commutative.

We have not been able to determine to what extent the special identities and quasi-identities given in 3.7 are reflected in special properties of congruence-permutable WBSO varieties. For example, if we had replaced the quasi-identity $1 \rightarrow x = 1$ implies $x = 1$ in Definition 2.1 by the stronger identity $1 \rightarrow x = x$, then we would have obtained a stricter notion of WBSO variety for which, in view of 3.7(iii), Theorem 3.6 would continue to hold. However, we know of no nice characterization of WBSO varieties in this stricter sense comparable to the one given in Theorem 2.6.

In the next section we present a very general method for constructing algebras that generate congruence-permutable EDPC varieties (Corollary 4.5) and congruence-permutable WBSO varieties (Theorem 4.6). It provides counter-examples to many of the conjectures about such algebras that naturally arise from considering discriminator varieties and the more familiar varieties of algebraic logic. For example, Bulman-Fleming and Werner [4] have shown that each member of a discriminator variety is e-regular for every element e, and, moreover, a Gödel equivalence operation relative to e is uniformly definable in terms of the QD term function (the normal transform) with e as parameter; see (4) of Section 2. We shall see in Section 4 that this result does not extend to congruence-permutable WBSO varieties in general, even those that are in a natural sense closest to discriminator varieties. (See the remarks following Theorem 4.6 below.)

It follows from the existence of a weak relative pseudo complementation and weak meet (3.5(i), (ii)) that, if $\mathfrak{A}$ is a member of congruence-permutable EDPC variety, then principal congruences of the form $\Theta(a, e)$ for fixed e are closed under both join and dual relative pseudo complementation, and hence form a subalgebra $\mathfrak{Cp}_e\mathfrak{A}$ of $\mathfrak{Cp}\mathfrak{A}$, regardless of whether $\mathfrak{A}$ is e-regular. In fact, it follows from 3.1(i) that $\Phi * \Theta(a, e)$ is contained in $\mathfrak{Cp}_e\mathfrak{A}$ for every $a \in A$ and every $\Phi \in Cp\mathfrak{A}$. I.e., $\mathfrak{Cp}_e\mathfrak{A}$ is what is called a *total* subalgebra of $\mathfrak{Cp}\mathfrak{A}$ (see [17]). Moreover, the mapping $a \mapsto \Theta(a, e)$ is a homomorphism from $\langle A, \cdot_e, \rightarrow_e, e\rangle$ onto $\mathfrak{Cp}_e\mathfrak{A}$. If $\approx_e$ is the relation-kernel of this homomorphism, $a \approx_e b$ iff $\Theta(a, e) = \Theta(b, e)$ iff both $\Theta(a, e) * \Theta(b, e) = I$ and $\Theta(b, e) * \Theta(a, e) = I$ iff both $a \rightarrow_e b = e$ and $b \rightarrow_e a = e$. We summarize these results in the following

THEOREM 3.8. *Assume $\mathfrak{A}$ is a member of a variety with* EDPC *and permuting congruences. For every $e \in A$ let*

$$Cp_e\mathfrak{A} = \{\Theta(a, e) : a \in A\}.$$

Then $Cp_e\mathfrak{A}$ is the universe of a subalgebra $\mathfrak{Cp}_e\mathfrak{A}$ of $\mathfrak{Cp}\mathfrak{A}$, and, if q is a QD *polynomial for $H_\omega\mathfrak{A}$ and $\to_e$ and $\cdot_e$ are defined as in* (4), *then*

$$\langle A, \cdot_e, \to_e, e\rangle/{\approx_e} \cong \mathfrak{Cp}_e\mathfrak{A}$$

where $\approx_e$ is a congruence relation on $\langle A, \cdot_e, \to_e, e\rangle$ defined by $a \approx_e b$ if $a \to_e b = e$ and $b \to_e a = e$.

Observe that $\mathfrak{Cp}_e\mathfrak{A} = \mathfrak{Cp}\mathfrak{A}$ iff $\mathfrak{A}$ is e-regular.

Consider for example a Brouwerian semilattice $\mathfrak{A} = \langle A, \cdot, \to, 1\rangle$ together with the QD term q defined in (2). After some computation the definitions (4) can be reduced to the formulas

$$a \to_e b = ((((a \to e) \cdot (e \to a) \cdot b) \to e) \to e) \cdot ((e \to a) \to b),$$
$$a \cdot_e b = ((((a \to e) \cdot (e \to a) \cdot b) \to a) \to a) \cdot ((a \to e) \to b),$$

and $\mathfrak{Cp}_e\mathfrak{A}$ corresponds under the natural isomorphism between $\mathfrak{Cp}\mathfrak{A}$ and $\mathfrak{A}$ to the subalgebra $\{(a \to e) \cdot (e \to a) : a \in A\}$ of $\mathfrak{A}$. We note again $\cdot_1$ and $\to_1$ coincide wtih and $\to$, and $\approx_1$ is the identity relation. If $\mathfrak{A}$ has a smallest element 0, and $\mathfrak{A}$ is linearly ordered or more generally 0 is meet irreducible, then $a \neq 0$ implies $(a \to 0) \cdot (0 \to a) = 1$. Thus in this case $\mathfrak{Cp}_0\mathfrak{A}$ contains just two elements, the identity and universal congruences.

If $\mathfrak{A}$ is a member of a discriminator variety then $\mathfrak{Cp}_e\mathfrak{A} = \mathfrak{Cp}\mathfrak{A}$ for every $e \in A$; see Bulman-Fleming and Werner [4] and the remarks following Lemma 2.7.

If a WBSO variety is Fregean (see the remarks after 2.7), then it must have permuting congruences since, as noted previously, its $(\cdot, \to, 1)$-reduct is a class of Brouwerian semilattices. Underlying this fact is a close connection between congruence permutability and the characteristic property of the weak meet operation (see (3) of Section 2).

THEOREM 3.9. *Assume the variety $\mathcal{V}$ is Fregean with respect to* 1 *(so in particular it is* 1*-regular). If every compact congruence of every member of $\mathcal{V}$ is principal, then $\mathcal{V}$ has permuting congruences.*

Proof. Let $\mathfrak{F}$ be the free algebra of $\mathcal{V}$ with a denumerable number of generators $x, y, z_0, z_1, \ldots$. Let $t(x, y, z_0, \ldots, z_{m-1})$ be a term such that

$$\Theta(x, 1) \vee \Theta(y, 1) = \Theta(t(x, y, z_0, \ldots, z_{m-1}), 1),$$

and let

$$x \cdot y = t(x, y, x, \ldots, x).$$

By a well known argument (see for instance [18, the proof of Theorem 8]) it can be shown that, for every $\mathfrak{A} \in \mathcal{V}$ and all $a, b \in A$,

$$\Theta_{\mathfrak{A}}(a, 1) \vee \Theta_{\mathfrak{A}}(b, 1) = \Theta_{\mathfrak{A}}(a \cdot b, 1). \tag{9}$$

Thus $x \cdot y$ is a weak meet term with respect to 1. The existence of a weak meet term, together with the assumption that 1-ideals behave properly, implies the existence of a Gödel equivalence term $x \Delta y$ with respect to 1 (see the proof of 3.6).

Recall that the identity $x \Delta x = 1$ holds for any Gödel equivalence term, and, since $\mathcal{V}$ is Fregean with respect to 1, the identities $x \Delta 1 = x$ and $1 \Delta y = y$ also hold. Define

$$m(x, y, z) = ((x \Delta y) \Delta z) \cdot (x \Delta (y \Delta z)).$$

Then by (9) we have for all $\mathfrak{A} \in \mathcal{V}$ and $a, b \in A$

$$\begin{aligned} \Theta_{\mathfrak{A}}(m(a, a, b), 1) &= \Theta_{\mathfrak{A}}((a \Delta a) \Delta b, 1) \vee \Theta_{\mathfrak{A}}(a \Delta (a \Delta b), 1) \\ &= \Theta_{\mathfrak{A}}(b, 1) \vee \Theta_{\mathfrak{A}}(a, a \Delta b) \\ &= \Theta_{\mathfrak{A}}(b, 1); \end{aligned}$$

the last equality holds since $a = a \Delta 1 \equiv a \Delta b (\Theta(b, 1))$. Thus, since $\mathcal{V}$ is Fregean, $m(a, a, b) = b$, and by a symmetric argument $m(a, b, b) = a$. Thus $m(x, y, z)$ is a Mal'cev term for $\mathcal{V}$.

Combining this result with Theorem 3.5(ii) we immediately get the following

COROLLARY 3.10. *Assume $\mathcal{V}$ is Fregean with respect to* 1 *and has* EDPC. *Then $\mathcal{V}$ has a weak meet term with respect to* 1 *iff $\mathcal{V}$ is congruence-permutable.*

Brouwerian semilattices satisfy the hypothesis of the corollary. They obviously have a weak meet so by the corollary they are congruence-permutable, a well known fact we noted previously. On the other hand, the variety of Hilbert algebras (which consists of all subalgebras of $(\rightarrow, 1)$-reducts of Brouwerian semilattices) also satisfies the hypothesis of the corollary. But it clearly does not have a weak meet term, and hence must necessarily fail to be congruence-permutable.

The assumption that $\mathcal{V}$ be Fregean is essential for the conclusions of both the theorem and the corollary. Since Nelson algebras form a WBSO variety, they have a weak meet term and every compact congruence is principal. But they fail to be congruence-permutable. On the other hand, in Section 4 below we will construct an algebra $\mathfrak{A}$ that generates a congruence-permutable EDPC variety and yet has compact congruences that are not principal.

In the case of interior algebras we cannot use 3.10 directly to infer congruence-permutability from the existence of a weak meet term. But we can use it indirectly by applying it to the Boolean algebra reduct. Observe that in the case of Nelson algebras the analogus reduction leads to DeMorgan algebras, another non-Fregean variety with EDPC.

4. Congruence relative Stone varieties

There is a natural way of arranging the class of varieties with EDPC into a hierarchy, the so-called *Brouwerian hierarchy*, corresponding to the lattice of varieties of Brouwerian semilattices. The position of any given variety $\mathcal{V}$ within the hierarchy is determined by the variety of Brouwerian semilattices generated by all the semilattices of compact congruences of members of $\mathcal{V}$. At the lowest non-trivial level, the one corresponding to generalized Boolean algebras (the unique minimal variety of Brouwerian semi-lattices) we have exactly the semisimple varieties with EDPC. These have been shown to coincide with filtral varieties in Fried, Grätzer, and Quackenbush [9]. In the presence of congruence permutability they turn out to coincide with discriminator varieties. (For a more detailed discussion of the Brouwerian hierarchy see [1, Section 4].)

In this section we investigate those varieties that, in the congruence-permutable part of the Brouwerian hierarchy, lie just above discriminator varieties.

A Brouwerian semilattice that is the subdirect product of a family of chains is called a *relative Stone lattice*. The class of all relative Stone lattices forms a variety denoted by $\mathscr{C}$. It can be axiomatized, relative to Brouwerian semilattices, by any one of the following four identities:

$$((x \to y) + z) + ((y \to x) + z) = 1,$$

$$((x \to z) \cdot (y \to z)) \to (x + y \to z) = 1, \tag{1}$$

$$(x + y) + z = x + (y + z), \tag{2}$$

$$((x \to y) + z) + (((x \to y) \to y) + z) = 1, \tag{3}$$

where $+$ is defined by

$$x+y=((x\to y)\to y)\cdot((y\to x)\to x).$$

It is easy to check that the identities $x+y=y+x$ and $x\leq x+y$ hold in any Brouwerian semilattice. Thus relative to the axioms of Brouwerian semilattices (1) is equivalent to the condition that $a+b$ is the least upper bound of a and b. Consequently, every relative Stone lattice is a relatively pseudo complemented lattice.

Brouwerian semilattices that are subdirect products of chains have been studied by Nemitz and Whaley [24] who, in particular, obtained the characterizations (1) and (2) above. It follows from results of Hecht and Katrinák [14] that they are equationally definitionally equivalent with what are commonly called relative Stone lattices (or algebras) in the literature. The following basic facts about relative Stone lattices are established in Nemitz and Whaley [24].

$\mathscr{C}$ is generated by any infinite chain and hence also by the set of all finite chains. The subdirectly irreducibles of $\mathscr{C}$ are exactly the chains with a dual atom. If $\mathscr{C}_n$ denotes the subvariety of $\mathscr{C}$ generated by a chain of n-elements, then $\mathscr{C}_n$ is axiomatized relative to $\mathscr{C}$ by the identity

$$(x_1\to x_2)+(x_2\to x_3)+\cdots+(x_n\to x_{n+1})=1.$$

The subdirectly irreducibles of $\mathscr{C}_n$ are exactly the chains of length $\leq n$. It follows immediately from these results that $\mathscr{C}_n\subset\mathscr{C}_m$ whenever $n<m$, and every proper subvariety of $\mathscr{C}$ is of the form $\mathscr{C}_n$ for some n with $1\leq n<\omega$. Observe that $\mathscr{C}_2$ is the variety of generalized Boolean algebras.

A variety $\mathscr{V}$ with EDPC is called a *congruence relative Stone variety* if $\mathfrak{Cp}\mathfrak{A}\in\mathscr{C}$ for every $\mathfrak{A}\in\mathscr{V}$, or, equivalently, if every subdirectly irreducible of $\mathscr{V}$ has a linearly ordered congruence lattice. $\mathscr{V}$ is a *congruence relative Stone variety of length* $\leq n$ if $\mathfrak{Cp}\mathfrak{A}\in\mathscr{C}_n$ for every $\mathfrak{A}\in\mathscr{V}$. Congruence relative Stone varieties of length ≤ 2 coincide with filtral varieties. The paradigm for a congruence relative Stone variety at the $\mathscr{C}_n$-level of the congruence-permutable Brouwerian hierarchy is the variety $\mathscr{C}_n$ itself.

If $\mathfrak{A}$ is a member of a congruence relative Stone variety, then $Cp\mathfrak{A}$ is closed under intersection. In fact, let $q=\langle q_i : i\leq n\rangle$ be a QD system of terms for $H_\omega\mathfrak{A}$ such that, for all $a, b, c, d\in A$,

$$\bigvee_{\text{even } i<n}\Theta(q_i(a,b,c,d),q_{i+1}(a,b,c,d))=\Theta(a,b)*\Theta(c,d)^2$$

where $\Theta(a,b)*\Theta(c,d)^2=(\Theta(a,b)*\Theta(c,d))*\Theta(c,d)$. (By Theorem 1.7 such a q

always exists.) Then

$$\Theta(a, b) \cap \Theta(c, d) = \Phi(a, b, c, d) \vee \Phi(c, d, a, b) \tag{4}$$

where $\Phi(x, y, z, w) = \bigvee_{\text{even } i<n} \Theta(q_i(x, y, z, w), q_{i+1}(x, y, z, w))$.

In the sequel the congruence relative Stone variety $\mathcal{V}$ we consider will always be congruence-permutable, and the algebra $\mathfrak{A}$ will always be an arbitrary member of such a variety. (Hence $\mathcal{V}$ is a discriminator variety if $\mathfrak{Cp}\mathfrak{A}$ is a generalized Boolean algebra.) Furthermore, q will always be a QD term for $H_\omega\mathfrak{A}$ that satisfies the following condition for all $a, b, c, d \in A$ (Theorem 3.3.(iii)):

$$\Theta(q(a, b, c, d), c) = \Theta(a, b) * \Theta(c, d)^2. \tag{5}$$

Then as a particular case of (4) we have

$$\Theta(a, b) \cap \Theta(c, d) = \Theta(q(a, b, c, d), c) \vee \Theta(q(c, d, a, b), a).$$

Theorem 3.8 can be improved in the expected way for congruence relative Stone varieties. Consider an arbitrary element e of $\mathfrak{A}$ and define for all $a, b \in A$

$$a +_e b = ((a \to_e b) \to_e b) \cdot_e ((b \to_e a) \to_e a)$$

where $\to_e$ and $\cdot_e$ are the weak relative pseudo complementation and weak meet operations defined in (4) of Section 3. Recall that $Cp_e\mathfrak{A} = \{\Theta(a, e) : a \in A\}$. The proof of the following is obvious.

THEOREM 4.1. *Assume $\mathfrak{A}$ is a member of a congruence relative Stone variety with permuting congruences. Let e be an arbitrary element of $\mathfrak{A}$. Then for all a, b*

$$\Theta(a +_e b, e) = \Theta(a, e) \cap \Theta(b, e). \tag{6}$$

Hence $Cp_e\mathfrak{A}$ is closed under intersection, as well as join and dual relative pseudo complementation, and

$$\langle A, +_e, \cdot_e, \to_e, e\rangle / \approx_e \cong \langle Cp_e\mathfrak{A}, \cap, \wedge, *, I\rangle \tag{7}$$

where $\approx_e$ is the congruence relation defined by $a \approx_e b$ if $a \to_e b = e$ and $b \to_e a = e$.

A binary term $+_e$ that satisfies (6) for all $a, b \in A$ is called a *weak join* (*with respect to* e).

If e has well behaved ideals, then every compact congruence of $\mathfrak{A}$ is principal, and then the quotient algebra on the right-hand side of (7) is isomorphic to the lattice of all principal congruences. As previously noted Bulman-Fleming and Werner [4] have shown that, if $\mathfrak{A}$ belongs to a discriminator variety, $\mathfrak{A}$ is e-regular for every element e, and $\mathfrak{A}$ has a Gödel equivalence term Δ_e definable in terms of q (the normal transform term in the discriminator case) with e as parameter:

$$x\Delta_e y = q(x, y, e, q(x, e, y, x)).$$

(Compare formula (5) of Section 2.) To see this observe first of all that by (4) of Section 3, (5), and 3.5(ii) we have $\Theta(a\Delta_e b, e) = \Theta(a, b) * (\Theta(a, e) \vee \Theta(b, e))^2$. Recall that every generalized Boolean algebra satisfies the identity $(x \to y) \to y = x + y$. Hence, if $\mathfrak{A}$ is a member of a discriminator variety,

$$\Theta(a\Delta_e b, e) = \Theta(a, b) \cap (\Theta(a, e) \vee \Theta(b, e)) = \Theta(a, b).$$

At higher levels of the Brouwerian hierarchy Δ_e does not give a Gödel equivalence term. For example, if the compact congruences of $\mathfrak{A}$ form a chain, and a (or equivalently b) fails to be congruent to e modulo $\Theta(a, b)$, then $\Theta(a\Delta_e b, e) = I_{\mathfrak{A}}$. Consequently, if $\mathcal{V}$ is not a discriminator variety, there is always an $\mathfrak{A} \in \mathcal{V}$ and $e \in A$ such that the congruence condition $\Theta(a\Delta_e b, e) = \Theta(a, b)$ fails for some $a, b \in A$. We shall construct below, following 4.6, a WBSO variety at the $\mathcal{C}_3$-level of the Brouwerian hierarchy in which no Gödel equivalent term (with respect to the constant 1) is definable in terms of q and 1.

Some characteristic properties of discriminator varieties can, however, be generalized in a natural way higher up in the Brouwerian hierarchy, at least at the permutable congruence relative Stone levels.

Let A be any non-empty set. Let $\mathfrak{L} = \langle L, \vee, \cap, I, A \times A\rangle$ be a complete 0, 1-sublattice of the lattice of equivalence relations on A. (By a *complete sublattice* we mean one closed under arbitrary intersection and join of equivalence relations.) For every pair a, b of elements of A let

$$\Theta_{\mathfrak{L}}(a, b) = \bigcap \{\Phi \in L : a \equiv b(\Phi)\}.$$

A quaternary operation q on A is called the $\mathfrak{L}$-*normal transform* if, for all $a, b, c, d \in A$,

$$q(a, b, c, d) = \begin{cases} d & \text{if} \quad c \equiv d(\Theta_{\mathfrak{L}}(a, b)) \\ c & \text{otherwise.} \end{cases}$$

Observe that in opposition to the definition of a QD term this definition completely determines q; conversely, q completely determines the lattice $\mathfrak{L}$. Recall that $\mathfrak{Co}\mathfrak{A}$ denotes the congruence lattice of $\mathfrak{A}$.

LEMMA 4.2. *Let $\mathfrak{A}$ be any algebra and q a quaternary term function of $\mathfrak{A}$. Then the conjunction of any two of the following conditions implies the third.*

(i) *q is a QD term for $H_\omega\mathfrak{A}$ satisfying the special condition of 3.3(iii);*

(ii) *q coincides with the $\mathfrak{Co}\mathfrak{A}$-normal transform on A;*

(iii) *$\Phi \vee \Psi = \Phi \cup \Psi$ for all $\Phi, \Psi \in Co\mathfrak{A}$.*

Proof. We observe first of all that (iii) is equivalent to the same condition holding for all principal congruences Φ, Ψ of $\mathfrak{A}$. It is also equivalent to the condition that $\langle Cp\mathfrak{A}, \vee, I\rangle$ be dually relatively pseudo complemented and, for all $a, b, c, d \in A$,

$$\Theta(a, b) * \Theta(c, d) = I \quad \text{or} \quad \Theta(a, b) * \Theta(c, d)^2 = I. \tag{8}$$

In order to see this assume (iii) holds. If $c \equiv d(\Theta(a, b))$, i.e., $\Theta(c, d) \subseteq \Theta(a, b)$, then $\Theta(a, b) * \Theta(c, d)$ exists and equals I. Suppose $c \not\equiv d(\Theta(a, b))$. Then for any congruence Φ, the equality $\Theta(a, b) \vee \Phi = \Theta(a, b) \cup \Phi$ implies $c \equiv d(\Theta(a, b) \vee \Phi)$ iff $c \equiv d(\Phi)$. Hence $\Theta(a, b) * \Theta(c, d)$ exists and equals $\Theta(c, d)$, and thus $\Theta(a, b) * \Theta(c, d)^2 = \Theta(c, d) * \Theta(c, d) = I$. So (8) holds, and from the fact that the dual relative pseudo complement of every pair of principal congruences exists, it follows that $\langle Cp\mathfrak{A}, \vee, I\rangle$ is dually relatively pseudo complemented (see for instance [18, Lemma 4].)

Assume conversely that (8) holds for all $a, b, c, d \in A$. Suppose

$$\langle e, f\rangle \in (\Theta(a, b) \vee \Theta(c, d)) \sim \Theta(a, b). \tag{9}$$

Since $\Theta(e, f) \nsubseteq \Theta(a, b)$ we can conclude from (8) (taking e and f respectively for c and d) that $\Theta(a, b) * \Theta(e, f)^2 = I$, and hence, in view of (9),

$$\Theta(e, f) \subseteq \Theta(a, b) * \Theta(e, f) \subseteq \Theta(c, d). \tag{10}$$

Thus $(\Theta(a, b) \vee \Theta(c, d)) \sim \Theta(a, b) \subseteq \Theta(c, d)$, which gives $\Theta(a, b) \vee \Theta(c, d) = \Theta(a, b) \cup \Theta(c, d)$. So condition (iii) holds.

Assume (i) and (iii). If $c \equiv d(\Theta(a, b))$, then $q(a, b, c, d) = d$ since q is a QD term function. Suppose $c \not\equiv d(\Theta(a, b))$. Then (8) gives $\Theta(a, b) * \Theta(c, d)^2 = I$, and thus $q(a, b, c, d) = c$ by (5).

Assume (i) and (ii) hold. If $\Theta(c, d) \subseteq \Theta(a, b)$, then $\Theta(a, b) * \Theta(c, d) = I$; otherwise $q(a, b, c, d) = c$, and hence, by 3.1(ii), $\Theta(a, b) * \Theta(c, d)^2 \subseteq \Theta(q(a, b, c, d), c) = I$. Thus (8) holds for all $a, b, c, d \in A$, and this gives (iii).

Assume finally that (ii) and (iii) hold. From (8) it follows easily that

$$\Theta(a, b) * \Theta(c, d) = \begin{cases} I & \text{if } c \equiv d(\Theta(a, b)) \\ \Theta(c, d) & \text{otherwise,} \end{cases} \tag{11}$$

$$\Theta(a, b) * \Theta(c, d)^2 = \begin{cases} \Theta(c, d) & \text{if } c \equiv d(\Theta(a, b)) \\ I & \text{otherwise.} \end{cases} \tag{12}$$

From (11) and (ii) we get that the second of the following two equivalences, and from (12) and (ii) we get the first.

$$c \equiv_{\Theta(a, b) * \Theta(c, d)^2} q(a, b, c, d) \equiv_{\Theta(a, b) * \Theta(c, d)} d.$$

Thus q is a QD term for $H_\omega \mathfrak{A}$ by Theorem 3.2(i),(ii) (since $\Theta(a, b) * \Theta(c, d)^2 \subseteq \Theta(a, b)$), and so (i) holds. This proves the lemma.

Observe that in order to infer (iii) from (i) and (ii) we needed only to assume q is a QD term for $H_\omega \mathfrak{A}$, and not that it satisfies in addition the special condition of 3.3(iii). As a consequence, if q is a QD term for $H_\omega \mathfrak{A}$, and (iii) is satisfied, in particular if $\mathfrak{Co}\mathfrak{A}$ is a chain, then q must necessarily satisfy the special condition of 3.3(iii).

The lemma allows us to characterize permutable congruence relative Stone varieties in a way very close in spirit to the usual definition of discriminator varieties; see Werner [28].

THEOREM 4.3. *Let $\mathcal{V}$ be a variety. The following are equivalent.*

(i) *$\mathcal{V}$ is a congruence relative Stone variety (of length $\leq n$) with permuting congruences.*

(ii) *There is a quaternary term q of $\mathcal{V}$ such that, for every subdirectly irreducible $\mathfrak{A} \in \mathcal{V}$, $\mathfrak{Co}\mathfrak{A}$ is a chain (of length $\leq n$) and q coincides with the $\mathfrak{Co}\mathfrak{A}$-normal transform on A.*

(iii) *There is a class $\mathcal{K}$ of algebras and a quaternary term q of $\mathcal{K}$ such that $\mathcal{V} = \mathrm{HSP}\mathcal{K}$ and, for every $\mathfrak{A} \in \mathcal{K}$, $\mathfrak{Co}\mathfrak{A}$ is a chain (of length $\leq n$) and q coincides with the $\mathfrak{Co}\mathfrak{A}$-normal transform on A.*

Proof. Suppose $\mathcal{V}$ is a congruence relative Stone variety (of length $\leq n$) with permuting congruences. By definition each subdirectly irreducible $\mathfrak{A}$ has a linearly ordered congruence lattice (of length $\leq n$), and, if q is a QD term for $\mathcal{V}$, then by 4.2 q is the $\mathfrak{Co}\mathfrak{A}$-normal transform on A. Thus (i) implies (ii) which in turn trivially implies (iii).

Suppose (iii) holds. By Lemma 4.2, q is a QD term for $H_\omega\mathcal{K}$, and hence, by 3.3, $\mathcal{V} = \mathrm{HSP}\mathcal{K}$ has EDPC and is congruence-permutable. It then follows directly from Theorem 0.8 that $\mathcal{V}$ is a congruence relative Stone variety (of length $\leq n$).

Another very characteristic property of discriminator varieties is the following: when we adjoin the normal transform to the fundamental operations of an arbitrary algebra $\mathfrak{A}$, the new algebra $(\mathfrak{A}, q)$ we get obviously generates a discriminator variety. But when we try to adjoin an $\mathfrak{L}$-normal transform where $\mathfrak{L}$ contains equivalence relations other than I and $A \times A$, we run into apparent difficulties. We may select any linearly ordered complete sub-lattice $\mathfrak{L}$ of $\mathfrak{Co}\mathfrak{A}$ and adjoin to the operations of $\mathfrak{A}$ the special normal transform determined by $\mathfrak{L}$. But we cannot immediately apply Lemma 4.2 to conclude that q is a QD term for $H_\omega(\mathfrak{A}, q)$ since it is not apparent, as it is in the case of the ordinary normal transform, that $\mathfrak{L}$ coincides with $\mathfrak{Co}(\mathfrak{A}, q)$. The next theorem establishes this result and, indeed, something even stronger.

THEOREM 4.4. *Let $\mathfrak{A}$ be any algebra and $\mathfrak{L}$ a complete sublattice of $\mathfrak{Co}\mathfrak{A}$ having the following two properties:*

(i) *the join semilattice of compact members of $\mathfrak{L}$ form a dual relative Stone lattice,*

(ii) *the members of $\mathfrak{L}$ permute.*

Then there exists a unique quaternary operation q on A such that $\mathfrak{L} = \mathfrak{Co}(\mathfrak{A}, q)$, and q is a QD term for $H_\omega(\mathfrak{A}, q)$ satisfying the condition of 3.3(iii).

Proof. Recall that the identity (3) characterizes congruence relative Stone lattices among the Brouwerian semilattices; in particular the identity $(x \rightarrow y) + ((x \rightarrow y) \rightarrow y = 1$ holds in any congruence relative Stone lattice. Thus, in view of 4.1, the premiss (i) implies that, for all $a, b, c, d \in A$,

$$\Theta_{\mathfrak{L}}(a, b) * \Theta_{\mathfrak{L}}(c, d) \cap \Theta_{\mathfrak{L}}(a, b) * \Theta_{\mathfrak{L}}(c, d)^2 = I. \tag{13}$$

In view of the premiss (ii) we can define $q(a, b, c, d)$ to be the unique element of A such that

$$c \equiv_{\Theta_{\mathfrak{L}}(a, b) * \Theta_{\mathfrak{L}}(c, d)^2} q(a, b, c, d) \equiv_{\Theta_{\mathfrak{L}}(a, b) * \Theta_{\mathfrak{L}}(c, d)} d. \tag{14}$$

Observe that since $\Theta_{\mathfrak{L}}(a, b) * \Theta_{\mathfrak{L}}(c, d)^2 \subseteq \Theta_{\mathfrak{L}}(a, b)$ we have

$$c \equiv_{\Theta_{\mathfrak{L}}(a,b)} q(a, b, c, d) \equiv_{\Theta_{\mathfrak{L}}(a, b) * \Theta_{\mathfrak{L}}(c, d)} d,$$

and thus $q(x, x, z, w) = z$ holds identically.

Let $\mathfrak{B} = (\mathfrak{A}, q)$. If $c \equiv d$ $(\Theta_{\mathfrak{L}}(a, b))$, then

$$c = q(a, a, c, d) \equiv_{\Theta_{\mathfrak{B}}(a, b)} q(a, b, c, d) = d.$$

Hence $\Theta_{\mathfrak{L}}(a, b) \subseteq \Theta_{\mathfrak{B}}(a, b)$. Suppose we can prove that each Φ in $\mathfrak{L}$ has the substitution property with respect to q. Then $\mathfrak{L}$ would be a sublattice of $\mathfrak{Co}\mathfrak{B}$. This would imply $\Theta_{\mathfrak{B}}(a, b) \subseteq \Theta_{\mathfrak{L}}(a, b)$, and thus $\Theta_{\mathfrak{L}}(a, b) = \Theta_{\mathfrak{B}}(a, b)$ for all $a, b \in A$. But this in turn would imply that $\mathfrak{L} = \mathfrak{Co}\mathfrak{B}$ since $\mathfrak{L}$ and $\mathfrak{Co}\mathfrak{B}$ are both complete sublattices of $\mathfrak{Co}\mathfrak{A}$. The last part of the conclusion of the theorem would then follow immediately from (14) by 3.2(i), (ii).

Let Φ be any member of $\mathfrak{L}$ and let $a, b, c, d, a', b', c', d' \in A$ such that $a \equiv a'$, $b \equiv b'$, $c \equiv c'$, $d \equiv d'$ modulo Φ. We must show

$$q(a, b, c, d) \equiv q(a', b', c', d')(\Phi). \tag{15}$$

Let $\Psi = \Theta_{\mathfrak{L}}(a, b) * \Theta_{\mathfrak{L}}(c, d)$ and $\Psi' = \Theta_{\mathfrak{L}}(a', b') * \Theta_{\mathfrak{L}}(c', d')$. Since $\Theta_{\mathfrak{L}}(c', d') \subseteq \Theta_{\mathfrak{L}}(a', b') \vee \Psi \vee \Phi$ we have $\Psi' \subseteq \Psi \vee \Phi$, and so by symmetry

$$\Psi' \vee \Phi = \Psi \vee \Phi. \tag{16}$$

Using this identity we get the inclusion $\Theta_{\mathfrak{L}}(c', d') \subseteq \Psi' \vee (\Psi * \Theta_{\mathfrak{L}}(c, d)) \vee \Phi$ and also the symmetric inclusion. Thus

$$\Psi' * \Theta_{\mathfrak{L}}(c', d') \vee \Phi = \Psi * \Theta_{\mathfrak{L}}(c, d) \vee \Phi. \tag{17}$$

By (14),

$$q(a, b, c, d) \equiv_{\Psi * \Theta_{\mathfrak{L}}(c, d)} c \equiv_{\Phi} c' \equiv_{\Psi' * \Theta_{\mathfrak{L}}(c', d')} q(a', b', c', d').$$

Thus by (17) we have

$$q(a, b, c, d) \equiv q(a', b', c', d')(\Psi * \Theta_{\mathfrak{L}}(c, d) \vee \Phi). \tag{18}$$

By (14), $q(a, b, c, d) \equiv_{\Psi} d \equiv_{\Phi} d' \equiv_{\Psi'} q(a', b', c', d')$. Thus by (16) we also have

$$q(a, b, c, d) \equiv q(a', b', c', d')(\Psi \vee \Phi). \tag{19}$$

By the premiss (i) the join semilattice of compact members of $\mathfrak{L}$ is dually relatively pseudo complemented. It is well known that this implies $\mathfrak{L}$ is distributive. (See for instance [18, the proof of Corollary 6].) Combining the distributivity of $\mathfrak{L}$ with (13) we get $\Phi = (\Psi * \Theta_{\mathfrak{L}}(c, d) \vee \Phi) \cap (\Psi \vee \Phi)$. Thus (15) follows immediately from (18) and (19), and the proof is complete.

By taking the set of operations of $\mathfrak{A}$ to be empty we obtain the following concrete congruence-lattice representation result.

COROLLARY 4.5. *Let A be any set and $\mathfrak{L}$ any complete sublattice of the lattice of equivalence relations that satisfies conditions* 4.4(*i*), (*ii*). *Then $\mathfrak{L}$ is the congruence lattice of a member of a congruence-permutable variety with* EDPC.

Several natural questions arise as to how this result might be improved. For instance, does the conclusion of 4.5 continue to hold if condition 4.4(i) is weakened to the requirement that the join semilattice of compact members of $\mathfrak{L}$ be dually relatively pseudo complemented? Does it continue to hold if in place of 4.4(ii) we require only the existence of an n such that all the members of $\mathfrak{L}$ are n-permutable? Condition 4.4(ii) cannot be done away with entirely however. For this would allow us to conclude that every algebraic lattice $\mathfrak{L}$ that satisfies the purely lattice-theoretic condition 4.4(i) is strongly representable. But Pudlák and Tůma [25] have shown that an algebraic lattice is strongly representable only if it is completely distributive.

It turns out that Theorem 4.4 and its corollary can in fact be somewhat improved. More precisely, the proof given is easily seen to apply to any complete sublattice $\mathfrak{L}$ of permuting relations for which the condition

$$(\Phi * \Psi) \cap (\Phi * \Psi) * \Psi = \mathrm{I} \tag{20}$$

is satisfied by all principal equivalences (but not necessarily all compact equivlances) of $\mathfrak{L}$. Hence, in view of the first part of the proof of 4.2, the conclusion of 4.4 applies to any $\mathfrak{L}$ with the property that $\Phi \vee \Psi = \Phi \cup \Psi$ holds for all $\Phi, \Psi \in L$. This result was found independently by Fried [8] who also obtained an intrinsic characterization of the lattices isomorphic to some lattice $\mathfrak{L}$ with this property; for a similar result see Korec [19].

There are lattices $\mathfrak{L}$ with the above property that fail to satisfy 4.4(i). Consider for example the set $A = \{0, 1, 2, 3, 4, 5, 6, 7\}$, and let Θ and Ξ be the equivalence relations on A whose associated partitions are respectively $\{\{0, 1, 2, 3\}, \{4, 5\}, \{6, 7\}\}$ and $\{\{0, 1\}, \{2, 3\}, \{4, 5, 6, 7\}\}$; observe that $\Theta \cup \Xi$ is an equivalence relation. Let $\mathfrak{L}$ be the sublattice of equivalence relations generated by Θ and Ξ;

its members are $I, \Theta, \Xi, \Theta \cap \Xi, \Theta \cup \Xi$, and $A \times A$: all of these are principal except $\Theta \cup \Xi$. Condition (20) is satisfied by all the principal members of $\mathfrak{L}$, but $\Theta * (\Theta \cup \Xi) \cap (\Theta * (\Theta \cup \Xi) * (\Theta \cup \Xi)) = \Xi \cap (\Xi * (\Theta \cup \Xi)) = \Xi \cap \Theta \neq \mathrm{I}$.

As an application of the corollary we shall construct an algebra $\mathfrak{A}$ that generates a congruence-permutable variety with EDPC but has the property that not all its compact congruences are principal. (See the remarks following Corollary 3.10.)

Let $A = \{a, b, c, d\}$, and let Θ, Φ, Ψ be the equivalence relations on A whose associated partitions are respectively $\{\{a\}, \{b\}, \{c, d\}\}$, $\{\{a, b\}, \{c\}, \{d\}\}$, and $\{\{a, b\}, \{c, d\}\}$. Then the set $\{\Theta, \Phi, \Psi, I, A \times A\}$ forms the universe of a complete sublattice of equivalence relations that satisfies conditions 4.4(i), (ii). By 4.5 there is a quaternary operation q on A such that $\mathrm{Co}\langle A, q\rangle = \{\Theta, \Phi, \Psi, I, A \times A\}$ and $\langle A, q\rangle$ generates a congruence-permutable variety with EDPC, in fact, at the $\mathscr{C}_3$-level of the Brouwerian hierarchy. Observe that Ψ is not principal.

A weaker analog of 4.4 is available for WBSO varieties.

THEOREM 4.6. *Let $\mathfrak{A}$ be any algebra and $\mathfrak{L}$ any complete sublattice of $\mathfrak{Co}\mathfrak{A}$ that satisfies the following conditions.*

(i) *$\mathfrak{L}$ is a chain,*

(ii) *$\mathfrak{L}$ is 1-regular for some element $1 \in A$.*

Then there exist a quaternary operation q and a binary operation Δ on A such that $\mathfrak{L} = \mathfrak{Co}(\mathfrak{A}, q, \Delta, 1)$, and $(\mathfrak{A}, q, \Delta, 1)$ generates a congruence-permutable WBSO *variety with* QD *term q (satisfying 3.3(iii)(8)) and Gödel equivalence term Δ.*

Proof. By 4.4 there is a quaternary operation q on A such that $\mathfrak{L} = \mathfrak{Co}(\mathfrak{A}, q)$ and q is a QD term for $H_\omega(\mathfrak{A}, q)$. Thus by 3.5(ii), $(\mathfrak{A}, q)$ has a weak meet term function with respect to the element 1. Combining this with the premise (ii) it is straightforward to show that, for each pair $a, b \in A$, there exists a $c \in A$ such that $\Theta_{\mathfrak{L}}(c, 1) = \Theta_{\mathfrak{L}}(a, b)$. (Compare the proof of 3.6.) Define Δ such that $\Theta_{\mathfrak{L}}(a \Delta b, 1) = \Theta_{\mathfrak{L}}(a, b)$ for all $a, b \in A$, and, in addition, $\Theta_{\mathfrak{L}}(a, b) = \Theta_{\mathfrak{L}}(c, d)$ implies $a \Delta b = c \Delta d$. In order to prove the theorem it is enough, in view of 3.6, to show that each Φ of $\mathfrak{L}$ has the substitution property with respect to of Δ, and hence that $\mathfrak{L} = \mathfrak{Co}(\mathfrak{A}, q, \Delta, 1)$.

Assume Φ is a relation of $\mathfrak{L}$, $a \equiv_\Phi a'$, and $b \equiv_\Phi b'$. Suppose first of all that

$$\Theta_{\mathfrak{L}}(a \Delta b, a' \Delta b') \subseteq \Theta_{\mathfrak{L}}(a, b) \cap \Theta_{\mathfrak{L}}(a', b'). \tag{21}$$

Then $a' \Delta b' \equiv 1$ $(\Theta_{\mathfrak{L}}(a, b))$ since $\Theta_{\mathfrak{L}}(a, b) = \Theta_{\mathfrak{L}}(a \Delta b, 1)$. Thus $\Theta_{\mathfrak{L}}(a', b') \subseteq \Theta_{\mathfrak{L}}(a, b)$, and so by symmetry $\Theta_{\mathfrak{L}}(a', b') = \Theta_{\mathfrak{L}}(a, b)$. Hence by hypothesis $a \Delta b = a' \Delta b'$ and, *a fortiori*, $a \Delta b \equiv a' \Delta b' (\Phi)$. So we may assume that (21) fails, i.e., since $\mathfrak{L}$ is a chain, that either $\Theta_{\mathfrak{L}}(a, b) \subset \Theta_{\mathfrak{L}}(a\Delta b, a'\Delta b')$ or $\Theta_{\mathfrak{L}}(a', b') \subset \Theta_{\mathfrak{L}}(a\Delta b, a'\Delta b')$. Without loss of generality we assume the former. Thus, since $\Theta_{\mathfrak{L}}(a', b') \subseteq \Theta_{\mathfrak{L}}(a, b) \vee \Phi$, and hence $a\Delta b \equiv 1 \equiv a'\Delta b'$ $(\Theta(a, b) \vee \Phi)$, we get $\Theta_{\mathfrak{L}}(a\Delta b, a'\Delta b') \subseteq \Theta_{\mathfrak{L}}(a, b) \vee \Phi$. Therefore, since $\mathfrak{L}$ is a chain and $\Theta_{\mathfrak{L}}(a\Delta b, a'\Delta b') \nsubseteq \Theta_{\mathfrak{L}}(a, b)$, we must have $\Theta_{\mathfrak{L}}(a\Delta b, a'\Delta b') \subseteq \Phi$. This completes the proof.

Finally, as an application of the last theorem, we construct an algebra $\mathfrak{A}$ that generates a congruence-permutable WBSO variety (at the $\mathscr{C}_3$-level of the Brouwerian hierarchy in fact) in which the Gödel equivalence operation is not definable in terms of the QD term and the constant 1 (see the remarks following 3.7).

Let $A = \{0, 1, 2, 3, 4\}$, and let Φ be the equivalence relation on A corresponding to the partition $\{\{0\}, \{1, 2\}, \{3, 4\}\}$. Then $\{\Phi, \mathrm{I}, A \times A\}$ forms the universe of a complete sublattice of equivalence relations that satisfies conditions 4.6(i)–(ii). Thus there are operations q and Δ on A such that $Co\langle A, q, \Delta, 1\rangle = \{\Phi, I, A \times A\}$, and $\langle A, q, \Delta, 1\rangle$ generates a congruence-permutable WBSO variety. Since $\mathfrak{Co}\langle A, q, \Delta, 1\rangle$ is a chain and q is a QD term satisfying 3.3(iii)(8), q must be the $(\mathfrak{Co}\langle A, q, \Delta, 1\rangle)$-normal transform on A by 4.2. Consequently the set $\{1, 3, 4\}$ is closed under any term function constructed from q and 1. But clearly $3\Delta 4$ must take the value 2, and the same is true of any Gödel equivalence operation on $\langle A, q, \Delta, 1\rangle$.

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