

ON EIGENVALUES OF DIFFERENTIABLE POSITIVE DEFINITE KERNELS

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If a positive definite kernel $K(x, y)$ has the p th order partial derivative $(\partial^p/\partial y^p)K(x, y)$ continuous on the square $[0, 1]^2$, we show that the eigenvalues of the integral operator generated by $K(x, y)$ are asymptotically $o(1/n^{p+1})$. We also obtain the anticipated asymptotic estimate when $(\partial^p/\partial y^p)K(x, y)$ satisfies further a Lipschitz condition in y of order $0 < \alpha \leq 1$. These results, which extend some classical estimates of I. Fredholm and H. Weyl under the additional positive definiteness assumption, are based on two interesting inequalities of K. Fan.

1. INTRODUCTION

Let $K(x, y)$ belong to $L^2[0, 1]^2$ such that $K(x, y) = \overline{K(y, x)}$ almost everywhere on the square $[0, 1]^2$. Then the integral operator with the kernel $K(x, y)$ defined on the Hilbert space $L^2[0, 1]$ by

$$Kf(x) = \int_0^1 K(x, y)f(y) dy$$

is compact Hermitian, and so has a sequence $\{\lambda_n(K)\}$ of real eigenvalues which are as usual arranged in the order of decreasing modulus and counted according to multiplicities. We assume further that

$$\int_0^1 \int_0^1 K(x, y)f(x)\overline{f(y)} dx dy \geq 0$$

for $f \in L^2[0, 1]$ so that K is also positive definite. Extending a classical result of Weyl [11] it has been shown that if $r \geq 0$ is an integer and the symmetric derivative

$$K_{rr}(x, y) \equiv \frac{\partial^{2r}}{\partial x^r \partial y^r} K(x, y)$$

exists and is either continuous or has partial derivative with respect to one of the variables continuous on $[0, 1]^2$, then

$$(1) \quad \lambda_n(K) = o(1/n^{p+1}) \quad \text{as } n \rightarrow \infty,$$

where $p = 2r$ or $p = 2r + 1$ according to whether the former or the latter is the case (see Ha [9] and Reade [10]). Moreover, if the symmetric derivative $K_{rr}(x, y)$ itself or its partial derivative in y satisfies further a Lipschitz condition in y of order $0 < \alpha \leq 1$, then

$$(2) \quad \lambda_n(K) = O(1/n^{p+\alpha+1}) \quad \text{as } n \rightarrow \infty$$

(see Cochran-Lukas [2] and also Fredholm [7]). We recall that a kernel $H(x, y)$ is said to satisfy a Lipschitz condition in y of order $0 < \alpha \leq 1$ if

$$|H(z, x) - H(z, y)| \leq A(z)|x - y|^\alpha \quad \text{for } x, y, z \in [0, 1],$$

where $A \in L^2[0, 1]$.

In this paper we obtain further results along those lines and prove that if $p \geq 1$ is an integer and a positive definite Hermitian kernel $K(x, y)$ has the p th order partial derivative $(\partial^p/\partial y^p)K(x, y)$ continuous on $[0, 1]^2$, then the estimate (1) remains valid. Moreover, we show that if $(\partial^p/\partial y^p)K(x, y)$ satisfies further a Lipschitz condition in y of order $0 < \alpha \leq 1$, then the estimate (2) is also true. These results, which complement those cited above, are based on two interesting inequalities of Fan [5, 6].

2. TWO INEQUALITIES OF K. FAN

We assume throughout this section that $K(x, y)$ is a positive definite Hermitian kernel continuous on $[0, 1]^2$. We fix an arbitrarily chosen integer $N \geq 1$. For $1 \leq n \leq N$ let $I_n = [\frac{n-1}{N}, \frac{n}{N}]$ and let

$$\mathbb{I}_n(x) = \begin{cases} 1 & \text{if } x \in I_n \\ 0 & \text{otherwise.} \end{cases}$$

We define

$$K_N(x, y) = \sum_{n=1}^N \mathbb{I}_n(x)K(x, y)\mathbb{I}_n(y)$$

which is clearly positive definite Hermitian.

We first prove the following majorization (3) of the eigenvalues of K_N by that of K whose matrix version is obtained in Fan [6].

Lemma 1. *Let K and K_N be given as above. Then for any integer $m \geq 1$*

$$(3) \quad \sum_{j=1}^m \lambda_j(K_N) \leq \sum_{j=1}^m \lambda_j(K).$$

Proof. For each integer $n \geq 1$ we divide the square $I_i \times I_j$ for $1 \leq i, j \leq N$ into n^2 small squares of equal size. We then define two kernels $K^{(n)}(x, y)$ and $K_N^{(n)}(x, y)$ which have constant values on each small square that equal, respectively, to the values of $K(x, y)$ and $K_N(x, y)$ at the center of the small square. It follows from Fan's majorization theorem for Hermitian matrices cited above (see also [1, p. 50]) that for $n \geq 1$

$$(4) \quad \sum_{j=1}^m \lambda_j(K_N^{(n)}) \leq \sum_{j=1}^m \lambda_j(K^{(n)}).$$

We observe that the eigenvalues of $K^{(n)}$ and $K_N^{(n)}$ as integral operators and as matrices differ from each other by the same weight factor for the same n , and so (4) holds in either case. Clearly $\{K^{(n)}\}$ converges to K and $\{K_N^{(n)}\}$ converges to K_N in the operator norm as $n \rightarrow \infty$, and so (3) is proved by applying Lemma 5 in [4, p. 1091].

Lemma 2. *Let K and K_N be given as above. Then for any integer $m \geq 1$*

$$(5) \quad \sum_{n=m+1}^{\infty} \lambda_n(K) \leq \sum_{n=m+1}^{\infty} \lambda_n(K_N).$$

Proof. By the well-known Mercer's theorem (see also [8, p.115]) both K and K_N are operators of trace class and

$$\begin{aligned} \sum_{n=1}^{\infty} \lambda_n(K) &= \int_0^1 K(x, x) dx \\ &= \int_0^1 K_N(x, x) dx = \sum_{n=1}^{\infty} \lambda_n(K_N). \end{aligned}$$

Thus (5) follows from (3).

Now we invoke another inequality of Fan [5] to obtain a formula which is useful for the asymptotic estimates of $\{\lambda_n(K)\}$.

Lemma 3. *Let $r \geq 0$ be an integer, $B \geq 0$. If $H(x, y)$ is a Hermitian kernel which differs from $K_N(x, y)$ by a degenerate kernel of rank $\leq r$ and satisfies*

$$\lambda_n(H) \geq -B$$

for $n \geq 1$, then

$$(6) \quad \sum_{n=N+r+1}^{\infty} \lambda_n(K) \leq \sum_{n=1}^N N \int_{I_n} \int_{I_n} [H(x, x) - H(x, y)] dx dy + 2rB.$$

Proof. Clearly

$$(7) \quad \lambda_{n+r}(K_N) \leq |\lambda_n(H)|$$

for $n \geq 1$ (see [8, p. 29]). Moreover, H is an operator of trace class with at most r negative eigenvalues (see [2, Lemma 1]) and so

$$(8) \quad \sum_{n=1}^{\infty} |\lambda_n(H)| - 2rB \leq \sum_{n=1}^{\infty} \lambda_n(H) = \int_0^1 H(x, x) dx.$$

The inequality of Fan cited above (see also Lemma 4.1 in [8, p. 47]) implies in the present situation that for any orthonormal family $\{\phi_1, \dots, \phi_N\}$ in $L^2[0, 1]$

$$\sum_{n=1}^N |\lambda_n(H)| \geq \sum_{n=1}^N \int_0^1 \int_0^1 H(x, y) \phi_n(x) \overline{\phi_n(y)} dx dy.$$

Choosing in particular $\phi_n = \sqrt{N} \mathbb{I}_n$ for $1 \leq n \leq N$ we have

$$(9) \quad \sum_{n=1}^N |\lambda_n(H)| \geq \sum_{n=1}^N N \int_{I_n} \int_{I_n} H(x, y) dx dy.$$

From (7), (8) and (9) it follows that

$$(10) \quad \begin{aligned} \sum_{n=N+r+1}^{\infty} \lambda_n(K_N) &\leq \sum_{n=N+1}^{\infty} |\lambda_n(H)| \\ &\leq \int_0^1 H(x, x) dx - \sum_{n=1}^N N \int_{I_n} \int_{I_n} H(x, y) dx dy + 2rB \\ &\leq \sum_{n=1}^N N \int_{I_n} \int_{I_n} [H(x, x) - H(x, y)] dx dy + 2rB. \end{aligned}$$

Finally we combine (5) and (10) to obtain (6).

The formula (10) is of some interest of its own right. We observe that when $r = 0$ it reduces to

$$\sum_{n=N+1}^{\infty} \lambda_n(K_N) \leq \sum_{n=1}^N N \int_{I_n} \int_{I_n} [K(x, x) - K(x, y)] dx dy$$

which improves the one obtained in Reade [10].

3. THE MAIN RESULTS

We shall continue with the notations given in §2, and write for short

$$K^{(j)}(x, y) \equiv \frac{\partial^j}{\partial y^j} K(x, y)$$

for $0 \leq j \leq p$.

Theorem 1. *If $K(x, y)$ is a positive definite Hermitian kernel such that the partial derivative $\frac{\partial^p}{\partial y^p} K(x, y)$ exists and is continuous on $[0, 1]^2$, and satisfies a Lipschitz condition in y of order $0 < \alpha \leq 1$, that is,*

$$\left| \frac{\partial^p}{\partial y^p} K(z, x) - \frac{\partial^p}{\partial y^p} K(z, y) \right| \leq A(z)|x - y|^\alpha \quad \text{for } x, y, z \in [0, 1],$$

where $A \in L^2[0, 1]$, then

$$(11) \quad \lambda_n(K) = O(1/n^{p+\alpha+1}) \quad \text{as } n \rightarrow \infty.$$

Proof. We fix $N \geq 1$, denote the center of I_n by c_n and define

$$(12) \quad H(x, y) = K_N(x, y) - \frac{1}{2} \sum_{j=0}^p \left[K^{(j)}(x, c_n)(y - c_n)^j + \overline{K^{(j)}(y, c_n)}(x - c_n)^j \right] / j!$$

for $x, y \in I_n$, $1 \leq n \leq N$ and $H(x, y) = 0$ otherwise. Clearly $H(x, y)$ is a Hermitian kernel and for $1 \leq n \leq N$

$$(13) \quad \begin{aligned} |H(x, y)| &\leq |K^{(p)}(x, \xi) - K^{(p)}(x, c_n)| |y - c_n|^p / 2p! \\ &\quad + |K^{(p)}(y, \eta) - K^{(p)}(y, c_n)| |x - c_n|^p / 2p! \\ &\leq \frac{1}{2p!} [A(x) + A(y)] N^{-(p+\alpha)} \end{aligned}$$

for $x, y \in I_n$, where ξ, η lie between y and c_n , x and c_n , respectively. Thus we also have

$$\left| \int_0^1 \int_0^1 H(x, y) f(x) \overline{f(y)} dx dy \right| \leq C_0 N^{-(p+\alpha)} \int_0^1 |f(x)|^2 dx$$

for $f \in L^2[0, 1]$, here and in what follows C_0 and C denote positive constants independent of the choice of N , and so

$$(14) \quad |\lambda_n(H)| \leq C_0 N^{-(p+\alpha)}$$

for $n \geq 1$. Hence H satisfies the hypothesis of Lemma 3 with $r = 2p + 2$ and $B = C_0 N^{-(p+\alpha)}$. By (6)

$$\sum_{n=N+2p+3}^{\infty} \lambda_n(K) \leq C N^{-(p+\alpha)}.$$

Since $N \geq 1$ is arbitrary, (11) follows.

Theorem 2. If $K(x, y)$ is a positive definite Hermitian kernel such that the partial derivative $\frac{\partial^p}{\partial y^p} K(x, y)$ exists and is continuous on $[0, 1]^2$, then

$$(15) \quad \lambda_n(K) = o(1/n^{p+1}) \quad \text{as } n \rightarrow \infty.$$

Proof. For a given $\epsilon > 0$ we choose $N \geq 1$ so large that

$$|K^{(p)}(z, x) - K^{(p)}(z, y)|/p! \leq \epsilon$$

for $x, y, z \in I_n$, $1 \leq n \leq N$. We define a Hermitian kernel $H(x, y)$ as in (12). Then it follows analogously to (13) and (14) that

$$|H(x, y)| \leq \frac{\epsilon}{N^p}$$

for $x, y \in I_n$, $1 \leq n \leq N$, and

$$|\lambda_n(H)| \leq \frac{\epsilon}{N^p}$$

for $n \geq 1$. Hence H satisfies the hypothesis of Lemma 3 with $r = 2p + 2$ and $B = \epsilon N^{-p}$. Using (6) again

$$\sum_{n=N+2p+3}^{\infty} \lambda_n(K) \leq \epsilon CN^{-p}$$

for $N \geq 1$ large enough. Since $\epsilon > 0$ is arbitrary, (15) follows.

As noted in Cochran-Lukas [2] and Reade [10], the estimates (11) and (15) would remain valid if the Hermitian kernel $K(x, y)$ is assumed to have at most finitely many negative eigenvalues instead of positive definiteness, since the positive part of K satisfies the hypothesis of Theorem 1 or 2 whenever K does itself.

4. AN EXAMPLE

We consider a selfadjoint eigenvalue problem

$$(16) \quad \begin{cases} L\psi = \mu\psi \\ B\psi = 0 \end{cases}$$

with homogeneous boundary condition on $[0, 1]$, where L is an m th order ordinary differential operator with sufficiently smooth coefficients (see [3], Chap. 7). If μ is not an eigenvalue of (16), a Hermitian kernel $G(x, y)$ known as the Green's function of the problem (16) can be constructed such that the eigenvalues of (16) are precisely the reciprocals of that of the kernel $G(x, y)$. It is known that the partial derivative $\frac{\partial^{m-2}}{\partial y^{m-2}} G(x, y)$ is continuous on

$[0, 1]^2$. But $\frac{\partial^{m-1}}{\partial y^{m-1}}G(x, y)$ is continuous on each of the triangles $x \leq y$ and $x \geq y$ and has jump discontinuities along the diagonal $x = y$. Thus $\frac{\partial^{m-1}}{\partial y^{m-1}}G(x, y)$ is bounded on $[0, 1]^2$. Keeping the previous notations we have by the mean value theorem for $N \geq 1$

$$|G^{(m-2)}(z, x) - G^{(m-2)}(z, y)| \leq CN^{-1}$$

for $x, y, z \in I_n$, $1 \leq n \leq N$, where as before $C > 0$ is a constant independent of the choice of N . If we assume further that all but a finite number of the eigenvalues of G are of the same sign, then by Theorem 1 the eigenvalues of the kernel $G(x, y)$ are asymptotically $O(1/n^m)$ which gives also an asymptotic estimate for the eigenvalues of the problem (16). It is easy to show by examples that this result, though somewhat rough, is the best possible in terms of the powers of n .

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1991 *Mathematics Subject Classification*. 45P05, 47B38.

Submitted: January 27, 1998