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SINGULAR INTEGRAL OPERATORS IN A WEIGHTED L^2-SPACE

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Cauchy singular integral operators are characterized as operators in a weighted L^2-space. The integral operator arises from a singular integral equation with variable coefficients. An appropriate weight function associated with the singular integral operator is constructed, and the set of polynomials orthogonal with respect to this weight function is defined. The action of the operator on polynomial sets is studied, and the definition of the operator is extended to a weighted L^2-space. In this space, the operator is shown to be bounded, and, in some cases, isometric. Formulas are developed for the composition of the singular integral operator and its one sided inverse.

0. INTRODUCTION

We consider the singular integral equation

$$a(x)\phi(x) + \frac{1}{\pi}\int_{-1}^{1} \frac{M(x,t)\phi(t)}{t-x}dt = f(x), \quad -1 < x < 1, \tag{0.1}$$

given, for example, in [M](Section 106). In this equation, a(x), M(x,t), and f(x) are known functions, and $\phi(x)$ is the unknown function. The left hand side of (0.1) can be decomposed, and the equation rewritten as

$$a(x)\phi(x) + \frac{1}{\pi}\int_{-1}^{1} \frac{b(t)\phi(t)}{t-x}dt + \int_{-1}^{1} k(x,t)\phi(t)dt = f(x), \quad -1 < x < 1, \tag{0.2}$$

where

$$b(t) = M(t,t)$$

and

$$k(x,t) = \frac{1}{\pi}\frac{M(x,t) - M(t,t)}{t-x}.$$

The integral in (0.1) and the first integral in (0.2) are understood to be Cauchy principal value integrals.

When a and b are constant functions, it is well known (cf.,[Er]) that a singular integral operator associated with the first two terms of (0.2) (the *dominant* part of the equation) maps a certain set of Jacobi polynomials to another set of Jacobi polynomials. This fact has been exploited in developing techniques for the numerical solution of (0.2) in this case.

In [E] and [W], it was shown that a similar mapping result holds in the case when a(x) and b(x) are non-constant functions. When the appropriate weight function is considered, it can be shown that the singular integral operator associated with the dominant part of (0.2) maps one set of orthogonal polynomials to another set of orthogonal polynomials. However, both [E] and [W] make certain assumptions about the coefficient function b (due to its role in the construction of the weight function). In the present paper, we construct a weight function and the associated orthogonal polynomial set, with the only assumptions on b(x) being Hölder continuity (the usual assumption in the analytical theory) and that it change sign at no more than finitely many points in (-1,1). The mapping result holds in this case as well.

In Section 5, the definition of the singular integral operator is extended to a weighted L^2-space and some operator theoretic results are given. These results are generalizations of those given in [W]. It is hoped that this theory will aid in the development of numerical methods for the variable coefficient equation (0.2), as was done for special cases in [W].

1. THE COEFFICIENT FUNCTIONS

In equation (0.2), we assume that a,b, and k are real valued and, following [M], we assume also that a(x) and b(x) are Hölder continuous in [-1,1]. We require also that $a^2(x) + b^2(x)$ does not vanish for $x \in [-1,1]$. With this condition in mind, we assume without loss of generality that

$$a^2(x) + b^2(x) = 1, \quad x \in [-1,1].$$

Solutions to (0.2) are sought in the space H* of functions Hölder continuous in every closed subinterval of (-1,1) and of the form

$$\frac{\phi^*(x)}{|x - c|^{\alpha}}, \quad 0 \le \alpha < 1$$

near an end $c = \pm 1$, where ϕ^* is Hölder continuous in $[-1,1]$. The real valued function f is assumed to be in H^* as well.

We assume also that the function $b(x)$ changes sign at no more than finitely many points $t_1 < t_2 < \ldots < t_N$ in $(-1,1)$. This assumption is necessary because of the role played by $b(x)$ in the construction of weight functions for orthogonal polynomial sequences. Thus, the function

$$g(t) = \frac{b(t)}{(t - t_1)\ldots(t - t_N)}$$

is of one sign in $(-1,1)$. Furthermore, it follows from the Hölder continuity of b that g is integrable in $[-1,1]$. In fact, for t sufficiently close to t_j, we have

$$|g(t)| < C|t - t_j|^{\mu-1}$$

for each $j = 1,\ldots,N$, where μ $(0 < \mu \le 1)$ is the Hölder index of b.

Thus, we can write

$$b(t) = p(t)g(t) \tag{1.1}$$

where $g(t)$ is integrable and of one sign in $(-1,1)$ and $p(t)$ is a monic (lead coefficient equal to one) polynomial of exact degree $N \ge 0$, all of whose zeros are simple and lie in $(-1,1)$. Note also that g is bounded near the ends ± 1, since these points are not considered as candidates for $t_1,\ldots,t_N$.

2. ANALYTICAL SOLUTION OF THE DOMINANT EQUATION

Define the operator U on H^* by

$$U\phi(x) \equiv a(x)\phi(x) + \frac{p(x)}{\pi}\int_{-1}^{1} \frac{g(t)\phi(t)}{t - x}\,dt \tag{2.1}$$

where p and g are defined by (1.1). The domain of definition of U will eventually be extended to a weighted L^2-space. We now consider the so called *dominant equation*

$$U\phi(x) = f(x) \tag{2.2}$$

where, as in the previous section, f is a known real valued function in the class H^*. Define also the sectionally holomorphic function

$$\Psi(z) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{g(t)\phi(t)}{t - z}\, dt. \tag{2.3}$$

Applying the Plemelj formulas to (2.3), one can formulate the equation

$$(a + ib)\Psi^{+} - (a - ib)\Psi^{-} = g\cdot f. \tag{2.4}$$

Equation (2.4) can then be written as a nonhomogeneous Hilbert problem:

$$\Psi^{+}(x) = G(x)\Psi^{-}(x) + \frac{g(x)}{a(x) + ib(x)} f(x), \tag{2.5}$$

where

$$G(x) = \frac{a(x) - ib(x)}{a(x) + ib(x)}.$$

Note that G depends only on the Hölder continuous functions a and b, and, in fact, is the same G one would obtain when analyzing the dominant equation associated with the usual decomposition of (0.1). Thus, the solution X(z) of the homogeneous Hilbert problem

$$X^{+}(x) = G(x)X^{-}(x)$$

is also the same as that obtained when analyzing the usual dominant equation. The index κ for our problem is also the same (see [M, p.232] for a discussion of the determination of the index).

A solution of (2.5) vanishing at infinity has the form

$$\Psi(z) = \frac{X(z)}{2\pi i} \int_{-1}^{1} \frac{g(t)}{a(t) + ib(t)} \frac{f(t)}{X^{+}(t)(t - z)}\, dt + X(z)P(z). \tag{2.6}$$

Here, P(z) is a polynomial of degree not greater than κ-1. We define the *fundamental function* Z(x) by

$$Z(x) = (a(x) + ib(x))X^{+}(x). \tag{2.7}$$

Once again, this is the same as the fundamental function associated with the usual dominant equation, and so has the usual properties of being real valued, nonvanishing in (-1,1), and bounded in each closed subinterval of (-1,1). The function 1/Z has the same properties.

From (2.3) and (2.6), using the Plemelj formulas and recalling the relationship (1.1) among b, g, and p, one obtains

$$\phi(x) = a(x)f(x) - \frac{p(x)Z(x)}{\pi} \int_{-1}^{1} \frac{g(t)f(t)}{Z(t)(t - x)}\, dt + Z(x)p(x)\tilde{P}(x) \tag{2.8}$$

as a solution of the dominant equation (2.2). Here, $\tilde{P}$ is an arbitrary polynomial of degree not greater than κ-1 (if $\kappa-1 < 0$, then $\tilde{P} \equiv 0$).

Define the operator V on H* by

$$V\psi(x) \equiv a(x)\psi(x) - \frac{p(x)Z(x)}{\pi} \int_{-1}^{1} \frac{g(t)\psi(t)}{Z(t)(t-x)}\,dt. \qquad (2.9)$$

It then follows that

(i) if $\kappa \geqslant 0$ then all solutions of (2.2) are given by

$$\phi(x) = Vf(x) + Z(x)p(x)\pi_{\kappa-1}(x) \qquad (2.10)$$

where $\pi_{\kappa-1}$ is an arbitrary polynomial of degree $\leqslant \kappa-1$ ($\pi_{-1} \equiv 0$).

(ii) if $\kappa < 0$ then a solution of (2.2) exists if and only if the conditions

$$\int_{-1}^{1} \frac{g(t)}{Z(t)} t^j f(t)\,dt = 0, \; j = 0,1,\dots,-\kappa-1 \qquad (2.11)$$

are satisfied. In this case, the unique solution is given by

$$\phi(x) = Vf(x).$$

(See [M], Sections 107 and 108.)

3. THE WEIGHT FUNCTIONS

We shall be interested in the action of the operators U and V on certain sequences of orthogonal polynomials. As weight functions for these orthogonal polynomials, we take gZ and g/Z. Recall that g is integrable and of one sign in (-1,1), and bounded near the ends ±1. The functions Z and 1/Z are integrable and nonvanishing in (-1,1) and bounded in each closed subinterval of (-1,1). Thus, the products gZ and g/Z are integrable and of one sign in (-1,1). The standard theory of orthogonal polynomials (cf., [S]) therefore applies to polynomials orthogonal with respect to these functions as weight functions.

In this section, we shall examine some properties of these weight functions. Throughout, p(x) is the monic polynomial of degree $N \geqslant 0$ defined by (1.1).

LEMMA 3.1 *If* $\kappa > 0$ *then*

$$U(Z\cdot p\cdot \pi_{\kappa-1}) = 0$$

where $\pi_{\kappa-1}$ *is any polynomial of degree not greater than* $\kappa-1$.

PROOF: This follows immediately from equation (2.10) with $f \equiv 0$. In particular, solutions of $U\phi = 0$ are given by $\phi = Z\cdot p\cdot \pi_{\kappa-1}$. □

LEMMA 3.2 *If* $\kappa < 0$ *then*

$$V(p\cdot\pi_{-\kappa-1}) = 0$$

where $\pi_{-\kappa-1}$ *is any polynomial of degree not greater than* $-\kappa-1$.

PROOF: This result is obtained by considering the homogeneous equation $V\psi = 0$. An analysis similar to that done in Section 2 shows that this equation has solutions $\psi = p\cdot\pi_{-\kappa-1}$.□

Denote by $L^2(g/Z)$ and $L^2(gZ)$ the spaces of functions defined and measurable in $[-1,1]$ for which, respectively, the quantity

$$\|\phi\|_{g/Z} \equiv \left(\int_{-1}^{1} \phi^2(t)\,|g(t)/Z(t)|\,dt\right)^{1/2} \tag{3.1}$$

or

$$\|\phi\|_{gZ} \equiv \left(\int_{-1}^{1} \phi^2(t)\,|g(t)Z(t)|\,dt\right)^{1/2}$$

is finite. We denote the corresponding inner products by $(\cdot,\cdot)_{g/Z}$ and $(\cdot,\cdot)_{gZ}$.

The following theorem establishes a relationship between the index κ and N, the number of times b changes sign in $(-1,1)$.

THEOREM 3.1 *When* $\kappa > 1$,

$$(p,t^m)_{gZ} = 0 \tag{3.2}$$

for $m = 0,1,\dots,\kappa-2$. *When* $\kappa < -1$,

$$(p,t^m)_{g/Z} = 0 \tag{3.3}$$

for $m = 0,1,\dots,-\kappa-2$. *It follows that we must have*

$$|\kappa| \leqslant N + 1 \tag{3.4}$$

ie., the index can be no larger in absolute value than $N+1$, *where* $N = \deg(p)$ *is the number of times b changes sign in* $(-1,1)$. *Equation (3.4) holds for all possible values of the index, including* -1, 0, *and* 1.

PROOF: Suppose $\kappa > 1$. We have

$$U(Z(t)p(t)t^{\kappa-1})(x) = x^{\kappa-1}U(Z(t)p(t))(x) + \frac{p(x)}{\pi}\int_{-1}^{1} g(t)Z(t)p(t)r(x,t)\,dt \tag{3.5}$$

where $r(x,t) = (t^{\kappa-1} - x^{\kappa-1})/(t-x)$

$$= x^{\kappa-2} + x^{\kappa-3}t + \dots + xt^{\kappa-3} + t^{\kappa-2}.$$

From Lemma 3.1, the left side of (3.5) is zero, and so is

$U(Z(t)p(t))(x)$. Thus, we get

$$0 = \sum_{m=0}^{\kappa-2} \frac{p(x)x^{\kappa-2-m}}{\pi} (p,t^m)_{gZ}$$

for all x in $[-1,1]$. The coefficient of each power of x in this polynomial must be 0, ie., (3.2) holds for $m = 0,1,\ldots,\kappa-2$.

Equation (3.3) is proved similarly when $\kappa < -1$ by applying Lemma 3.2 to $V(p(t)t^{-\kappa-1})(x)$. Since p cannot be orthogonal to itself in either inner product, (3.2) implies $\kappa-2 \leqslant N-1$, and (3.3) implies $-\kappa-2 \leqslant N-1$. In either case, (3.4) follows. The inequality is trivially true for $\kappa = -1,0$, or 1. □

We now introduce the orthogonal polynomial sequences related to the weight functions mentioned above. Denote by $\{P_n\}_{n=0}^{\infty}$ the sequence of monic polynomials orthogonal with respect to the weight function gZ, and by $\{Q_n\}_{n=0}^{\infty}$ the sequence of monic polynomials orthogonal with respect to g/Z. Since we are on the finite interval $[-1,1]$, $\{P_n\}$ forms a basis for $L^2(gZ)$ and $\{Q_n\}$ is a basis for $L^2(g/Z)$ ([S], Theorem 3.1.5). It is not difficult to show that the set of functions $\{P_n \cdot Z\}_{n=0}^{\infty}$ is also a basis for $L^2(g/Z)$ ([W]).

4. U AND V AS OPERATORS ON POLYNOMIALS

We first consider the action of U and V on the basis sets $\{P_nZ\}$ and $\{Q_n\}$. Later, we will extend these operators to all of $L^2(g/Z)$. Throughout this section, we adopt the convention that if n is a negative integer, then a polynomial of "degree n" is identically zero.

LEMMA 4.1 $U(pP_nZ)(x)$ *is a monic polynomial of degree* $N+n-\kappa$ *of the form* $p(x)h_{n-\kappa}(x)$, *where* $h_{n-\kappa}(x)$ *is a monic polynomial of degree* $n-\kappa$.

PROOF: The monic orthogonal polynomials $\{P_n\}$ satisfy a three term recurrence relation of the form

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= (x-\alpha_0)P_0(x) \\ P_{n+1}(x) &= (x-\alpha_n)P_n(x) - \beta_n P_{n-1}(x) \qquad (4.1) \\ & n = 1,2,\ldots \end{aligned}$$

where α_n, $n \geqslant 0$, are real (possibly zero) constants and β_n, $n \geqslant 1$, are positive constants.

Note that

$$U(tp(t)P_n(t)Z(t))(x) = xU(pP_nZ)(x) + \frac{p(x)}{\pi}(p,P_n)_{gZ}. \tag{4.2}$$

Combining (4.1) and (4.2) we obtain a nonhomogeneous recurrence relation for the functions $U(pP_nZ)$:

$$\begin{aligned} U(pP_1Z)(x) &= (x - \alpha_0)U(pP_0Z)(x) + \frac{p(x)}{\pi}(p,P_0)_{gZ} \\ U(pP_{n+1}Z)(x) &= (x - \alpha_n)U(pP_nZ)(x) - \beta_nU(pP_{n-1}Z)(x) \\ &\quad + \frac{p(x)}{\pi}(p,P_n)_{gZ} \\ &\quad n = 1,2,\ldots . \end{aligned} \tag{4.3}$$

In the case when $\kappa \geqslant 1$, we can apply a result of Dow and Elliott ([D], Theorem 3.1) to obtain

$$\frac{1}{\pi}\int_{-1}^{1} \frac{g(t)p(t)P_\kappa(t)Z(t)}{t - x}\,dt = -a(x)P_\kappa(x)Z(x) + 1,$$

where we have used the fact that $g(x)p(x) = b(x)$. Thus,

$$U(pP_\kappa Z)(x) = p(x). \tag{4.4}$$

From Lemma 3.1, we have $U(pP_nZ) = 0$ for $n = 0,1,\ldots,\kappa-1$. Substituting these values, along with (4.4), into (4.3), we obtain that $U(pP_nZ)$ is a monic polynomial of degree $N+n-\kappa$, with $p(x)$ as a factor, for $n = \kappa,\kappa+1,\ldots$.

When $\kappa \leqslant 0$, we use the same result of Dow and Elliott to show that

$$U(pP_0Z)(x) = p(x)h_{-\kappa}(x), \tag{4.5}$$

where $h_{-\kappa}(x)$ is a monic polynomial of degree $-\kappa$. Combining (4.5) and (4.3) we get that $U(pP_nZ)$ is a monic polynomial of degree $N+n-\kappa$ with $p(x)$ as a factor for $n \geqslant 0$. □

THEOREM 4.1 $U(P_nZ)(x)$, *$n \geqslant 0$, is a polynomial of exact degree $\max(n-\kappa, N-1-n)$. If $n-\kappa > N-1-n$, this polynomial is monic.*

PROOF: $U(P_nZ)$ can be computed as

$$U(P_nZ)(x) = a(x)P_n(x)Z(x) + \frac{1}{\pi}\int_{-1}^{1} \frac{p(t)P_n(t)Z(t)g(t)}{t - x}\,dt + I_Z(P_n)(x) \tag{4.6}$$

where, for a polynomial h, we define

$$I_Z(h)(x) \equiv \frac{1}{\pi}\int_{-1}^{1} \frac{p(x) - p(t)}{t - x} h(t)Z(t)g(t)dt. \quad (4.7)$$

Using the fact that p is a monic polynomial of degree N, we can write

$$I_Z(P_n)(x) = -\frac{1}{\pi}\sum_{j=0}^{N-1} x^{N-1-j}(a_j, P_n)_{gZ} \quad (4.8)$$

where $a_j(t)$ is a monic polynomial of degree j (if N = 0, I_Z is identically zero). For j < n, the terms in this sum are zero by orthogonality. The first nonzero term corresponds to j = n, so that $I_Z(P_n)$ is a polynomial of degree at most N-1-n. The coefficient of x^{N-1-n} is $-\frac{1}{\pi}(a_n, P_n)_{gZ}$, which is nonzero, so that $I_Z(P_n)$ in fact has exact degree N-1-n.

The second term on the right side of (4.6) can be written as

$$\frac{1}{p(x)}\{U(pP_nZ)(x) - a(x)p(x)P_n(x)Z(x)\}.$$

Thus, using Lemma 4.1, we can write (4.6) as

$$U(P_nZ)(x) = h_{n-\kappa}(x) + I_Z(P_n)(x), \quad (4.9)$$

where $h_{n-\kappa}$ is a monic polynomial of degree $n-\kappa$. When $n-\kappa \neq N-1-n$, the theorem follows immediately by comparing the two polynomials on the right side of (4.9). If $n-\kappa = N-1-n$, $U(P_nZ)$ is still a polynomial of *exact* degree $n-\kappa = N-1-n$. This follows from results to be established later. □

Theorem 4.1 completely characterizes the action of U on the set of functions $\{P_nZ\}_{n=0}^{\infty}$, which forms a basis for $L^2(g/Z)$. We now determine the action of V on the set of polynomials $\{Q_n\}_{n=0}^{\infty}$ which also forms a basis for $L^2(g/Z)$.

LEMMA 4.2 $V(pQ_n)(x)$ *is a function of the form* $p(x)Z(x)r_{n+\kappa}(x)$, *where* $r_{n+\kappa}$ *is a monic polynomial of degree* $n+\kappa$.

PROOF: The proof is similar to that of Lemma 4.1. The functions $V(pQ_n)$ are shown to satisfy a nonhomogeneous three term recurrence relation, using the recurrence relation satisfied by the orthogonal polynomials $\{Q_n\}$. When $\kappa \leq -1$, Lemma 3.2 implies $V(pQ_n) = 0$ for $n = 0,1,\ldots,-\kappa-1$. A result of Dow and Elliott [D, Theorem 3.2] shows that $V(pQ_{-\kappa})(x) = p(x)Z(x)$, and the theorem follows. When $\kappa \geq 0$, the same result of Dow and Elliott gives $V(pQ_0)(x) = p(x)Z(x)r_\kappa(x)$, where $r_\kappa(x)$ is a monic polynomial of degree κ. The lemma follows by combining this with the recurrence relation for $V(pQ_n)$. □

THEOREM 4.2 *$Z^{-1}(x)V(Q_n)(x)$, $n \geq 0$, is a polynomial of exact degree $\max(n+\kappa, N-1-n)$. If $n+\kappa > N-1-n$, this polynomial is monic.*

PROOF: The proof easily obtained by mimicing the proof of Theorem 4.1, using Lemma 4.2. □

Denote by Π the space of all polynomials defined on $[-1,1]$, and by ΠZ the space of functions of the form $\phi(x)Z(x)$, where $\phi \in \Pi$. Note that Π and ΠZ are both dense subspaces of $L^2(g/Z)$.

THEOREM 4.3 *Let ϕ and ψ be elements of Π. Then*

$$(U(\phi Z),\psi)_{g/Z} = (\phi Z,\tfrac{1}{p}V(p\psi))_{g/Z} \tag{4.10}$$

and

$$(\phi Z,V(\psi))_{g/Z} = (\tfrac{1}{p}U(p\phi Z),\psi)_{g/Z}. \tag{4.11}$$

PROOF: By Theorem 4.1, $U(\phi Z) \in \Pi$ and by Theorem 4.2, $V(\psi) \in \Pi Z$. Also, by Lemma 4.1, $\frac{1}{p}U(p\phi Z)$ is an element of Π and by Lemma 4.2, $\frac{1}{p}V(p\psi)$ is an element of ΠZ. Thus, all terms in (4.10) and (4.11) are finite quantities.

We can compute

$$(U(\phi Z),\psi)_{g/Z} =$$

$$\int_{-1}^{1}\{a(x)\phi(x)Z(x) + \frac{p(x)}{\pi}\int_{-1}^{1} \frac{g(t)\phi(t)Z(t)}{t-x}\,dt\}\psi(x)(g(x)/Z(x))\,dx$$

$$= \int_{-1}^{1}a(x)\psi(x)\phi(x)Z(x)(g(x)/Z(x))\,dx + \frac{1}{\pi}\int_{-1}^{1}\psi(x)p(x)(g(x)/Z(x))\int_{-1}^{1}\frac{g(t)\phi(t)Z(t)}{t-x}\,dt\,dx. \tag{4.12}$$

At this point, we would like to change the order of integration in the second term on the right. The general theory for the interchange of a regular integral and a singular integral given, for example, in [T]requires that the functions involved belong to certain L^p-spaces for $p > 1$. It is not clear that we have both gZ and g/Z belonging to L^p-spaces for $p > 1$. However, we do have the following lemma:

LEMMA 4.3 *Suppose* f(x) *has the form*

$$f(x) = (1 - x)^{\alpha}(1 + x)^{\beta}\theta(x) \tag{4.13}$$

where $\theta(x)$ *is bounded in* $[-1,1]$ *and* $-1 < \alpha,\beta < 1$, *and suppose* h(x) *has the form*

$$h(x) = (1 - x)^{-\alpha}(1 + x)^{-\beta}\xi(x) \tag{4.14}$$

where $\xi(x)$ *is integrable in* $[-1,1]$ *with isolated singularities at* $t_1 < t_2 < \ldots < t_N$ *in* $(-1,1)$ *and*

$$|\xi(t)| \leqslant c|t - t_j|^{\eta}, \tag{4.15}$$

$-1 < \eta$, *in a neighborhood of* t_j. *Then*

$$\int_{-1}^{1} f(x) \int_{-1}^{1} \frac{h(t)}{t - x}\, dtdx = \int_{-1}^{1} h(t) \int_{-1}^{1} \frac{f(x)}{t - x}\, dxdt. \tag{4.16}$$

We postpone the rather technical proof of this lemma until we have completed the proof of Theorem 4.3. Resuming the proof of the theorem, we can write

$$Z(x) = (1 - x)^{\alpha}(1 + x)^{\beta}w(x)$$

where w is a nonvanishing, bounded function in $[-1,1]$ and $-1 < \alpha,\beta < 1$ ([M, §107]). Thus, if we put

$$f(x) = \psi(x)p(x)g(x)/Z(x)$$
$$= \psi(x)b(x)/Z(x)$$

and

$$h(x) = \phi(x)g(x)Z(x)$$

then f and h satisfy the hypothesis of the lemma and we can change the order of integration in (4.12). Thus, the right side of (4.12) can be written as

$$\int_{-1}^{1} a(x)\psi(x)\phi(x)Z(x)(g(x)/Z(x))dx$$
$$+ \frac{1}{\pi}\int_{-1}^{1} \phi(t)g(t)Z(t)\int_{-1}^{1} \frac{\psi(x)p(x)g(x)}{Z(x)(t - x)}\, dxdt$$
$$= \int_{-1}^{1}\{a(x)\psi(x) - \frac{Z(x)}{\pi}\int_{-1}^{1} \frac{\psi(t)p(t)g(t)}{Z(t)(t - x)}\, dt\}\phi(x)Z(x)(g(x)/Z(x))dx$$
$$= (\phi Z, \frac{1}{p}V(p\psi))_{g/Z}.$$

This proves (4.10). Equation (4.11) is proved similarly with another application of Lemm 4.3. □

PROOF OF LEMMA 4.3: By hypothesis, unbounded behavior of the function f, defined by (4.13), can occur only at the endpoints ±1. Let us first consider the case $-1 < \alpha,\beta < 0$, so that f is unbounded near +1 and -1, and is bounded in every closed subinterval of (-1,1). The function h, defined by (4.14), is bounded at both endpoints ±1. Let δ_1, δ_2 be points in (-1,1) such that $-1 < \delta_1 < t_1$ and $t_N < \delta_2 < 1$. The right side of (4.16) can be written as

$$\int_{-1}^{1} f(x) \int_{-1}^{\delta_1} \frac{h(t)}{t - x}\, dtdx + \int_{-1}^{1} f(x) \int_{\delta_1}^{\delta_2} \frac{h(t)}{t - x}\, dtdx + \int_{-1}^{1} f(x) \int_{\delta_2}^{1} \frac{h(t)}{t - x}\, dtdx. \tag{4.17}$$

In the first term of this expression, h(t) can be replaced by a function $h_1(t)$ which agrees with h on $(-1,\delta_1)$ and is 0 on $(\delta_1,1)$. The domain of integration of the second integral in this term can then be extended to (-1,1).

Let $\gamma = \min(\alpha,\beta)$, so that $-1 < \gamma < 0$. Then $f \in L^q[-1,1]$ provided $\gamma q > -1$. Put $q = (1 - \frac{1}{\gamma})/2$ and $p = q/(q - 1)$. Then $1 < q < -\frac{1}{\gamma}$ and $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. We have $f \in L^q[-1,1]$ and $h_1 \in L^p[-1,1]$ and we can apply the results of Tricomi [T, Formula 11] to write this term as

$$\int_{-1}^{1} h_1(t) \int_{-1}^{1} \frac{f(x)}{t - x}\, dxdt = \int_{-1}^{\delta_1} h(t) \int_{-1}^{1} \frac{f(x)}{t - x}\, dxdt.$$

The third term in (4.17) can be treated in exactly the same way, since h is bounded in $[\delta_2,1]$.

To treat the second term in (4.17), choose points $\delta_1' \in (-1,\delta_1)$ and $\delta_2' \in (\delta_2,1)$ and write that term as

$$\int_{-1}^{\delta_1'} f(x) \int_{\delta_1}^{\delta_2} \frac{h(t)}{t - x}\, dtdx + \int_{\delta_1'}^{\delta_2'} f(x) \int_{\delta_1}^{\delta_2} \frac{h(t)}{t - x}\, dtdx + \int_{\delta_2'}^{1} f(x) \int_{\delta_1}^{\delta_2} \frac{h(t)}{t - x}\, dtdx. \tag{4.18}$$

Note that in the first and third terms of (4.18), the 'singular' integral is not singular, since x is not in the domain of integration (δ_1,δ_2) of t. Thus, by Fubini's Theorem, the order of integration for these ordinary integrals can be changed.

We can write the middle term of (4.18) as

$$\int_{-1}^{1} f_1(x)\int_{-1}^{1} \frac{h_2(t)}{t - x}\, dtdx$$

where f_1 is zero outside (δ_1', δ_2') and agrees with f in that interval, and similarly, h_2 is zero outside (δ_1, δ_2) and equals h in that interval. Observe that f_1 is bounded in the entire interval $[-1,1]$ and thus belongs to $L^p[-1,1]$ for any $p > 0$. From the condition (4.15) it is clear that $h_2 \in L^q[-1,1]$ provided $q\eta > -1$. If $\eta \geq 0$, then this condition is no restriction on q, and we can choose any $q > 1$ and $p = q/(q-1)$ so that $h_2 \in L^q[-1,1]$ and $f_1 \in L^p[-1,1]$. If $-1 < \eta < 0$, then put $q = (1 - \frac{1}{\eta})/2$ and $p = q/(q-1)$ so that, again, we have $h_2 \in L^q[-1,1]$ and $f_1 \in L^p[-1,1]$. In either case, we can again apply Tricomi's results to interchange the order of integration.

This establishes (4.16) in the case $-1 < \alpha,\beta < 0$. The cases $0 \leq \alpha,\beta < 1$ (h unbounded at both endpoints) and when α and β lie on different sides of 0 (h unbounded at one endpoint) are treated similarly. In the first of these cases, it is necessary to consider the minimum of η, α and β in order to find an appropriate q. In the second case, one must consider the minimum of η and one of α,β. □

The next lemma is a generalization of results given in [W; equations (3.2.3) and (3.2.8)]. It is interesting in its own right (an example is given in [W]), and it will also be used in the proof of the theorem which follows.

LEMMA 4.4 *When* $\kappa \geq 1$,

$$(p, t^{\kappa-1})_{gZ} = \pi \tag{4.19}$$

and when $\kappa \leq -1$,

$$(p, t^{-\kappa-1})_{g/Z} = -\pi. \tag{4.20}$$

PROOF: We have

$$U(Zpt^{\kappa})(x) = x^{\kappa}U(Zp)(x) + \frac{p(x)}{\pi}\int_{-1}^{1} \frac{g(t)Z(t)p(t)(t^{\kappa} - x^{\kappa})}{t - x}dt$$

$$= x^{\kappa}U(Zp)(x) + \frac{p(x)}{\pi}\{\sum_{j=0}^{\kappa-1} x^j\int_{-1}^{1} g(t)Z(t)p(t)t^{\kappa-1-j}dt\}.$$

By Lemma 3.1, the first term in the above expression is 0, and by Theorem 3.1, all terms in the summation, except that corresponding to j = 0, are 0 (if $\kappa = 1$, there is only this one term). Thus obtain

$$U(Zpt^{\kappa})(x) = \frac{p(x)}{\pi}(p,t^{\kappa-1})_{gZ}. \tag{4.21}$$

From Lemma 3.1 and the fact that P_κ is monic, it follows that the left side of (4.21) equals $U(ZpP_\kappa)(x)$, and, from (4.4), this is $p(x)$. Equation (4.19) follows.

Equation (4.20) is obtained similarly by looking at $V(pt^{-\kappa})(x)$, and applying Lemma 3.2, Theorem 3.1, and the fact that $V(pQ_{-\kappa})(x) = p(x)Z(x)$. □

THEOREM 4.4 *In the space* Π *we have*

$$U\circ V = \begin{cases} I & \text{if } \kappa \geqslant 0 \quad (4.22) \\ I + Y & \text{if } \kappa \leqslant -1 \quad (4.23) \end{cases}$$

and in the space ΠZ *we have*

$$V\circ U = \begin{cases} I & \text{if } \kappa \leqslant 0 \quad (4.24) \\ I - W & \text{if } \kappa \geqslant 1 \quad (4.25) \end{cases}$$

where I *is the identity operator and* Y *and* W *are linear operators such that*

$$Y(Q_n)(x) = 0 \quad \text{for } n \geqslant -\kappa \quad (\kappa \leqslant -1)$$

and

$$W(P_nZ)(x) = 0 \quad \text{for } n \geqslant \kappa \quad (\kappa \geqslant 1).$$

REMARK: Explicit representations of the operators Y and W will be obtained in the proof of this theorem.

PROOF: let $f \in \Pi$. The equation

$$U\phi = f \tag{4.26}$$

has the unique (in H*) solution $\phi = Vf \in \Pi Z$ when $\kappa = 0$. Thus $U\circ Vf = f$ for all $f \in \Pi$ in this case. When $\kappa \geqslant 1$, (4.26) has a family of solutions

$$\phi = Vf + \pi_{\kappa-1}\cdot Z\cdot p \tag{4.27}$$

where $\pi_{\kappa-1}$ is an arbitrary polynomial of degree $\leqslant \kappa-1$. Applying U to both sides of (4.27), and recalling that $\pi_{\kappa-1}\cdot Z\cdot p$ is in the null space of U when $\kappa \geqslant 1$ (Lemma 3.1), we obtain $U\circ Vf = f$ in this case also. This proves (4.22).

Now suppose ϕ is an element of ΠZ and $\kappa \geqslant 1$. Put

$f = U\phi$. Then ϕ can be written as in (4.27), and substituting for f, we obtain

$$\phi = V \circ U\phi + \pi_{\kappa-1} \cdot Z \cdot p. \tag{4.28}$$

The polynomial $\pi_{\kappa-1}$ depends on ϕ, and we can write

$$\pi_{\kappa-1}(x) = c_{\kappa-1}(\phi)x^{\kappa-1} + c_{\kappa-2}(\phi)x^{\kappa-2} + \ldots + c_0(\phi),$$

where we have indicated the dependence of the coefficients on ϕ. We now substitute this expression for $\pi_{\kappa-1}$ in (4.28) and take the inner product of both sides with Z to obtain:

$$(Z,\phi)_{g/Z} = (Z, V \circ U\phi)_{g/Z} + c_{\kappa-1}(\phi)(Z, Zpx^{\kappa-1})_{g/Z} + \ldots + c_0(\phi)(Z, Zp)_{g/Z}. \tag{4.29}$$

Since $\phi \in \Pi Z$, $U\phi$ is a polynomial and we can apply Theorem 4.3 to write the first term on the right as $(\frac{1}{p}U(pZ), U\phi)_{g/Z}$. But $U(pZ)$ is 0 for $\kappa \geqslant 1$, so this term vanishes. Also,

$$(Z, Zpx^j)_{g/Z} = (p, x^j)_{gZ}$$

and this is 0 for $j \leqslant \kappa-2$, and (by Lemma 4.4) equals π for $j = \kappa-1$. So from (4.29) we can solve for $c_{\kappa-1}(\phi)$ as:

$$c_{\kappa-1}(\phi) = \frac{1}{\pi}(Z,\phi)_{g/Z}.$$

Similarly, by taking inner products of both sides of (4.28) with xZ we can solve for $c_{\kappa-2}(\phi)$ as:

$$c_{\kappa-2}(\phi) = \frac{1}{\pi}(xZ,\phi)_{g/Z} - \frac{1}{\pi}c_{\kappa-1}(\phi)(x^{\kappa}, p)_{gZ}.$$

This process can be continued to find each of the $c_j(\phi)$'s in terms of the preceding $c_k(\phi)$'s $(k > j)$.

Alternatively, one can consider the $\kappa \times 1$ vector

$$c(\phi) \equiv (c_0(\phi), \ldots, c_{\kappa-1}(\phi))$$

as a solution of the linear system

$$c(\phi) = -\frac{1}{\pi} B \cdot c(\phi) + \frac{1}{\pi} r(\phi)$$

or

$$(I + \frac{1}{\pi} B)c(\phi) = \frac{1}{\pi} r(\phi) \tag{4.30}$$

where I is the $\kappa \times \kappa$ identity matrix and B is the $\kappa \times \kappa$ upper triangular matrix

$$\begin{bmatrix} 0 & (x^\kappa,p)_{gZ} & (x^{\kappa+1},p)_{gZ} & \cdots & (x^{2\kappa-2},p)_{gZ} \\ 0 & 0 & (x^\kappa,p)_{gZ} & \cdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ \vdots & \vdots & & & (x^{\kappa+1},p)_{gZ} \\ \vdots & \vdots & & & (x^\kappa,p)_{gZ} \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

and $r(\phi)$ is the $\kappa\times 1$ vector

$$((x^{\kappa-1}Z,\phi)_{g/Z},(x^{\kappa-2}Z,\phi)_{g/Z},\ldots,(Z,\phi)_{g/Z}).$$

Note that $I + \frac{1}{\pi}B$ is nonsingular (with determinant 1) so that the system (4.30) always has a unique solution for $c(\phi)$. Thus, we can write

$$c(\phi) = (I + \frac{1}{\pi}B)^{-1}r(\phi)$$

and we can define the operator W as

$$W(\phi)(x) \equiv \frac{1}{\pi} Z(x)p(x)\{(I + \frac{1}{\pi}B)^{-1}r(\phi)\cdot\tilde{x}_{\kappa-1}\} \tag{4.31}$$

where $\tilde{x}_{\kappa-1} \equiv (1,x,x^2,\ldots,x^{\kappa-1})$ is a $\kappa\times 1$ vector, and the dot $(\cdot)$ represents the ordinary scalar product for Euclidean vectors.

Substituting $W(\phi)$ for $\pi_{\kappa-1}\cdot Z\cdot p$ in (4.28), we obtain (4.25). Note that $r(P_nZ) = 0$ for $n \geqslant \kappa$ so that $W(P_nZ) = 0$ in this case.

When $f \in \Pi Z$, the equation

$$V\psi = f$$

has family of solutions

$$\psi = Uf + \pi_{-\kappa-1}\cdot p \tag{4.32}$$

when $\kappa \leqslant 0$, where $\pi_{-\kappa-1}$ is an arbitrary polynomial of degree $\leqslant -\kappa-1$. Applying V to both sides of (4.32), and using Lemma 3.2, we obtain $f = V\circ Uf$. This proves (4.24).

Finally, if we consider $\psi \in \Pi$ in the case $\kappa \leqslant -1$, and put $f = V\psi$, then ψ can be written as in (4.32). The polynomial $\pi_{-\kappa-1}$ depends on ψ, and we can write (4.32) as

$$\psi = U\circ V\psi + p\cdot(c_{-\kappa-1}(\psi)x^{-\kappa-1} + \ldots + c_0(\psi)). \tag{4.33}$$

Proceeding as before, we can solve for the constants $c_j(\psi)$ by taking $(\cdot,\cdot)_{g/Z}$ inner products of both sides of (4.33) with appropriate functions, in this case $1, x, \ldots, x^{-\kappa-1}$, and using Theorem 4.3. This leads to a linear system

$$c(\psi) = \frac{1}{\pi} A \cdot c(\psi) - \frac{1}{\pi} s(\psi) \qquad (4.34)$$

for the $(-\kappa)\times 1$ vector

$$c(\psi) \equiv (c_0(\psi), \ldots, c_{-\kappa-1}(\psi)),$$

where A is the $(-\kappa)\times(-\kappa)$ matrix

$$\begin{bmatrix} 0 & (x^{-\kappa},p)_{g/Z} & (x^{-\kappa+1},p)_{g/Z} & \cdots\cdots & (x^{-2\kappa-2},p)_{g/Z} \\ 0 & 0 & (x^{-\kappa},p)_{g/Z} & \cdots\cdots & \cdot \\ \vdots & \vdots & & \ddots & \vdots \\ \vdots & \vdots & & \ddots & (x^{-\kappa+1},p)_{g/Z} \\ \vdots & \vdots & & & (x^{-\kappa},p)_{g/Z} \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

and $s(\psi)$ is the $(-\kappa)\times 1$ vector

$$((x^{-\kappa-1},\psi)_{g/Z}, (x^{-\kappa-2},\psi)_{g/Z}, \ldots, (1,\psi)_{g/Z}).$$

The matrix $(I - \frac{1}{\pi} A)$ is nonsingular (with determinant 1), so the solution of (4.34) can be written as

$$c(\psi) = -\frac{1}{\pi} (I - \frac{1}{\pi} A)^{-1} s(\psi),$$

and we define the operator Y as

$$Y(\psi)(x) \equiv \frac{1}{\pi} p(x)\{(I - \frac{1}{\pi} A)^{-1} s(\psi) \cdot \tilde{x}_{-\kappa-1}\} \qquad (4.35)$$

where the dot product and the $(-\kappa)\times 1$ vector $\tilde{x}_{-\kappa-1}$ have the same meaning as in (4.31). Equation (4.33) can then be written as

$$\psi = U \circ V\psi - Y(\psi),$$

from which we obtain (4.23). This completes the proof of Theorem 4.4. □

The following theorem establishes the fundamental mapping properties of U and V on the basis sets $\{P_n Z\}$ and $\{Q_n\}$, respectively. In the statement of the theorem, the notation $\lceil r \rceil$ represents the smallest integer which is greater than or equal to the quantity r.

THEOREM 4.5 *For* $j \geq \left\lceil \frac{N+\kappa}{2} \right\rceil$,

$$U(P_j Z)(x) = Q_{j-\kappa}(x) \tag{4.36}$$

and for $j \geq \left\lceil \frac{N-\kappa}{2} \right\rceil$,

$$V(Q_j)(x) = P_{j+\kappa}(x) Z(x). \tag{4.37}$$

PROOF: When $j \geq \left\lceil \frac{N+\kappa}{2} \right\rceil$, we have $j-\kappa \geq N-j$, so that $j-\kappa > N-j-1$ in this case. Thus, from Theorem 4.1, $U(P_j Z)$ is a monic polynomial of degree $j-\kappa$. We can write

$$U(P_j Z)(x) = Q_{j-\kappa}(x) + b_{j-\kappa-1} Q_{j-\kappa-1}(x) + \ldots + b_0 Q_0(x)$$

for certain constants $b_0, \ldots, b_{j-\kappa-1}$. But, for $k = 0, \ldots, j-\kappa-1$, we have

$$\begin{aligned} b_k \|Q_k\|^2_{g/Z} &= (U(P_j Z), Q_k)_{g/Z} \\ &= (P_j Z, \tfrac{1}{p} V(pQ_k))_{g/Z} \quad \text{(Theorem 4.3)} \\ &= (P_j Z, \tfrac{1}{p} \cdot p \cdot r_{k+\kappa} \cdot Z)_{g/Z} \\ &\qquad \text{(Lemma 4.2, where } r_{k+\kappa} \text{ is a monic polynomial of degree } k+\kappa) \\ &= (P_j, r_{k+\kappa})_{gZ} \\ &= 0. \end{aligned}$$

The last equality follows from the orthogonality of the P_j's, since $k+\kappa \leq j-\kappa-1+\kappa = j-1$. Therefore $b_k = 0$ for $k = 0, \ldots, j-\kappa-1$. This proves (4.36).

Equation (4.37) can be proved by following a similar argument, using Theorem 4.2. Alternatively, one can apply V to both sides of (4.36) and employ Theorem 4.4. □

Theorem 4.5 represents a slight improvement over Theorems 3.2 and 3.4 of [E], in so much as our equations (4.36) and (4.37) are shown to hold for a larger subset of the bases $\{P_n Z\}$ and $\{Q_n\}$. In [E], these equations are proved for $j \geq \max\{N, \kappa\}$. Since $|\kappa| \leq N+1$, it is easy to show that $\left\lceil \frac{N+\kappa}{2} \right\rceil$ and $\left\lceil \frac{N-\kappa}{2} \right\rceil$ are always less than or equal to $\max\{N, \kappa\}$. If $|\kappa|$ is much smaller that N, then $\left\lceil \frac{N+\kappa}{2} \right\rceil$ and $\left\lceil \frac{N-\kappa}{2} \right\rceil$ are significantly smaller than $\max\{N, \kappa\}$.

COROLLARY 4.1 *For* $j \geq \left\lceil \frac{N+\kappa}{2} \right\rceil$ *we have*

$$\|P_j\|_{gZ} = \|Q_{j-\kappa}\|_{g/Z}.$$

PROOF: Applying Theorem 4.5, for $j \geq \left\lceil \frac{N+\kappa}{2} \right\rceil$, we have

$$\begin{aligned} \|Q_{j-\kappa}\|^2_{g/Z} &= (U(P_jZ), Q_{j-\kappa})_{g/Z} \\ &= (P_jZ, \tfrac{1}{p}V(pQ_{j-\kappa}))_{g/Z} \quad \text{(Theorem 4.3)} \\ &= (P_jZ, \tfrac{1}{p}\cdot p\cdot r_jZ)_{g/Z} \\ &\quad \text{(Lemma 4.4, where } r_j \text{ is a monic polynomial of degree } j) \\ &= (P_j, r_j)_{gZ} \\ &= \|P_j\|^2_{gZ} \ . \end{aligned}$$

The final equality follows from the orthogonality of the P_j's and the fact that both P_j and r_j are monic. □

We now define the sequences of orthonormal polynomials $\{p_n\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$ by

$$p_n = P_n/\|P_n\|_{gZ} \tag{4.38}$$

and

$$q_n = Q_n/\|Q_n\|_{g/Z} \ . \tag{4.39}$$

The next result is a generalization of Theorem 3.1.2 of [W]. It follows easily from Theorem 4.5 and Corollary 4.1.

THEOREM 4.6 *For* $j \geq \left\lceil \frac{N+\kappa}{2} \right\rceil$,

$$U(p_jZ)(x) = q_{j-\kappa}(x) \tag{4.40}$$

and for $j \geq \left\lceil \frac{N-\kappa}{2} \right\rceil$,

$$V(q_j)(x) = p_{j+\kappa}(x)Z(x) . \tag{4.41}$$

5. U AND V AS BOUNDED OPERATORS IN $L^2(g/Z)$

We now extend the definitions of the operators U and V to all of $L^2(g/Z)$, and show that these are in fact bounded operators in that space. The set $\{p_nZ\}_{n=0}^{\infty}$, where p_n is defined by (4.38), is an orthonormal basis in $L^2(g/Z)$. Consider the following elements of $L^2(g/Z)$:

$$f = \sum_{j=0}^{\infty} x_jp_jZ \tag{5.1}$$

and

$$f_n = \sum_{j=0}^{n} x_jp_jZ$$

where

$$x_j = (f, p_j Z)_{g/Z}.$$

For each n, $U(f_n)$ is a well defined element of $L^2(g/Z)$. We now define

$$U(f) \equiv \lim_{n\to\infty} U(f_n)$$
$$= \sum_{j=0}^{\infty} x_j U(p_j Z), \quad (5.2)$$

where the limit is understood to be the strong limit in the Hilbert space $L^2(g/Z)$.

To show that the right side of (5.2) in fact defines an element of $L^2(g/Z)$, we show that the sequence $U(f_n)$ is a strongly convergent Cauchy sequence in that space. For $n > m \geqslant \left\lceil \frac{N+\kappa}{2} \right\rceil$, we have

$$\| U(f_n) - U(f_m)\|^2_{g/Z} = \| \sum_{j=m+1}^{n} x_j U(p_j Z)\|^2_{g/Z}$$
$$= \| \sum_{j=m+1}^{n} x_j q_{j-\kappa}\|^2_{g/Z}$$
$$= \sum_{j=m+1}^{n} |x_j|^2.$$

This expression converges to 0 as $m,n \to \infty$, since the series (5.1) converges in $L^2(g/Z)$. Thus, $\{U(f_n)\}$ is a strongly convergent sequence in $L^2(g/Z)$, and (5.2) defines an operator U on all of $L^2(g/Z)$.

In a similar way, one can extend the definition of V to all of $L^2(g/Z)$ by using the orthonormal basis $\{q_n\}_{n=0}^{\infty}$. For the element

$$h = \sum_{j=0}^{\infty} y_j q_j$$

in $L^2(g/Z)$, where

$$y_j = (h, q_j)_{g/Z},$$

define

$$V(h) = \sum_{j=0}^{\infty} y_j V(q_j). \quad (5.3)$$

As above, one can show that the sequence of partial sums is a Cauchy sequence, so that (5.3) defines an element of $L^2(g/Z)$.

THEOREM 5.1 *U and V are bounded operators on* $L^2(g/Z)$.

PROOF: Define a functional Φ_k on $L^2(g/Z)$ by

$$\Phi_k(f) = (Uf, q_k)_{g/Z}.$$

Then, if we put $x_j = (f, p_j Z)_{g/Z}$, we have

$$\Phi_k(f) = \lim_{n\to\infty} \sum_{j=0}^{n} x_j (U(p_j Z), q_k)_{g/Z}$$
$$= \sum_{j=0}^{\eta+k} a_{jk} x_j$$

where

$$a_{jk} = (U(p_j Z), q_k)_{g/Z}$$

and

$$\eta = \left\lceil \frac{N+\kappa}{2} \right\rceil.$$

The final expression for $\Phi_k(f)$ follows from the fact that, for $j > \eta+k$,

$$(U(p_j Z), q_k)_{g/Z} = (q_{j-\kappa}, q_k)_{g/Z}$$
$$= 0$$

since $j-\kappa > \eta+k-\kappa \geqslant k$.

We also define the functional ρ_n, $n = 0,1,2,\ldots,$ by

$$\rho_n(f) \equiv \left(\sum_{k=0}^{n} |\Phi_k(f)|^2 \right)^{1/2}.$$

ρ_n is a convex functional (see [A]) and is bounded for each n:

$$\rho_n(f) = \left(\sum_{k=0}^{n} \left| \sum_{j=0}^{\eta+k} a_{jk} x_j \right|^2 \right)^{1/2}$$
$$= \left(\sum_{k=0}^{n} \left(\sum_{j=0}^{\eta+k} a_{jk}^2 \right) \left(\sum_{j=0}^{\eta+k} x_j^2 \right) \right)^{1/2}$$
$$\leqslant A_n \| f \|_{g/Z}$$

where A_n is a constant which does not depend on f. Also, since

$$\rho_n(f) = \left(\sum_{k=0}^{n} |(Uf, q_k)_{g/Z}|^2 \right)^{1/2}$$
$$\leqslant \| Uf \|_{g/Z},$$

the sequence $\{\rho_n(f)\}_{n=0}^{\infty}$ is bounded for each $f \in L^2(g/Z)$. It now follows from a lemma given in [A] that the functional ρ defined by

$$\rho(f) = \sup \rho_n(f)$$

is a convex continuous functional, ie., we have

$$\rho(f) \leqslant A\|f\|_{g/Z}$$

for some positive constant A. But we also have

$$\begin{aligned}\rho(f) &= \lim_{n\to\infty} \rho_n(f)\\ &= \|Uf\|_{g/Z}.\end{aligned}$$

Thus, U is a bounded operator on $L^2(g/Z)$.

In a similar way, one can show that V is a bounded operator on $L^2(g/Z)$ by considering the linear functionals

$$\Psi_k(h) = (Vh, p_k Z)_{g/Z}$$

and the convex functionals

$$\sigma_n(h) = \left(\sum_{k=0}^{n} |\Psi_k(h)|^2\right)^{1/2}. \quad \square$$

In some cases, U and V turn out to be isometric operators on $L^2(g/Z)$. In fact, if $\kappa = -N$ or $-N-1$, we have

$$\left\lceil\frac{N+\kappa}{2}\right\rceil = 0,$$

so that equation (4.40) of Theorem 4.6 holds for all $j \geqslant 0$. Thus, if we apply U to a typical element f (given by (5.1)) we obtain

$$\begin{aligned}\|Uf\|^2_{g/Z} &= \lim_{n\to\infty} \Big\|\sum_{j=0}^{n} x_j U(p_j Z)\Big\|^2_{g/Z}\\ &= \lim_{n\to\infty} \Big\|\sum_{j=0}^{n} x_j q_{j-\kappa}\Big\|^2_{g/Z}\\ &= \sum_{j=0}^{\infty} |x_j|^2\\ &= \|f\|^2_{g/Z}.\end{aligned}$$

Thus, U is an isometry when $\kappa = -N$ or $-N-1$. Similarly, one can show that V is an isometry on $L^2(g/Z)$ in the case when $\kappa = N$ or $N+1$. Also, in the special case when $N = 0$ and $\kappa = 0$, it is not difficult to show that the range space of both U and V is all of $L^2(g/Z)$, so that these operators are in fact unitary in this case (see [W]).

We now extend the results of Theorems 4.3 and 4.4 to U and V as operators on $L^2(g/Z)$. Let Π_n denote the subspace of polynomials of degree $\leqslant n$, and let $\Pi_n Z$ denote the subspace of functions of the form $\pi(x)Z(x)$, where $\pi \in \Pi_n$.

THEOREM 5.2 *In* $L^2(g/Z)$, *the adjoint of* U *is* $p^{-1}\cdot V\cdot p$ *and the adjoint of* V *is* $p^{-1}\cdot U\cdot p$.

PROOF: For $f,h \in L^2(g/Z)$, there are functions $f_n \in \Pi_n Z$ and $h_n \in \Pi_n$ such that $f_n \to f$ and $h_n \to h$ in $L^2(g/Z)$. We then have

$$\begin{aligned}(U(f),h)_{g/Z} &= \lim_{n\to\infty} (U(f_n),h_n)_{g/Z} \\ &= \lim_{n\to\infty} (f_n,\frac{1}{p} V(ph_n))_{g/Z} \\ &= (f,\frac{1}{p} V(ph))_{g/Z} .\end{aligned}$$

Thus, $p^{-1}Vp$ is the adjoint of U, and the statement for V is proved similarly. □

The following theorem is established in the same way, by considering limits of sequences of functions in Π_n and $\Pi_n Z$, and using Theorem 4.4.

THEOREM 5.3 *In* $L^2(g/Z)$

$$U\circ V = \begin{cases} I & \textit{when } \kappa \geq 0 \\ I + Y & \textit{when } \kappa \leq -1 \end{cases}$$

and

$$V\circ U = \begin{cases} I & \textit{when } \kappa \leq 0 \\ I - W & \textit{when } \kappa \geq 1 \end{cases}$$

where the operators Y *and* W *are as in Theorem 4.4.* □

As our final result, we relate the classical Muskhelishvili notion of the index κ to the more modern concept of the index of an operator as defined, for example, in [TL, p. 253]. The index of an operator A on a Hilbert space X is defined to be

$$\text{Index (A)} \equiv \text{Dimension of kernel (A) - Dimension of (X/Range(A))}.$$

Our Hilbert space is $L^2(g/Z)$. Let $\langle\phi_1,\ldots,\phi_k\rangle$ denote the linear manifold in $L^2(g/Z)$ spanned by the elements $\phi_1,\ldots,\phi_k$. In the case $\kappa \geq 0$, we have, from Lemma 3.1,

$$\text{kernel (U)} = \langle p, p\cdot x, p\cdot x^2,\ldots,p\cdot x^{\kappa-1}\rangle$$

so that the dimension of the kernel of U is κ. Also, for ψ in $L^2(g/Z)$, we have $V\psi \in L^2(g/Z)$, and from Theorem 5.3,

$$U\circ V\psi = \psi$$

in the case $\kappa \geq 0$. Thus,

$$\text{Range } (U) = L^2(g/Z)$$

and the dimension of $(L^2(g/Z)/\text{Range}(U))$ is 0. So we have

$$\text{Index } (U) = \kappa - 0 = \kappa.$$

When $\kappa < 0$, the dimension of the kernel of U is 0. Also, if $\pi_{-\kappa-1}$ is a polynomial of degree $\leqslant -\kappa-1$, we have, for any $\phi \in L^2(g/Z)$,

$$\begin{aligned}(\pi_{-\kappa-1}, U\phi)_{g/Z} &= (\tfrac{1}{p} V(p\pi_{-\kappa-1}), \phi)_{g/Z} \\ &= (0,\phi)_{g/Z} \\ &= 0.\end{aligned}$$

Thus, $\pi_{-\kappa-1}$ is orthogonal to the range of U, and we have

$$\text{Range } (U) \subset L^2(g/Z)/\langle 1,x,\ldots,x^{-\kappa-1}\rangle.$$

Now suppose $\psi \in L^2(g/Z)/\langle 1,x,\ldots,x^{-\kappa-1}\rangle$. Then $Y\psi = 0$, where Y is the operator defined by Theorem 4.4. Thus, we have

$$U\circ V\psi = \psi$$

in this case, so that $\psi \in \text{Range } (U)$. Therefore,

$$\text{Range } (U) = L^2(g/Z)/\langle 1,x,\ldots,x^{-\kappa-1}\rangle,$$

and the dimension of $L^2(g/Z)/\text{Range}(U)$ is $-\kappa$. So, we have

$$\text{Index } (U) = 0 - (-\kappa) = \kappa$$

when $\kappa < 0$. Thus, in every case, the classical index κ agrees with the operator theoretic index of U. One can also compute Index (V) as above. The result is

$$\text{Index } (V) = -\text{Index } (U)$$

which also agrees with classical results. □

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