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BOUNDEDNESS OF OPTIMAL MATRICES IN EXTREMAL MULTIGRAPH AND DIGRAPH PROBLEMS

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We consider multigraphs in which any two vertices are joined by at most q edges, and study the Turán-type problem for a given family of forbidden multigraphs. In the case $q=2$, answering a question of Brown, Erdős and Simonovits, we obtain an explicit upper bound on the size of the matrix generating an asymptotical solution of the problem. In the case $q>2$ we show that some analogous statements do not hold, and so disprove a conjecture of Brown, Erdős and Simonovits.

1. Introduction

Brown, Erdős and Simonovits published a sequence of papers on the solution of Turán-type problems for digraphs and multigraphs (see e.g. [1, 2, 3]). The aim of the present paper is to prove a conjecture of theirs, according to which the size of some matrices generating the asymptotically extremal graph sequences is bounded by an explicit function depending only on the forbidden graph. As they pointed out in [1, p.82], this would imply the possibility of algorithmic solution of digraph extremal problems. Later they gave an algorithmic solution to the digraph extremal problems, avoiding this question [3]. We shall first give an "effective" upper bound on the size of these matrices and then use this result to derive the main results of [3] in a shorter and simpler way. Brown, Erdős and Simonovits also conjectured that most of their results hold even for the case of directed graphs when two vertices are allowed to be joined by more than 1 arc of a given direction. We shall give a construction disproving this in some cases.

We fix a positive integer q and consider finite *multigraphs* (without loops) in which any two vertices are joined by at most q edges. The word "multigraph" will be replaced by "graph". A *simple graph* is a graph in which any two vertices are independent or joined by 1 edge. The number of vertices and the number of edges of a graph G will be denoted by $v(G)$ and $e(G)$, respectively (edges are counted with their multiplicities).

We consider the following general problem called Turán-type problem. Given a family $\mathcal{L}$ of graphs where $\forall L\in\mathcal{L}$: $e(L)>0$ (graphs from $\mathcal{L}$ we call *prohibited*). How many edges can be in a graph with n vertices such does not contain a subgraph

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isomorphic to some $L \in \mathscr{L}$? This maximal number of edges will be denoted by $\mathrm{ex}(n;\mathscr{L})$.

From the averaging argument of Katona, Nemetz and Simonovits [6] it follows that $\mathrm{ex}(n;\mathscr{L})/\binom{n}{2}$ is a monotone non-increasing function of n and there exists the limit

$$\gamma(\mathscr{L}) = \lim_{n\to\infty} \frac{\mathrm{ex}(n;\mathscr{L})}{\binom{n}{2}}.$$

Let $\mathcal{M}$ be the class of symmetric matrices whose elements are non-negative numbers not greater than q. Let $A=(a_{ij})$ be a $(r\times r)$-matrix from $\mathcal{M}$. For any partition $n=x_1+\ldots+x_r$ into nonnegative integers define a graph $A(x_1,\ldots,x_r)$ as follows: n vertices are partitioned into r classes $V_1,\ldots,V_r$ where $|V_i|=x_i$; any pair of distinct vertices $v\in V_i$, $v'\in V_j$ are joined by a_{ij} edges. The class of all graphs $A(x_1,\ldots,x_r)$ (for various n and various partitions) and all their subgraphs will be denoted by $\mathscr{C}(A)$. Using the terminology of [2] we may say that $\mathscr{C}(A)$ is the class of *A-colorable* graphs. If no graph $A(x_1,\ldots,x_r)$ contains subgraphs belonging to $\mathscr{L}$ (i.e. $\mathscr{C}(A)\cap\mathscr{L}=\emptyset$) we call A *admissible* for $\mathscr{L}$. The subclass of matrices which are admissible for a family $\mathscr{L}$ of forbidden subgraphs will be denoted by $\mathscr{A}(\mathscr{L})$.

Unlike [3] we allow diagonal entries to be equal q. However, no matrix having such a value in the diagonal is admissible for $\mathscr{L}\neq\emptyset$.

Following [3] we define the density

$$(1)\quad g(A) = \max\{\mathbf{u}A\mathbf{u}^* \mid \mathbf{u}=(u_1,\ldots,u_r),\ u_1+\ldots+u_r=1,\ u_i\geq 0\ (i=1,\ldots,r)\}$$

where the transpose operation is denoted by $*$. The $(r-1)$-dimensional polytope $u_1+\ldots+u_r=1$, $u_i\geq 0$ $(i=1,\ldots,r)$ in $\mathbb{R}^r$ is called the *standard simplex*. A vector $\mathbf{u}$ at which the maximum in (1) is attained, is called an *optimum vector* of A. The matrix A is *dense* if for any proper principal submatrix A', $g(A')<g(A)$; equivalently (for $r>1$), if any optimum vector belongs to the interior of the standard simplex.

The subclass of $\mathcal{M}$ consisting of dense matrices will be denoted by $\mathscr{D}$.

The following identity explains the significance of the definitions above:

$$(2)\qquad \gamma(\mathscr{L}) = \sup\{g(A)\ :\ A\in\mathscr{A}(\mathscr{L})\} = \sup\{g(A)\ :\ A\in\mathscr{A}(\mathscr{L})\cap\mathscr{D}\}.$$

This identity follows from Theorem 2.6 [9] and from Lemma 6 [3].

A dense matrix A at which the supremum in (2) is attained is called an *optimal matrix* for $\mathscr{L}$. The asymptotic solution of the Turán-type problem for $q=1$ (simple graphs) was obtained in [5]. The general case was considered by Brown, Erdős and Simonovits [1, 2, 3].

Theorem A [1]. *In the case $q=2$ for any $\mathscr{L}$ there exists an optimal matrix.*

Theorem B [2]. *For any $A\in\mathscr{D}$ there exists a finite family $\mathscr{L}$ such that A is the unique optimal matrix for $\mathscr{L}$.*

In section 2 below we shall obtain the explicit upper bound on the size of the dense matrix of Theorem A (see our Theorem 3). In particular, for the case of a finite family $\mathscr{L}$ it answers a question by Brown, Erdős and Simonovits:

Theorem 1. *Let $q=2$, $k=\min\{v(L) : L\in\mathcal{L}\}$ and $k'=\max\{v(L) : L\in\mathcal{L}\}$. Then any optimal matrix for $\mathcal{L}$ has at most $(R(k,k+6)-1)(k'-1)$ rows where $R(k,k+6)$ is the Ramsey number.*

Our Theorem 3 implies Theorem A and the main results of [3], namely, the following theorems:

Theorem C [3]. *In the case $q=2$ for any real γ there exist only finitely many dense matrices A such that $g(A)=\gamma$.*

Theorem D [3]. *In the case $q=2$ the set of attained densities $\{\gamma : \gamma=g(A)\}$ is well ordered (under the usual ordering of the reals).*

Theorem E [3] (Compactness property). *In the case $q=2$ for every infinite family $\mathcal{L}$ there exists a finite subfamily $\mathcal{L}'\subseteq\mathcal{L}$ for which $\gamma(\mathcal{L}')=\gamma(\mathcal{L})$.*

Theorem F [3]. *Let $q=2$ and let a subroutine ("oracle") be given for deciding for the family $\mathcal{L}$ and any dense matrix A, whether of not A is admissible for $\mathcal{L}$. There exists a finite algorithm (independent of $\mathcal{L}$ except that it uses the subroutine) which determines all dense matrices A which are optimal for $\mathcal{L}$.*

To prove Theorem 3 we shall use the fact that dense matrices cannot contain certain principal submatrices (see Theorem 2 and Examples 1–4).

It was conjectured in [3] that Theorems C, D, E, F remain valid for any $q\geq 3$. In section 3 we shall construct counterexamples demonstrating that neither Theorem C, nor some plausible analogues of Theorem 1 can hold for this case.

In section 4 we shall generalize our results to directed graphs.

2. Some properties of dense matrices

Theorem 2. *A matrix A is dense iff*
(a) *A is non-singular, and all components of the vector $\mathbf{e}A^{-1}$ are positive;*
and
(b) *A is of negative type, i.e. $\mathbf{x}A\mathbf{x}^*<0$ holds for any vector $\mathbf{x}$ such that $\mathbf{x}\neq\mathbf{0}$ and $\mathbf{x}\mathbf{e}^*=0$ with $\mathbf{e}=(1,1,\dots,1)$.*

Proof. The necessity of condition (a) was proved in [2, Lemma 2]. We first establish the necessity of condition (b). Suppose, (b) does not hold. Thus these exists a vector $\mathbf{x}$ such that $\mathbf{x}\mathbf{e}^*=0$, $\mathbf{x}\neq\mathbf{0}$, $\mathbf{x}A\mathbf{x}^*\geq 0$. Let $\mathbf{y}$ be an interior point of the standard simplex. Choose $\alpha>0$ and $\beta>0$ such that $\mathbf{y}+\alpha\mathbf{x}$ and $\mathbf{y}-\beta\mathbf{x}$ belong to the boundary of the standard simplex. We have

$$\beta(\mathbf{y}+\alpha\mathbf{x})A(\mathbf{y}+\alpha\mathbf{x})^*+\alpha(\mathbf{y}-\beta\mathbf{x})A(\mathbf{y}-\beta\mathbf{x})^*=$$
$$(\beta+\alpha)\mathbf{y}A\mathbf{y}^*+\alpha\beta(\beta+\alpha)\mathbf{x}A\mathbf{x}^*\geq(\beta+\alpha)\mathbf{y}A\mathbf{y}^*.$$

If $\mathbf{y}$ is an optimal vector then $\mathbf{y}+\alpha\mathbf{x}$ or $\mathbf{y}-\beta\mathbf{x}$ is optimal too. Hence A is not dense.

Now we prove that conditions (a) and (b) are sufficient. Let A satisfy (a) and (b). Put $\gamma=\mathbf{e}A^{-1}\mathbf{e}^*$, $\mathbf{y}=\gamma^{-1}(\mathbf{e}A^{-1})$. So $\mathbf{y}\mathbf{e}^*=1$ and all components of $\mathbf{y}$ are positive. Hence $\mathbf{y}$ belongs to the interior of the standard simplex. We must check that $\mathbf{y}$ is the unique optimal vector for A. Choose an arbitrary point $\mathbf{z} \neq \mathbf{y}$ from the standard simplex. Put $\mathbf{x} = \mathbf{z} - \mathbf{y}$. Since $\mathbf{xe}^* = 0$ and $\mathbf{x} \neq \mathbf{0}$, we get $xAx^* < 0$. Thus

$$\mathbf{z}A\mathbf{z}^* - \mathbf{y}A\mathbf{y}^* = (\mathbf{y} + \mathbf{x})A(\mathbf{y}^* + \mathbf{x}^*) - \mathbf{y}A\mathbf{y}^* = \mathbf{x}(A\mathbf{y}^*) + (\mathbf{y}A)\mathbf{x}^* + \mathbf{x}A\mathbf{x}^* =$$
$$= \gamma^{-1}\mathbf{xe}^* + \gamma^{-1}\mathbf{ex}^* + \mathbf{x}A\mathbf{x}^* = \mathbf{x}A\mathbf{x}^* < 0$$

Remarks. (1) If a matrix A is of negative type then all its principal submatrices are of negative type, too.

(2) Let a matrix A be of negative type. When we replace all diagonal entries of A by 0, we get a matrix of negative type.

(3) It is known [8] that matrix $A = \|a_{ij}\|$ of size r with zero diagonal is of negative type iff there is a linearly independent system of r points $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_r$ in the Hilbert space such that

$$a_{ij} + a_{ji} = \|\mathbf{y}_i - \mathbf{y}_j\|^2 \qquad \text{with} \quad i,\ j = 1, 2, \ldots, r.$$

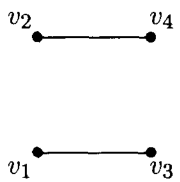
Note that the density $\varrho(A)$ is the squared radius of the $(r-1)$-dimensional sphere containing these points. Condition (a) of Theorem 2 states that the convex hull of $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_r$ contains the center $\mathbf{y}$ of the sphere as an interior point. Then the maximum in (1) is attained at the point $\mathbf{u} = (u_1, u_2, \ldots, u_r)$ such that $\mathbf{y} = u_1\mathbf{y}_1 + u_2\mathbf{y}_2 + \ldots + u_r\mathbf{y}_r$.

Examples. The adjacency matrices of the following graphs are not of negative type.

1. If the graph consists of two independent vertices then its adjacency matrix is the zero matrix.

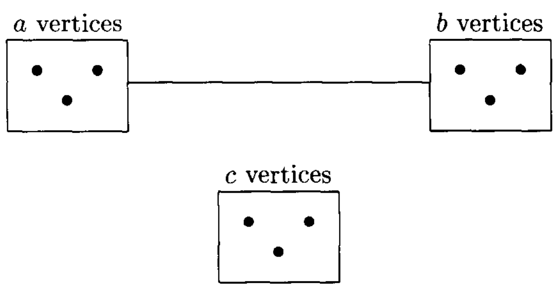
In the graphs described in Examples 2–4, any pair of vertices is joined either by 1 or 2 edges. In the corresponding figures, for every pair of vertices, one edge connecting these vertices is omitted. So if any two vertices are joined in the figure, then they are joined by 2 edges in the graph, else they are joined by only 1 edge in the graph.

2. E_4:



Put $x_1 = x_3 = 1$, $x_2 = x_4 = -1$. Then $\mathbf{x}A\mathbf{x}^* = 0$.

3. $E_{a,b,c}$: (where $c(ab-1) \geq 2ab + a + b$):

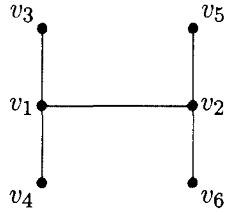


Let each vertex from the group of a vertices have a weight $c(b+1)$, each vertex from the group of b vertices have a weight $c(a+1)$ and each vertex from the group of c vertices have a weight $-(a(b+1)+b(a+1))$. Then

$$\mathbf{x}A\mathbf{x}^* = c(2ab + a + b)(c(ab-1) - (2ab + a + b)) \geq 0.$$

We shall use these graphs only for $a=1$ when we require $c(b-1) \geq 3b+1$. In particular, the last inequality holds with $b \geq 2$, $c \geq 4$, $b+c \geq 9$.

4. E_6:



Put $x_1 = x_3 = x_4 = 1$, $x_2 = x_5 = x_6 = -1$. Then $\mathbf{x}A\mathbf{x}^* = 0$.

Let us call the m-th and n-th rows of a matrix A *equivalent* if $a_{mm} = a_{nn} = 0$, $a_{mn} = a_{nm} = 1$, $a_{im} = a_{in} = a_{mi} = a_{ni}$ for each i where $i \neq m$, $i \neq n$. This relation is transitive and consequently it is an equivalence relation. Hence the set of all rows of A is partitioned into *equivalence classes*.

Theorem 3. *Let $q=2$ and A be a dense matrix not having principal submatrices of size k all off-diagonal entries of which are equal to 2. Then the number of equivalence classes of rows of A is less than the Ramsey number $R(k, \max\{k+3, 9\})$.*

We shall say that vertices v' and v'' of a graph are symmetric if for any other vertex v the number of edges joining v and v' equals the number of edges joining v and v''.

The proof of Theorem 3 is based on the following statement.

Lemma 1. *Let $q=2$ and G be a graph in which any two vertices are joined by 1 or 2 edges and any two symmetric vertices are joined by 2 edges. If G contains no E_4, E_6 and $E_{1,b,c}$ (with $b \geq 2$, $c \geq 4$, $b+c \geq 9$) as induced subgraphs and contains no complete k-vertex subgraphs with double edges then $v(G) < R(k, \max\{k+3, 9\})$.*

Proof. Let S be a maximal subset of vertices in which any two vertices are joined by 1 edge only. Suppose $|S| = s$, $s \geq 9$. Our aim is to show that $s < k+3$.

For any vertex v we denote

$$S_v = \{u \in S \ :\ u \text{ and } v \text{ are joined by 2 edges}\}, \qquad \overline{S}_v = S \setminus S_v.$$

Choose a subset T of vertices such that $T \cap S = \emptyset$, $\forall v, v' \in T$: $S_v \neq S_{v'}$, $\forall v \notin S$ $\exists v' \in T$: $S_v = S_{v'}$. Denote $T_i = \{v \in T : |S_v| = i\}$. If $v \in T_i$ then G contains $E_{1,i,s-i}$ as an induced subgraph. Hence $T_i = \emptyset$ for every $i = 2, 3, \ldots, s-4$.

Suppose v_1, $v_2 \in T$, $S_{v_1} \setminus S_{v_2} \neq \emptyset$, $S_{v_2} \setminus S_{v_1} \neq \emptyset$. Choose $v_3 \in S_{v_1} \setminus S_{v_2}$, $v_4 \in S_{v_2} \setminus S_{v_1}$. Since G does not contain E_4 as an induced subgraph, v_1 and v_2 are joined by 2 edges.

Suppose $|T_1| \geq 2$. Choose a_1, $b_1 \in T_1$. So a_1 and b_1 are joined by 2 edges. Let $S_{b_1} = \{a_2\}$. Since $|S| \geq 9$, there exist c_1, c_2, c_3, c_4, c_5, c_6, c_6, $c_7 \in S \setminus (S_{a_1} \cup S_{b_1})$. The vertices a_1, a_2, b_1, c_1, c_2, c_3, c_4, c_5, c_6, c_7 induce the subgraph $E_{2,1,7}$. Since this subgraph is excluded, $|T_1| \leq 1$.

Let

$$T'_{s-2} = \{v \in T_{s-2} \mid \forall v' \in T_{s-3} \ : \ S_{v'} \not\subset S_v\},$$
$$T'_{s-1} = \{v \in T_{s-1} \mid \forall v' \in T_{s-3} \cup T_{s-2} \ : \ S_{v'} \not\subset S_v\},$$
$$T' = T_{s-3} \cup T'_{s-2} \cup T'_{s-1},$$
$$S' = \bigcup\{\overline{S}_v \mid v \in T_{s-3} \cup T_{s-2} \cup T_{s-1}\},$$
$$S'' = \bigcup\{S_v \mid v \in T_1\}.$$

It follows that

$$S' = \bigcup\{\overline{S}_v \mid v \in T'\},$$
$$|S''| = |T_1| \leq 1.$$

Note that $S_v \setminus S_{v'} \neq \emptyset$ and $S_{v'} \setminus S_v \neq \emptyset$ for any v, $v' \in T'$. Hence any two vertices from T' are joined by 2 edges, Therefore $|T'| \leq k-1$. Note that any two vertices from $S \setminus (S' \cup S'')$ are symmetric. Hence

$$|S \setminus (S' \cup S'')| \leq 1,$$
$$|S'| \geq s - 1 - |S''| \geq s - 2.$$

If $|S'| = s-2$, there exists a vertex $u \in S \setminus S'$ which is connected to any vertex $v \in T'$ by 2 edges. Hence in this case $|T'| \leq k-2$. Thus

$$|T'| \leq k - (s - |S'|) = |S'| + k - s.$$

Suppose v_1, $v_2 \in T$, $|S_{v_1} \setminus S_{v_2}| \geq 2$, $|S_{v_2} \setminus S_{v_1}| \geq 2$. In this case v_1 and v_2 are joined by 2 edges. Choose v_3, $v_4 \in (S_{v_1} \setminus S_{v_2})$, v_5, $v_6 \in (S_{v_2} \setminus S_{v_1})$, and obtain that v_1, v_2, v_3, v_4, v_5, v_6 induce the subgraph E_6. Since this subgraph is excluded, therefore either $|S_{v_1} \setminus S_{v_2}| \leq 1$ or $|S_{v_2} \setminus S_{v_1}| \leq 1$ for any v_1, $v_2 \in T$. In particular,

$$\forall v, v' \in T_{s-3} \ : \ |\overline{S}_v \setminus \overline{S}_{v'}| \geq 2,$$
$$\forall v \in T_{s-3} \ \forall v' \in T'_{s-2} \ : \ |\overline{S}_v \setminus \overline{S}_{v'}| \geq 1,$$
$$\forall v, v' \in T'_{s-2} \ : \ |\overline{S}_v \setminus \overline{S}_{v'}| \geq 1.$$

This implies that

$$\left|\bigcup\{\overline{S}_v \mid v \in T_{s-3} \cup T'_{s-2}\}\right| \leq 2 + |T_{s-3}| + |T'_{s-2}|.$$

Thus

$$|S'| \leq 2 + |T_{s-3}| + |T'_{s-2}| + |T'_{s-1}| = |T'| + 2 \leq |S'| + k - s + 2$$

and we obtain $s \leq k+2$. Hence $|S| < \max\{9, k+3\}$.

So G contains neither an induced complete subgraph of order $\max\{9,k+3\}$ (whose edges have multiplicity 1) nor a complete subgraph of order k whose edges have multiplicity 2. Therefore $v(G) < R\{k, \max\{k+3,9\}\}$. ∎

Proof of Theorem 3. Since $A = a_{ij}$ is dense, one is of negative type. Suppose, there exist m, n such that $a_{mm} + a_{nn} > 0$, $a_{nn} \geq a_{mm}$, $a_{mn} = a_{nm} = 1$, $a_{im} = a_{in} = a_{mi} = a_{ni}$ for each i where $i \neq m$, $i \neq n$. Omitting the m-th row and the m-th column of A we obtain the submatrix A'. It is easy to see that $A \in \mathscr{C}(A')$, hence $g(A) \leq g(A')$. This inequality contradicts the assumption that A is dense. Therefore, if we replace all diagonal entries of A by 0 and obtain a matrix A_1, then any two equivalent rows of A_1 are equivalent in A, too. Thus A and A_1 have identical equivalence classes of rows. Choose one row from each equivalence class of rows of A_1 to obtain a principal submatrix A_2. The number of rows of A_2 is equal to the number of equivalence classes of rows of A. Note that A_2 is the adjacency matrix of a certain graph G. Since any two rows of A_2 are not equivalent, G has no two symmetric vertices joined by 1 edge only. Since A_2 is of negative type (see Remarks 1–2), G does not contain induced subgraphs from Examples 1–4 (in particular, G does not contain a pair of independent vertices). Now it is enough to apply Lemma 1. ∎

Next we demonstrate that Theorems 1, A, C, D, E, F are indeed corollaries of our Theorem 3.

Consider the linearly ordered set $\mathbb{N}_\infty = \{1, 2, \ldots, \infty\}$. Since $\mathbb{N}_\infty$ is well ordered, the partially ordered set $\mathbb{N}_\infty^r = \mathbb{N}_\infty \otimes \mathbb{N}_\infty \otimes \ldots \otimes \mathbb{N}_\infty$ is well ordered too. It is easy to prove (using induction on r) that every antichain in $\mathbb{N}_\infty^r$ is finite.

Consider any matrix B of size r with zero diagonal. Choose any element $\mathbf{n} = (n_1, n_2, \ldots, n_r) \in \mathbb{N}_\infty^r$. For each $i = 1, 2, \ldots, r$ replace the diagonal entry of the i-th row by 1 if $n_i = \infty$, otherwise replace the i-th row (and the i-th column) by n_i equivalent rows (and n_i equivalent columns). Let the obtained matrix be denoted by $B[\mathbf{n}]$.

Theorem 3 implies

Corollary 1. *Let $q = 2$ and A be a dense matrix not having principal submatrices of size k all off-diagonal entries of which are equal to 2. Then A may be represented in the form $A = B[\mathbf{n}]$ where $r(B) \leq R(k, \max\{k+3,9\})$.* ∎

Proof of Theorem 1. The case $k = 2$ is trivial. Suppose, $k \geq 3$. Let $\mathscr{L}$ be finite. It is easy to see that if $B[n_1, n_2, \ldots, n_r]$ is admissible for $\mathscr{L}$ and $\forall L \in \mathscr{L}$: $v(L) \leq n_1$ then $B[\infty, n_2, \ldots, n_r]$ is also admissible for $\mathscr{L}$. Hence we may consider only $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{N}_\infty^r$ such that either $n_i = \infty$ or $n_i < k' = \max\{v(L) : L \in \mathscr{L}\}$ for each $i = 1, 2, \ldots, r$. Therefore, any row of B corresponds to no more than $k' - 1$ rows of $A = B[\mathbf{n}]$. So Corollary 1 implies Theorem 1. ∎

Proof of Theorem C. Put $k = [2/(2-\gamma)] + 1$. If a matrix A has a submatrix of size k whose off-diagonal entries are equal to 2, then $g(A) \geq (2k-2)/k > \gamma$. Hence, according to Corollary 1, if A is dense and $g(A) = \gamma$, then $A = B[\mathbf{n}]$ with $r = r(B) \leq R(k, \max\{k+3,9\})$, $\mathbf{n} \in \mathbb{N}_\infty^r$. The set of such matrices B is finite. For any fixed matrix B, if $\mathbf{n}_1 \leq \mathbf{n}_2$, $\mathbf{n}_1 \neq \mathbf{n}_2$, $\mathbf{n}_1$, $\mathbf{n}_2 \in \mathbb{N}_\infty^r$ and $B[\mathbf{n}_1]$ is dense, then $g(B[\mathbf{n}_1]) <$

$g(B[\mathbf{n}]_2])$. Since any antichain in $\mathbb{N}_\infty^r$ is finite, the set of $\mathbf{n}\in\mathbb{N}_\infty^r$ such that $B[\mathbf{n}]$ is dense and $g(B[\mathbf{n}])=\gamma$, is also finite. ∎

Proof of Theorem D. We must verify that for any subclass $\mathscr{D}'\subseteq\mathscr{D}$ there exists a matrix $A_0\in\mathscr{D}$ with $g(A_0)=\min\{g(A):A\in\mathscr{D}'\}$. If $g(A)=2$ for each $A\in\mathscr{D}'$, then A_0 may be an arbitrary matrix from $\mathscr{D}'$. Suppose, there exists $A_1\in\mathscr{D}'$ with $g(A_1)=\gamma<2$. Put $k=[2/(2-\gamma)]+1$. If a matrix A has a submatrix of size k whose off-diagonal entries are equal to 2, then $g(A)\geq(2k-2)/k>\gamma$. Put $\mathscr{D}''=\{A\in\mathscr{D}:g(A)\leq\gamma\}$. Note that $\mathscr{D}''$ is not empty. According to Corollary 1, every matrix $A\in\mathscr{D}''$ may be represented in the form $A=B[\mathbf{n}]$ with $r=r(B)<R(k,\max\{k+3,9\})$, $\mathbf{n}\in\mathbb{N}_\infty^r$. Note that $\mathbf{n}_1\leq\mathbf{n}_2$ implies $g(B[\mathbf{n}_1])\leq g(B[\mathbf{n}_2])$. Since $\mathbb{N}_\infty^r$ is well ordered, the set of the densities $\Gamma(B)=\{\gamma=g(B[\mathbf{n}]):\mathbf{n}\in\mathbb{N}_\infty^r\}$ is also well ordered. The number of rows of B is bounded, hence the set of such matrices B is finite. Therefore, $\Gamma_k=\cup\{\Gamma(B)\,|\,r(B)<R(k,\max\{k+3,9\})\}$ is well ordered.

Since $\{g(A):A\in\mathscr{D}''\}$ is a non-empty subset of Γ_k, there exists a matrix $A_0\in\mathscr{D}''$ such that

$$g(A_0)=\min\{g(A)\ :\ A\in\mathscr{D}''\}=\min\{g(A)\ :\ A\in\mathscr{D}'\}.$$ ∎

Proof of Theorem E. In fact, Theorem E is equivalent to Theorem D. Here we show that Theorem D implies Theorem E. Since the set of attained densities is well ordered, we may choose $\varepsilon>0$ such that there does not exist a matrix A whose density satisfies $\gamma(\mathscr{L})<g(A)\leq\gamma(\mathscr{L})+\varepsilon$. Choose n such that $\mathrm{ex}(n,\mathscr{L})/\binom{n}{2}\leq\gamma(\mathscr{L})+\varepsilon$. Put $\mathscr{L}'=\{L\in\mathscr{L}:v(L)\leq n\}$. Clearly, $\gamma(\mathscr{L}')\geq\gamma(\mathscr{L})$. It remains to show that $\gamma(\mathscr{L}')\leq\gamma(\mathscr{L})$. Note that $\mathrm{ex}(n;\mathscr{L}')=\mathrm{ex}(n;\mathscr{L})$ and

$$\gamma(\mathscr{L}')\leq\mathrm{ex}(n;\mathscr{L}')/\binom{n}{2}=\mathrm{ex}(n;\mathscr{L})/\binom{n}{2}<\gamma(\mathscr{L})+\varepsilon.$$

Since $g(A)<\gamma(\mathscr{L})+\varepsilon$ implies $g(A)\leq\gamma(\mathscr{L})$, hence (see (2)): $\gamma(\mathscr{L}')\leq\gamma(\mathscr{L})$. ∎

A subset P of a partially ordered set is called a *lower ideal*, if $n\in P$, $n'\leq n$ implies $n'\in P$. A lower ideal P in $\mathbb{N}_\infty^r$ is called closed, if for any component-wise convergent sequence of elements of P, the limit element also belongs to P. To prove Theorem F we need the following.

Lemma 2 [10, Lemma 17]. *The number of the maximal elements of any closed lower ideal in $\mathbb{N}_\infty^r$ is finite. Moreover, there exists a finite procedure to determine all maximal elements of an arbitrary closed lower ideal P in $\mathbb{N}_\infty^r$ which uses "oracle" for deciding whether P contains a given element $\mathbf{n}\in\mathbb{N}_\infty^r$.* ∎

Proof of Theorem F. Let A_k be a matrix of size k whose off-diagonal entries are equal to 2 and the diagonal entries are zeros. Testing A_2, then A_3, A_4 and so on, we find k such that A_k is not addmissible for $\mathscr{L}$. According to Corollary 1, every matrix A which is admissible for $\mathscr{L}$, may be represented in the form $A=B[\mathbf{n}]$ with $r=r(B)<R(k,\max\{k+3,9\})$, $\mathbf{n}\in\mathbb{N}_\infty^r$. The set of these matrices B is finite. For any fixed matrix B, the subset $P_b=\{\mathbf{n}\in\mathbb{N}_\infty^r:B[\mathbf{n}]\in\mathscr{A}(\mathscr{L})\}$ is a lower ideal and the subset Q_B of all its maximal elements may be determined by a finite procedure.

If $\mathbf{n}_1 \in P_B \setminus Q_B$, there exists $\mathbf{n}_2 \in Q_B$ such that $\mathbf{n}_1 \leq \mathbf{n}_2$. If $B[\mathbf{n}_1]$ is dense, then $g(B[\mathbf{n}_1]) \leq g(B[\mathbf{n}_2])$. Put

$$\mathscr{B} = \bigcup \{B[\mathbf{n}] \ : \ \mathbf{n} \in Q_B,\ r(B) < R(k, \max\{k+3, 9\})\}.$$

So $\mathscr{B}$ is finite and can be determined by a finite procedure. According to (2), we have

$$\gamma(\mathscr{L}) = \sup\{g(A) \ : \ A \in \mathscr{D} \cap \mathscr{A}(\mathscr{L})\} = \sup\{g(A) \ : \ A \in \mathscr{B}\}$$

and

$$\forall\, A \in (\mathscr{D} \cap \mathscr{A}(\mathscr{L})) \setminus \mathscr{B} \ : \ g(A) < \sup\{g(A') \mid A' \in \mathscr{B}\}.$$

Thus the problem is reduced to checking all matrices from $\mathscr{B}$ and selecting those matrices whose densities are maximal. ∎

Theorem F implies Theorem A.

2. The case $q \geq 3$. Infinite series of graphs with identical densities.

Theorem 4. *Let G be a connected simple graph in which the degree of any vertex equals $q-1$. Join any two vertices of G by $q-1$ additional edges. As a result we obtain a multigraph G^q. Then G^q is dense and $g(G^q) = q-1$.*

Proof. Put $r = v(G)$. Let $B = (b_{ij})$ be the adjacency matrix of G and A be the adjacency matrix of G^q. If $\mathbf{x} = (x_1, \ldots, x_r)$, $x_1 + \ldots + x_r = 1$ then

$$\begin{aligned}\mathbf{x}A\mathbf{x}^* &= (q-1)((x_1 + \ldots + x_r)^2 - x_1^2 - x_2^2 - \ldots - x_r^2) - \mathbf{x}B\mathbf{x}^* = \\ &= (q-1) - (q-1)(x_1^2 + \ldots + x_r^2) + \sum_{i=1}^{r}\sum_{j=1}^{r} b_{ij}x_i x_j = \\ &= (q-1) - \frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r} b_{ij}(x_i - x_j)^2 \leq q-1.\end{aligned}$$

Since G is connected, equality holds iff $x_1 = \ldots = x_r = 1/r$ (note that this point belongs to the interior of the standard simplex). Thus G^q is dense. ∎

Remarks. (4) The graphs G^q with arbitrary large number of vertices exist for every $q \geq 3$. In particular, if $q = 3$, then G is a cycle of an arbitrary length.

(5) Let L_q be a graph with vertices $v_0, v_1, \ldots, v_q$, where v_0 and v_i are joined by q edges, v_i and v_j are joined by $q-1$ edges ($i, j = 1, 2, \ldots, q$; $i \neq j$). Similarly to the proof of Theorem 4, it is easy to show that $g(\{L_q\}) = q-1$. In the case $q \geq 3$ every graph G^q described in Theorem 4 is optimal for $\mathscr{L} = \{L_q\}$. Hence, there can not exist any analogue for our Theorems 1 and 3.

(6) The matrix $A = [q-1]$ of size 1 and graphs G^q from Theorem 4 give a counterexample for the statement of Main Lemma of [3] in the case $q \geq 3$.

4. Digraphs

In this section we consider *digraphs* (i.e. *oriented multigraphs*) without loops where any vertices u and v are joined by at most q arcs oriented from u to v and at most q arcs oriented from v to u.

The notation $v(G)$, $e(G)$ for the numbers of vertices and arcs and the notation $\mathrm{ex}(n;\mathscr{L})$ are preserved. However we define

$$\gamma(\mathscr{L}) = \lim_{n\to\infty} \frac{\mathrm{ex}\,(n;\mathscr{L})}{n(n-1)}.$$

The square matrices $A=(a_{ij})$ considered here are allowed to be non-symmetric. Their diagonal entries may have values $0, 1, 2, \ldots, q$ and off-diagonal entries may have values $0, 2, 4, \ldots, 2q$. The definition of a graph $A(x_1,\ldots,x_r)$ needs some correction. Each class V_i is linearly ordered; if u precedes v in this ordering, then u and v are joined by a_{ii} arcs oriented from u to v. Every vertex of V_i is joined to every vertex of V_j by $a_{ij}/2$ arcs oriented towards V_j ($i, j=1, 2, \ldots, r$; $i\neq j$).

The definitions of an admissible matrix, the density of a matrix, an optimum vector, a dense matrix and an optimal matrix are the same. In the considered case Theorems A–F are valid, too. (cf. [1, 2, 3]). Our aim is to show that the main results of sections 1–3 are also valid.

Theorem 2′. *A matrix A is dense iff*
(a) *All components of vector* $\mathbf{e}\hat{A}^{-1}$ *are positive;*
and
(b) $\forall \mathbf{x}$ $(\mathbf{x}\mathbf{e}^*=0,\ \mathbf{x}\neq\mathbf{0})$: $\mathbf{x}A\mathbf{x}^*=\mathbf{x}\hat{A}\mathbf{x}^*<0$,
where $\mathbf{e}=(1,1,\ldots,1)$, $\hat{A}=(A+A^*)/2$. ∎

The proof of Theorem 2′ completely coincides the proof of Theorem 2 with the replacement of $\mathbf{e}A^{-1}$ by $\mathbf{e}(\hat{A})^{-1}$.

Theorem 4′. *Let G be a connected simple non-oriented graph in which the degree of any vertex equals $k-1$. Choose an arbitrary orientation for every edge of G and join any two vertices of G by $k-1$ additional arcs. As a result we obtain the oriented multigraph G^k. Then G^k is dense and $g(G^k)=k-1$.* ∎

The proof of Theorem 4′ is completely analogous to the proof of Theorem 4. It is clear that we may choose a certain orientation of arcs such that every two vertices of G^k are joined by at most $(k+1)/2$ arcs of each orientation. Thus our Remarks 4–6 remain valid for oriented multigraphs with $q\geq 2$.

The *underlying multigraph* $\hat{G}$ of a digraph G is obtained by suppressing the orientations. The multiplicity of an edge equals the sum of the multiplicities of the corresponding arcs of both orientations (cf. [3]). So, if $\hat{A}=(A+A^*)/2$, then $\hat{A}(x_1,\ldots,x_r)$ is the underlying multigraph of $A(x_1,\ldots,x_r)$.

If G is a digraph in which any two vertices are joined by at most 1 arc of each orientation (the case $q=1$ for digraphs), then any two vertices of its underlying multigraph are joined by at most two edges (the case $q=2$ for non-oriented multigraphs).

We shall not use the concept of equivalent rows of a matrix. However we say that a subset I of rows of a matrix $A=(a_{ij})$ is a *quasi-equivalence class* if the values of a_{ij} and a_{ji} with $i \in I$, $j \notin I$ depend on j only (but do not depend on i) and for any distinct rows $i, j \in I$: $a_{ij}+a_{ji}=2$, $a_{ii}+a_{jj}=0$. In particular, any single row forms a quasi-equivalence class of size 1.

Theorem 3′. *Let $q=1$ and A be a dense matrix which has no principal submatrix of size k whose all off-diagonal entries equal 2. Then the number of quasi-equivalence classes of rows of A is not larger than $t2^{t-1}$ with $t=R(k,\max\{k+3,9\})-1$.*

Proof. Denote by c the number of equivalence classes of rows in $\hat{A}$. Each of these classes may be decomposed into most 2^{c-1} quasi-equivalence classes of rows in A. According to Theorem 3, $c \le R(k,\max\{k+3,9\})-1$. ∎

In contrast to the non-oriented case, principal submatrices, which are induced by quasi-equivalence classes of rows in a dense matrix, do not have any fixed form. However, in the oriented case Theorem 3′ also implies analogues of Theorems 1, A, C, D, E, F. To prove it we use a fact that any tournament with 2^{s-1} vertices must contain an acyclic subtournament with s vertices (cf. [4]). For instance, we get the following statement.

Theorem 1′. *Let $q=1$, $k=\min\{v(L) : L \in \mathcal{L}\}$ and $k'=\max\{v(L) : L \in \mathcal{L}\}$. Then any optimal matrix for $\mathcal{L}$ has at most $t2^{t-1}(2^{k'-1}-1)$ rows where $t=R(k,k+6)-1$.*

Proof. Consider a dense matrix A which is admissible for $\mathcal{L}$. By Theorem 3′, the set of all rows of A may be partitioned into at most $t2^{t-1}$ quasi-equivalence classes. Suppose any class consists of $2^{k'-1}$ or more rows. Then some k' rows in this class may be rearranged to form a top-triangle principal submatrix of size k' whose diagonal entries are equal to 0 and all entries above the diagonal are equal to 2. Omit all rows of the quasi-equivalence class involved and the corresponding columns except for one row and the corresponding column and replace its diagonal entry by 1. As a result we obtain a matrix A' which is also admissible for $\mathcal{L}$, but $g(A') > g(A)$. It contradicts the assumption that A is dense. Therefore, to determine the supremum in (2) it is sufficient to consider only dense matrices A with at most $t2^{t-1}$ quasi-equivalence classes and at most $2^{k'-1}$ rows in each class. The number of such matrices is finite. ∎

The obtained bound on the number of rows of the matrix can be improved. It follows from [7] that the factor 2^{t-1} can be replaced by $(7/8)\cdot 2^{t-1}$.

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