Integr. equ. oper. theory 41 (2001) 123-125
0378-620X/01/010123-3 $ 1.50+0.20/0

Integral Equations and Operator Theory

ONE MORE PROOF OF THE BORODIN-OKOUNKOV FORMULA FOR TOEPLITZ DETERMINANTS

A. Böttcher

Recently, Borodin and Okounkov [2] established a remarkable identity for Toeplitz determinants. Two other proofs of this identity were subsequently found by Basor and Widom [1], who also extended the formula to the block case. We here give one more proof, also for the block case. This proof is based on a formula for the inverse of a finite block Toeplitz matrix obtained in the late seventies by Silbermann and the author.

Given an $N \times N$ matrix function f in L^∞ on the complex unit circle $\mathbf{T}$ with Fourier coefficients $\{f_n\}_{n\in\mathbf{Z}}$, the Toeplitz operator $T(f)$ and the two Hankel operators $H(f)$ and $H(\tilde{f})$ are defined by the infinite (block) matrices

$$T(f) = (f_{j-k})_{j,k=0}^{\infty}, \quad H(f) = (f_{j+k+1})_{j,k=0}^{\infty}, \quad H(\tilde{f}) = (f_{-j-k-1})_{j,k=0}^{\infty}.$$

These matrices induce bounded operators on $\ell^2(\mathbf{Z}_+, \mathbf{C}^N)$. Let $T_n(f) = (f_{j-k})_{j,k=0}^{n-1}$ and $D_n(f) = \det T_n(f)$. We denote by P_n the orthogonal projection of $\ell^2(\mathbf{Z}_+, \mathbf{C}^N)$ onto the subspace $\ell^2(\{0, \ldots, n-1\}, \mathbf{C}^N)$ and we put $Q_n = I - P_n$. Let finally $K_{2,2}^{1/2,1/2}$ be the Krein algebra of all matrix functions f in L^∞ on $\mathbf{T}$ for which $\sum_{n\in\mathbf{Z}}(|n|+1)\|f_n\|^2 < \infty$, where $\|\cdot\|$ is any matrix norm.

Suppose a is a matrix function in $K_{2,2}^{1/2,1/2}$ and a admits (left and right Wiener-Hopf) factorizations $a = u_- u_+$ and $a = v_+ v_-$ where $u_+, v_+, \overline{u}_-, \overline{v}_-$ and the inverses of these matrix functions belong to $K_{2,2}^{1/2,1/2} \cap H^\infty$. Put $b = v_- u_+^{-1}$ and $c = u_-^{-1} v_+$. Then the operator $H(b)H(\tilde{c})$ is of trace class and the operator $I - H(b)H(\tilde{c})$ is invertible. The Borodin-Okounkov formula (à la Widom) says that

$$D_n(a) = G(a)^n \frac{\det(I - Q_n H(b) H(\tilde{c}) Q_n)}{\det(I - H(b)H(\tilde{c}))} \tag{1}$$

for all $n \geq 1$, where $G(a) = \exp \int_0^{2\pi} \log\det a(e^{i\theta})d\theta$. It is well known that we also have

$$1/\det(I - H(b)H(\tilde{c})) = 1/\det T(b)T(c) = \det T(a)T(a^{-1}) \tag{2}$$

and that (2) equals $\exp \sum_{k=1}^{\infty} k(\log a)_k (\log a)_{-k}$ in the scalar case ($N = 1$), where $(\log a)_j$ is the jth Fourier coefficient of $\log a$. In what follows we abbreviate $H(b)H(\tilde{c})$ to K.

Our proof is based on the observation that $T_n(a)$ is invertible if and only if $I - Q_nKQ_n$ is invertible and that in this case

$$T_n^{-1}(a) = T_n(u_+^{-1})\left(I - P_nT(c)Q_n(I - Q_nKQ_n)^{-1}Q_nT(b)P_n\right)T_n(u_-^{-1}). \tag{3}$$

If n is large enough then $\|Q_nKQ_n\| < 1$, because K is compact and $Q_n \to 0$ strongly. Hence, for sufficiently large n we can write (3) as

$$T_n^{-1}(a) = T_n(u_+^{-1})\left(I - P_nT(c)Q_n\sum_{j=0}^{\infty}(Q_nKQ_n)^jQ_nT(b)P_n\right)T_n(u_-^{-1}),$$

and in exactly this form the identity was proved in [3, p. 188] (also see [4, p. 443]).

Formula (1) is immediate from (3): passing to determinants in (3) and taking into account that $\det(I + AB) = \det(I + BA)$ and $\det[(I + A)^{-1}] = 1/\det(I + A)$, we get

$$\begin{aligned}\frac{1}{D_n(a)} &= \frac{1}{G(a)^n}\det\left(I - Q_nT(b)P_nT(c)Q_n(I - Q_nKQ_n)^{-1}\right)\\ &= \frac{1}{G(a)^n}\frac{\det(I - Q_nKQ_n - Q_nT(b)P_nT(c)Q_n)}{\det(I - Q_nKQ_n)},\end{aligned} \tag{4}$$

and since $I - Q_nKQ_n - Q_nT(b)P_nT(c)Q_n$ equals

$$I - Q_n(T(bc) - T(b)T(c))Q_n - Q_nT(b)P_nT(c)Q_n = P_n + Q_nT(b)Q_nT(c)Q_n$$

(note that $H(b)H(\tilde{c}) = T(bc) - T(b)T(c)$ and $bc = I$), we see that the numerator in (4) is $\det(P_n + Q_nT(b)Q_nT(c)Q_n) = \det T(b)T(c)$.

Here is, for the reader's convenience, a proof of (3). Let A be an invertible operator and let P and Q be complementary projections. It is well known that the compression $PAP|\operatorname{Im} P$ of A to the range of P is invertible if and only if $QA^{-1}Q|\operatorname{Im} Q$ is invertible, in which case

$$(PAP)^{-1}P = PA^{-1}P - PA^{-1}Q(QA^{-1}Q)^{-1}QA^{-1}P. \tag{5}$$

Thus, $T_n(a)$ is invertible if and only if $Q_nT^{-1}(a)Q_n|\operatorname{Im} Q_n$ is invertible, which in turn is equivalent to the invertibility of

$$\begin{aligned}&Q_nT(v_-)Q_n\,Q_nT^{-1}(a)Q_n\,Q_nT(v_+)Q_n|\operatorname{Im} Q_n\\ &= Q_nT(v_-)Q_nT(u_+^{-1})T(u_-^{-1})Q_nT(v_+)Q_n|\operatorname{Im} Q_n\\ &= Q_nT(v_-)T(u_+^{-1})T(u_-^{-1})T(v_+)Q_n|\operatorname{Im} Q_n\\ &= Q_nT(b)T(c)Q_n|\operatorname{Im} Q_n = Q_n|\operatorname{Im} Q_n - Q_nKQ_n|\operatorname{Im} Q_n.\end{aligned}$$

Clearly, the last operator is invertible if and only if so is $I - Q_nKQ_n$. Now suppose that $T_n(a)$ is invertible. From (5) and the preceding computations we obtain

$$\begin{aligned}&T_n^{-1}(a) - P_nT^{-1}(a)P_n\\ &= P_nT^{-1}(a)Q_n(Q_nT^{-1}(a)Q_n)^{-1}Q_nT^{-1}(a)P_n\\ &= P_nT(u_+^{-1})P_nT(u_-^{-1})Q_nT(v_+)Q_n(I - Q_nKQ_n)^{-1}Q_nT(v_-)Q_nT(u_+^{-1})P_nT(u_-)P_n\\ &= T_n(u_+^{-1})P_nT(c)Q_n(I - Q_nKQ_n)^{-1}Q_nT(b)P_nT_n(u_-^{-1}),\end{aligned}$$

and this is (3).

References

[1] E.L. Basor and H. Widom: On a Toeplitz determinant identity of Borodin and Okounkov. *Integral Equations Operator Theory* **37** (2000), 397-401.

[2] A. Borodin and A. Okounkov: A Fredholm determinant formula for Toeplitz determinants. *Integral Equations Operator Theory* **37** (2000), 386-396.

[3] A. Böttcher and B. Silbermann: Notes on the asymptotic behavior of block Toeplitz matrices and determinants. *Math. Nachr.* **98** (1980), 183-210.

[4] A. Böttcher and B. Silbermann: *Analysis of Toeplitz Operators.* Springer-Verlag, Berlin, Heidelberg, New York 1990.

Fakultät für Mathematik
Technische Universität Chemnitz
09107 Chemnitz, Germany
aboettch@mathematik.tu-chemnitz.de

MSC 2000: 47B35

Submitted: December 18, 2000