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# Some Remarks on the Paper of Callias

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**Abstract.** This paper discusses the topological setting of the preceding paper by Callias. In particular, an alternate way of deriving his results is outlined.

## 1.

The preceding paper brings a nice proof, along new lines, of the index theorem for a special class of elliptic operators on $\mathbb{R}^n$.

The purpose of this note is to point out other methods which could have been used to get at the same result, and at the same time to explain the topological setting of Callias's formula.

We first recall the analytic conditions guaranteeing that the operators considered by Callias are Fredholm, hence have finite index. (Sources are discussed in a remark at the end of this note.) Consider the space

$$\Gamma = \Gamma(\mathbb{R}^n; V)$$

of $V$-valued smooth functions on $\mathbb{R}^n$, where $V$ is a finite dimensional vector space over $\mathbb{C}$. A differential operator on $\Gamma$ then has the form

$$D = \sum_{|\alpha| \leqq m} a_\alpha(x) \left(\frac{\partial}{\partial x}\right)^\alpha, \tag{1.1}$$

where the $a_\alpha(x)$ are smooth functions from $\mathbb{R}^n$ to the endomorphisms of $V$. It is assumed that the derivatives of $a_\alpha$ decay at $\infty$:

$$D^\beta a_\alpha(x) = 0(|x|^{-|\beta|}) \quad \text{as} \quad x \to \infty . \tag{1.2}$$

Then the $D$ in (1.1) induces a *Fredholm operator*

$$D : H_m(\mathbb{R}^n; V) \to L^2(\mathbb{R}^n; V)$$

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if and only if the total symbol

$$\sigma_D(x, \xi) = \sum_{\alpha \leq m} a_\alpha(x)\,\xi^\alpha \tag{1.3}$$

satisfies the condition:

*On large enough spheres*

$$\Sigma(x_i^2 + \xi_i^2) = K \tag{1.4}$$

*the symbol $\sigma_D(x, \xi)$ is nonsingular.*

It follows that $D$ determines a well defined homotopy class of maps

$$\sigma_D : S^{2n-1} \to \mathrm{Aut}(V), \tag{1.5}$$

of this "$(2n-1)$-sphere at $\infty$" into the full linear group of Automorphisms of $V$, and a complete solution of the index problem in this setting would be a formula of the type:

$$\mathrm{index}(D) = \int_{S^{2n-1}} \sigma_D^* \omega, \tag{1.6}$$

where $\omega$ is a well-defined closed $(2n-1)$-form on $\mathrm{Aut}\, V$, and $\sigma_D^*$ denotes the pullback of this form to $S^{2n-1}$.

Put differently, such a formula implies that the index of $D$ is *purely a function of the homology class of the $\sigma_D$-image of $S^{2n-1}$.*

Now the group $\mathrm{Aut}(V)$ has a potentially nontrivial $\omega$ to fit into (1.6), only when $\dim_{\mathbb{C}} V \geq n$, and then such an $\omega$ must be homologous to a multiple of the following class:

Let us choose a base $v_1, \ldots, v_n$ in $V$, and for every $g \in \mathrm{Aut}(V)$ define:

$$Z(g) = \text{Matrix of } g \text{ relative to the basis chosen}. \tag{1.7}$$

Then $Z$ is a well-defined matrix-valued function on $G = \mathrm{Aut}(V)$ and

$$\Theta = Z^{-1} dZ \tag{1.8}$$

is a well-defined matrix of 1-forms on $G$. The formula $Z(g \cdot g') = Z(g) \cdot Z(g')$, then shows that $\Theta$ goes into itself under the map $L_g : g' \to gg'$, and indeed the $n^2$ components $\Theta_j^i$ of $\Theta$ *form a base* for the *left invariant* 1-*forms on* $G$. In any case we can consider the differential forms:

$$\omega_k = \mathrm{Trace}\{\underbrace{\Theta \wedge \ldots \wedge \Theta}_{k}\}, \tag{1.9}$$

and it is a fundamental theorem of the subject that:

**Theorem.** *The DeRham cohomology of* $G = \mathrm{Aut}\, V$, $\dim_{\mathbb{C}} V = m$, *is an exterior algebra with generators* $\omega_1, \omega_3, \ldots, \omega_{2m-1}$,

$$H^*(G) = E(\omega_1, \ldots, \omega_{2m-1}). \tag{1.10}$$

This theorem follows essentially from the fact that the cohomology of a compact group $U$ is already computed by the *left and right invariant* forms on $U$, and that $\mathrm{Aut}(V)$ has the cohomology of its maximal compact subgroup, which may be taken as the subgroup $U(V)$ preserving a fixed hermitian structure on $G$.

Now granting (1.10) we see that every class in $H^{2n-1}(G)$ is represented by a form of the type:

$$\omega = \mathrm{const}\cdot\omega_{2n-1} + \text{products of lower } \omega\text{'s}. \tag{1.11}$$

But if one integrates a product $u \wedge v$, with both $du=0$ and $dv=0$, $\dim u>0$, $\dim v>0$, over a sphere one obtains zero because a sphere carries cohomology only in the extreme dimensions. Hence any candidate for $\omega$ in (1.1) can be taken to be of the form:

$$\omega = \mathrm{const}\cdot\omega_{2n-1}. \tag{1.12}$$

Note by the way that

$$\begin{aligned}\omega_1 &= \mathrm{trace}\, Z^{-1}dZ \\ &= d\log\{\det Z\},\end{aligned} \tag{1.13}$$

so that $\frac{\omega_1}{2\pi i}$, when pulled back by a map

$$\sigma_D: S^1 \to \mathrm{Aut}(V)$$

just counts the winding number of the curve traced out by $\det\sigma$ in $\mathbb{C}$. Thus the higher $\omega_k$, when properly renormalized should be thought of as higher analogues of the winding number, and in fact one of the consequences of the periodicity theorem for the Unitary group, implies the following result:

**Theorem.** *The homotopy group* $\pi_{2n-1}\{\mathrm{Aut}(V)\}$ *is isomorphic to the integers* $\mathbb{Z}$ *provided* $\dim V \geqq n$, *and such an isomorphism is obtained by assigning to a map*

$$f: S^{2n-1} \to \mathrm{Aut}(V)$$

*the integer*

$$\int_{S^n} f^*\omega \quad \textit{where} \quad \omega = \left(\frac{i}{2\pi}\right)^{2n} \frac{(n-1)!}{(2n-1)!}\,\omega_{2n-1}.$$

By the way it is not supposed to be *obvious* why this integral should turn out to be an integer, and in fact there are cycles $C$ in $\mathrm{Aut}(V)$ on which this form would *not* be an integer. It is only on the *spherical cycles* that it is integer valued.

With these remarks understood the best of all theorems in this context would be:

**Theorem A.** *Let*

$$D: \Gamma(V) \to \Gamma(V)$$

*be a differential operator satisfying* (1.2) *and* (1.4). *Then*

$$\mathrm{index}(D) = \pm\left(\frac{i}{2\pi}\right)^n \frac{(n-1)!}{(2n-1)!} \int_{S^{2n-1}} \sigma_D^*\omega_{2n-1}. \tag{1.14}$$

In fact, a formula proved by Fedosov [2] has been extended by Hörmander [3] to a result which implies (1.14). Theorem 7.3 in [3] leads easily to

$$\operatorname{index}(D) = -\left(\frac{i}{2\pi}\right)^n \frac{(n-1)!}{(2n-1)!} \int_{S^{2n-1}} \operatorname{Tr}(\sigma_D^{-1} d\sigma_D)^{2n-1}. \tag{1.15}$$

It is therefore of interest to relate (1.15) with the formula of Callias's paper.

Recall first of all that Callias deals with a vector space $V$ which is naturally the tensor product of two vector spaces

$$V = V' \otimes V'', \quad \dim V' = p, \dim V'' = m$$

and that relative to this decomposition, and some preliminary normalizations, his symbols take the form

$$\sigma(x, \xi) = \delta(\xi) \otimes 1 + i 1 \otimes U(x), \tag{1.16}$$

where $\delta = \Sigma \delta^i \xi_i$, is linear in $\xi$, and the two functions $\delta$ and $U$ satisfy the conditions

$$\delta^2(\xi) = |\xi|^2, U^2(x) = 1 \quad |x| \geqq 1, \tag{1.17}$$

Furthermore, $U(x)$ is homogeneous of degree 0 in $|x| \geqq 1$, and both $\delta(\xi)$ and $U(x)$ are unitary on the respective spheres

$$S_\xi = \{|\xi|^2 = 1\} \quad \text{and} \quad S_x = \{|x|^2 = 1\}.$$

From these conditions it follows easily that $\sigma$ is nonsingular on $|x|^2 + |\xi|^2 \geqq 1$, so that it is sufficient to study $\sigma$ on the unit sphere $S^{2n-1} = \{|x|^2 + |\xi|^2 = 1\}$.

Now consider the map

$$S_\xi^{n-1} \times I \times S_x^{n-1} \xrightarrow{j} S^{2n-1} \quad I = [0, \pi/2]$$

which sends $(\xi, t, x)$ to $\xi \cos t + x \sin t$. This map is then onto and $1-1$ except for $t=0$, when the $x$-coordinate becomes immaterial, and $t = \pi/2$, when the $\xi$ coordinate becomes immaterial. Modulo these identifications the L.H.S. becomes the *join* of $S^{n-1}$ with $S^{n-1}$, denoted by $S^{n-1} * S^{n-1}$ in topology, so that $j$ verifies the well known topological fact that

$$S^p * S^q = S^{p+q+1}.$$

Now because $j$ is $1-1$ except for a set of measure 0, an integration of a differential form over $S^{2n-1}$ can be carried out by first pulling the form up to $S^{n-1} \times I \times S^{n-1}$ and then integrating over the factors one by one. Thus in our case, this amounts to the following recipe:

Set $\quad z = \delta \otimes \cos t + i \sin t \otimes U,$

and then integrate

$$\omega^* = \operatorname{trace}\,(z^{-1} dz)^{2n-1} \tag{1.18}$$

over $S^{n-1} \times I \times S^{n-1}$.

Let us now prove that at least modulo constants, this procedure produces the desired formula: Indeed one has:

$$z^{-1}\cdot dz = (\delta\cos t - i\sin t U)(d\delta\cos t + i\sin t dU) + dt(-\delta\sin t + i\cos t dU)$$
$$= \delta d\delta\cos^2 t + \sin^2 t + UdU + i(\sin t\cos t)(\delta dU - Ud\delta) + idtU\delta\,,$$

and in view of the relations $U^2 = 1$, $\delta^2 = 1$ we have

$$U\cdot dU + dU\cdot U = 0$$
$$\delta\cdot d\delta + d\delta\cdot\delta = 0\,.$$

Thus in our computations $U$ and $\delta$ commute, but $U$ and $dU$'s, $\delta$ and $d\delta$'s, anticommute.

Hence remembering that our integrand must contain precisely one $dt$ and $(n-1)$ $d\delta$ and $(n-1)\,dU$ factors we see that after the $t$-integration the integral must take the form:

$$\text{const}\cdot \int_{S^{n-1}} \text{trace}\, U(dU)^{n-1}\cdot \int_{S^{n-1}} \text{trace}\,\delta(d\delta)^{n-1}\,.$$

The constant here is of course in principle computable from (1.18) but involves some algebra which is, it seems, beyond our collective abilities.

We will therefore sketch an alternative derivation which hopefully might at the same time serve as an introduction to $K$-theory for physicists.

## 2.

First some notation. We will assume that all our vector spaces $V$, etc. have a fixed hermitian structure and then write $\boldsymbol{U}(V)$ for the group of isometries of $V$. We also write $\boldsymbol{G}_n(V)$ for the Grassmann variety of $n$-planes in $V$ and in the hope of making this space more palatable to physicists we define it as follows:

*$\boldsymbol{G}_n(V)$ is the subset of $\boldsymbol{U}(V)$ characterized by the condition that:*

$$u^2 = 1 \textit{ and } u \text{ has } n \text{ eigenvalues equal to } +1\,. \tag{2.1}$$

Thus $\boldsymbol{G}_n(V)$ becomes a component of the solution set of the equation

$$u^2 = 1$$

in $\boldsymbol{U}(V)$. It is also an orbit of $\boldsymbol{U}$ acting on itself by inner automorphisms.

If $u \in \boldsymbol{G}_n(V)$ we write $P(u)$ for the Eigenspace of $u$ associated to the eigenvalue $+1$, so that the $1-1$ correspondence

$$u \leftrightarrow P(u)$$

gives an identification of our $\boldsymbol{G}_n(V)$ with the more standard view of the Grassmannian as the *space of n-dimensional subspaces of V.*

In any case the family of planes $P(u)$ naturally defines *an n-plane bundle P*, over $\boldsymbol{G}_n(V)$, and this bundle is of fundamental importance in all bundle theory.

Obviously the Eigenspace $N(u)$ associated to $-1$, also defines a bundle $N$ over $\boldsymbol{G}_n(V)$ and their direct sum is manifestly the trivial bundle $V$ over $\boldsymbol{G}_n(V)$:

$$P \oplus N = V\,.$$

We now define a map

$$G_n(V')*G_m(V'')\xrightarrow{\alpha}U_{nm}(V'\otimes V'') \tag{2.3}$$

which underlies Callias's computations. The map (2.4) is given by

$$(u,\theta,v)\mapsto u\otimes\cos\theta+i\sin\theta\otimes v \qquad 0\leqq\theta\leqq\pi/2. \tag{2.4}$$

It is easy to check that the R.H.S. is unitary if $u^2=1$ and $v^2=1$, as we are assuming, and also that (2.4) makes the join identification. Notice now that an immediate consequence of (2.4) – and in fact this is a better way of thinking of this map – is the following: *the image of $(u,\theta,v)$ under $\alpha$, is the unitary transformation which, restricted to various subspaces of $V'\otimes V''$, is given by:*

$$\alpha(u,\theta,v)\left|\begin{array}{l} P(u)\otimes P(v)=e^{i\theta}\\ P(u)\otimes N(v)=e^{-i\theta}\\ N(u)\otimes P(v)=-e^{-i\theta}\\ N(u)\otimes N(v)=-e^{i\theta}.\end{array}\right. \tag{2.5}$$

In any case, the basic topological lemma which transforms Theorem A into the Callias formula is the following one.

**Lemma.** *Let*

$$S^a\xrightarrow{f}G_p(V') \quad \text{and} \quad S^b\xrightarrow{g}G_q(V'')$$

*be smooth maps of spheres, and let h be the composition*

$$S^a*S^b\to G_p(V')*G_q(V'')\xrightarrow{\alpha}U(V'\otimes V''). \tag{2.6}$$

*Then*

$$\int_{S^a*S^b}h^*\omega=-\int_{S^a}\operatorname{trace}\operatorname{ch}f^{-1}P\cdot\int_{S^b}\operatorname{trace}\operatorname{ch}f^{-1}P.$$

Here $\omega$ is the generalized winding class of the index theorem, $f^{-1}$ denotes the pull-back of bundles, and $\operatorname{ch}E$ denotes the *Chern character* of the bundle $E$.

Recall here that this $\operatorname{ch}(E)$ is a mixed differential form of even degree[1]:

$$\operatorname{ch}(E)=\operatorname{ch}_0(E)+\operatorname{ch}_1(E)+\dots\operatorname{ch}_i(E)+\dots$$

with $\operatorname{ch}_i(E)$ a $2i$ form, which is well defined once a notion of parallel transport is chosen on $E$, and then has the following functorial properties:

$$\operatorname{ch}(\text{trivial bundle})=\dim(\text{bundle}), \tag{2.8}$$

$$\operatorname{ch}(f^{-1}E)=f^*\operatorname{ch}(E), \tag{2.9}$$

$$\operatorname{ch}(E\oplus F)=\operatorname{ch}(E)+\operatorname{ch}(F), \tag{2.10}$$

$$\operatorname{ch}(E\otimes F)=\operatorname{ch}(E)\cdot\operatorname{ch}(F). \tag{2.11}$$

If $\dim V=2$ and $P$ is the "positive" bundle over $G_1(V)\simeq S^2$. Then

$$\operatorname{ch}(P)=1-\lambda$$

1 See for instance [1]

where $\lambda$ is a 2 form with integral 1:

$$\int_{S^2} \lambda = 1 .$$

Before we give some indications concerning the proof of this lemma, let us demonstrate how the computations of Sect. IV in the preceeding paper follow from it.

Let

$$\boldsymbol{S}_q : V \mapsto \boldsymbol{S}_q(V) \tag{2.12}$$

denote the operation of taking the $q$-th Symmetric power of a Vector space. Then $\boldsymbol{S}_q$ satisfies the additivity formula

$$\boldsymbol{S}_q(V' \oplus V'') = \sum_{i+j=q} \boldsymbol{S}_i(V') \otimes \boldsymbol{S}_j(V'') . \tag{2.13}$$

This formula enables one to write down many maps from

$$\boldsymbol{G}_r(V) \quad \text{to} \quad \boldsymbol{G}_s(\boldsymbol{S}_q(V)) .$$

For instance, take $\dim V = 2$, so that $\boldsymbol{G}_1(V) = S^2$. Now let $H$ be a 1-subspace of $V$ and $H^\perp$, its orthogonal complement.

Then the assignment, say,

$$H \mapsto S_3(H) \otimes 1 \oplus S_2(H) \otimes H^\perp$$

clearly defines a map

$$f : \boldsymbol{G}_1(V) \mapsto \boldsymbol{G}_2\{S_3(V)\}$$

and the formula (2.13) – read backward! – gives the pull back of $P$ under $f$:

$$f^{-1}P = P^3 \otimes 1 \oplus P^2 \otimes N .$$

Hence, for instance, using $(2\cdot 8), \ldots, (2\cdot 11)$,

$$\begin{aligned} \mathrm{ch}(f^{-1}P) &= (1-\lambda)^3 + (1-\lambda)^2(1+\lambda) \\ &= 1 - 3\lambda + 1 - \lambda = 2 - 4\lambda \end{aligned}$$

and so

$$\int_{S^2} \mathrm{ch} f^{-1} P = -4 .$$

In fact it should be clear now that (2.13) implies the following proposition.

**Proposition.** *Consider the bundles* $P^i \otimes N^j$ $i+j \leq q$ *over* $\boldsymbol{G}_1(V)$, $\dim V = 2$. *If we select* $r$ *of these say* $P^{i_\alpha} \otimes N^{j_\alpha}$ $\alpha = 1, \ldots, r$, *then there is a unique map of*

$$\boldsymbol{G}_1(V) \xrightarrow{f} \boldsymbol{G}_r\{\boldsymbol{S}_q(V)\}$$

*with*

$$f^{-1}P = \sum_{\alpha=1}^{r} P^{i_\alpha} \otimes N^{j_\alpha} . \tag{2.14}$$

*Furthermore,*

$$\int_{S^2} \operatorname{ch} f^{-1} P = \sum_{\alpha=1}^{n} (j_\alpha - i_\alpha).$$

It is now easy to verify that the map $\phi$ of Sec. IV, say for $m=0$, and $q$ odd, is described by the $f$ associated to the bundles

$$P^i \otimes N^j, i+j=q \quad i<j$$

whence

$$\int_{S^2} \operatorname{ch}(f^{-1} P) = \left(\frac{q+1}{2}\right)^2,$$

in agreement with (4.6) where $q=(T+1)/2$.

Finally, a word about the proof of our Lemma. For this purpose one needs a notion closely related to the join construction, called *suspension* and usually denoted by $\Sigma$:

If $X$ is a space, then $\Sigma X$ is the space $X \times [0,1]$ with $X \times 0$ identified to a point and $X \times 1$ identified to another point:

$$\Sigma X = X \times I/\sim\,;\ (x,0) \sim (x',0) \qquad (x,1) \sim (x',1). \tag{2.16}$$

It is clear then, that

$$\Sigma X = X * S^0, \quad \text{and} \quad \Sigma S^n = S^{n+1}. \tag{2.17}$$

Further, for any map

$$f: X \to Y$$

one has a suspension:

$$\Sigma f: \Sigma X \to \Sigma Y. \tag{2.18}$$

This operation enters into our considerations because there is a map

$$\sigma: \Sigma \boldsymbol{U}(V) \to \boldsymbol{G}_n(V \otimes E) \dim V = n, \dim E = 2, \tag{2.19}$$

fundamental in the homotopy theory of $\boldsymbol{U}(V)$ in that it induces an isomorphism

$$\pi_k\{\boldsymbol{U}(V)\} \to \pi_{k+1}\{\boldsymbol{G}_n(V \otimes E)\} \tag{2.20}$$

for $k \leqq n$. This arrow is of course induced by sending a map

$$f: S^k \to \boldsymbol{U}(V)$$

to

$$\sigma \circ \Sigma \circ f: S^{k+1} \to \boldsymbol{G}_n(V \otimes E).$$

Now this $\sigma$ is best thought of as assigning to $A \in \boldsymbol{U}(V)$ the graph of $(tA): t \in [0, \infty]$.

$$\sigma(A,t) = \operatorname{graph}(tA) \in G_n(V \oplus V). \tag{2.21}$$

Precisely, if we chose a basis in a 2-dim Vector space $E$ say $e_1$ and $e_2$, then this $\sigma$ is given by:

$$(A,t)\mapsto\{v\otimes e_1+tAv\otimes e_2\}\subset V\otimes E\,,\qquad v\in V. \tag{2.22}$$

Thus $(A,0)$ goes to $V\otimes e_1$ and $(A,\infty)$ to $V\otimes e_2$, so that (2.11) does perform the suspension identifications.

Now our final aim will be to understand the map

$$\Sigma[G_p(V')*G_q(V'')]\xrightarrow{\sigma\circ\Sigma\alpha}G_{pq}(V'\otimes V''\otimes E)$$

in its effect on the bundle $P$, and for this purpose we will study the pull-back of $P$ under $\sigma\circ\Sigma\alpha$ as a sub-bundle of the trivial bundle

$$V\otimes W\otimes E\,.$$

But combining (2.5) with (2.22) clearly solves this problem. Indeed over the point $(A,\theta,B,t)$ we find the subspace:

$$\begin{aligned}&P(A)\oplus P(B)\oplus(e_1+te^{i\theta}e_2)\\&P(A)\oplus N(B)\oplus(e_1+te^{-i\theta}e_2)\\&N(A)\oplus P(B)\oplus(e_1-te^{-i\theta}e_2)\\&N(A)\oplus N(B)\oplus(e_1-te^{i\theta}e_2)\,.\end{aligned} \tag{2.23}$$

At this stage it is natural to introduce the complex number $z=te^{i\theta}$, and to define the line bundle

$$H\subset S^2\times E$$

by letting the fiber over $z\in S^2=\mathbb{C}\cup\infty$ be the line

$$H_z=\{e_1+ze_2\}\,.$$

With these conventions we can identify the two suspension parameters $t$ and $\theta$ with the set $Q$ indicated below:

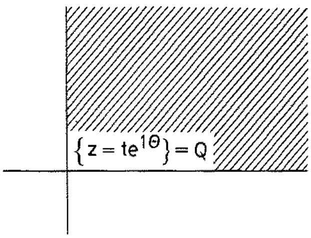


**Fig. 1.**

and $\sigma\circ\Sigma(\alpha)$ then appears as a map

$$G_p(V')\otimes G_q(V'')\times Q\to G_{pq}(V'\otimes V''\otimes E)\,.$$

Further, this map has the property, that it pulls back the *bundle* $P$ *of the* R.H.S. *to the bundle:*

$$P\otimes P\otimes H\oplus P\otimes N\otimes\bar{H}\oplus N\otimes P\otimes(-1)\bar{H}\oplus N\otimes N\otimes(-1H) \tag{2.24}$$

over the L.H.S., where now $\bar{H}$ is the pull-back of $H$ under the map $z \to \bar{z}$, and $(-1)H$ the pull-back of $H$ under the map $z \to -z$, etc. Indeed all this is just a trivial rewriting of (2.13)!

But now we are essentially done.

The pertinent diagrams are (2.25) and (2.26).

We wish to compute the map in $H^{a+b+1}$ induced by the arrows on a certain class $\boldsymbol{\omega}$ – "the winding class" in $H^{a+b+1}(\boldsymbol{U})$,

$$S^a * S^b \to \boldsymbol{G}' * \boldsymbol{G}'' \xrightarrow{\alpha} \boldsymbol{U}, \tag{2.25}$$

but instead we compute *the map in* $H^{a+b+2}$ of the suspension of (2.25). This is admissible because *suspension is an isomorphism in cohomology raising the dimension by* 1.

Thus we are now to compute the induced map of

$$\Sigma S^a * S^b \to \Sigma(\boldsymbol{G}' * \boldsymbol{G}'') \to \Sigma U. \tag{2.26}$$

*But now, by definition if you wish, the suspension of* $\omega$ *in* $H^{a+b+2}(\Sigma \boldsymbol{U})$ *is precisely the pull-back of* $\mathrm{ch}_{a+b+2}(P)$ *under our map*

$$\sigma \cdot \Sigma \boldsymbol{U} \to \boldsymbol{G}.$$

Finally replacing the suspensions by their $Q$-models, we see that what really has to be computed is the effect of the sequence:

$$S^a \times S^b \times Q \to \boldsymbol{G}' \times \boldsymbol{G}'' \times Q \to \Sigma \boldsymbol{U} \to \boldsymbol{G}$$

on $\mathrm{ch}_{a+b+2}(P)$ over $\boldsymbol{G}$. But now (2.24) and our rules for ch essentially enable one to identify this pull-back. Indeed all we need is the following refinement of the property (2.15):

The form $\lambda = \mathrm{ch}_1(P)$ over $G_1(V) = S^2$ *can be taken to be invariant under the group of rotations of* $S^2$.

From this it follows that

$$\mathrm{ch}(H) = \mathrm{ch}\{(-1)H\} = 1 - \lambda$$
$$\mathrm{ch}(\bar{H}) = \mathrm{ch}\{(-1)\bar{H}\} = 1 + \lambda$$

because the $H$ of (2.24) is clearly the $P$ of (2.15) and the other bundles above are induced from $H = P$ by either rotations or reflections.

With this understood and using the identities

$$\mathrm{ch}(P) + \mathrm{ch}(N) = n,$$

we can rewrite ch of (2.24) in terms of the $P$'s alone, to obatin

$$\mathrm{ch}\{(2.24)\} = -4\,\mathrm{ch}\, P \otimes \mathrm{ch}\, P' \otimes \lambda + \dots,$$

where all the remaining terms have a 0-dimensional term in one of the places of the tensor product.

Now in our final integration these all disappear and the leading term, when integrated over $Q$, contributes $\frac{1}{4}$ because $Q$ is just one quarter of $S^2$ and $\int_{S^2} \lambda = -1$. The Lemma now follows, and we are done.

*Remark on the Fredholm Property.* The fact that the operator $D$ in (1.1) is a Fredholm operator under conditions (1.2) and (1.4) is proved in [3], as part of an elaborate theory. More direct proofs were given independently by Taylor [5], and in [4], with the decay condition (1.2) replaced by a weaker condition

$$D^{\beta}a_{\alpha}\to 0 \quad \text{as} \quad x\to\infty, \quad \text{for} \quad \alpsa\neq 0. \tag{1.2'}$$

These papers also prove that (1.4) is *necessary* for a Fredholm operator. They do not explicitly discuss systems, but they easily could. Callias refers to [4] because we discovered only recently that the other two references contain what he needs.

It would be worthwhile to make all these results more accessible. Reese Prosser is working on such a paper.

The known proofs of the index formula (1.15) do not seem to work when (1.2) is weakened to (1.2′), but it would be surprising if (1.15) where *not* true in that case, too.

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