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# An elementary proof of the Eberlein-Šmulian Theorem and the Double Limit Criterion

By

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**0. Introduction and notations.** In the sequel let $E$ denote a normed vector space and Ball$(E)$ its closed unit ball, furthermore $E^*, E^{**}, E^{***}, \ldots$ its sequence of topological dual spaces. For a nonvoid set $X$ and a nonvoid set $F$ of realvalued functions on $X$ let $\sigma(X, F)$ denote the weakest topology on $X$ in which all functions of $F$ are continuous. A sequence $(x_n)_n$ in $X$ converges to an $x \in X$ in $\sigma(X, F)$ iff $(f(x_n))_n$ converges to $f(x)$ in $\mathbb{R}$ for every $f \in F$. In particular $\sigma(E, E^*)$ is the usual weak topology on $E$ and $\sigma(E^*, E)$ the weak* topology on $E^*$.

In this paper we present an elementary and transparent proof of the Eberlein-Šmulian theorem for normed vector spaces (cp. e.g. [3], p. 145–149) and of the Double Limit Criterion for Banach spaces (cp. e.g. [3], p. 157). The essential idea (Lemma 1.1) will be the simple fact that if $\emptyset \neq A \subset E$ and $\emptyset \neq T \subset E^*$ have the interchangeable double limit property, then the convergence of a sequence $(a_n)_n$ in $A$ to some $a \in E$ in $\sigma(E, T)$ implies the convergence of $(a_n)_n$ to $a$ even in $\sigma(E, \bar{T}^{\sigma(E^*, E)})$. By combining this fact with an inductive construction we can prove (Lemma 1.2) that each $a \in \bar{A}^{\sigma(E, T)}$ is the limit of a sequence $(a_n)_n$ in $A$ even in the topology $\sigma(E, \bar{T}^{\sigma(E^*, E)})$. With these two lemmata the abovementioned theorems can be easily proved by the usual embedding of $E$ into $E^{**}$ and with standard tools like the Banach-Alaoglu and bipolar theorem. We think that the proof presented here is simpler and more natural than Whitley's proof [6] of the Eberlein-Šmulian theorem, which H. B. Cohen [1] and J. D. Pryce [5] refined to obtain the present version of that theorem. Furthermore our approach leads at the same time to the Double Limit Criterion for Banach spaces.

**1. The two main lemmata.** We fix nonvoid subsets $A \subset E$ and $T \subset E^*$. Let $\bar{\mathbb{R}}$ denote the two point compactification of $\mathbb{R}$.

We say that $A$ and $T$ have the interchangeable double limit property (abbreviated by IDLP) in $\bar{\mathbb{R}}$ if for every pair of sequences $(\varphi_l)_l$ in $T$ and $(a_n)_n$ in $A$ for which the iterated limits $\lim_{l \to \infty} \lim_{n \to \infty} \varphi_l(a_n)$ and $\lim_{n \to \infty} \lim_{l \to \infty} \varphi_l(a_n)$ both exist in $\bar{\mathbb{R}}$, these limits are equal.

**1.1. Lemma.** *Let $A$ and $T$ have the* IDLP *in $\bar{\mathbb{R}}$. Then every sequence $(a_n)_n$ in $A$ which converges to some $a \in E$ in $\sigma(E, T)$ converges to $a$ even in $\sigma(E, \bar{T}^{\sigma(E^*, E)})$.*

Proof. Let $\varphi \in \bar{T}^{\sigma(E^*, E)}$. Choose for each $l \in \mathbb{N}$ a $\varphi_l \in T$ such that

$$|\varphi_l(a_k) - \varphi(a_k)| < \frac{1}{l} \quad \text{for } k = 1, \ldots, l \quad \text{and} \quad |\varphi_l(a) - \varphi(a)| < \frac{1}{l}.$$

It follows that $\lim_{l \to \infty} \varphi_l(a) = \varphi(a)$ and hence $\lim_{l \to \infty} \lim_{n \to \infty} \varphi_l(a_n) = \lim_{l \to \infty} \varphi_l(a) = \varphi(a)$. On the other hand we have $\lim_{l \to \infty} \varphi_l(a_n) = \varphi(a_n)\ \forall n \in \mathbb{N}$. Now the IDLP of $A$ and $T$ in $\mathbb{R}$ implies that $\lim_{n \to \infty} \lim_{l \to \infty} \varphi_l(a_n) = \lim_{n \to \infty} \varphi(a_n)$ exists in $\mathbb{R}$ and is $= \varphi(a)$, because each convergent subsequence of $\left(\lim_{l \to \infty} \varphi_l(a_n)\right)_n$ must have the limit $\varphi(a)$. This is the assertion.

**1.2. Lemma.** *Let $A$ and $T$ have the* IDLP *in $\mathbb{R}$. Then for each $a \in \bar{A}^{\sigma(E, T)}$ there exists a sequence $(a_n)_n$ in $A$ which converges to $a$ in $\sigma(E, \bar{T}^{\sigma(E^*, E)})$.*

Proof. Fix $a \in \bar{A}^{\sigma(E, T)}$. We put $a =: a_0$ and construct inductively for $n = 1, 2, \ldots$ a sequence $(\varphi_k^n)_k$ in $T$ and a point $a_n \in A$ with the properties

(i) $\{\varphi_k^n : k \in \mathbb{N}\} \mid U_{n-1}$ is a countable dense subset of $T \mid U_{n-1} \subset U_{n-1}^*$ in $\sigma(U_{n-1}^*, U_{n-1})$, where $U_{n-1} := \text{Linear Span}\{a_0, \ldots, a_{n-1}\}$;

(ii) $|\varphi_k^l(a) - \varphi_k^l(a_n)| < \frac{1}{n} \quad \forall k, l = 1, \ldots, n.$

In each induction step (i) can be achieved since $\dim U_{n-1} < \infty$, and then (ii) can be achieved since $a \in \bar{A}^{\sigma(E, T)}$. Now let

$$D := \{\varphi_k^n : n, k \in \mathbb{N}\} \quad \text{and} \quad U := \text{Linear Span}\{a_n : n = 0, 1, 2, \ldots\}.$$

Then

(0) $A \cap U$ and $D \mid U$ have the IDLP in $\mathbb{R}$;

(1) $D \mid U \subset T \mid U \subset U^*$ is $\sigma(U^*, U)$ dense in $T \mid U$;

(2) $(a_n)_n$ converges to $a$ in $\sigma(U, D \mid U)$.

In view of Lemma 1.1 $(a_n)_n$ converges to $a$ in $\sigma(U, T \mid U)$, that is in $\sigma(E, T)$; and then (again by Lemma 1.1) even in $\sigma(E, \bar{T}^{\sigma(E^*, E)})$.

**1.3. Corollary.** *Let $A$ and* Ball$(E^*)$ *have the* IDLP *in $\mathbb{R}$. Then for each $\xi \in \bar{A}^{\sigma(E^{**}, E^*)} \subset E^{**}$ there exists a sequence $(a_n)_n$ in $A$ which converges to $\xi$ in the weak topology $\sigma(E^{**}, E^{***})$ (to reduce notation we consider $E$ as a subset of its bidual $E^{**}$ via the canonical embedding).*

Proof. We consider $A \subset E^{**} := S$ and $T := \text{Ball}(E^*) \subset E^{***} = S^*$. Since $A$ and $T$ have the IDLP in $\mathbb{R}$, Lemma 1.2 furnishes for $\xi \in \bar{A}^{\sigma(S, T)}$ a sequence $(a_n)_n$ in $A$ which converges to $\xi$ in the topology $\sigma(S, \bar{T}^{\sigma(S^*, S)})$. But $\bar{T}^{\sigma(S^*, S)} = \overline{\text{Ball}(E^*)}^{\sigma(E^{***}, E^{**})} = \text{Ball}(E^{***})$ in view of the bipolar theorem (cp, e.g. [4], Theorem 6.4), so that the assertion follows.

**2. The Eberlein-Šmulian Theorem and the Double Limit Criterion.** In the following we use the usual notions of compactness (cp. e.g. [3], p. 145).

2.1. Remark. If $\emptyset \neq A \subset E$ is relatively countably compact in $\sigma(E, E^*)$, then $A$ and $\mathrm{Ball}(E^*)$ have the IDLP in $\bar{\mathbb{R}}$ (and even in $\mathbb{R}$ since $A$ and $\mathrm{Ball}(E^*)$ are both norm bounded).

Proof. We follow the usual argument. Let $(\varphi_l)_l$ in $\mathrm{Ball}(E^*)$ and $(a_n)_n$ in $A$ be sequences such that the double limits $\lim_{l\to\infty} \lim_{n\to\infty} \varphi_l(a_n)$ and $\lim_{n\to\infty} \lim_{l\to\infty} \varphi_l(a_n)$ both exist. Let $a_\infty \in E$ be a cluster point of $(a_n)_n$ in $\sigma(E, E^*)$ and $\varphi_\infty \in \mathrm{Ball}(E^*)$ be a cluster point of $(\varphi_l)_l$ in $\sigma(E^*, E)$, the existence of which is guaranteed by the Banach-Alaoglu theorem. One verifies that the two double limits both have the value $\varphi_\infty(a_\infty)$ and hence are equal.

**2.2. The Eberlein-Šmulian Theorem.** *For a nonvoid subset $A \subset E$ the following properties relative to the weak topology $\sigma(E, E^*)$ are equivalent.*

(i) $\bar{A}^{\sigma(E,E^*)}$ *is countably compact.*
(i′) *$A$ is relatively countably compact.*
(ii) $\bar{A}^{\sigma(E,E^*)}$ *is sequentially compact.*
(ii′) *$A$ is relatively sequentially compact.*
(iii) $\bar{A}^{\sigma(E,E^*)}$ *is compact.*

*In this case $A$ and $\mathrm{Ball}(E^*)$ have the* IDLP. *Hence (after* 1.2) *for each $a \in \bar{A}^{\sigma(E,E^*)}$ there exists a sequence $(a_n)_n$ in $A$ which converges to $a \in E$ in $\sigma(E, E^*)$.*

Proof. The following implications are clear:

$$\begin{array}{ccccc} \text{(iii)} & \Rightarrow & \text{(i)} & \Leftarrow & \text{(ii)} \\ & & \Downarrow & & \Downarrow \\ & & \text{(i')} & \Leftarrow & \text{(ii')}. \end{array}$$

Now by Remark 2.1 $A$ and Ball $(E^*)$ have the IDLP in all these cases. Therefore it remains to prove (i′) ⇒ (iii) and (i) ⇒ (ii).

(i′) ⇒ (iii). By Remark 2.1 $A$ and $\mathrm{Ball}(E^*)$ have the IDLP, hence by Corollary 1.3 there exists for $\xi \in \bar{A}^{\sigma(E^{**},E^*)} \subset E^{**}$ a sequence $(a_n)_n$ in $A$ which converges to $\xi$ in $\sigma(E^{**}, E^{***})$. Since $A$ is relatively countably compact in $\sigma(E, E^*)$ it follows that $\xi \in E$. Hence $\bar{A}^{\sigma(E^{**},E^*)} \subset E$ and therefore $\bar{A}^{\sigma(E^{**},E^*)} = \bar{A}^{\sigma(E,E^*)}$. Now $A$ is norm bounded, so that by the Banach-Alaoglu theorem $\bar{A}^{\sigma(E^{**},E^*)}$ is $\sigma(E^{**}, E^*)$ compact.

It follows that $\bar{A}^{\sigma(E,E^*)}$ is $\sigma(E, E^*)$ compact.

(i) ⇒ (ii). By Remark 2.1 $\bar{A}^{\sigma(E,E^*)}$ and $\mathrm{Ball}(E^*)$ have the IDLP. Now let $(a_n)_n$ be a sequence in $\bar{A}^{\sigma(E,E^*)}$ and $a_\infty \in \bar{A}^{\sigma(E,E^*)}$ be a cluster point of $(a_n)_n$ in $\sigma(E, E^*)$. Let $U := \text{Linear Span}\{a_\infty, a_1, \ldots, a_n, \ldots,\}$ and $D \subset \mathrm{Ball}(U^*)$ be a countable dense subset of $\mathrm{Ball}(U^*)$ in $\sigma(U^*, U)$. To obtain such a subset choose for each $n \in \mathbb{N}$ a countable subset $D_n \subset \mathrm{Ball}(U^*)$ with $D_n | U_n := \text{Linear Span}\{a_\infty, a_1, \ldots, a_n\}$ dense in $\mathrm{Ball}(U^*) | U_n = \mathrm{Ball}(U_n^*)$ with respect to $\sigma(U_n^*, U_n)$, and then put $D := \bigcup_{n\in\mathbb{N}} D_n$. Since $D$ is countable there exists a subsequence $(a_{n(k)})_k$ of $(a_n)_n$ which converges to $a_\infty$ in $\sigma(U, D)$. Since $\bar{A}^{\sigma(E,E^*)} \cap U$ and $D$ have the IDLP it follows from Lemma 1.1 that $(a_{n(k)})_k$ converges to $a_\infty$ in $\sigma(U, \mathrm{Ball}(U^*))$, that is in $\sigma(E, E^*)$.

**2.3. The Double Limit Criterion for Banach spaces.** *Let $E$ be a Banach space. For a nonvoid subset $A \subset E$ the properties* (i)–(iii) *relative to the weak topology $\sigma(E, E^*)$ occurring in Theorem* 2.2 *are equivalent to* (iv): *$A$ is norm bounded and $A$ and* Ball$(E^*)$ *have the* IDLP.

Proof. In view of Theorem 2.2 it suffices to show (iv) $\Rightarrow$ (iii). By Corollary 1.3 there exists for $\xi \in \bar{A}^{\sigma(E^{**}, E^*)}$ a sequence $(a_n)_n$ in $A$ which converges to $\xi$ in $\sigma(E^{**}, E^{***})$. By the bipolar theorem $\xi$ is in the norm closure of the convex hull of $A$ in $E^{**}$. Since $E$ is a Banach space it is norm closed in $E^{**}$, and therefore $\xi \in E$. The assertion follows as in the proof of (i') $\Rightarrow$ (iii) in Theorem 2.2.

In conclusion we remark, that the method presented above can be used to prove more general results concerning the so called "Angelic Spaces" (cp. e.g. [2], p. 30). The author will treat this in another paper.

## References

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