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ALGEBRAS GENERATED BY IDEMPOTENTS AND THE SYMBOL CALCULUS FOR SINGULAR INTEGRAL OPERATORS

Steffen Roch and Bernd Silbermann

It is proved that in Banach algebras generated by two idempotents and, perhaps, by a certain flip operator the standard identity F_4 is fulfilled. The maximal ideal space of such algebras is determined and the corresponding symbol is given. By means of local techniques these results are applied to obtain a symbol calculus for singular integral operators with Carleman shift (changing the orientation) in weighted Banach spaces.

0. INTRODUCTION

In the late sixties, the C*-algebra generated by two idempotents p and q when the spectrum of pqp is the interval $[0,1]$ was studied by several authors from an operator theoretic point of view. We only mention the papers of P. Halmos [H] and G. K. Pedersen [Pe]. Perhaps, R. G. Douglas was the first who recognized that these results combined with certain local techniques lead to a symbol calculus for singular integral operators with piecewise continuous coefficients. S. C. Power succeeded in applying such ideas to the study of Fredholm properties of Hankel operators and Fourier integral operators with piecewise continuous generating functions (cf. [P1 - P3]). Recently, B. Silbermann [S] also utilized such ideas for describing the C*-algebra generated by Toeplitz and Hankel operators with piecewise quasicontinuous coefficients. Moreover, this approach immediately yields a symbol for singular integral operators with Carleman shift changing the orientation on the Hilbert space L^2. Note that questions of this kind were previously studied in L^p-spaces with Kvedelidze weight (in the case of piecewise continuous coefficients) using quite different methods. Here the pioneer work of I. Z. Gohberg and N. Ya. Krupnik should be quoted (see [GK 2]).

See also the paper of M. Costabel, who tried to simplify some of Gohberg and Krupnik's arguments.
We shall show that in the above mentioned C*-algebra techniques the underlying C*-algebras can be replaced by Banach algebras and we shall demonstrate how these results apply to the theory of singular integral operators.
The paper is organized as follows. Its first part deals with Banach algebras generated by two idempotent elements respective by two idempotents and a certain flip element. It turns out that the results obtained are widely analogous to their C*-algebra versions. On the other hand, the methods used here are partially related to Krupnik's book [K] and are quite different from the standard ones known from the C*-theory.
In the second part we apply these results to determine a "local" symbol for the Fredholm property in Banach algebras generated by singular integral operators with piecewise continuous coefficients and by a Carleman shift changing the orientation. Then local techniques will be employed to construct a "global" symbol. In particular, we apply this approach to algebras of operators defined on the weighted Banach spaces $l^{p,\gamma}$, $H_o^{\mu}(\varrho)$ and $L^p(\Gamma,\varrho)$.
As far as we know for the first two of the mentioned spaces the results seem to be new.

1. STANDARD IDENTITIES AND MAXIMAL IDEALS

For the reader's convenience and to fix notations we record some results from Krupnik's book [K].
Let $\mathfrak{R}$ be an algebra with unity e, $G\mathfrak{R}$ its group of invertible elements and $M(\mathfrak{R})$ the set of its two-sided maximal ideals.
Given $a_1,\dots,a_m \in \mathfrak{R}$ define the standard polynomial (of order m) by

$$F_m(a_1,\dots,a_m) = \sum_{\sigma\in S_m} (-1)^{\sigma} a_{\sigma(1)}\cdots a_{\sigma(m)} . \tag{1}$$

Herein S_m denotes the symmetric group and $(-1)^{\sigma}$ refers to the sign of the permutation $\sigma \in S_m$. The algebra $\mathfrak{R}$ is said to fulfil the standard identity of order m (in that case we shall write $\mathfrak{R} \in F_m$) if $F_m(a_1,\dots,a_m) = 0$ for arbitrarily taken $a_1,\dots,a_m \in \mathfrak{R}$.

THEOREM 1. Let $\mathfrak{R} \in F_{2n}$. Then

(a) for $M \in M(\mathfrak{R})$ the quotient algebra $\mathfrak{R}/M$ is isomorphic to $\mathbb{C}^{l\times l}$ with some $l = l(M) \leq n$;

(b) if η_M is the canonical homomorphism $\mathfrak{R} \to \mathfrak{R}/M$, if ζ_M is the isomorphism $\mathfrak{R}/M \to \mathbb{C}^{l\times l}$, and if $\nu_M = \zeta_M\eta_M$, then $x \in \mathfrak{R}$ is in $G\mathfrak{R}$ if and only if $\det \nu_M(x) \neq 0$ for all $M \in M(\mathfrak{R})$;

(c) the radical $R(\mathfrak{R})$ coincides with the intersection of all two-sided maximal ideals of $\mathfrak{R}$.

Let $M_l(\mathfrak{R})$ $(l=1,\ldots,n)$ stand for the set of all maximal ideals M of $\mathfrak{R}$ with $\mathfrak{R}/M \cong \mathbb{C}^{l\times l}$. Let $\varepsilon > 0$, $x_1,\ldots,x_r \in \mathfrak{R}$, $M_o \in M_l(\mathfrak{R})$, and put

$$U_{x_1,\ldots,x_r,\varepsilon}(M_o) = \{M \in M_l(\mathfrak{R}) : \|\nu_M(x_k) - \nu_{M_o}(x_k)\|_{L(\mathbb{C}^l)} < \varepsilon, \quad k = 1,\ldots,r\}.$$

The sets U form an open neighborhood base of M_o. These neighborhoods determine the so-called Gelfand topology on $M(\mathfrak{R})$, which is the coarsest topology so that for each $x \in \mathfrak{R}$ the function $\operatorname{smb} x : M \mapsto \nu_M(x) \in C^{l(M)\times l(M)}$ is continuous. Note that $M(\mathfrak{R})$ provided with its Gelfand topology is Hausdorff but, in general, not compact (see section 2).

THEOREM 2. Let $\mathfrak{R} \in F_{2n}$ be a Banach algebra with unity e and let $\mathfrak{C}$ be a (closed) subalgebra lying in the center of $\mathfrak{R}$. Then each maximal ideal in the Shilov boundary of $\mathfrak{C}$ is contained in a certain two-sided maximal ideal of $\mathfrak{R}$.

If $n = 1$, i.e. if $\mathfrak{R}$ is a commutative Banach algebra, Theorem 2 is well-known. A proof for the commutative case is in [GRS], for $n > 1$ only minor modifications of this proof are needed.

2. ALGEBRAS GENERATED BY TWO IDEMPOTENTS

Let $\mathfrak{B}$ be a Banach algebra with unity element e. If there are elements $a_1,\ldots,a_r \in \mathfrak{B}$ such that the algebra $\mathbf{alg_o}(a_1,\ldots,a_r)$ of all <u>finite</u> sums of products of $a_1,\ldots,a_r$ is dense in $\mathfrak{B}$, then we say that $a_1,\ldots,a_r$ generate $\mathfrak{B}$ and write $\mathfrak{B} = \operatorname{alg}(a_1,\ldots,a_r)$.

Throughout this section let $\mathfrak{B} = \operatorname{alg}(e,p,q)$ where p and q are idempotents in $\mathfrak{B}$, i.e. $p^2 = p$ and $q^2 = q$. The following observation is the basis for our objective to construct a matrix-valued symbol which determines the invertibility in $\mathfrak{B}$.

THEOREM 3. $alg(e,p,q) \in F_4$.

PROOF. By the continuity of the mapping $(a_1,\dots,a_4) \mapsto F_4(a_1,\dots,a_4)$ it suffices to show that $F_4(a_1,\dots,a_4) = 0$ for $a_i \in alg_0(e,p,q)$. Each $a \in alg_0(e,p,q)$ can be written as

$$a = pf_1(x)p + pf_2(x)q + qf_3(x)p + qf_4(x)q + \beta_5 q + \beta_6 e \tag{2}$$

where $x := pqp$, $x_i^0 := p$, f_i are algebraic polynomials in x and β_5 , β_6 complex numbers. Hence, a is a linear combination of the terms

$$px^\alpha p,\ px^\alpha q\ ,\ qx^\alpha p\ ,\ qx^\alpha q\ ,\ (\text{for } \alpha \geq 0),\ q \text{ and } e \tag{3}$$

Further, each $a \in \mathfrak{B}$ can be written as

$$a = pap + pa(e-p) + (e-p)ap + (e-p)a(e-p). \tag{4}$$

Taking into account (3) and (4) it follows that each $a \in alg_0(e,p,q)$ is a linear combination of terms of the form e, $A_1^\alpha := px^\alpha p$, $A_2^\alpha := px^\alpha q(e-p)$, $A_3^\alpha := (e-p)qx^\alpha p$, and $A_4^\alpha = (e-p)qx^\alpha q(e-p)$ with $\alpha \geq -1$, where we ddfine $x^{-1} := e$.

The following table shows how to compute the products $A_i^\alpha \cdot A_j^\beta$

A_i^α \ A_j^β	A_1^β	A_2^β	A_3^β	A_4^β
A_1^α	$A_1^{\alpha+\beta}$	$A_2^{\alpha+\beta}$	0	0
A_2^α	0	0	$A_1^{\alpha+\beta+1}-A_1^{\alpha+\beta+2}$	$A_2^{\alpha+\beta+1}-A_2^{\alpha+\beta+2}$
A_3^α	$A_3^{\alpha+\beta}$	$A_4^{\alpha+\beta}$	0	0
A_4^α	0	0	$A_3^{\alpha+\beta+1}-A_3^{\alpha+\beta+2}$	$A_4^{\alpha+\beta+1}-A_4^{\alpha+\beta+2}$

Let $A_i = \{A_i^\alpha : \alpha \geq -1\}$. By the multilinearity of F_4 it suffices to verify that $F_4(a_1,\dots,a_4) = 0$ if each a_j belongs to one of the A_i's.

If two elements (say, a_1 and a_2) lie in the same A_i then $F_4(a_1,\dots,a_4) = 0$. Indeed, let for instance $a_1, a_2 \in A_1$. Then divide F_4 into the sum $\sum_1 - \sum_2$ where $\sum_1$ is the sum of all products $a_{\sigma(1)} \cdots a_{\sigma(4)}$ with $\sigma^{-1}(1) < \sigma^{-1}(2)$ and $\sum_2$ is the sum of all products with $\sigma^{-1}(1) > \sigma^{-1}(2)$. Each product $r = a_{\sigma(1)}\cdots a_{\sigma(4)} \in \Sigma_1$ corresponds with a product $\bar{r}$ in Σ_2 which is obtained from r by interchanging the elements a_1 and a_2. We point out that $r - \bar{r} = 0$. If the elements a_1 and a_2 stand in r side-by-side then $r = \bar{r}$ since a_1 and a_2 commute. If there is exactly one element (say, a_3) standing in r between a_1 and a_2 then $r = 0$ and $\bar{r} = 0$ for $a_3 \in A_i$ $(i > 1)$ (cp. the table). If $a_3 \in A_1$ then a_1 , a_2 , a_3 commute and, hence, $r - \bar{r} = 0$. Finally assume that there are two elements standing between a_1 and a_2 , and let b be their product. By the table, b is in A_i or in $\underline{A_i} - A_i$ for some $i \geq 1$. Thus, as in the previous step, $r = \bar{r} = 0$, or the elements a_1, b, a_2 commute pairwise, and so $r - \bar{r} = 0$. Analogously the cases when $a_1, a_2 \in A_i$, $i > 1$, can be checked.

What remains to prove is that $F_4(a_1,\ldots,a_4) = 0$ when $a_i \in A_i$ for $i = 1,\ldots,4$. In this case we obtain

$$F_4(a_1,a_2,a_3,a_4) = - a_1a_2a_4a_3 + a_2a_4a_3a_1 + a_3a_1a_2a_4 - a_4a_3a_1a_2$$

(note that the other 20 products occuring in (1) vanish), and an easy computation yields the assertion.

REMARK. If p and q do not commute then $\mathrm{alg}(e,p,q) \notin F_3$, as the example $F_3(p,q,e-p) = pq - qp$ shows.

By Theorem 1 there exists an at most 2×2 symbol on $M(\mathcal{B})$. Before we explain the structure of $M(\mathcal{B})$ and the explicit form of the symbol we quote an elementary lemma which describes those matrices in $\mathbb{C}^{2\times 2}$ which are idempotents

LEMMA 1. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{C}^{2\times 2}$. The matrix A is idempotent if and only if one of the following conditions is fulfilled:

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.1}$$

$$A = \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ b & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 \\ b & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix} \tag{5.2}$$

with some $b \in \mathbb{C}$, $b \neq 0$.

There exist a $g \neq 0$ such that

$$A = \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} a & \sqrt{a(1-a)} \\ \sqrt{a(1-a)} & 1-a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & g^{-1} \end{pmatrix}, \tag{5.3}$$

where $\sqrt{}$ stands for the main branch of the root function. The proof is a straightforward computation.

Let (H1) denote the following hypothesis:

(H1) The spectrum $\sigma_{\mathfrak{B}}(pqp)$ is connected and

$$\{0,1\} = \sigma_{\mathfrak{B}}(p) = \sigma_{\mathfrak{B}}(q) \subseteq \sigma_{\mathfrak{B}}(pqp) .$$

THEOREM 4. (a) If (H1) is fulfilled then $M_1(\mathfrak{B})$ consists of exactly four ideals. These are clos id(p,q), clos id(p,e-q), clos id(e-p,q) and clos id(e-p,e-q), where "clos id" stands for the smallest closed two-sided ideal in $\mathfrak{B}$ generated by the elements quoted in parentheses.

(b) If $M \in M_2(\mathfrak{B})$ then there exist an invertible matrix $E \in \mathbb{C}^{2\times 2}$ and a complex number a (both depending on M) such that

$$(\text{smb } p)(M) := E^{-1}\nu_M(p)E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} ,$$

$$(\text{smb } q)(M) := E^{-1}\nu_M(q)E = \begin{pmatrix} a & \sqrt{a(1-a)} \\ \sqrt{a(1-a)} & 1-a \end{pmatrix},$$

$$(\text{smb } e)(M) := E^{-1}\nu_M(e)E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

PROOF. (a) If $M \in M_1(\mathfrak{B})$ then $\nu_M(p)$ and $\nu_M(q)$ can take the values 0, 1 only, so that at most the four ideals quoted exist.

Consider M = clos id(p,q) (i.e. $\nu_M(p) = \nu_M(q) = 0$). First we show that M is a proper ideal in $\mathfrak{B}$: Assume the contrary, i.e. that $e \in M$. By (2) we can represent e as

$$e = \lim pf_1^{(n)}(x)p + pf_2^{(n)}(x)q + qf_3^{(n)}(x)p + qf_4^{(n)}(x)q + \beta_5^{(n)}q .$$

Multiplying this equation from both sides by p gives $p = \lim pf^{(n)}(x)p$, where $f^{(n)}$ are polynomials in $x = pqp$. Thus, $p \in \text{alg}(pqp)$.

By (H1), $\sigma_{\mathfrak{B}}(p) = \{0,1\}$. Let Ω be a disjoint union of two sufficiently small open disks D_0, D_1 with $0 \in D_0$, $1 \in D_1$. The theorem on the upper semicontinuity of spectra in Banach algebras (see [R], Theorem 10.20) involves that for each polynomial $f(x) \in \text{alg}(p\ q\ p)$ which is sufficiently close to p, $\sigma_{\mathfrak{B}}(f(pqp)) \subseteq \Omega$. This is a contradiction since $\sigma_{\mathfrak{B}}(pqp)$ was assumed to be connected, what implies the connectedness of $\sigma_{\mathcal{B}}(f(pqp))$ by the spectral mapping theorem for polynomials. Hence, M is a proper ideal. It remains to prove that $\mathcal{B}/M \cong \mathbb{C}^1$. Let η_M denote the canonical homomorphism of $\mathcal{B}$ onto $\mathcal{B}/M$. For $a \in alg_0(e,p,q)$ the identity (2) gives $\eta_M(a) = \alpha\eta_M(e)$, and this immediately gives the assertion since $\eta_M(e) \neq 0$.

(b) Let $M \in M_2(\mathcal{B})$. The eigenvalues of the matrix $\nu_M(p) \in \mathbb{C}^{2\times 2}$ must be 0 and 1 since otherwise $\nu_M(p) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ or $\nu_M(p) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (cp. (5.1) in Lemma 1), both contradicting $\mathcal{B}/M \cong \mathbb{C}^{2\times 2}$. Hence there is an invertible matrix D transforming $\nu_M(p)$ into its Jordan canonical form: $D^{-1}\nu_M(p)D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. If $D^{-1}\nu_M(q)D$ were of the form (5.1) or (5.2) (see Lemma 1) this would also contradict the fact that $\mathcal{B}/M \cong \mathbb{C}^{2\times 2}$. Hence, by (5.3), there is a matrix $G = \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}$ with $g \neq 0$ such that

$$G^{-1}D^{-1}\nu_M(q)DG = \begin{bmatrix} a & \sqrt{a(1-a)} \\ \sqrt{a(1-a)} & 1-a \end{bmatrix}$$

with an $a \in \mathbb{C}$.

Now put $E = DG$ and notice that $E^{-1}\nu_M(p)E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $E^{-1}\nu_M(e)E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ to finish the proof.

For $a \in \mathcal{B}$, we define the <u>symbol of a</u> at $M \in M_2(\mathcal{B})$ by $(\text{smb } a)(M) := E^{-1}\nu_M(a)E$; E the matrix occuring in (b).

Our next concern is the determination of the maximal ideal space of $\mathcal{B}$. To that end put

$$\mathcal{B}_p := alg(p,pqp), \quad \mathcal{B}_q := alg(q,qpq)$$

$$\mathcal{B}_{e-p} := alg(e-p,(e-p)(e-q)(e-p))$$

$$\mathcal{B}_{p,e-p} := alg(e,pqp + (e-p)(e-q)(e-p)).$$

LEMMA 2. (a) $\mathcal{B}_{p,e-p}$ is a subalgebra of the center of $\mathcal{B}$.

(b) Let $a \in \mathcal{B}$. Then $pap \in \mathcal{B}_p$ and $(e-p)a(e-p) \in \mathcal{B}_{e-p}$.

PROOF. (a) Obviously, e and p commute with $pqp + (e-p)(e-q)(e-p)$, and a simple computation shows that

$$q(pqp+(e-p)(e-q)(e-p)) = (pqp+(e-p)(e-q)(e-p))q = qpq.$$

(b) Let $a \in \mathcal{B}$. Approximate a by $a_n \in alg_o(e,p,q)$. The representation (2) yields immediately that $pa_np \in \mathcal{B}_p$. Hence, $\lim pa_np = pap \in \mathcal{B}_p$. Analogously, $(e-p)a(e-p) \in \mathcal{B}_{e-p}$.

Besides the hypotheses (H1) we need the following one:
(H2) $\sigma_{\mathcal{B}}(b) = \sigma_{\mathcal{B}_{p,e-p}}(b)$, where we put $b = pqp+(e-p)(e-q)(e-p)$ for brevity.

LEMMA 3. a)

$$\sigma_{\mathcal{B}}(b) = \sigma_{\mathcal{B}_p}(pqp) \cup \sigma_{\mathcal{B}_{e-p}}((e-p)(e-q)(e-p)) \,. \tag{6}$$

b) If (H1) is fulfilled then

$$\sigma_{\mathcal{B}_p}(pqp) \quad = \quad \sigma_{\mathcal{B}}(pqp) \,. \tag{7}$$

c) If the hypotheses (H1) and (H2) are fulfilled then

$$\sigma_{\mathcal{B}_{p,e-p}}(b) = \sigma_{\mathcal{B}_p}(pqp) \,.$$

PROOF. a) Let $\lambda \notin \sigma_{\mathcal{B}}(b)$. Then there is an $a \in \mathcal{B}$ such that $a(b-\lambda e) = e$. Multiplying this from both sides by p we obtain $pap(pqp-\lambda p) = e$. By Lemma 2, $pap \in \mathcal{B}_p$. Hence, $\lambda \notin \sigma_{\mathcal{B}_p}(pqp)$. Analogously we have $\lambda \notin \sigma_{\mathcal{B}_{e-p}}((e-p)(e-q)(e-p))$. Now let $\lambda \notin \sigma_{\mathcal{B}_p}(pqp) \cup \sigma_{\mathcal{B}_{e-p}}((e-p)(e-q)(e-p))$, and let pap resp. $(e-p)c(e-p)$ be the inverses of $pqp - \lambda p$ resp. $(e-p)(e-q)(e-p) - \lambda(e-p)$ in $\mathcal{B}_p$ resp. $\mathcal{B}_{e-p}$. Obviously, $pap + (e-p)c(e-p)$ is the inverse of $b - \lambda e$ in $\mathcal{B}$, and this gives (6).

b) Let $\lambda \notin \sigma_{\mathcal{B}_p}(pqp) \cup \{0\}$ and let pap be the inverse of $pqp - \lambda p$. Then $pap - \lambda^{-1}(e-p)$ is the inverse of $pqp - \lambda e$ in $\mathcal{B}$, i.e. $\lambda \notin \sigma_{\mathcal{B}}(pqp)$. On the other hand, if $\lambda \notin \sigma_{\mathcal{B}}(pqp)$ and if a is the inverse of $pqp - \lambda e$ in $\mathcal{B}$ then $pap(pqp - \lambda p) = p$, and by Lemma 2, pap is the inverse of $(pqp - \lambda p)$ in $\mathcal{B}_p$, i.e. $\lambda \notin \sigma_{\mathcal{B}_p}(pqp)$. Moreover, in this case $\lambda \neq 0$ since otherwise the equality $a \cdot pqp = e$ would imply that $e-p = 0$ what contradicts (H1).

c) In the first step we prove that for arbitrary idempotents p, q the equality

$$\sigma_{\mathcal{B}_p}(pqp) \cup \{0,1\} = \sigma_{\mathcal{B}_{e-p}}((e-p)(e-q)(e-p)) \cup \{0,1\} \qquad (8)$$

holds. Obviously, (8) is valid if $p,q \in \{0,e\}$. Let $p,q \neq 0,e$. In what follows we need the well-known equality

$$\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\} , \qquad (9)$$

holding for elements A, B of an arbitrary algebra. Thus,

$$\begin{aligned}
&\sigma_{\mathcal{B}_{e-p}}((e-p)(e-q)(e-p)) \cup \{0,1\} = \\
&= \sigma_{\mathcal{B}_{e-p}}((e-p)-(e-p)q(e-p)) \cup \{0,1\} \\
&= 1 - (\sigma_{\mathcal{B}_{e-p}}((e-p)q(e-p)) \cup \{0,1\}) \\
&= 1 - (\sigma_{\mathcal{B}}((e-p)q(e-p)) \cup \{0,1\}) && \text{by (7)} \\
&= 1 - (\sigma_{\mathcal{B}}(q(e-p)q) \cup \{0,1\}) && \text{by (9)} \\
&= 1 - (\sigma_{\mathcal{B}_q}(q(e-p)q) \cup \{0,1\}) && \text{by (7)} \\
&= \sigma_{\mathcal{B}_q}(qpq) \cup \{0,1\} \\
&= \sigma_{\mathcal{B}}(qpq) \cup \{0,1\} && \text{by (7)} \\
&= \sigma_{\mathcal{B}}(pqp) \cup \{0,1\} && \text{by (9)} \\
&= \sigma_{\mathcal{B}_p}(pqp) \cup \{0,1\} && \text{by (7)} .
\end{aligned}$$

Now we can conclude as follows: By (6) and (H2),

$$\sigma_{\mathcal{B}_{p,e-p}}(b) = \sigma_{\mathcal{B}_p}(pqp) \cup \sigma_{\mathcal{B}_{e-p}}((e-p)(e-q)(e-p)) . \qquad (10)$$

By (7) and (H1),

$$\{0,1\} \in \sigma_{\mathcal{B}_p}(pqp) , \qquad (11)$$

and by (11) and (8)

$$\sigma_{\mathcal{B}_{e-p}}((e-p)(e-q)(e-p)) \subseteq \sigma_{\mathcal{B}_p}(pqp) .$$

Lemma 3 is completely proved.

COROLLARY 1. The maximal ideal space of $\mathcal{B}_{p,e-p}$ is homeomorphic to the spectrum $\sigma_{\mathcal{B}_p}(pqp)$. I.e., by (H1), it is connected, and $0,1 \in M(\mathcal{B}_{p,e-p})$.

The fact that $M(\mathcal{B}_{p,e-p})$ is homeomorphic to the spectrum of pqp is extremely favourable since pqp is the "simplest Toeplitz operator", and for Toeplitz operators numerous properties of their spectra are known.

Now we are in a position to identify the maximal ideal space of $\mathcal{B}$:

THEOREM 5. Let (H1) and (H2) be fulfilled.
(a) If $M \in M(\mathcal{B})$ then $M \cap \mathcal{B}_{p,e-p}$ is a maximal ideal of $\mathcal{B}_{p,e-p}$.
(b) Every maximal ideal m of $\mathcal{B}_{p,e-p}$ with $m \neq 0,1$ is contained in exactly one maximal ideal M of $\mathcal{B}$ which, moreover, belongs to $M_2(\mathcal{B})$. Furthermore, $M_2(\mathcal{B})$ is homeomorphic to $\sigma_{\mathcal{B}_p}(pqp) \setminus \{0,1\}$.
(c) Each of the points 0 and $1 \in \sigma_{\mathcal{B}_p}(pqp) = M(\mathcal{B}_{p,e-p})$ is contained in exactly two ideals both belonging to $M_1(\mathcal{B})$. Thereby, $\{1\} \subset \text{clos id}(p,q)$, $\{1\} \subset \text{clos id}(e-p,e-q)$,
$\{0\} \subset \text{clos id}(p,e-q)$ and $\{0\} \subset \text{clos id}(e-p,q)$.
Here, $\{0\}$ and $\{1\}$ refer to the ideals corresponding with the points 0 and 1, respectively.

PROOF. (a) See [A].
(b) Let $\mathfrak{M}$ be the set of all maximal ideals of $\mathcal{B}_{p,e-p}$ which are contained in a maximal ideal of $\mathcal{B}$. Assume that there is an $m^* \in M(\mathcal{B}_{p,e-p}) \setminus \mathfrak{M}$. Let $b = pqp + (e-p)(e-q)(e-p)$. Then $b - m^*(b)e \in m^*$, i.e. $b - m^*(b)e$ is not invertible in $\mathcal{B}_{p,e-p}$. By (H2), $b - m^*(b)e$ is not invertible in $\mathcal{B}$; thus there exists a maximal ideal $M \in M(\mathcal{B})$ such that $b - m^*(b)e \in M$. By (a) we can find a maximal ideal m $(= M \cap \mathcal{B}_{p,e-p})$ of $\mathcal{B}_{p,e-p}$ such that $b - m^*(b)e \in m$. This implies that $m^*(b) = m(b)$, and since b generates the algebra $\mathcal{B}_{p,e-p}$, this gives $m^*(a) = m(a)$ for each $a \in \mathcal{B}_{p,e-p}$, whence $m^* = m$. Thus, each maximal ideal of $\mathcal{B}_{p,e-p}$ is extendible to a maximal ideal of $\mathcal{B}$.

Now let $m \in M(\mathcal{B}_{p,e-p}) \setminus \{\{0\},\{1\}\}$. The characterization of the ideals in $M_1(\mathcal{B})$ given in Theorem 4 shows that these maximal ideals can only generate the points $\{0\},\{1\}$ of $M(\mathcal{B}_{p,e-p})$. Hence, if m is extendible to the matrix ideal $M \in M(\mathcal{B})$ then $\mathcal{B}/M \cong \mathbb{C}^{2\times 2}$. Again by Theorem 4, the symbols of p, q on $\mathcal{B}(M)$ are $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ respective $\begin{pmatrix} a & \sqrt{a(1-a)} \\ \sqrt{a(1-a)} & 1-a \end{pmatrix}$.
If there were another maximal ideal $\tilde{M} \in M_2(\mathcal{B})$ with $m \subseteq \tilde{M}$, then $(\text{smb } p)(\tilde{M}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $(\text{smb } q)(\tilde{M}) = \begin{pmatrix} a & \sqrt{\tilde{a}(1-\tilde{a})} \\ \sqrt{\tilde{a}(1-\tilde{a})} & 1-\tilde{a} \end{pmatrix}$. This yields

$$(\text{smb } b)(M) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad (\text{smb } b)(\tilde{M}) = \begin{pmatrix} \tilde{a} & 0 \\ 0 & \tilde{a} \end{pmatrix},$$

and so we obtain $a = \tilde{a} = m$, i.e. the symbols on M and $\tilde{M}$ coincide, i.e. $M = \tilde{M}$.

Now we are going to prove that $M_2(\mathcal{B})$ and $\sigma_{\mathcal{B}_p}(pqp) \setminus \{0,1\}$ are homeomorphic.
Since the topologies on $M(\mathcal{B}_{p,e-p})$ respective on $M(\mathcal{B})$ are generated by all elements (symbols) of $\mathcal{B}_{p,e-p}$ respective of $\mathcal{B}$, the topology on $\sigma_{\mathcal{B}_p}(pqp) \setminus \{0,1\} = M(\mathcal{B}_{p,e-p}) \setminus \{\{0\},\{1\}\}$ is at most coarser than that on $M_2(\mathcal{B})$.
On the other hand, let for $M \in M_2(\mathcal{B})$

$$(\text{smb } p)(M) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (\text{smb } q)(M) = \begin{bmatrix} a(M) & \sqrt{a(M)(1-a(M))} \\ \sqrt{a(M)(1-a(M))} & 1-a(M) \end{bmatrix}.$$

Hence, $(\text{smb } b)(M) = \begin{pmatrix} a(M) & 0 \\ 0 & a(M) \end{pmatrix}$, and by the hypothesis (H1), $a(M)$ is the value of the Gelfand transform of b at the ideal $m \in \sigma_{\mathcal{B}_{p,e-p}}(b) \setminus \{0,1\}$ which corresponds with M. So, $a(M)$ is nothing else than the identical mapping of $M(\mathcal{B}_{p,e-p}) \setminus \{\{0\},\{1\}\}$ onto itself; this means that a is continuous on $M(\mathcal{B}_{p,e-p}) \setminus$ $\setminus \{\{0\},\{1\}\}$. Therefore, each 2x2-symbol of an element of $\mathcal{B}$ is a continuous matrix function on $M(\mathcal{B}_{p,e-p}) \setminus \{\{0\},\{1\}\}$, and thus, the topology on $M(\mathcal{B}_{p,e-p})\setminus\{\{0\},\{1\}\}$ is finer than that on $M_2(\mathcal{B})$. Part (b) is proved.

(c) Let, for instance, $M = \text{clos id }(p,q)$.

Then $(\text{smb } b)(M) = 0\cdot0\cdot0 + 1\cdot1\cdot1 = 1$, i.e. $\{1\} \subseteq M$. Analogously one checks the other cases. The proof of Theorem 5 is complete.

By Theorem 5, we can consider the symbol smb a $(a \in \mathcal{B})$ as a function defined on $M_1 \cup (\sigma_{\mathcal{B}}(pqp)\setminus\{0,1\})$. The continuous extension of smb a onto $\sigma_{\mathcal{B}}(pqp)$ gives

$$(\text{smb } p)(0) = (\text{smb } p)(1) = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$$

and $$(\text{smb } q)(0) = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}, \quad (\text{smb } q)(1) = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}.$$

A comparision with Theorem 4 shows that for $a \in \mathcal{B}$ $(\text{smb } a)(0)$ is invertible if and only if the images of a in $\mathcal{B}/\text{clos id}(p,e-q)$ and in $\mathcal{B}/\text{clos id}(e-p,q)$ are invertible, and that $(\text{smb } a)(1)$ is invertible if and only if the images of a in $\mathcal{B}/\text{clos id}(p,q)$ and in $\mathcal{B}/\text{clos id}(e-p,e-q)$ are invertible. Therefore, we can extend smb a formally onto $\sigma_{\mathcal{B}}(pqp)$, and we consider smb a as the continuous matrix function given on $\sigma_{\mathcal{B}}(pqp)$ by

$$(\text{smb } p)(x) = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \quad (\text{smb } q)(x) = \begin{bmatrix} x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1-x \end{bmatrix}$$

and $(\text{smb } e)(x) = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ where the square root is unterstood in the sense of the main branch.

COROLLARY 2 (C^*-algebra version, cf. [P3]).

Let $\overline{e}$, $\overline{p}$ and $\overline{q} \in C([0,1], \mathbb{C}^{2\times2})$ be defined by

$$\overline{e}(x) = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \quad \overline{p}(x) = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} \text{ and } \overline{q}(x) = \begin{bmatrix} x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1-x \end{bmatrix},$$

and let e, p and q be self adjoint elements of a certain C^*-algebra $\mathcal{B}$ with $\sigma_{\mathcal{B}}(pqp) = [0,1]$ and $p^2 = p$, $q^2 = q$. Then the C^*-algebras $\text{alg}(e,p,q)$ and $\text{alg}(\overline{e},\overline{p},\overline{q})$ are isometrically isomorphic, and the isomorphism transforms e,p,q into $\overline{e},\overline{p},\overline{q}$, respectively.

The proof results from the Theorems 4 and 5 by using some simple C^*-algebra arguments. Notice that Corollary 2 is essentially due to Halmos ([H], see also [P2]), who gave a proof when p and q

are self adjoint projection operators on a Hilbert space.

We conclude this section by mentioning two sufficient conditions for (H2) to be fulfilled.

PROPOSITION 1. Each of the following conditions implies (H2): (a) $\mathbb{C}\setminus\sigma_{\mathcal{B}}(pqp + (e-p)(e-q)(e-p))$ is connected.

(b) $M(\mathcal{B}_{p,e-p})$ coincides with its Shilov boundary (i.e. $\sigma_{\mathcal{B}_{p,e-p}}(pqp + (e-p)(e-q)(e-p))$ coincides with its topological boundary).

PROOF. (a) See [R], Theorem 10.18.

(b) Immediate from Theorem 2.

3. ALGEBRAS GENERATED BY TWO IDEMPOTENTS AND ONE FLIP

Let $\mathfrak{C}$ be the Banach algebra $alg(e,p,q,j)$ with the unit e where

$$p^2 = p, \quad q^2 = q, \quad j^2 = e \quad \text{and} \quad jpj = e-p, \quad jqj = e-q. \tag{12}$$

We are going to construct a symbol related to the invertibility in $\mathfrak{C}$.

THEOREM 6. $alg(e,p,q,j) \in F_4$.

PROOF. This proof is an extended copy of the proof of Theorem 3. As in the latter, each element $a \in alg_0(e,p,q,j)$ is a (finite) linear combination of items of the form

$$\begin{aligned}
A_1^\alpha &= p\,x^\alpha\,p, & A_2^\alpha &= p\,x^\alpha\,qjp\\
A_3^\alpha &= p\,x^\alpha\,q(e-p), & A_4^\alpha &= p\,x^\alpha\,j(e-p)\\
A_5^\alpha &= (e-p)q\,x^\alpha\,p, & A_6^\alpha &= (e-p)q\,x^\alpha\,qjp\\
A_7^\alpha &= (e-p)q\,x^\alpha\,q(e-p), & A_8^\alpha &= (e-p)q\,x^\alpha\,j(1-p),
\end{aligned} \tag{13}$$

where $x = pqp$, $x^0 := p$, $x^{-1} = e$.

The following table shows how to compute the products $A_i^\alpha A_j^\beta$. Put $\gamma = \alpha + \beta$.

A_i^α \ A_j^β	A_1^β	A_2^β	A_3^β	A_4^β	A_5^β	A_6^β	A_7^β	A_8^β
A_1^α	A_1^γ	A_2^γ	A_3^γ	A_4^γ	0	0	0	0
A_2^α	A_2^γ	$A_1^{\gamma+2}-A_1^{\gamma+1}$	$A_4^{\gamma+2}-A_4^{\gamma+1}$	A_3^γ	0	0	0	0
A_3^α	0	0	0	0	$A_1^{\gamma+1}-A_1^{\gamma+2}$	$A_2^{\gamma+1}-A_2^{\gamma+2}$	$A_3^{\gamma+1}-A_3^{\gamma+2}$	$A_4^{\gamma+1}-A_4^{\gamma+2}$
A_4^α	0	0	0	0	$-A_2^\gamma$	$A_1^{\gamma+1}-A_1^{\gamma+2}$	$A_4^{\gamma+1}-A_4^{\gamma+2}$	$-A_3^\gamma$
A_5^α	A_5^γ	A_6^γ	A_7^γ	A_8^γ	0	0	0	0
A_6^α	A_6^γ	$A_5^{\gamma+2}-A_5^{\gamma+1}$	$A_8^{\gamma+2}-A_8^{\gamma+1}$	A_7^γ	0	0	0	0
A_7^α	0	0	0	0	$A_5^{\gamma+1}-A_5^{\gamma+2}$	$A_6^{\gamma+1}-A_6^{\gamma+2}$	$A_7^{\gamma+1}-A_7^{\gamma+2}$	$A_8^{\gamma+1}-A_8^{\gamma+2}$
A_8^α	0	0	0	0	$-A_6^\gamma$	$A_5^{\gamma+1}-A_5^{\gamma+2}$	$A_8^{\gamma+1}-A_8^{\gamma+2}$	$-A_7^\gamma$

Let $A_i = \{A_i^\alpha : \alpha \geq -1\}$ $(i=1,\ldots,8)$ and $B_i = A_{2i-1} \cup A_{2i}$ $(i=1,\ldots,4)$. If two elements (say a_1 and a_2) lie in the same B_i then $F_4(a_1,a_2,a_3,a_4) = 0$. This can be proved in the same way as the corresponding assertion in the proof of Theorem 3. Hence we can restrict ourselves to the case when $a_i \in B_i$. A straightforward evaluation of the resulting 16 terms $F_4(a_1,\ldots,a_4)$ shows that $F_4(a_1,\ldots,a_4) = 0$ if $a_i \in B_i$, hence $\mathfrak{C} \in F_4$.

Note that there is at least one more proof of Theorem 6, which runs as follows:

The mapping $F' : \mathfrak{C} \to \mathfrak{C}^{2\times 2}$ given for $a \in \mathfrak{C}$ by

$$F'(a) = \begin{pmatrix} pap & pa(e-p) \\ (e-p)ap & (e-p)a(e-p) \end{pmatrix}$$

is a homomorphism of $\mathfrak{C}$ into $\mathfrak{C}^{2\times 2}$ with a trivial kernel, and there are constants $c_1, c_2 > 0$ such that

$$c_1\|a\| \leq \|F'(a)\| \leq c_2\|a\| .$$

Put $F(a) = \begin{pmatrix} e & 0 \\ 0 & j \end{pmatrix} F'(a) \begin{pmatrix} e & 0 \\ 0 & j \end{pmatrix}$, where the mapping $\begin{pmatrix} e & 0 \\ 0 & j \end{pmatrix} : \mathfrak{C}^{2\times 2} \to \mathfrak{C}^{2\times 2}$ is obviously invertible. Since

$$\begin{pmatrix} e & 0 \\ 0 & j \end{pmatrix}\begin{pmatrix} pap & pa(e-p) \\ (e-p)ap & (e-p)a(e-p) \end{pmatrix}\begin{pmatrix} e & 0 \\ 0 & j \end{pmatrix} = \begin{pmatrix} pap & pajp \\ pjap & pjajp \end{pmatrix} ,$$

F is a continuous homomorphism of $\mathfrak{C}$ into the algebra $(p\mathfrak{C}p)^{2\times 2}$. Notice that a is invertible in $\mathfrak{C}$ if and only if $F(a)$ is invertible in $(p\mathfrak{C}p)^{2\times 2}$ (with the unit $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$). The second proof is finished by the observation that $p\mathfrak{C}p$ is a commutative Banach algebra, which follows from $p\mathfrak{C}p = \text{alg } B_1$ (see (13) and the table after (13)).

The analogue of Theorem 4 reads as follows:

THEOREM 7. $M(\mathfrak{C}) = M_2(\mathfrak{C})$, and for each $M \in M_2(\mathfrak{C})$ there is an invertible matrix $E \in \mathbb{C}^{2\times 2}$ and a complex number a both depending on M such that

$$(\text{smb } e)(M) = E^{-1}\nu_M(e)E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ,$$

$$(\text{smb } p)(M) = E^{-1}\nu_M(p)E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} ,$$

$$(\text{smb } q)(M) = E^{-1}\nu_M(q)E = \begin{pmatrix} a & \sqrt{a(1-a)} \\ \sqrt{a(1-a)} & 1-a \end{pmatrix}$$

and either

$$(\mathrm{smb}\ j)(M) = E^{-1}\nu_M(j)E = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \text{ or}$$

$$(\mathrm{smb}\ j)(M) = E^{-1}\nu_M(j)E = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} .$$

REMARK. For $a \in \mathfrak{C}$ we call $(\mathrm{smb}\ a)(M) := E^{-1}\nu_M(a)E$ the symbol of a at M. Deviating from the situation in section 2, the "axioms" (12) do not determine a unique symbol. This is consequence of the fact that (12) remains invariant if j is replaced by $-j$. We shall see later which kind of additional information allows to overcome this difficulty.

PROOF. A little thought shows that there are no complex numbers a, b, c such that

$$a^2 = a, \quad b^2 = b, \quad c^2 = 1, \quad \text{and} \quad cac = 1-a, \quad cbc = 1-b.$$

Hence, $M_1(\mathfrak{C})$ is empty.
Let $M \in M(\mathfrak{C}) = M_2(\mathfrak{C})$. Since $\nu_M^2(j) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ the assumptions that $\nu_M(p) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ respective $\nu_M(p) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ would imply that

$$\nu_M(j)\nu_M(p)\nu_M(j) = \nu_M(p) \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \nu_M(p).$$

Hence, in accordance with Lemma 1, the eigenvalues of $\nu_M(p)$ are 0 and 1. Let D denote the (invertible) matrix transforming $\nu_M(p)$ into its Jordan canonical form: $D^{-1}\nu_M(p)D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and let $D^{-1}\nu_M(j)D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The identities $j^2 = e$ and $jpj = e-p$ imply immediately that $a = d = 0$ and $c = b^{-1}$, i.e. that $D^{-1}\nu_M(j)D = \begin{pmatrix} 0 & b \\ b^{-1} & 0 \end{pmatrix}$ with some $b \in \mathbb{C} \setminus \{0\}$. A simple computation shows that if $D^{-1}\nu_M(q)D$ were of the form (5.2) then $jqj = e-q$ is violated. Hence there is a matrix $G = \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}$ $(g \in \mathbb{C},\ g \neq 0)$ such that $G^{-1}D^{-1}\nu_M(q)DG = \begin{bmatrix} a & \sqrt{a(1-a)} \\ \sqrt{a(1-a)} & 1-a \end{bmatrix}$ with some $a \in \mathbb{C}$.

Notice that transformation by G does not change the structure of p respective j:

$$G^{-1}D^{-1}\nu_M(p)DG = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$$

$$G^{-1}D^{-1}\nu_M(j)DG = \begin{pmatrix}0 & bg\\ (bg)^{-1} & 0\end{pmatrix} .$$

To guarantee the identity $jqj = e-q$ it is necessary that $(bg)^2 = -1$, i.e. either $bg = i$ or $bg = -i$. Now set $E = DG$ to finish the proof.

LEMMA 4. a) The algebra $\mathcal{B}_{p,e-p}$ lies in the centre of $\mathfrak{C}$.

b) The algebras $\mathcal{B}_{p,e-p}$ and $\mathcal{B}_p$ are isomorphic, and hence, the maximal ideal space of $\mathcal{B}_{p,e-p}$ is homeomorphic to $\sigma_{\mathcal{B}}(pqp)$.

PROOF. a) Apply Lemma 1,b and the identity

$$jpqp = (e-p)(e-q)(e-p)j .$$

b) Obviously,

$$S : \mathcal{B}_{p,e-p} \to \mathcal{B}_p , \quad a \mapsto pap$$

defines a homomorphism from $\mathcal{B}_{p,e-p}$ into $\mathcal{B}_p$, and $S^{-1}:\mathcal{B}_p \to \mathcal{B}_{p,e-p}$, $b \mapsto b + jbj$ is its inverse.

THEOREM 8. The maximal ideal space $M(\mathfrak{C})$ is homeomorphic to $M(\mathcal{B}_{p,e-p}) \cong \sigma_{\mathcal{B}_p}(pqp)$, and the symbol smb a (for $a \in \mathfrak{C}$) is given at $x \in \sigma_{\mathcal{B}_p}(pqp)$ by

$$(\text{smb } e)(x) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$(\text{smb } p)(x) = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$$

$$(\text{smb } q)(x) = \begin{bmatrix} x & \sqrt{x(1-x)}\\ \sqrt{x(1-x)} & 1-x\end{bmatrix}$$

and either $(\text{smb } j)(x) = \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}$ or $(\text{smb } j)(x) = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$.

We omit the proof since it is only a slight modification of the proof of Theorem 5.

One possibility to decide whether the symbol of j is $\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}$ or $\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$ is the following:

By Theorem 8, $\mathrm{smb}(ipqjp)(x) = \begin{pmatrix} \pm\sqrt{x(1-x)} & 0 \\ 0 & 0 \end{pmatrix}$, i.e. the spectrum of $ipqjp$ in $\mathfrak{C}$ equals either

$$\{y \in \mathbb{C} : \ y = +\sqrt{x(1-x)}\ , \ \ x \in \sigma_{\mathfrak{B}_p}(pqp)\}$$

or

$$\{y \in \mathbb{C} : \ y = -\sqrt{x(1-x)}\ , \ \ x \in \sigma_{\mathfrak{B}_p}(pqp)\}\ ,$$

where $\sqrt{\ }$ refers to the main branch. If the spectrum $\sigma_{\mathfrak{B}_p}(pqp)$ is given then the knowledge of only one suitable point of $\sigma_{\mathfrak{C}}(ipqjp)$ would allow to decide which sign(+ or −) holds and so to make the symbol unique.

As a simple consequence we mention a C*-algebra version of Theorem 8 (see [P2]).

COROLLARY 3. Let $\overline{e},\overline{p},\overline{q},\overline{j} \in C([0,1], \mathbb{C}^{2\times 2})$ be defined by $\overline{e}(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\overline{p}(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\overline{q}(x) = \begin{bmatrix} x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1-x \end{bmatrix}$ and

$\overline{j}(x) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$,

and let e,p,q,j be self adjoint elements of a certain C*-algebra $\mathfrak{C}$ which fulfil (12) and

$$ipqjp \geq 0 \quad \text{and} \quad \sigma_{\mathfrak{C}}(pqp) = [0,1]\ . \qquad (13)$$

Then the C*-algebras $\mathrm{alg}(e,p,q,j)$ and $\mathrm{alg}(\overline{e},\overline{p},\overline{q},\overline{j}) = C([0,1], \mathbb{C}^{2\times 2})$ are isometrically isomorphic, and the isomorphism transforms e,p,q,j into $\overline{e},\overline{p},\overline{q},\overline{j}$, respectively.

PROOF. The relations (13) guarantee that $\mathrm{smb}\ j = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, and that the mapping smb is isometric follows from simple C*-algebra arguments.

We conclude this section with a few remarks on the uniqueness of the symbols obtained. To that end we assume the symbols $\mathrm{smb}\ e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathrm{smb}\ p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ to be fixed and ask how the symbols smb q, smb j could look like.

PROPOSITION 2. a) The general form of the symbol on $\mathrm{alg}(e,p,q)$ is (for fixed smb e and smb p) given by

$(\mathrm{smb}\ q)(x) = \begin{bmatrix} x & b(x)\sqrt{x(1-x)} \\ b^{-1}(x)\sqrt{x(1-x)} & 1-x \end{bmatrix}$, where $b(x) : \sigma_{\mathfrak{B}_p}(pqp) \to \mathbb{C}$

is an arbitrary invertible function.
b) The general form of the symbol on $\mathrm{alg}(e,p,q,j)$ is (for fixed smb e and smb p) given by

$$(\mathrm{smb}\ q)(x) = \begin{pmatrix} x & b\sqrt{x(1-x)} \\ b^{-1}\sqrt{x(1-x)} & 1-x \end{pmatrix}$$

and $(\mathrm{smb}\ j)(x) = \begin{pmatrix} 0 & \pm\frac{1}{z} \\ \pm z & 0 \end{pmatrix}$ where $b^4 = 1$ and $z^2 = -b^2$.

The simple proof is omitted.

4. TOEPLITZ AND HANKEL OPERATORS ON WEIGHTED l^p-SPACES

In this and the next sections the general results of the sections 2 and 3 will be applied to give matrix-valued symbols related to the Fredholm property of Toeplitz and Hankel operators and of singular integral operators with a Carleman shift changing the orientation and with piecewise continuous coefficients. Let

$$l^{p,\gamma} := \{x = \{x_i\}_{i=0}^{\infty} : \|x\|_{p,\gamma}^p = \sum_{i=0}^{\infty} |x_i|^p (i+1)^{p\gamma} < \infty\}$$

and $$\widetilde{l^{p,\gamma}} := \{x = \{x_i\}_{i=-\infty}^{\infty} : \|x\|_{p,\gamma}^p = \sum_{i=-\infty}^{\infty} |x_i|^p (|i|+1)^{p\gamma} < \infty\}$$

and put $l^p := l^{p,0}$ and $\widetilde{l^p} := \widetilde{l^{p,0}}$. The operators

$$P : \widetilde{l^{p,\gamma}} \to \widetilde{l^{p,\gamma}} : \{x_i\}_{i=-\infty}^{\infty} \longmapsto \{\ldots,0,0,x_0,x_1,\ldots\}$$

and $$J : \widetilde{l^{p,\gamma}} \to \widetilde{l^{p,\gamma}} : \{x_i\}_{i=-\infty}^{\infty} \longmapsto \{x_{-i-1}\}_{i=-\infty}^{\infty}$$

are bounded and $P^2 = P$, $J^2 = I$ (I the identity operator).
Given $a \in L^{\infty}(\mathbb{T})$ with Fourier coefficients $\{a_n\}_{n=-\infty}^{\infty}$ we define the operator $M(a)$ on the space of all sequences in $\widetilde{l^{p,\gamma}}$ with a finite support by

$$M(a) : \{x_i\}_{i=-\infty}^{\infty} \longrightarrow \{\sum_{j=-\infty}^{\infty} a_{i-j}x_j\}_{i=-\infty}^{\infty} .$$

If $M(a)$ can be extended to a bounded operator on $\widetilde{l^{p,\gamma}}$ we call $M(a) \in L(\widetilde{l^{p,\gamma}})$ the multiplication operator with symbol $a(t)$ and write $a \in M_{p,\gamma}$.

Now define $M_{\langle p,\gamma\rangle}$ as follows:

$$M_{\langle 2,0\rangle} = M_{2,0} \quad (= L^\infty(\mathbb{T})),$$

$$M_{\langle 2,\gamma\rangle} = \{a \text{ in } M_{2,\tilde{\gamma}} \text{ for all } \tilde{\gamma} \text{ in some neighborhood of } \gamma\} \quad \text{if } \gamma \neq 0,$$

$$M_{\langle p,0\rangle} = \{a \text{ in } M_{\tilde{p},0} \text{ for all } \tilde{p} \text{ in some neighborhood of } p\} \quad \text{if } p \neq 2,$$

$$M_{\langle p,\gamma\rangle} = \{a \text{ in } M_{\tilde{p},\tilde{\gamma}} \text{ for all } (\tilde{p},\tilde{\gamma}) \text{ in some neighborhood of } (p,\gamma)\} \quad \text{if } p \neq 2,\ \gamma \neq 0.$$

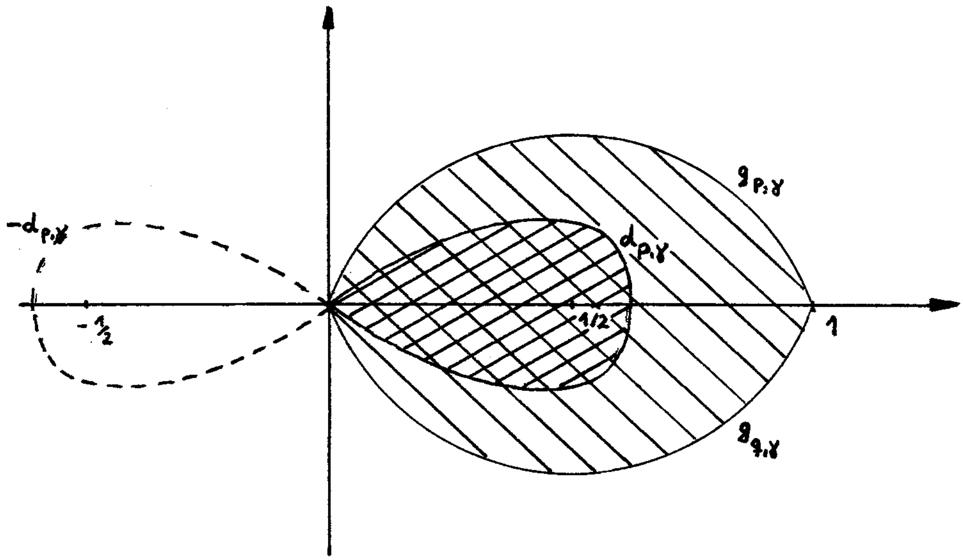
For $a \in M_{p,\gamma}$ the operator $T(a) := PM(a)P : l^{p,\gamma} \to l^{p,\gamma}$ is called Toeplitz and the operator $H(a) : PM(a)JP : l^{p,\gamma} \to l^{p,\gamma}$ Hankel (here we identify im P and $l^{p,\gamma}$).
The function χ which is 1 on the upper half circle and 0 on the lower belongs to $M_{\langle p,\gamma\rangle}$ for all $1 < p < \infty$ and $-\frac{1}{p} < \gamma < \frac{1}{q}$ (see [V, Lemma 10]). Obviously, $\chi^2 = \chi$.

Let $\vartheta_{p,\gamma} = \pi - 2\pi(\frac{1}{p} + \gamma)$ and define, for $s \in [0,1]$,

$$g_{\vartheta_{p,\gamma}}(s) := g_{p,\gamma}(s) := \begin{cases} s & \text{if } \vartheta_{p,\gamma} = 0 \\ \dfrac{\sin \vartheta_{p,\gamma} s \exp(i\vartheta_{p,\gamma} s)}{\sin \vartheta_{p,\gamma} \exp(i\vartheta_{p,\gamma})} & \text{otherwise.} \end{cases}$$

If s runs from 0 to 1 then $g_{p,\gamma}(s)$ runs in $\mathbb{C}$ for $\frac{1}{2} < \frac{1}{p} + \gamma$ (resp. $\frac{1}{p} + \gamma < \frac{1}{2}$) along a circular arc joining 0 to 1 and located on the left (resp. right) of the line segment $[0,1]$, and the segment is seen from each point of the arc under the angle $2\pi(\frac{1}{p} + \gamma)$ if $\frac{1}{p} + \gamma < \frac{1}{2}$ respective $2\pi(1 - \frac{1}{p} - \gamma) = 2\pi(\frac{1}{q} - \gamma)$ if $\frac{1}{2} < \frac{1}{p} + \gamma$.

PROPOSITION 3. a) The spectrum $\sigma(T(\chi))$ equals the lentiform domain bounded by the arcs $g_{p,\gamma}([0,1])$ and $g_{q,-\gamma}([0,1])$.
b) The spectrum $\sigma(iH(\chi))$ equals the drop-shaped domain bounded by the curve $d_{p,\gamma} = \sqrt{g_{p,\gamma}(1-g_{p,\gamma})}\,([0,1])$.



PROOF. a) Well-known (see [VK] or [BS]).
b) Since $\mathbb{C} \setminus \sigma(T(\chi))$ is connected, the spectrum of $T(\chi)$ in $L(l^{p,\gamma})$ is the same as its spectrum in $alg(P,\chi,J,I)$. By Theorem 8 the spectrum of $iH(\chi)$ in $alg(P,\chi,J,I)$ is either the domain bounded by $d_{p,\gamma}$ or the domain bounded by $-d_{p,\gamma}$.

Let first $\gamma = 0$ and $1 < p \leq 2$. Then l^p is continuously embedded into the Hilbert space l^2. The adjoint operator of $iH(\chi)$ (considered in l^p) is again $iH(\chi)$ (considered in l^q). This shows that the operator $iH(\chi)$ is bibounded (see [GK], chapter V) and that $iH(\chi)$ coincides with its classical adjoint $(iH(\chi))^+$. By Theorem V. 3.2 of [GK] the spectrum $\sigma_{l^2}(iH(\chi))$ is contained in $\sigma_{l^p}(iH(\chi))$. Since $\sigma_{l^2}(iH(\chi))$ is the straight line $[0,\frac{1}{2}]$ (see [P1]), the spectrum of $iH(\chi)$ in $alg(P,\chi,J,I)$ with respect to l^p must be located on the right of the imaginary axis, i.e. it is the domain bounded by $d_{p,0}$. Note that this spectrum coincides with the spectrum of $iH(\chi)$ in $L(l^{p,0})$. This is due to the equality $(iH(\chi))^2 = T(\chi) - T^2(\chi)$, to the fact that the spectra of $T(\chi)$ in $L(l^{p,\gamma})$ and in $alg(P,\chi,J,I)$ coincide, and to the spectral mapping theorem for polynomials.
Taking adjoints it is easily seen that the assertion remains true for $\gamma = 0$ and $2 < p < \infty$, too.

Now let $\gamma \neq 0$. Since $(l^{p,\gamma})^* \cong l^{q,-\gamma}$ (with $p^{-1}+q^{-1}=1$), by the Stein-Weiss interpolation theorem (see [BL], 5.5.4) there exists a p_o lying between p and q such that the space l^{p_o} is interpolated by $l^{p,\gamma}$ and $l^{q,-\gamma}$. Let $\mu \in \mathbb{R}$, and assume $iH(\chi) - \mu I$ to be invertible in $L(l^{p,\gamma})$. Then $(iH(\chi) - \mu I)^* =$ $= iH(\chi) - \mu I$, and since the action of $iH(\chi) - \mu I$ and $(iH(\chi) - \mu I)^*$ on finitely supported sequences is the same on both $l^{p,\gamma}$ and $l^{q,-\gamma}$, the inverses of $iH(\chi) - \mu I$ in $l^{p,\gamma}$ respectively in $l^{q,-\gamma}$ have the same action on a dense subset of $l^{p,\gamma}$ respectively of $l^{q,-\gamma}$. Hence, by the Stein-Weiss theorem the operator which acts on the finitely supported sequences in $l^{p_o,o}$ as $iH(\chi) - \mu I$, extends continuously to an invertible operator on $l^{p_o,o}$. Consequently, the real points of the spectrum of $iH(\chi)$ on $l^{p_o,o}$ must belong to the spectrum of $iH(\chi)$ on $l^{p,\gamma}$. As we have shown above, the spectrum of $iH(\chi)$ on $l^{p_o,o}$ is located on the right of the imaginary axis. So the spectrum on $l^{p,\gamma}$ must lie at the right, too, what finishes the proof.

COROLLARY 4. The maximal ideal space of $alg(P,\chi,J,I)$ (in $l^{p,\gamma}$) is homeomorphic to the domain bounded by $g_{p,\gamma}([0,1])$ and $g_{q,-\gamma}([0,1])$, and a symbol for the invertibility in $alg(P,\chi,J,I)$ is given by

$$(\text{smb } P)(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad (\text{smb } \chi)(x) = \begin{bmatrix} a & \sqrt{x(1-x)} \\ \sqrt{x\,(1-x)} & 1-x \end{bmatrix},$$

$$(\text{smb } J)(x) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \text{ and } (\text{smb } I)(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now we are going to extend these results to a symbol calculus for the Fredholm property of operators belonging to the algebra generated in $L(l^{p,\gamma})$ by I, P, J and by the piecewise continuous functions f which are multipliers on $l^{p,\gamma}$, i.e. $f \in PC \cap M_{\langle p,\gamma \rangle} =: PC_{\langle p,\gamma \rangle}$.
Let $\mathcal{A}$ stand for $alg(I,P,J,PC_{\langle p,\gamma \rangle})$, $LC(l^{p,\gamma})$ for the ideal of the compact operators, π for the canonical homomorphism from $L(l^{p,\gamma})$ onto $L(l^{p,\gamma})/ LC(l^{p,\gamma})$ and $\mathcal{A}^\pi$ for the quotient algebra $\mathcal{A}/LC(l^{p,\gamma})$. The algebra $\mathcal{A}^\pi$ contains a copy of the algebra

$\mathcal{C}(\mathbb{T}) \cap M_{\langle p,\gamma \rangle}$ in their center (where $\mathcal{C}(\mathbb{T})$ stands for the set of the continuous functions f with $f(t) = f(t^{-1})$) the maximal ideal space of which is homeomorphic to the upper half unit circle $\mathbb{T}_+ := \{z \in \mathbb{T} : \text{Im } z \geq 0\}$. For given $x \in \mathbb{T}_+$, denote by J_x the smallest closed two-sided ideal of A^π containing x, put $\mathcal{A}_x^\pi := \mathcal{A}^\pi / J_x$, and write Φ_x^π for the canonical homomorphism from $\mathcal{A}^\pi$ onto $\mathcal{A}_x^\pi$.

THEOREM 9. Let $A \in \mathcal{A}$. Then

$$\sigma_{\mathcal{A}^\pi}(\pi(A)) = \bigcup_{x \in \mathbb{T}_+} \sigma_{\mathcal{A}_x^\pi}(\Phi_x^\pi(A)) .$$

Compare [A] for a proof.

PROPOSITION 4. The local algebra $\mathcal{A}_1^\pi$ is generated by the idempotents $\Phi_1^\pi(P)$, $\Phi_1^\pi(\chi)$ and by the flip $\Phi_1^\pi(J)$, which can occupy the place of p, q and j in (12), respectively. The local spectra $\sigma_{\mathcal{A}_1^\pi}(\Phi_1^\pi(P\chi P))$ and $\sigma_{\mathcal{A}_1^\pi}(\Phi_1^\pi(iP\chi JP))$ equal the curves $g_{p,\gamma}[0,1]$ and $d_{p,\gamma}$, respectively.

PROOF. The first assertion is immediately clear. For a proof of the second assertion let $a \in PC_{\langle p,\gamma \rangle}$ denote a function which is continuous at each point $t \neq 1$, which only takes values lying on the arc $g_{p,\gamma}([0,1])$, and with $a(1-0) = 0$, $a(1+0) = 1$, and let $b \in PC_{\langle p,\gamma \rangle}$ denote a function which is continuous at each point $t \neq 1$, which only takes values lying on the straight line $[0,1]$, and with $b(1-0) = 0$, $b(1+0) = 1$. Since the essential spectrum of $T(a)$ equals the arc $g_{p,\gamma}([0,1])$ (see [VK]), the spectra of $\pi(T(a))$ in $L(l^{p,\gamma})/LC(l^{p,\gamma})$ respective in $\mathcal{A}^\pi$ coincide. Thus, the local spectrum $\sigma_{\mathcal{A}_1^\pi}(\Phi_1^\pi(T(a))$ is contained in $g_{p,\gamma}([0,1])$. On the other hand, $\sigma_{\mathcal{A}_1^\pi}(\Phi_1^\pi(T(b)) = \sigma_{\mathcal{A}_1^\pi}(\Phi_1^\pi(T(a))$ and $\sigma_{\mathcal{A}_x^\pi}(\Phi_x^\pi(T(b)) = b(x)$. Hence, $\sigma_{\mathcal{A}^\pi}(\pi(T(b))$ is contained in $g_{p,\gamma}([0,1]) \cup [0,1]$. Since the "global" essential spectrum of $T(b)$ coincides with $g_{p,\gamma}([0,1]) \cup [0,1]$, the inclusion

$\sigma_{\mathcal{A}_1^\pi}(\Phi_1^\pi(T(a)) \supseteq g_{p,\gamma}([0,1])$ follows. This shows the second assertion, and the third one follows immediately from $\sigma_{\mathcal{A}_1^\pi}(\Phi_1^\pi(T(\chi)) = g_{p,\gamma}([0,1])$, Theorem 8, and from the fact that the spectrum $\sigma_{L(l^{p,\gamma})}(iH(\chi))$ is located on the right of the imaginary axis (Proposition 3), what finishes the proof.

Thus, Theorem 8 provides a symbol for the invertibility in the quotient algebra $\mathcal{A}_1^\pi$. Similarly one obtains a symbol in $\mathcal{A}_{-1}^\pi$.

Now let $x \in \mathbb{T}_+$, $\operatorname{Im} x > 0$. A moment's thought shows that the corresponding local algebras $\mathcal{A}_x^\pi$ are not generated by only two projections and a flip. For that reason we explain a scheme which allows to eleminate the flip: Let $\mathfrak{C}$ be an algebra with identity e and with elements p, q, j satisfying $p^2 = p$, $p + q = e$, $j^2 = e$ and $jpj = q$. As in the second proof of Theorem 6, the mapping

$$F : \mathfrak{C} \to (p\mathfrak{C}p)^{2\times 2}, \quad a \mapsto \begin{pmatrix} pap & pajp \\ pjap & pjajp \end{pmatrix}$$

proves to be a continuous homomorphism, and $a \in \mathfrak{C}$ is invertible if and only if $F(a) \in (p\mathfrak{C}p)^{2\times 2}$ is invertible (in a similar form this approach was proposed by Krupnik [K], § 23).

Next assume that $\mathfrak{C}$ is generated by the flip j and by another algebra $\mathcal{L}$ (containing e and p) with the property that $j\mathcal{L}j \subseteq \mathcal{L}$ and that $pap = ap = pa$ for $a \in \mathcal{L}$. Then we can write each element $a \in \mathfrak{C}$ in the form $a = a_1 + a_2 j$ with $a_1, a_2 \in \mathcal{L}$, and this gives

$$F(a) = \begin{bmatrix} pa_1p + pa_2jp & pa_1jp + pa_2p \\ pja_1p + pja_2jp & pja_1jp + pja_2p \end{bmatrix}$$

$$= \begin{bmatrix} pa_1p & pa_2p \\ pja_2jp & pja_1jp \end{bmatrix} = \begin{bmatrix} pa_1p & pa_2p \\ p\tilde{a}_2p & p\tilde{a}_1p \end{bmatrix},$$

where we put $\tilde{a}_i := ja_ij \in \mathcal{L}$. What results is that the mapping F is a continuous homomorphism of $\mathfrak{C}$ into the algebra $(p\mathcal{L}p)^{2\times 2}$, and that $a \in \mathfrak{C}$ is invertible if and only if $F(a) \in (p\mathcal{L}p)^{2\times 2}$ is invertible.

To reify this scheme, let $f \in M_{\langle p,\gamma\rangle}$ denote a continuous function with $f(x) = 1$ which vanishes outside a small neighborhood of x and for which $f(t) \cdot f(\frac{1}{t}) = 0$ holds for all $t \in \mathbb{T}$. Put $p = \Phi_x^\pi(T(f))$, $q = e-p$, and $j = \Phi_x^\pi(J)$. Further, identify $\mathbb{C}$ and $\mathcal{A}_x^\pi$ and put $\mathcal{L} := \mathcal{B}_x^\pi = \mathcal{B}^\pi/J_x$ where $\mathcal{B}$ stands for the algebra $\mathcal{B} = \mathrm{alg}(P, PC_{\langle p,\gamma\rangle})$. Obviously, $p \in \mathcal{B}_x^\pi$, and the hypotheses of the scheme are fulfilled. Thus, $\mathcal{A}_x^\pi$ is homomorphic to a subalgebra of $(p\mathcal{B}_x^\pi p)^{2\times 2}$; hence, we need a symbol calculus for $p\mathcal{B}_x^\pi p$. Notice that the algebra $p\mathcal{B}_x^\pi p$ is generated by the two idempotents $p\Phi_x^\pi(P)p$ and $p\Phi_x^\pi(\chi_x)p$ where χ_x denotes the characteristic functions of the arc joining x and -1 on the upper half unit circle, and the same considerations as those in the proof of Proposition 4 yield that the local spectrum of $p\Phi_x^\pi(P\chi_x P)p$ in the algebra $p\mathcal{B}_x^\pi p$ coincides with the arc $g_{p,\gamma}([0,1])$. Hence, the spectrum of $p\Phi_x^\pi(P\chi_x P)p$ in the algebra $\mathrm{alg}(p, p\Phi_x^\pi(P\chi_x P)p)$ is $g_{p,\gamma}([0,1])$, too, and this shows that the spectrum of $p\Phi_x^\pi(P\chi_x P)p$ in $\mathrm{alg}(p\Phi_x^\pi(P)p\ ,\ p\Phi_x^\pi(P\chi_x P)p)$ equals $g_{p,\gamma}([0,1])$. Analogously, the spectrum of $p\Phi_x^\pi((I-P)(I-\chi_x)(I-P))p$ in $\mathrm{alg}(p\Phi_x^\pi(I-P)p,\ p\Phi_x^\pi((I-P)(I-\chi_x)(I-P))p)$ is also $g_{p,\gamma}([0,1])$. Lemma 3a shows that then the spectrum of $p\Phi_x^\pi(P\chi_x P+(I-P)(I-\chi_x)(I-P))p$ in $p\mathcal{B}_x^\pi p$ is nothing else than the arc $g_{p,\gamma}([0,1])$. Consequently, (H1) and (H2) are fulfilled (see Proposition 1), and Theorem 5 applies to give the following result:

PROPOSITION 5. For $x \in \mathbb{T}_+$, $\mathrm{Im}\, x > 0$, there is a 4x4-symbol for $\mathcal{A}_x^\pi$ which is given at $t \in g_{p,\gamma}([0,1])$ by

$$\Phi_x^\pi(P) \longleftrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Phi_x^\pi(J) \longleftrightarrow \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\text{and}\quad \Phi_x^\pi(\chi_x) \longleftrightarrow \begin{bmatrix} t & \sqrt{t(1-t)} & 0 & 0 \\ \sqrt{t(1-t)} & 1-t & 0 & 0 \\ 0 & 0 & 1-t & -\sqrt{t(1-t)} \\ 0 & 0 & -\sqrt{t(1-t)} & t \end{bmatrix}.$$

PROOF. The representations for $\Phi_x^\pi(P)$ and $\Phi_x^\pi(\chi_x)$ follow easily from Theorem 8. For $\Phi_x^\pi(J)$ note that $\Phi_x^\pi(J) \longleftrightarrow \begin{pmatrix} 0 & p \\ p & 0 \end{pmatrix}$ by the scheme, and that p is the identity element in $p\mathcal{B}_x^\pi p$.

REMARK. If only a symbol for operators from $\mathrm{alg}(P, PC_{\langle p,\gamma\rangle})$ is seeked then the 4-dimensional symbol given in Proposition 5 reduces to two 2-dimensional symbols which may be assumed to be given at x and x^{-1}, respectively.

THEOREM 10. A symbol for $\mathcal{A}^\pi = \mathrm{alg}(I,P,J,PC_{\langle p,\gamma\rangle})/ /LC(\widetilde{l^{p,\gamma}})$ can be given as follows. For $(x,t)\in \mathbb{T}_+ \times g_{p,\gamma}([0,1])$ let $(\mathrm{smb}\ P)(x,t)$ be

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ if } x = \pm 1, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ if } \operatorname{Im} x > 0,$$

define $(\mathrm{smb}\ J)(x,t)$ as

$$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \text{ if } x = \pm 1, \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \text{ if } \operatorname{Im} x > 0,$$

and for a piecewise continuous function a let $(\mathrm{smb}\ a)(x,t)$ be equal to

$$\begin{bmatrix} a(x+0)t + a(x-0)(1-t) & (a(x+0)-a(x-0))\sqrt{t(1-t)} \\ (a(x+0)-a(x-0))\sqrt{t(1-t)} & a(x+0)(1-t) + a(x-0)t \end{bmatrix} \text{ if } x = \pm 1$$

and equal to $\begin{pmatrix} X & 0 \\ 0 & \tilde{X} \end{pmatrix}$ with

$$X = \begin{bmatrix} a(x+0)t + a(x-0)(1-t) & (a(x+0)-a(x-0))\sqrt{t(1-t)} \\ (a(x+0)-a(x-0))\sqrt{t(1-t)} & a(x+0)(1-t) + a(x-0)t \end{bmatrix},$$

$$\tilde{X} = \begin{bmatrix} a(x^{-1}+0)(1-t)+a(x^{-1}-0)t & (a(x^{-1}-0)-a(x^{-1}+0))\sqrt{t(1-t)} \\ (a(x^{-1}-0)-a(x^{-1}+0))\sqrt{t(1-t)} & a(x^{-1}+0)t+a(x^{-1}-0)(1-t) \end{bmatrix}$$

in case $\operatorname{Im} x > 0$.

The proof follows immediately from Propositions 4 and 5 and from Allen's local principle (Theorem 9).

COROLLARY 5. Let $A \in \mathcal{A} = \operatorname{alg}(I,P,J,PC_{\langle p,\gamma\rangle})$. Then

$\sigma_{L(\widetilde{l^{p,\gamma}})/LC(\widetilde{l^{p,\gamma}})}(\pi(A)) = \sigma_{\mathcal{A}^\pi}(\pi(A))$.

PROOF. Approximate A by operators A_n which are finite sums of products of the operators P,J,I and of multiplication operators $M(a)$ when a is piecewise polynomial and has only a finite number of discontinuities. The spectrum of A_n^π in $\mathcal{A}^\pi$ consists by Theorem 10 of a (finite) set S of curves which can impossibly be obtained as the union of another set of curves S' with some bounded and connected components of $\mathbb{C} \setminus S'$. Hence, the essential spectra of A_n^π in $\mathcal{A}^\pi$ and in $L(\widetilde{l^{p,\gamma}})/LC(l^{p,\gamma})$ coincide, respectively.

We need the following elementary fact: If x is an invertible element of a Banach algebra and if $\|x-y\| < \frac{1}{\|x^{-1}\|}$ then y is invertible and

$$\|y^{-1}\| < \frac{\|x^{-1}\|}{1-\|x^{-1}\|\,\|x-y\|} \,. \tag{14}$$

Since $\sigma_{L(\widetilde{l^{p,\gamma}})/LC(\widetilde{l^{p,\gamma}})}(\pi(A)) \subseteq \sigma_{\mathcal{A}^{\pi}}(\pi(A))$, assume to finish the proof that there is a $\lambda \in \mathbb{C}$ such that $\pi(A-\lambda)$ is invertible in $L(\widetilde{l^{p,\gamma}})/LC(\widetilde{l^{p,\gamma}})$, but not in $\mathcal{A}^{\pi}$. By (14), $\pi(A_n-\lambda)$ is then invertible in $L(\widetilde{l^{p,\gamma}})/LC(\widetilde{l^{p,\gamma}})$ for n large enough, and

$$\|(\pi(A_n-\lambda))^{-1}\| < \frac{\|(\pi(A-\lambda))^{-1}\|}{1-\|(\pi(A-\lambda))^{-1}\|\,\|\pi(A-A_n)\|} . \tag{15}$$

By what has been shown above, $\pi(A_n-\lambda)$ is invertible in $\mathcal{A}^{\pi}$ for n large enough. The estimation (15) guaranties that there is an n_o such that

$$\|\pi(A-A_{n_o})\| < \frac{1}{\|(\pi(A_{n_o}-\lambda))^{-1}\|} ,$$

but this implies via (14) the invertibility of $\pi(A-\lambda)$ in $\mathcal{A}^{\pi}$.

COROLLARY 6. Let $a \in PC_{\langle p,\gamma\rangle}$. Then the essential spectrum of the Hankel operator $H(a) : l^{p,\gamma} \to l^{p,\gamma}$ equals

$$(-i(a(1+0)-a(1-0))d_{p,\gamma}) \cup (-i(a(-1+0)-a(-1-0))d_{p,\gamma}) \cup$$

$$\cup \bigcup_{\operatorname{Im} x>0} \pm\sqrt{-(a(x+0)-a(x-0))(a(x^{-1}+0)-a(x^{-1}-0))}\; d_{p,\gamma} .$$

PROOF. The first two items result from Theorem 10 and Proposition 3; the third term is obtained from Theorem 10, which gives the matrix

$$\begin{bmatrix} 0 & 0 & 0 & (a(x+0)-a(x-0))\sqrt{t(1-t)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (a(x^{-1}-0)-a(x^{-1}+0))\sqrt{t(1-t)} & 0 & 0 & 0 \end{bmatrix}$$

as a symbol for $PaJP : l^{p,\gamma} \to l^{p,\gamma}$.

If $PaJP$ is considered as an operator acting from $l^{p,\gamma}$ to $l^{p,\gamma}$ this symbol reduces to

$$\begin{bmatrix} 0 & (a(x+0)-a(x-0))\sqrt{t(1-t)} \\ (a(x^{-1}-0)-a(x^{-1}+0))\sqrt{t(1-t)} & 0 \end{bmatrix} ,$$

and this gives immediately the assertion.

REMARKS. a) Power [P1,2] gave the description of the essential spectrum of a Hankel operator for the space $l^{2,0}$.
b) The results obtained in this section evidently carry over to the space $Fl^{p,\gamma}$ of all functions $f \in L^1(\mathbb{T})$, the sequence of the Fourier coefficients of which belongs to $l^{p,\gamma}$.
c) Now it is an easy matter to obtain Fredholm criteria, for instance for paired equations with Carleman shift

$$aP + bQ + (a'P + b'Q)J : l^{p,\gamma} \to l^{p,\gamma}$$

where $a,b,a',b' \in PC_{p,\gamma}$ but we renounce to do this.

5. SINGULAR INTEGRAL EQUATIONS ON WEIGHTED SPACES OF HÖLDER-CONTINUOUS FUNCTIONS

For $0 < \mu < 1$ we denote by H^μ the Banach space of all Hölder-continuous functions of degree μ on the unit circle $\mathbb{T}$, i.e. of all functions $f \in L^\infty(\mathbb{T})$ with

$$\|f\|_\mu := \|f\|_{L^\infty} + \sup_{\substack{t_1 \neq t_2 \\ t_1,t_2 \in \mathbb{T}}} \frac{|f(t_2)-f(t_1)|}{|t_2-t_1|^\mu} < \infty .$$

Let $t_1,\dots,t_n \in \mathbb{T}$. Let $H^\mu(t_1,\dots,t_n)$ denote the space of all functions $f \in L^\infty(\mathbb{T})$ which fulfil the Hölder condition of degree μ at $t \neq t_1,\dots,t_n$ and which may have jumps at $t_1,\dots,t_n$. Let $H^\mu_0(t_1,\dots,t_n)$ stand for the subspace of H^μ consisting of all functions vanishing at $t_1,\dots,t_n$.

Put $\varrho(t) := \prod_{k=1}^{n} |t-t_k|^{\alpha_k}$ with

$$\mu < \alpha_k < \mu + 1 \quad \text{for} \quad k = 1,\dots,n. \tag{16}$$

The set $H^\mu_0(\varrho) := \{f : \varrho f \in H^\mu_0(t_1,\dots,t_n)\}$ becomes a Banach space under the norm

$$\|f\|_{H^\mu_0(\varrho)} := \|\varrho f\|_{H^\mu} ,$$

and the singular integral operator S,

$$(Sf)(t) = \frac{1}{\pi i} \int_{\mathbb{T}} \frac{f(\tau)d\tau}{\tau - t} ,$$

is bounded on $H_0^\mu(\varrho)$ if and only if (16) holds (see [Du] or [GK], pp. 276–279). Put $P = \frac{1}{2}(I+S)$ and $Q = I-P$, and, for $f \in L^\infty(\mathbb{T})$, define $(Jf)(t) = f(\frac{1}{t})$. If the weight ϱ is symmetric with respect to the real axis, i.e. if

$$\varrho(t) = \prod_{k=1}^{m} |(t-t_k')(t-\overline{t_k'})|^{\alpha_k},$$

then J is bounded on $H_0^\mu(\varrho)$, and $J^2 = I$.

Let $$\vartheta_t = \begin{cases} 0 & \text{if } t \neq t_1,\dots,t_n \\ \pi-2\pi(\alpha_k-\mu) & \text{if } t = t_k \end{cases}$$

and, for $s \in [0,1]$, define $g_{\vartheta_t}(s) = g_t(s)$ as in section 4.

Further we put, for $t \in \mathbb{T}$,

$$d_t := \sqrt{g_t(1-g_t)}\ ([0,1]),$$

where the square root is understood in the sense of the main branch.

PROPOSITION 6. If the weight is specified to $\varrho(t) = |t-1|^{\alpha_1}|t+1|^{\alpha_1}$ and if χ stands for the characteristic function of the upper half unit circle, then

a) the spectrum $\sigma(P\chi P)$ in $L(H_0^\mu(\varrho))$ equals the lentiform domain bounded by the arcs $g_1([0,1])$ and $g_{-1}([0,1])$;

b) the spectrum $\sigma(iP\chi JP)$ in $L(H_0^\mu(\varrho))$ equals the drop-shaped domain bounded by d_1.

PROOF. a) See [GK], Chapter IX, Theorem 10.1.

b) Since H^μ is continuously embedded into $L^2(\mathbb{T})$, there is a $c > 0$ such that for $f \in H_0^\mu(\varrho)$

$$\|f\|_{H_0^\mu(\varrho)} = \|\varrho f\|_{H^\mu} \le c\|\varrho f\|_{L^2} \le c\|\varrho\|_\infty \|f\|_{L^2},$$

i.e. $H_0^\mu(\varrho)$ is continuously embedded into the Hilbert space $L^2(\mathbb{T})$. Moreover, the classical adjoint $(iP\chi JP)^+$ coincides with $iP\chi JP$ so that the same proof as that of Proposition 3 applies in this situation, too. It only remains to verify that the operator $iP\chi JP$ is positive when considered on $L^2(\mathbb{T})$. But this is easily shown (see [P2] for the similar proof related to l^2), and the proof is finished.

Our further considerations proceed as in the fourth section. Let $\mathcal{A}$ stand for the algebra generated in $L(H_0^\mu(\varrho))$ by P, J, I and $H^\mu(t_1,\ldots,t_n)$. Here and in what follows the weight ϱ is assumed to be symmetric with respect to the real axis. Finally, let $\mathcal{A}^\pi$ refer to the quotient algebra $\mathcal{A}/LC(H_0^\mu(\varrho))$.

Since for $f \in H^\mu$ the commutator $fP - Pf$ is compact on $H_0^\mu(\varrho)$, the center of $\mathcal{A}^\pi$ contains a copy of the algebra H^μ consisting of those $f \in H^\mu$ with $f(t) = f(\bar{t})$ for $t \in \mathbb{T}$. The maximal ideal space of the latter algebra is homeomorphic to $\mathbb{T}_+$, so that we can localize $\mathcal{A}^\pi$ relative to $\mathbb{T}_+$ by Theorem 9. As in section 4, we denote the local algebras by $\mathcal{A}_x^\pi$ $(x \in \mathbb{T}_+)$ and the homomorphisms from $\mathcal{A}^\pi$ into $\mathcal{A}_x^\pi$ by Φ_x^π.

PROPOSITION 7. Assume that $1 \in \{t_1,\ldots,t_n\}$ and let $\chi \in H^\mu(t_1,\ldots,t_k)$ denote a piecewise constant function with $\chi(1+0) = 1$, $\chi(1-0) = 0$. Then the local algebra $\mathcal{A}_1^\pi$ is generated by the idempotents $\Phi_1^\pi(P)$, $\Phi_1^\pi(\chi)$, and by the flip $\Phi_1^\pi(J)$. The local spectra $\sigma_{\mathcal{A}_1^\pi}(\Phi_1^\pi(P\chi P))$ and $\sigma_{\mathcal{A}_1^\pi}(\Phi_1^\pi(iP\chi JP))$ equal the curves $g_1([0,1])$ and d_1, respectively.

The proof is only a slight modification of that of Proposition 4.

If $1 \notin \{t_1,\ldots,t_n\}$ then the situation is essentially simpler: Indeed, since each multiplier is Hölder continuous at 1, the algebra $\mathcal{A}_1^\pi$ is generated by $p := \Phi_1^\pi(P)$ and $j := \Phi_1^\pi(J)$. Taking into account that $pj = j(e-p)$, the scheme presented after Proposition 4 yields immediately that $F(p) = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$, $F(j) = \begin{pmatrix} 0 & p \\ p & 0 \end{pmatrix}$ and $F(e) = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$ defines a symbol on $\mathrm{alg}(p,j,e)$. Since

$$F : \mathrm{alg}(p,j,e) \to (p\ \mathrm{alg}(p,e)p)^{2\times 2} = (\mathrm{alg}(p))^{2\times 2},$$

and since $\mathrm{alg}(p)$ is obviously isomorphic to $\mathbb{C}$, the correspondence

$$\Phi_1^\pi(P) \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\Phi_1^\pi(J) \longleftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\Phi_1^\pi(f) \longleftrightarrow \begin{pmatrix} f(1) & 0 \\ 0 & f(1) \end{pmatrix} \quad \text{for } f \in H^\mu(t_1,\ldots,t_n)$$

represents a symbol for A_1^π in the case when $1 \notin \{t_1,\dots,t_n\}$. We renounce to indicate the symbols for the local algebras A_x^π (Im $x > 0$) explicitely but prefer to give the following summarizing theorem

THEOREM 11. A symbol for $A^\pi = alg(P,J,I,H^\mu(t_1,\dots,t_n))/LC(H_0^\mu(\varrho))$ is given

a) if $x = \pm 1$ by

$$(\text{smb } P)(x,t) = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \quad (\text{smb } J)(x,t) = \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix},$$

$$(\text{smb } a)(x,t) = \begin{bmatrix} a(x+0)t+a(x-0)(1-t) & (a(x+0)-a(x-0))\sqrt{t(1-t)} \\ (a(x+0)-a(x-0))\sqrt{t(1-t)} & a(x+0)(1-t)+a(x-0)t \end{bmatrix},$$

where t runs through $g_{\vartheta_x}([0,1]) =: g_x([0,1])$

b) if Im $x > 0$ by

$$(\text{smb } P)(x,t) = \begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix}, \quad (\text{smb } J)(x,t) = \begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix},$$

$$(\text{smb } a)(x,t) = \begin{bmatrix} X & 0 \\ 0 & \tilde{X}\end{bmatrix} \quad \text{with}$$

$$X = \begin{bmatrix} a(x+0)t+a(x-0)(1-t) & (a(x+0)-a(x-0))\sqrt{t(1-t)} \\ (a(x+0)-a(x-0))\sqrt{t(1-t)} & a(x+0)(1-t)+a(x-0)t \end{bmatrix}$$

$$\tilde{X} = \begin{bmatrix} a(x^{-1}+0)(1-t)+a(x^{-1}-0)t & (a(x^{-1}-0)-a(x^{-1}+0))\sqrt{t(1-t)} \\ (a(x^{-1}-0)-a(x^{-1}+0))\sqrt{t(1-t)} & a(x^{-1}+0)t+a(x^{-1}-0)(1-t) \end{bmatrix}$$

where t runs through $g_x([0,1])$.

PROOF. a) If $x \in \{t_1,\dots,t_n\}$, the assertion is a consequence of Proposition 7. If $x \notin \{t_1,\dots,t_n\}$ then a must be continuous at x, and the symbol quoted above reduces to

$$\text{smb } P = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \quad \text{smb } J = \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}, \quad \text{smb } a = \begin{pmatrix}a(x) & 0\\ 0 & a(x)\end{pmatrix}. \qquad (17)$$

The invertible transformation $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \mapsto \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -i\end{pmatrix}$ applied to the matrices (17) gives exactly the symbol at $x = \pm 1$ obtained after Proposition 7.

b) If $x \in \{t_1, \ldots, t_n\}$, the proof runs completely parallel to that of Proposition 5. For $x \notin \{t_1, \ldots, t_n\}$ one can argue combining the arguments given after Proposition 7 and part a) of this proof.

REMARKS. a) The analogues of the Corollaries 5 and 6 hold in the Hölder space case, too.

b) The results remain valid if the unit circle $\mathbb{T}$ is replaced by any system Γ of piecewise Lyapunow curves the singular points of which belong to the set $\{t_1, \ldots, t_n\}$ of the zeros of the weight ϱ, and if J is replaced by a Carleman shift changing the orientation of Γ.

To that end one makes Γ to a closed piecewise Lyapunow curve by filling in straight lines between the endpoints of the single curves, and then one considers only multipliers which are identically 1 on these lines (see [GK] for details).

c) For the algebra generated by P and $H^\mu(t_1, \ldots, t_n)$, (i.e. if the flip J is absent) the results of this section were obtained by Duducava (see his survey [Du] for further references).

6. SINGULAR INTEGRAL EQUATIONS ON WEIGHTED L^p-SPACES

For given $t_1, \ldots, t_n \in \mathbb{T}$ put $\varrho(t) = \prod_{k=1}^{n} |t - t_k|^{\beta_k}$ $(t \in \mathbb{T})$ and let $L^p(\varrho)$ stand for the Banach space of all functions f with $\|f\|_{L^p(\varrho)} := \|\varrho f\|_{L^p} < \infty$. The singular integral operator S defined in section 5 is bounded on $L^p(\varrho)$ is and only if $-\frac{1}{p} < \beta_k < \frac{1}{q}$ $(\frac{1}{p} + \frac{1}{q} = 1)$ for $k = 1, \ldots, n$. Define P, Q, J as in section 5. If the weight ϱ is symmetric with respect to the real axis, the operator J is bounded on $L^p(\varrho)$ and $J^2 = I$ (I the identity).

$$\text{Let } \vartheta_t = \begin{cases} \pi(1 - \frac{2}{p}) & \text{if } t \notin \{t_1, \ldots, t_n\} \\ \pi(1 - \frac{2}{p} - \beta_k) & \text{if } t = t_k. \end{cases}$$

and for $s \in [0,1]$ define $g_{\vartheta_t}(s) =: g_t(s)$ and $d_t := \sqrt{g_t(1-g_t)}([0,1])$ as in section 4.

The same arguments as in section 5 lead us to the following theorem

THEOREM 12. For the algebra $A^{\pi} = \mathrm{alg}(I,P,J,PC)/LC(L^p(\varrho))$ the symbol has the same form as that given in Theorem 11. One only has to replace the g_{ϑ_x} in Theorem 11 by the g_{ϑ_x} defined above.

(See [GK], chapter IX, § 3 for the essential spectra of the special singular operators necessary for the proofs.)

REMARKS. a) This result is well-known. For the case that no flip occurs the proof is in [GK1]. For the general case see [GK2] and [C].

b) The results remain valid when replacing $\mathbb{T}$ by a system of piecewise Lyapunow curves Γ and J by a Carleman shift changing the orientation of Γ (see the concluding remarks of section 5).

c) The analogues of the Corollaries 5, 6 hold, too.

d) Taking into consideration Theorem 11.2 of Chapter IX of [GK1], an analogue of Theorems 11, 12 can be formulated for symmetric spaces $E(\mathbb{T})$ or $E(\Gamma)$ (see [GK1], IX).

CONCLUDING REMARK. Note that the considerations of the sections 4 - 6 also apply to the matrix case.

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Technische Universität Karl-Marx-Stadt
Sektion Mathematik
PSF 964
Karl-Marx-Stadt
DDR 9010

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