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A note on conservation principles in classical mechanics

B. Tabarrok, Victoria, British Columbia, **C. Tezer,** Ankara, Turkey, and **M. Stylianou,** Victoria, British Columbia

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Summary. In the following we examine the notion of conserved quantities in problems of elastostatics and elastodynamics. We show that the well known formulation which leads to $T + V =$ Constant can be generalised yielding conservation properties to some new and at times unexpected quantities. These new conservation properties may at times be the only means for verifying results obtained by numerical techniques.

1 Introduction

The first law of thermodynamics, namely the principle of conservation of energy, is a fundamental law governing all physical processes. In classical mechanics, where one is concerned essentially with the kinetic energy T and the potential energy V, the celebrated principle of conservation of energy, namely $T + V =$ Constant, assumes a much narrower interpretation. In this case the mathematical *form* of the work done becomes crucial for conservation of energy in a narrow sense. This mathematical *form* is related to the extremum properties of functions and is naturally dealt with under the subject of Calculus of Variations. From a variational perspective, conserved quantities are invariant under changes of some independent variables. An alternative interpretation is the inherent symmetry of the conserved quantities when evaluated for different values of the independent variable. Noether's theorem provides a framework for such invariant properties and is not limited to a specific physical law [1].

2 The simplest form

We start from the simplest expression for the extremum of a functional as follows:

$$\delta \int_{x_1}^{x_2} L(x, y, y_x)\, dx = 0, \tag{1}$$

where $y_x = dy/dx$. The extremising function $y(x)$ is governed by the Euler-Lagrange equation of this functional, namely

$$\frac{d}{dx}\left[\frac{\partial L}{\partial y_x}\right] - \frac{\partial L}{\partial y} = 0. \tag{2}$$

Consider now the total derivative of the integrand L with respect to the independent variable x. We may compute this as

$$\frac{dL}{dx} = \frac{\partial L}{\partial x} + \frac{\partial L}{\partial y} y_x + \frac{\partial L}{\partial y_x} y_{xx}. \tag{3}$$

From Eq. (2), $\frac{\partial L}{\partial y} y_x = \frac{d}{dx}\left[\frac{\partial L}{\partial y_x}\right] y_x$, and substituting this into Eq. (3) yields

$$\frac{d}{dx}\left[L - \frac{\partial L}{\partial y_x} y_x\right] = \frac{\partial L}{\partial x}. \tag{4}$$

Equations (2) and (4) provide alternative differential equations for the determination of the extremising function for the functional in Eq. (1) and in this case, of single independent variable and single extremising function, these alternative forms are equivalent. Each of these admits one order of integration under special conditions. Thus, if y does not appear in L, Eq. (2) can be integrated yielding

$$\frac{dL}{dy_x} = \text{Constant}. \tag{5}$$

On the other hand if x is absent (explicitly) in L then Eq. (4) yields upon integration

$$L - \frac{\partial L}{\partial y_x} y_x = \text{Constant}. \tag{6}$$

To consider the more familiar case of dynamics, let us examine Hamilton's principle in terms of generalised displacements q_i. In this case we have that

$$\delta \int_{t_1}^{t_2} L(t, q_i, \dot{q}_i)\, dt = 0 \quad i = 1, 2, \ldots n. \tag{7}$$

Apart from the change of the independent variable from x to t and increasing the number of extremising functions to n, the functional in Eq. (7) is similar to that in Eq. (1) and in this case we find a set of Euler-Lagrange equations as

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\right] - \frac{\partial L}{\partial q_i} = 0 \quad i = 1, 2 \ldots n. \tag{8}$$

Following essentially the procedure already outlined, we find that in this case the expression corresponding to Eq. (4) takes the following form:

$$\frac{d}{dt}\left[L - \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i\right] = \frac{\partial L}{\partial t}. \tag{9}$$

It is worth noting that although the extremising functions satisfy Eqs. (8) and (9), these equations are not equivalent in terms of information content. Specifically, Eq. (8) provides n coupled equations for determination of n extremising functions, whereas Eq. (9) provides only a single equation. Nevertheless, the notion of conservation flows from these equations. Thus if one of the q's, say q_j, is absent in L while its velocity is present (i.e. q_j is an ignorable coordinate), then Eq. (8)

yields for the j^{th} equation

$$\frac{\partial L}{\partial \dot{q}_j} = \text{Constant}. \tag{10}$$

Recalling that in dynamics $L = T^*(t, q_i, \dot{q}_i) - V(t, q_i)$, where T^* is the complementary kinetic energy, and

$$\frac{d}{d\dot{q}_j} T^* = P_j \tag{11}$$

is the generalised momentum, we note that Eq. (10) expresses the conservation of momentum for the j^{th} generalised coordinate. On the order hand if t does not appear explicitly in L, then Eq. (9) upon integration yields

$$L - \sum_{i=1}^{n} P_i \dot{q}_i = -H = \text{Constant}, \tag{12}$$

where H is the Hamiltonian of the system. It is interesting to note that the conservation of the Hamiltonian does not require a linear relationship between the momenta and velocities, that is Eq. (12) is valid for relativistic as well as Newtonian mechanics [2]. For scleronomous systems the Hamiltonian takes the following familiar form [3]:

$$-(T^* - V) + (T + T^*) = H, \tag{13}$$

and Eq. (12) becomes

$$T + V = \text{Constant}. \tag{14}$$

In this case the total *energy* is conserved. As further specialisation, we note that in Newtonian mechanics $T(P_i) = T^*(\dot{q}_i)$, and in this case one finds that $(T^* + V)$ is also conserved.

3 Higher order functionals

Consider next the extremisation of the following functional:

$$\delta \int_{x_1}^{x_2} L(x, y, y_x, y_{xx})\, dx = 0. \tag{15}$$

The Euler-Lagrange equation for this functional takes the following form:

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\left[\frac{\partial L}{\partial y_x}\right] + \frac{d^2}{dx^2}\left[\frac{\partial L}{\partial y_{xx}}\right] = 0. \tag{16}$$

We may write this equation as

$$\frac{d\hat{P}}{dx} = \frac{\partial L}{\partial y}, \tag{17}$$

where we now define the modified momentum $\hat{P}$ as

$$\hat{P} = \frac{\partial L}{\partial y_x} - \frac{dR}{dx}, \tag{18}$$

where we call $R = \dfrac{\partial L}{\partial y_{xx}}$ as the hypermomentum. Evidently for this type of problem if y does not appear explicitly in the Lagrangian L, the conservation of the modified momentum implies that

$$\frac{\partial L}{\partial y_x} - \frac{d}{dx}\left[\frac{\partial L}{\partial y_{xx}}\right] = \text{Constant}. \tag{19}$$

To develop the conservation equation for the case when x is not explicitly present in L, we determine the total derivative of L with respect to x. Thus we write

$$\frac{dL}{dx} = \frac{\partial L}{\partial x} + \frac{\partial L}{\partial y} y_x + \frac{\partial L}{\partial y_x} y_{xx} + \frac{\partial L}{\partial y_{xx}} y_{xxx}. \tag{20}$$

From Eq. (16) we have that

$$\frac{d}{dx}\left[\frac{\partial L}{\partial y_x} - \frac{d}{dx}\left[\frac{\partial L}{\partial y_{xx}}\right]\right] y_x = \frac{\partial L}{\partial y} y_x, \tag{21}$$

substituting this in Eq. (20) and simplifying we find

$$\frac{d}{dx}\left[L - y_x\left[\frac{\partial L}{\partial y_x} - \frac{d}{dx}\left[\frac{\partial L}{\partial y_{xx}}\right]\right] - y_{xx}\frac{\partial L}{\partial y_{xx}}\right] = \frac{\partial L}{\partial x},$$

or in terms of earlier defined quantities

$$\frac{d}{dx}[-L + y_x\hat{P} + y_{xx}R] = -\frac{\partial L}{\partial x}. \tag{22}$$

It is evident now that when x is not explicitly present in L the generalised Hamiltonian, namely

$$\hat{H} = -L + \hat{P}y_x + Ry_{xx}, \tag{23}$$

will be invariant in x. The above result can be readily generalised for the case of several extremising functions y_i. In this case the Euler-Lagrange equations emerge as

$$\frac{d\hat{P}_i}{dx} = \frac{\partial L}{\partial y_i} \quad i = 1, 2 \dots n, \tag{24}$$

and the modified Hamiltonian becomes

$$\hat{H} = -L + \sum_{i=1}^{n} (\hat{P}_i y_{i_x} + R_i y_{i_{xx}}). \tag{25}$$

3.1 Example 1: A prismatic beam

Consider a prismatic beam of constant length l and with a constant lateral load q per unit length subjected to a compressive axial force F. The governing equation and the boundary conditions for this problem can be found from

$$\delta \int_0^l (EIy_{xx}^2/2 - qy - F{y_x}^2/2)\,dx = 0, \tag{26}$$

where EI is the flexural rigidity of the beam and the three terms in the integrand express the strain energy of the beam in flexure, the potential energy of the distributed load and the potential energy of the compressive axial force F. In this case

$$R = EIy_{xx}, \tag{27}$$

$$\hat{P} = -Fy_x - \frac{d}{dy}[EIy_{xx}] \tag{28}$$

and the governing equation becomes

$$EIy_{xxxx} + Fy_{xx} = q, \tag{29}$$

which is the well known equation of the beam. If $q = 0$, i.e. y is absent in L, we find that $\hat{P}$ is conserved, that is

$$Fy_x + EIy_{xxx} = \text{Constant} \tag{30}$$

in the beam. Evidently this equation is correct as one can easily verify by differentiating through with respect to x and deriving the equilibrium equation for the beam under zero lateral load. In the functional in Eq. (26) x does not appear explicitly. Accordingly, in this case the generalised Hamiltonian as given in Eq. (23), must also be conserved. For this problem

$$\hat{H} = -EIy_{xx}^2/2 + qy + Fy_x{}^2/2 + y_x(-Fy_x - EIy_{xxx}) + y_{xx}(EIy_{xx}). \tag{31}$$

On simplifying we find

$$EIy_{xx}^2/2 + qy - Fy_x{}^2/2 - EIy_{xxx}y_x = \text{Constant}.$$

To verify this relationship, which incidentally is independent of the boundary conditions, we differentiate through with respect to x to find

$$y_x(EIy_{xxxx} + Fy_{xx} - q) = 0, \tag{32}$$

which will be readily recognised as the governing equilibrium equation for the beam.

4 Several independent variables

Consider next the following extremum problem:

$$\delta \int_\Omega L(x, y, w, w_x, w_y)\, dxdy = 0.$$

The Euler-Lagrange equation in this case may be written as follows:

$$\frac{\partial L}{\partial w} - \left[\frac{\partial}{\partial x}\right]\frac{\partial L}{\partial w_x} - \left[\frac{\partial}{\partial y}\right]\frac{\partial L}{\partial w_y} = 0. \tag{33}$$

At this juncture it is worth underscoring the meaning of various differential operators. The operators $\partial/\partial w$, $\partial/\partial w_x$, and $\partial/\partial w_y$ are partial derivatives, that is, for these operations all variables, save those with respect to which the differentiation is being performed, are held fixed. On the other hand the operators $\partial/\partial x$ and $\partial/\partial y$ are for all intents and purposes total derivatives except that for the former y is held fixed and for the latter x is fixed. To maintain clarity in the meaning of these operators we enclose them in [] brackets whenever a total derivative, in the sense just mentioned, is meant.

We now write the above Euler-Lagrange equation as

$$\left[\frac{\partial}{\partial x}\right] P_1 + \left[\frac{\partial}{\partial y}\right] P_2 = \frac{\partial L}{\partial w}, \tag{34}$$

where P_1 and P_2 are defined as

$$P_1 = \frac{\partial L}{\partial w_x} \quad P_2 = \frac{\partial L}{\partial w_y}. \tag{35}$$

Evidently when w does not appear in L, the notion of "Conservation of Momentum" takes the form

$$\left[\left[\frac{\partial}{\partial x}\right]\left[\frac{\partial}{\partial y}\right]\right]\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = 0. \tag{36}$$

It is interesting to see that in this case the "Conservation of Momentum" translates into the momentum vector $\lfloor P_1 \quad P_2 \rfloor$ being divergence free. To examine the case when x and/or y are absent (explicitly) in L, we will follow the already familiar procedure. Thus we evaluate the derivatives of L as follows:

$$\begin{aligned} \left[\frac{\partial}{\partial x}\right] L &= \frac{\partial L}{\partial x} + \frac{\partial L}{\partial w} w_x + \frac{\partial L}{\partial w_x} w_{xx} + \frac{\partial L}{\partial w_y} w_{yx} \\ \left[\frac{\partial}{\partial y}\right] L &= \frac{\partial L}{\partial y} + \frac{\partial L}{\partial w} w_y + \frac{\partial L}{\partial w_y} w_{yy} + \frac{\partial L}{\partial w_x} w_{xy}. \end{aligned} \tag{37}$$

Now isolating the terms $\frac{\partial L}{\partial w} w_x$ in Eq. (33) and using it in the first of Eqs. (37) we can write the following equation:

$$\left[\frac{\partial}{\partial x}\right]\left(-L + \frac{\partial L}{\partial w_x} w_x\right) + \left[\frac{\partial}{\partial y}\right]\left(\frac{\partial L}{\partial w_y} w_x\right) = -\frac{\partial L}{\partial x}. \tag{38}$$

Similarly from the second of Eqs. (37) and Eq. (33) we can derive the following equation:

$$\left[\frac{\partial}{\partial x}\right]\left(\frac{\partial L}{\partial w_x} w_y\right) + \left[\frac{\partial}{\partial y}\right]\left(-L + \frac{\partial L}{\partial w_y} w_y\right) = -\frac{\partial L}{\partial y}. \tag{39}$$

Equations (38) and (39) suggest a generalisation of the Hamiltonian function. To this end we write these equations as

$$\left[\left[\frac{\partial}{\partial x}\right]\left[\frac{\partial}{\partial y}\right]\right]\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} = -\left[\frac{\partial L}{\partial x}\ \frac{\partial L}{\partial y}\right], \tag{40}$$

where

$$\begin{aligned} H_{11} &= -L + P_x w_x \quad & H_{12} &= P_x w_y \\ H_{21} &= P_y w_x & H_{22} &= -L + P_y w_y. \end{aligned} \tag{41}$$

Here again we see that the absence of x and/or y (explicitly) in L leads to the Hamiltonian vectors $\lfloor H_{11} \;\; H_{21} \rfloor$ and/or $\lfloor H_{12} \;\; H_{22} \rfloor$ being divergence free.

4.1 *Example 2: A rectangular membrane*

Consider the well known membrane problem governed by

$$\delta \int_{\Omega} (\Gamma(w_x^2 + w_y^2)/2 - qw)\, dxdy = 0, \tag{42}$$

where Γ is the membrane tension per unit length, q is the applied pressure and w denotes the small deflection of the membrane surface. In this case

$$\begin{aligned} &P_1 = \Gamma w_x \quad P_2 = \Gamma w_y \\ &H_{11} = \Gamma(w_x^2 - w_y^2)/2 + qw \quad H_{12} = \Gamma w_x w_y \\ &H_{21} = \Gamma w_y w_x \quad H_{22} = \Gamma(w_y^2 - w_x^2)/2 + qw. \end{aligned} \tag{43}$$

For this problem w is present in L and hence the momentum vector is not divergence free. However both x and y are absent and hence the Hamiltonian vectors are divergence free. That is

$$\left[\left[\frac{\partial}{\partial x}\right]\left[\frac{\partial}{\partial y}\right]\right] \begin{bmatrix} \Gamma(w_x^2 - w_y^2)/2 + qw & \Gamma w_x w_y \\ \Gamma w_y w_x & \Gamma(w_y^2 - w_x^2)/2 + qw \end{bmatrix} = [0 \quad 0]. \tag{44}$$

Performing the indicated differentiations we find the following equations:

$$\begin{aligned} &w_x\{\Gamma(w_{xx} + w_{yy}) + q\} = 0 \\ &w_y\{\Gamma(w_{xx} + w_{yy}) + q\} = 0. \end{aligned} \tag{45}$$

The terms in the curly bracket will be recognised as the well known equilibrium equation for the membrane.

Next let us examine the consequence of the momentum and Hamiltonian vectors being divergence free. From Green's theorem we have that

$$\int_{\Omega} \left[\left[\frac{\partial}{\partial x}\right]\left[\frac{\partial}{\partial y}\right]\right] \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} d\Omega = \oint_S [n_x \;\; n_y] \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} dS, \tag{46}$$

where $\lfloor n_x \;\; n_y \rfloor$ is the outward unit normal vector to the bounding curve S of the region Ω as shown in Fig. 1. Using Green's theorem for a rectangular membrane, we can assert that whenever w is absent in L, the divergence free momentum vector implies that

$$\int_{y_1}^{y_2} P_1 dy|_{x=x_2} + \int_{x_2}^{x_1} P_2(-dx)|_{y=y_2} + \int_{y_2}^{y_1} (-P_1)\,(-dy)|_{x=x_1} + \int_{x_1}^{x_2} (-P_2)\, dx|_{y=y_1} = 0. \tag{47}$$

Similar expressions can be written for the two Hamiltonian vectors whenever x and y do not arise explicitly in L. We may deduce two invariant forms by considering an infinitesimal change in x, say. That is, if we let

$$x = x_1 \quad \text{and} \quad x_2 = x + \Delta x,$$

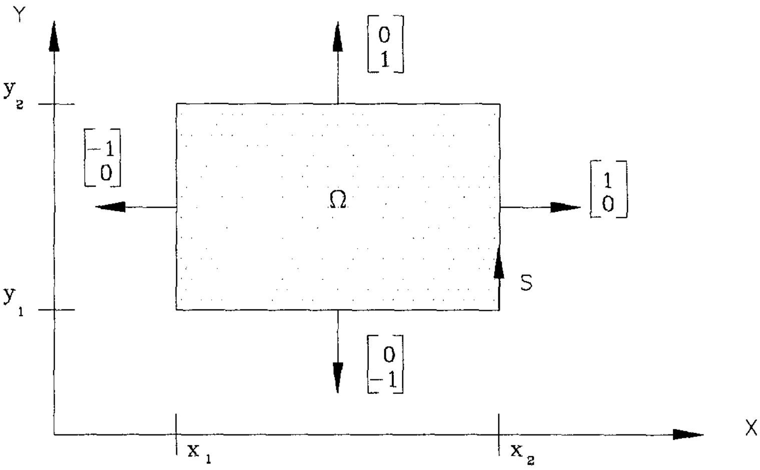

Fig. 1. Membrane's configuration space

Eq. (47) may be written as follows

$$\frac{d}{dx}\left[\int_{y_1}^{y_2} P_1 dy\right] = -P_2|_{y_1}^{y_2}. \tag{48}$$

If P_2 happens to have the same value at $y = y_1$ and $y = y_2$, then

$$\int_{y_1}^{y_2} P_1 dy = \text{Constant} \tag{49}$$

along x. We see that the result requires the satisfaction of some conditions along the *boundary* as well as within the domain. To illustrate this result let us return to the membrane problem. In this case the Hamiltonian vectors were divergence free. Thus we deduce for $\lfloor H_{11} \;\; H_{21} \rfloor$ the following result:

$$\frac{d}{dx}\left[\int_{y_1}^{y_2} (\Gamma(w_x^2 = w_y^2)/2 + qw)\, dy\right] = \Gamma w_x w_y|_{y_1}^{y_2}. \tag{50}$$

Now, for a rectangular membrane with all edges fixed, we note that w_x vanishes along the $y = y_1$ and $y = y_2$ edges. It follows then that

$$\int_{y_1}^{y_2} \big((\Gamma(w_x^2 - w_y^2))/2 + qw\big)\, dy = \text{Constant} \tag{51}$$

along x. In particular we can see that along the edge $x = x_1$, both w and w_y vanish whereas along the center line $w_x = 0$. Thus, we can deduce that for this case

$$\int_{y_1}^{y_2} (\Gamma/2)\, w_x^2 dy|_{x=x_1} = \int_{y_1}^{y_2} (qw - (\Gamma/2)\, w_y^2)\, dy\Big|_{x=\frac{x_1+x_2}{2}}. \tag{52}$$

This relationship reflects the inherent symmetry of the deflected shape of a rectangular membrane under a constant pressure. It is also evident that this result can be generalised for quadrilateral membranes with parallel curved edges.

To prove Eq. (51) let us differentiate through with respect to x. Then

$$\int_{y_1}^{y_2} \left(w_x(\Gamma w_{xx} + q) - \Gamma w_y w_{yx}\right) dy = 0. \tag{53}$$

But from the governing equilibrium equation we have that

$$w_x(\Gamma w_{xx} + q) = -\Gamma w_x w_{yy}. \tag{54}$$

Substituting from Eq. (54) into (53) we find that

$$\int_{y_1}^{y_2} \Gamma(w_x w_{yy} + w_y w_{yx})\, dy = 0, \quad \text{or} \quad \int_{y_2}^{y_1} \Gamma \frac{d}{dy}[w_x w_y]\, dy = 0, \tag{55}$$

which upon integration yields

$$\Gamma w_x w_y |_{y_1}^{y_2} = 0. \tag{56}$$

Equation (56) is true by virtue of w being equal to zero along the straight edges $y = y_1$ and $y = y_2$.

5 Several independent variables and several extremising functions

Consider the functional

$$\delta \int_{\Omega} L(x, y, u, u_x, u_y, v, v_x, v_y)\, dxdy = 0, \tag{57}$$

where x and y are the independent variables and $u = u(x, y)$ and $v = v(x, y)$ the dependent. In this case we obtain the Euler-Lagrange equations as follows:

$$\begin{aligned} \frac{\partial L}{\partial u} - \left[\frac{\partial}{\partial x}\right]\left(\frac{\partial L}{\partial u_x}\right) - \left[\frac{\partial}{\partial y}\right]\left(\frac{\partial L}{\partial u_y}\right) &= 0 \\ \frac{\partial L}{\partial v} - \left[\frac{\partial}{\partial x}\right]\left(\frac{\partial L}{\partial v_x}\right) - \left[\frac{\partial}{\partial y}\right]\left(\frac{\partial L}{\partial v_y}\right) &= 0. \end{aligned} \tag{58}$$

These equations suggest the following definitions for modified momentum vectors:

$$\begin{aligned} P_{11} &= \frac{\partial L}{\partial u_x} \quad P_{12} = \frac{\partial L}{\partial u_y} \\ P_{21} &= \frac{\partial L}{\partial v_x} \quad P_{22} = \frac{\partial L}{\partial v_y}. \end{aligned} \tag{59}$$

Then Eq. (58) may be expressed as

$$\left[\left[\frac{\partial}{\partial x}\right]\left[\frac{\partial}{\partial y}\right]\right]\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[\frac{\partial L}{\partial u}\ \frac{\partial L}{\partial v}\right]. \tag{60}$$

Thus, when u and v are not present in L, the vectors $\lfloor P_{11}\ \ P_{21} \rfloor$ and/or $\lfloor P_{12}\ \ P_{22} \rfloor$ become divergence free. As far as the Hamiltonian vectors are concerned, the governing equations remain as given in Eq. (40). The only change is in the definition of the components for the Hamiltonian vectors. Thus, in this case we find

$$
\begin{aligned}
H_{11} &= -L + P_{11}u_x + P_{vx}v_x \quad & H_{12} &= P_{11}u_y + P_{21}v_y \\
H_{21} &= P_{12}u_x + P_{22}v_x & H_{21} &= -L + P_{22}v_y + P_{12}u_y .
\end{aligned}
\tag{61}
$$

6 Several independent variables and higher order functionals

Consider the functional

$$
\delta \int_{\Omega} L(t, x, y, y_t, y_x, y_{tx}, y_{xt}, y_{tt}, y_{xx})\, dt dx = 0, \tag{62}
$$

where t and x are the independent variables and $y = y(t, x)$ the dependent. The Euler-Lagrange equation for this functional takes the following form:

$$
\begin{aligned}
&\frac{\partial L}{\partial y} - \left[\frac{\partial}{\partial t}\right]\left(\frac{\partial L}{\partial y_t}\right) - \left[\frac{\partial}{\partial x}\right]\left(\frac{\partial L}{\partial y_x}\right) + \left[\frac{\partial^2}{\partial t^2}\right]\left(\frac{\partial L}{\partial y_{tt}}\right) \\
&\quad + \left[\frac{\partial^2}{\partial x^2}\right]\left(\frac{\partial L}{\partial y_{xx}}\right) + 2\left[\frac{\partial^2}{\partial t \partial x}\right]\left(\frac{\partial L}{\partial y_{tx}}\right) = 0.
\end{aligned}
\tag{63}
$$

If we now define the modified momenta as

$$
\begin{aligned}
\hat{P}_1 &= \frac{\partial L}{\partial y_t} - \left[\frac{\partial}{\partial t}\right] R_{11} - \left[\frac{\partial}{\partial x}\right] R_{12} \\
\hat{P}_2 &= \frac{\partial L}{\partial y_x} - \left[\frac{\partial}{\partial t}\right] R_{21} - \left[\frac{\partial}{\partial x}\right] R_{22},
\end{aligned}
\tag{64}
$$

where

$$
\begin{aligned}
R_{11} &= \frac{\partial L}{\partial y_{tt}} \quad & R_{12} &= \frac{\partial L}{\partial y_{tx}} \\
R_{21} &= \frac{\partial L}{\partial y_{xt}} & R_{22} &= \frac{\partial L}{\partial y_{xx}}
\end{aligned}
\tag{65}
$$

are the hypermomenta, the Euler-Lagrange equation (63) may be written as

$$
\left[\left[\frac{\partial}{\partial t}\right]\left[\frac{\partial}{\partial x}\right]\right]\begin{bmatrix}\hat{P}_1 \\ \hat{P}_2\end{bmatrix} = \frac{\partial L}{\partial y}. \tag{66}
$$

Thus, when y is absent in L, the modified momentum vector $\lfloor \hat{P}_1 \ \hat{P}_2 \rfloor$ becomes divergence free. Similarly, the absence of t and/or x in L leads to the Hamiltonian vectors $\lfloor H_{11} \ H_{21} \rfloor$ and/or $\lfloor H_{12} \ H_{22} \rfloor$ being divergence free. The governing equations for the Hamiltonian vectors remain as given in Eq. (40) where only the components of the Hamiltonian vectors need to be changed to the following definitions:

$$
\begin{aligned}
H_{11} &= -L + \hat{P}_1 y_t + R_{11}y_{tt} + R_{12}y_{tx} \quad & H_{12} &= \hat{P}_1 y_x + R_{11}y_{xt} + R_{12}y_{xx} \\
H_{21} &= \hat{P}_2 y_t + R_{21}y_{tt} + R_{22}y_{tx} & H_{22} &= -L + \hat{P}_2 y_x + R_{21}y_{xt} + R_{22}y_{xx}.
\end{aligned}
\tag{67}
$$

6.1 Example 3: An axially moving beam

Consider the axially moving slender beam of Fig. 2, governed by [4]

$$\delta \int_{\Omega} \left(\frac{\varrho A}{2} (y_t^2 + 2\dot{l} y_t y_x + \dot{l}^2 y_x^2) - \frac{EI}{2} y_{xx}^2 + \frac{\varrho A}{2} \dot{l}^2 \right) dt dx = 0, \tag{68}$$

where ϱA is the material density per unit length, EI is the flexural rigidity of the beam, and l is the instantaneous length of the beam – a given function of time. The three terms in the integrand express the transverse complementary kinetic energy, the potential energy, and the longitudinal complementary kinetic energy. In this case the hypermomenta (65) and modified momenta (64) are given by

$$\begin{aligned} &R_{11} = 0 && R_{12} = 0 \\ &R_{21} = 0 && R_{22} = -EIy_{xx} \\ &\hat{P}_1 = \varrho A(y_t + \dot{l} y_x) && \hat{P}_2 = \varrho A \dot{l}(y_t + \dot{l} y_x) + EIy_{xxx}, \end{aligned} \tag{69}$$

and using the Euler-Lagrange equation (66) the governing equation becomes

$$\varrho A(y_{tt} + 2\dot{l} y_{tx} + \dot{l}^2 y_{xx} + \ddot{l} y_x) + EIy_{xxxx} = 0. \tag{70}$$

Since y is absent in L, the modified momentum is conserved and the corresponding vector $\lfloor \hat{P}_1 \;\; \hat{P}_2 \rfloor$ is divergence free, that is,

$$\left[\left[\frac{\partial}{\partial t} \right] \left[\frac{\partial}{\partial x} \right] \right] \begin{bmatrix} \hat{P}_1 \\ \hat{P}_2 \end{bmatrix} = 0. \tag{71}$$

Using Green's theorem for the axially moving beam with the region Ω, bounding curve S, and the outward unit normal vectors as shown in Fig. 3, we can state that the divergence free modified momentum vector implies that

$$\begin{aligned} &\int_0^{l(t_2)} \hat{P}_1 dx|_{t=t_2} + \int_{t_2}^{t_1} \left(\frac{1}{\sqrt{(1+\dot{l}^2)}} \right) (-\dot{l}\hat{P}_1 + \hat{P}_2)\left(-\sqrt{(1+\dot{l}^2)}\right) dt \Big|_{x=l(t)} \\ &\qquad + \int_{l(t_1)}^{0} (-\hat{P}_1)(-dx)|_{t=t_1} + \int_{t_1}^{t_2} (-\hat{P}_2)\, dt|_{x=0} = 0. \end{aligned} \tag{72}$$

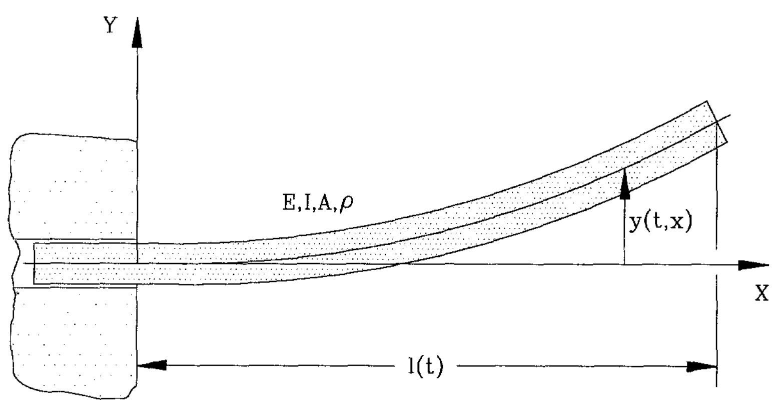

Fig. 2. Axially moving beam

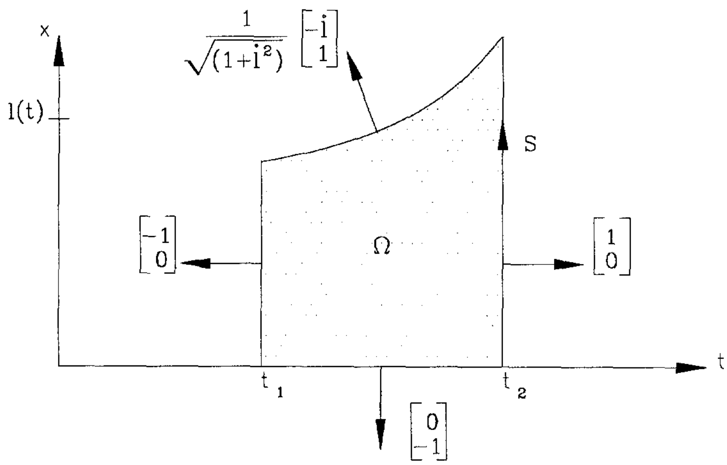

Fig. 3. Configuration space of axially moving beam

The term $(-\sqrt{(1-\dot{l}^2)})$ appears in the calculation of the unit normal of the curved edge *and* in the Jacobian of the transformation from the straight line t to the curved line.

Now, considering an infinitesimal change in t with

$$t = t_1 \quad \text{and} \quad t_2 = t + \Delta t, \tag{73}$$

Eq. (72) may be written as follows:

$$\frac{d}{dt}\left[\int_0^{l(t)} \hat{P}_1 dx\right] = \hat{P}_2|_{x=0} - (-\dot{l}\hat{P}_1 + \hat{P}_2)|_{x=l(t)}. \tag{74}$$

Making use of Eqs. (69) and the boundary conditions of the axially moving beam, namely

$$y_t = y_x = 0 \qquad \text{at} \quad x = 0$$

$$y_{xx} = y_{xxx} = 0 \qquad \text{at} \quad x = l(t), \tag{75}$$

Eq. (74) becomes

$$\frac{d}{dt}\left[\int_0^{l(t)} \varrho A\{y_t + \dot{l}y_x\}\, dx\right] = EIy_{xxx}|_{x=0}. \tag{76}$$

Noting that the term in the curly brackets corresponds to the transverse velocity of points along the beam, the above equation states that the rate of change of transverse momentum is equal to the shear force evaluated at the base of the axially moving beam.

Time appears explicitly in L (within the given function of time $l^2(t)$), hence the Hamiltonian vector $\lfloor H_{11} \;\; H_{21} \rfloor$ is not divergence free, that is,

$$\left[\left[\frac{\partial}{\partial t}\right]\left[\frac{\partial}{\partial x}\right]\right]\begin{bmatrix}H_{11}\\H_{21}\end{bmatrix} = -\varrho A\ddot{l}[(y_t + \dot{l}y_x)\, y_x + \dot{l}], \tag{77}$$

where the modified Hamiltonian vector components, from Eq. (67), become for the axially moving beam

$$\begin{aligned}
H_{11} &= \frac{\varrho A}{2}(y_t^2 - \dot{l}^2 y_x^2) + \frac{EI}{2} y_{xx}^2 - \frac{\varrho A \dot{l}^2}{2} \\
H_{12} &= \varrho A(y_t + \dot{l} y_x)\, y_x \\
H_{21} &= \varrho A \dot{l}(y_t + \dot{l} y_x)\, y_t + EI y_{xxx} y_t - EI y_{xx} y_{tx} \\
H_{22} &= \frac{\varrho A}{2}(\dot{l}^2 y_x^2 - y_t^2) - \frac{EI}{2} y_{xx}^2 + EI y_{xxx} y_x - \frac{\varrho A}{2} \dot{l}^2 .
\end{aligned} \tag{78}$$

Following the same procedure as that with the modified momentum vector, we can state that the modified Hamiltonian vector $\lfloor H_{11} \;\; H_{21} \rfloor$ implies that

$$\frac{d}{dt}\left[\int_0^{l(t)} H_{11} dx\right] = H_{21}|_{x=0} - (-\dot{l} H_{11} + H_{21})|_{x=l(t)} - \int_\Omega \varrho A \ddot{l}[(y_t + \dot{l} y_x)\, y_x + \dot{l}]\, d\Omega . \tag{79}$$

We will now study the above expression in the context of different forms of the given function $l(t)$. If the beam's length is constant, i.e. $\dot{l}(t) = \ddot{l}(t) = 0$, using Eq. (78) and the system boundary conditions (Eqs. (75)), the above expression becomes

$$\frac{d}{dt}\left[\int_0^{l(t)} \left(\frac{EI}{2} y_{xx}^2 + \frac{\varrho A}{2} y_t^2\right) dx\right] = 0, \tag{80}$$

namely, a statement of conservation of energy. If, on the other hand, the beam is axially moving at a constant velocity, i.e. $\dot{l}(t) = v$ and $\ddot{l}(t) = 0$, using Eq. (78) and the system boundary conditions, Eq. (79) becomes an invariant statement for the modified Hamiltonian component H_{tt}, namely

$$\frac{d}{dt}\left[\int_0^{l(t)} H_{11} dx\right] = -\dot{l}\left[\frac{\varrho A}{2}\{y_t + v y_x\}^2 + \frac{\varrho A v^2}{2}\right]\Bigg|_{x=l(t)} . \tag{81}$$

Finally, for the general case where neither $\dot{l}$, nor $\ddot{l}$ are equal to zero, and using the expressions of Hamiltonian vectors, Eq. (79) becomes

$$\frac{d}{dt}\left[\int_0^{l(t)} H_{11} dx\right] = -\dot{l}\left[\frac{\varrho A}{2}\{y_t + \dot{l} y_x\}^2 + \varrho A \dot{l}^2\right]\Bigg|_{x=l(t)} - \int_\Omega \varrho A \ddot{l}[(y_t + \dot{l} y_x)\, y_x + \dot{l}]\, d\Omega . \tag{82}$$

Another invariant result is obtained by noting that x is absent in L, hence the modified Hamiltonian vector $\lfloor H_{12} \;\; H_{22} \rfloor$ is divergence free, that is,

$$\left[\left[\frac{\partial}{\partial t}\right]\left[\frac{\partial}{\partial x}\right]\right]\begin{bmatrix} H_{12} \\ H_{22} \end{bmatrix} = 0. \tag{83}$$

Following the same procedure as that with the modified momentum vector, we can assert that the divergence-free modified Hamiltonian vector $\lfloor H_{12} \;\; H_{22} \rfloor$ implies that

$$\frac{d}{dt}\left[\int_0^{l(t)} H_{12} dx\right] = H_{22}|_{x=0} - (-\dot{l} H_{12} + H_{22})|_{x=l(t)} . \tag{84}$$

Making use of Eq. (78) and the system boundary conditions in Eqs. (75), the above expression becomes

$$\frac{d}{dt}\left[\int_0^{l(t)} \varrho A\{y_t + \dot{l}y_x\}\, y_x dx\right] = -\frac{EI}{2}\, y_{xx}^2\Big|_{x=0} + \frac{\varrho A}{2}\{y_t + \dot{l}y_x\}^2\Big|_{x=l(t)}. \tag{85}$$

Equation (85) states that the rate of change of the component of the transverse momentum in the longitudinal direction is equal to the difference of the potential energy per unit length at the base and the transverse complementary kinetic energy per unit length at the tip of the axially moving beam. The governing equation for the axially moving beam, Eq. (70), can also be obtained from any one of the three Eqs. (71), (77), and (83), by simply carrying out the differentiation and simplifying the resulting expression.

The correctness of Eqs. (76), (80), (81), (82), and (85), can be verified by direct differentiation of their left hand sides. Since the differentiation of their left hand sides is with respect to time and the upper limit of the integral is also a function of time, we need to use the Leibniz rule to carry out the differentiation. For example, carrying out the differentiation of the left hand side of Eq. (85) we have

$$\frac{d}{dt}\left[\int_0^{l(t)} \varrho A\{y_t + \dot{l}y_x\}\, y_x dx\right] = \int_0^{l(t)} \varrho A[(y_{tt} + \ddot{l}y_x + 2\dot{l}y_{xt})\, y_x + (y_t y_{xt})]\, dx + \varrho A\dot{l}(y_t + \dot{l}y_x)\, y_x|_{x=l(t)}. \tag{86}$$

But from the governing equation (70) we have that

$$\varrho A(y_{tt} + \ddot{l}y_x + 2\dot{l}y_{tx}) = -EIy_{xxxx} - \varrho A\dot{l}^2 y_{xx}, \tag{87}$$

which when substituted in Eq. (86) yields

$$\frac{d}{dt}\left[\int_0^{l(t)} \varrho A\{y_t + \dot{l}y_x\}\, y_x dx\right] = \int_0^{l(t)} (-EIy_{xxxx}y_x - \varrho A\dot{l}^2 y_{xx}y_x + \varrho A y_t y_{xt})\, dx + \varrho A\dot{l}(y_t + \dot{l}y_x)\, y_x|_{x=l(t)}. \tag{88}$$

Noting that

$$\varrho A\dot{l}^2 y_{xx}y_x = \frac{\varrho A}{2}\,\dot{l}^2\,\frac{d}{dx}(y_x{}^2) \quad \text{and} \quad \varrho A y_t y_{xt} = \frac{\varrho A}{2}\frac{d}{dx}(y_t{}^2), \tag{89}$$

Eq. (88) can be simplified to

$$\frac{d}{dt}\left[\int_0^{l(t)} \varrho A\{y_t + \dot{l}y_x\}\, y_x dx\right] = \int_0^{l(t)} (-EIy_{xxxx}y_x)\, dx + \frac{\varrho A}{2}(y_t + \dot{l}y_x)^2\Big|_{x=l(t)}. \tag{90}$$

We now use integration by parts on the integral on the right hand side of the above expression to get

$$\int_0^{l(t)} (-EIy_{xxxx}y_x)\, dx = -\int_0^{l(t)} \frac{d}{dt}(EIy_{xxxx}y_x)\, dx + \int_0^{l(t)} (EIy_{xxx}y_{xx})\, dx, \tag{91}$$

or

$$\int_0^{l(t)} (-EIy_{xxxx}y_x)\, dx = -(EIy_{xxxx}y_x)|_0^{l(t)} + \int_0^{l(t)} (EIy_{xxx}y_{xx})\, dx. \tag{92}$$

Using the boundary conditions (75), Eq. (92) becomes

$$\int_0^{l(t)} (-EIy_{xxxx}y_x)\,dx = \int_0^{l(t)} (EIy_{xxx}y_{xx})\,dx. \tag{93}$$

Finally, noting that

$$EIy_{xxx}y_{xx} = \frac{EI}{2}\frac{d}{dx}(y_{xx}^2), \tag{94}$$

Eq. (92) further simplifies to

$$\int_0^{l(t)} (-EIy_{xxxx}y_x)\,dx = -\frac{EI}{2}\,y_{xx}^2\Big|_{x=0}. \tag{95}$$

Substituting Eq. (95) into Eq. (90) we obtain

$$\frac{d}{dt}\left[\int_0^{l(t)} \varrho A\{y_t + \dot{l}y_x\}\,y_x dx\right] = \frac{\varrho A}{2}\{y_t + \dot{l}y_x\}^2\Big|_{x=l(t)} - \frac{EI}{2}\,y_{xx}^2\Big|_{x=0} \tag{96}$$

which is identical to Eq. (85).

For non-conservative mechanical systems, such as the axially moving beam, in the absence of energy conservation principles, invariant expressions such as Eqs. (76), (81), and (85) are valuable results for verifying numerical time integration algorithms [5].

7 Concluding comments

In the foregoing we have reviewed the celebrated statement on conservation of energy, in classical mechanics, within the framework of calculus of variations. From such a setting and by not making a particular distinction between time and space variables we have developed other conservation statements for different functionals and we have illustrated them by problems from elastodynamics. Such invariant forms are manifestations of inherent symmetries in the system being studied. They are therefore of intrinsic interest on physical grounds and they provide useful checks for soundness of numerical algorithms devised to solve the system equations [5].

The technique we have used to arrive at the new invariant forms can be readily generalised to higher order functionals. For instance it would be relatively straightforward to derive corresponding results for problems of plate and shell dynamics.

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Authors' addresses: Prof. B. Tabarrok and M. Stylianou, Department of Mechanical Engineering, University of Victoria, P.O. Box 3055, Victoria V8W 3P6, B. C., Canada, Assoc. Prof. C. Tezer, Department of Mathematics, Middle East Technical University, Ankara, Turkey