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Iterated skew polynomial rings of Krull dimension two

By

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Introduction. Smith [4] proved that the universal enveloping algebra $U(sl(2, \mathbb{C}))$ of the Lie algebra $sl(2, \mathbb{C})$ has Krull dimension two. This algebra, the algebras similar to it studied by Smith in [5] and the quantum enveloping algebra $U_q(sl(2, \mathbb{C}))$ belong to the class of iterated skew polynomial rings $R = A[y; \alpha][x; \alpha^{-1}, \delta]$ over a commutative ring A which we studied in [1] and [2]. Smith's proof that $\operatorname{Kdim} U(sl(2, \mathbb{C}))$ is two makes strong use of the fact that, over $U(sl(2, \mathbb{C}))$, all finite-dimensional modules are semisimple. In [2, Theorem 3.7] we extended Smith's result to certain of the iterated skew polynomial rings R but were unable to avoid including this condition on the finite-dimensional modules as one of the hypotheses. However not all the algebras of [5] have this property. The purpose of this note is to remove this hypothesis and thereby to extend Smith's result to all the algebras of [5] and their quantum analogues described in [1, 2.4]. Our approach is much as in [2] but we refine the proof in such a way that it applies more generally.

Notation. Here we give details of and notation for the iterated skew polynomial rings R referred to above. More complete details, including justification for the statements made, may be found in [1, Section 1].

Let A be a commutative domain. In this paper we assume that A is a finitely generated algebra over an algebraically closed field k. Let α be a k-automorphism of A and let $u \in A$ be such that $\alpha(u) \neq u$. Form the skew polynomial ring $A[y; \alpha]$ and extend α to $A[y; \alpha]$ by setting $\alpha(y) = y$. There is an α^{-1}-derivation δ of $A[y; \alpha]$ such that $\delta(A) = 0$ and $\delta(y) = u - \alpha(u)$. The ring R is the iterated skew polynomial ring $A[y; \alpha][x; \alpha^{-1}, \delta]$. Thus R consists of polynomials in y and x over A subject to the relations $xy - yx = u - \alpha(u)$ and, $\forall a \in A$, $ya = \alpha(a)y$ and $xa = \alpha^{-1}(a)x$. The sets $\{y^i\}_{i \geqq 1}$ and $\{x^i\}_{i \geqq 1}$ are right and left Ore sets in R and we denote the localizations by R_y and R_x respectively. By z we denote the element $xy - u = yx - \alpha(u)$ which is central in R. The localization $R_y \cong S[z]$, the ordinary polynomial ring over the skew Laurent polynomial ring $S = A[y, y^{-1}; \alpha]$ and $R_x \cong A[x, x^{-1}; \alpha^{-1}][z] \cong R_y$.

For the case where $R = U(sl(2, \mathbb{C}))$, the elements x and y are commonly written as e and f respectively, $A = \mathbb{C}[h]$, $\alpha(h) = h + 2$, $u = -\frac{1}{4}(h-1)^2$ and $z = \frac{1}{4}(\Omega + 1)$, where Ω is the Casimir element. For other examples, including $U_q(sl(2, \mathbb{C}))$, see [1].

If X is a right R-module then, for $c = x$ or y, the R_c-module $X \otimes R_c$ will be denoted X_c and if X is torsion with respect to $\{c^i\}_{i \geqq 1}$ then we say that X is *c-torsion*. If X is both y-torsion and x-torsion we say that X is *xy-torsion*.

If the only ideals of A invariant under α are A and 0 then A is said to be *α-simple.* M. Holland, unpublished, has shown that, for A as above, if $\operatorname{Kdim} A = 1$ and A is α-simple for some α then $A \cong k[t]$ or $A \cong k[t, t^{-1}]$, whence A is a principal ideal domain.

Lemma 1. *Suppose that A is α-simple and a principal ideal domain. For every maximal ideal M of A and every $\lambda \in k$, the R-module $R/(MR + (z + \lambda)R)$ has finite length.*

Proof. Let $J = MR + (z + \lambda)R$. For $i \geqq 0$, let $w_i = y^i + J$ and let $v_i = x^i + J$. (Thus $w_0 = v_0 = 1 + J$.) The elements w_i and v_i span R/J over k. As A is an α-simple domain the maximal ideals $\alpha^i(M)$, $i \in \mathbb{Z}$, are distinct, otherwise a finite product of them would be invariant under α. Each w_i is annihilated by $\alpha^{-i}(M)$ and each v_i is annihilated by $\alpha^i(M)$. Consequently, if F is any nonempty finite subset of $\{v_i, w_i : i \geqq 0\}$ then there exist $a \in A$ and $f \in F$ such that $ea = 0$ for all $e \in F \setminus \{f\}$ and a is invertible modulo $\operatorname{ann}_A f$. Hence every non-zero submodule of R/J must contain w_i or v_i for some $i \geqq 0$. For $i \geqq 1$, $w_i y = w_{i+1}$ and $w_i x = y^{i-1}(yx) + J = w_{i-1}(z + \alpha(u)) = w_{i-1}(\alpha(u) - \lambda)$. Similarly, for $i \geqq 1$, $v_i x = v_{i+1}$ and $v_i y = v_{i-1}(u - \lambda)$. Therefore every non-zero submodule of R/J is of the form $\sum_{i \geqq j} w_i k$, where $j \geqq 1$ and $\alpha^j(u) - \lambda \in M$ or is of the form $\sum_{i \geqq j} v_i k$, where $j \geqq 1$ and $\alpha^{1-j}(u) - \lambda \in M$, or is a sum of two submodules, one of each of the preceding forms. Since only finitely many of the principal ideals $\alpha^i(M)$, $i \in \mathbb{Z}$, can contain $u - \lambda$, it follows that R/J has only finitely many submodules. □

Lemma 2. *Suppose that A is α-simple and a principal ideal domain. Every finitely generated xy-torsion right R-module has finite length.*

Proof. Suppose this to be false. Then there is a counterexample of the form $X = R/J$, where J is a right ideal of R, maximal with respect to being xy-torsion of infinite length. For every non-zero submodule I/J of R/J, R/I has finite length so I/J has infinite length.

Let $d \geqq 1$ be minimal such that there exists $0 \neq w \in X$ with $wx = 0 = wy^d$. By [1, 1.9 (i)], $xy^d - y^d x = (u - \alpha^d(u))\, y^{d-1}$. By the choice of d, $w(u - \alpha^d(u)) = 0$. There exist factors p, q of $u - \alpha^d(u)$, with p irreducible, such that $wqp = 0$ but $wq \neq 0$. Let $v = wq$. Then $\operatorname{ann}_A v$ is the maximal ideal $M = pA$, and $vx = 0 = vy^d$.

There exists $\lambda \in k$ such that $u - \lambda \in M$. Then $v(z + \lambda) = v(xy - u + \lambda) = 0$. Thus $MR + (z + \lambda)R \subseteqq \operatorname{ann}_R v$ and, by Lemma 1, $vR \cong R/\operatorname{ann}_R v$ has finite length. But we have seen above that every non-zero submodule of X has infinite length. As $v \neq 0$, this is a contradiction. □

Theorem 3. *Suppose that A is α-simple and a principal ideal domain. Let X be a finitely generated right R-module such that both X_y and X_x have finite length. Then X has finite length.*

Proof. As in the proof of [2, 3.7], it suffices to prove this in the case where X_y and X_x are each either zero or simple. The case where both are zero is dealt with by Lemma 2.

Consider the case where $X_x = 0$ and X_y is simple. As $R_y \cong S[z]$ has centre $k[z]$ and as S is simple, every nonzero primitive ideal of R_y intersects $k[z]$ in a maximal ideal. Furthermore R_y is a constructible k-algebra in the sense of [3] and, as its centre is $k[z]$, it is, by [3, 9.4.21], not primitive. Hence there exists $\lambda \in k$ such that $X_y(z+\lambda) = 0$. As $z+\lambda$ is central, $X(z+\lambda)$ is xy-torsion and so, by Lemma 2, has finite length. By [1, 1.11], $R/(z+\lambda)R$ is a domain and by [2, 4.6] it has a unique minimal non-zero ideal $I/(z+\lambda)R$. The ideal $I/(z+\lambda)R$ is idempotent and, by [2, 4.6 (ii)], has finite codimension. Then X/XI, being finitely generated over the finite-dimensional k-algebra R/I, has finite length over R/I and hence over R. If $XI = X(z+\lambda)$ then X has finite length so we can assume that $XI \neq X(z+\lambda)$. Let N be a submodule of X with $X(z+\lambda) \subseteqq N \subset XI$ and with XI/N simple. Suppose that XI/N is y-torsion, hence xy-torsion. By [1, 3.10], $y^d, x^d \in \operatorname{ann}_R(XI/N)$ for some d. Therefore $\operatorname{ann}_{R/(z+\lambda)R}(XI/N)$ is nonzero and must contain $I/(z+\lambda)R$. Thus $XI^2 \subseteqq N$. As $I/(z+\lambda)R$ is idempotent and $X(z+\lambda) \subseteqq N$, $XI = X(I^2 + (z+\lambda)R) \subseteqq N$, a contradiction. Hence XI/N is not y-torsion. As X_y is simple, N must be y-torsion otherwise $N_y = X_y$ and X/N is y-torsion. Therefore N is xy-torsion and, by Lemma 2, has finite length. As X/XI, XI/N and N have finite length, so too does X. The case where $X_y = 0$ and X_x is simple is similar.

Now consider the case where X_y and X_x are both simple. As above, there exists $\lambda \in k$ such that $X_y(z+\lambda) = 0$. Then $X(z+\lambda)$ is y-torsion and $(X(z+\lambda))_x$ is 0 or simple. By the above $X(z+\lambda)$ has finite length. With I and N as above, so that $X(z+\lambda) \subseteqq N \subset XI$ and XI/N is simple, X/XI again has finite length. The possibility that XI/N is xy-torsion leads, using the idempotence of $I/(z+\lambda)R$, to a contradiction as before. Any other possibility leads, using the simplicity of X_y and X_x, to the conclusion that N_y and N_x are each zero or simple with at least one of them zero. By previously considered cases, N and, hence, X have finite length. □

Corollary 4. *Suppose that A is α-simple and a principal ideal domain. Then* $\operatorname{Kdim} R = 2$.

Proof. Using [3, 6.5.4 and 6.6.11] and applying [3, 6.1.17], with $\delta = \gamma = 1$, as in the proof of [2, 3.7], we have $\operatorname{Kdim} R \leqq \sup(\operatorname{Kdim} R_y, \operatorname{Kdim} R_x) = 2 \leqq \operatorname{Kdim} R$. □

The next result shows that Lemma 2 and Theorem 3 do not extend to α-simple rings A of Krull dimension $\geqq 2$.

Proposition 5. *Suppose that A is α-simple of Krull dimension $d \geqq 2$. If $u - \alpha(u)$ is not a unit then there is a finitely generated xy-torsion R-module of Krull dimension $\geqq d-1$.*

Proof. Let $I = (u - \alpha(u))A$. For all ideals J of A containing I, $(xR + yR + JR) \cap A = J$. Hence $\operatorname{Kdim}_R(R/(xR + yR + IR)) \geqq \operatorname{Kdim}_A(A/I) = d - 1$. □

Remark. Proposition 5 suggests that, in order to extend Corollary 4 to higher dimensions using this approach, one should show that if X is a finitely generated right R-module such that X_y and X_x both have Krull dimension $\leqq d-1$ then $\operatorname{Kdim} X \leqq d-1$.

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