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NOTE

SENSITIVITY VS. BLOCK SENSITIVITY OF BOOLEAN FUNCTIONS

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Sensitivity and block sensitivity are important measures of complexity of Boolean functions. In this note we exhibit a Boolean function of n variables that has sensitivity $O(\sqrt{n})$ and block sensitivity $\Omega(n)$. This demonstrates a quadratic separation of the two measures.

A Boolean function of n variables is a function $f:\{0,1\}^n \rightarrow \{0,1\}$.

The *sensitivity* (or "critical complexity") $s(f)$ of f is defined as follows ([5], [1]).

Let H be a subset of $\{1,2,...,n\}$. Let $f(H)$ denote the value of the function on the input where exactly the variables with indices in H have value 1. Let $\oplus$ denote the symmetric difference of two sets.

Definition 1. The sensitivity of a Boolean function f on input H, denoted by $s(f,H)$, is the number of indices i such that $f(H) \neq f(H \oplus \{i\})$. The sensitivity $s(f)$ of f is the maximum of $s(f,H)$ taken over all possible inputs.

It has been shown by H. U. Simon [5] that every function that depends on all its variables has sensitivity at least $\Omega(\log n)$. Simon and Cook, Dwork, Reischuk [1] found important relations between sensitivity and the parallel complexity of f.

N. Nisan [3] introduced a related measure of complexity called *block sensitivity*. He proved that a number of other complexity measures (deterministic, nondeterministic, and randomized decision tree complexity) are polynomially related to this measure. This enabled him to give a complete characterization of the parallel complexity of Boolean functions in the CREW model (concurrent read, exclusive write) as the logarithm of any of these equivalent measures (up to a bounded factor).

A further important complexity measure which has been shown to be polynomially related to block sensitivity is the degree of the (unique) real multilinear polynomial which extrapolates the Boolean function from the Boolean domain $\{0,1\}^n$ to $\mathbf{R}^n$ [4].

Block sensitivity has also turned out to be the most appropriate measure in other circumstances, including circuit reliability [2].

Definition 2. The block sensitivity of a Boolean function f on input H, denoted by $bs(f,H)$, is the largest number t such that there exist t disjoint sets of indices

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$H_1, H_2, \ldots, H_t$ such that $f(H) \neq f(H \oplus H_i)$ for $1 \leq i \leq t$. The block sensitivity $bs(f)$ of f is the maximum of $bs(f,H)$ taken over all possible inputs.

It is clear that $s(f) \leq bs(f)$. It is also easy to see that for monotone functions $s(f) = bs(f)$.

For an arbitrary Boolean function f, let $n(f)$ denote the number of variables f actually depends on. Then we have

$$c \log(n(f)) \leq s(f) \leq bs(f) \leq n(f).$$

(The first inequality follows from Simon's quoted result [5], where c is a positive constant.) This implies that $bs(f) \leq \exp(c' \cdot s(f))$ so the gap between sensitivity and block sensitivity is at most exponential. It is an open problem of considerable interest whether or not this gap is polynomially bounded.

Problem. Does there exist a constant $c > 0$ such that for all f,

$$bs(f) \leq (s(f))^c \ ?$$

The aim of the present note is to show that at least some gap exists; indeed we demonstrate a quadratic gap.

Theorem. *There exists an infinite family of Boolean functions f such that*

$$bs(f) = \Theta(s(f)^2).$$

More specifically, for every n that is an even perfect square, we construct a Boolean function f of n variables with $2bs(f) = s(f)^2 = n$.

Proof. Let Δ_i denote the interval $\Delta_i = \{(i-1)\sqrt{n}+1, \ldots, i\sqrt{n}\}$ $(i = 1, \ldots, \sqrt{n})$.

Let g_i denote the Boolean function defined as follows: $g_i(H) = 1$ exactly if $H \cap \Delta_i = \{2j-1, 2j\}$ for some j such that $2j \in \Delta_i$.

We define f to be the join of all the g_i:

$$f(H) = g_1(H) \vee \ldots \vee g_{\sqrt{n}}(H).$$

Clearly $bs(f) \geq n/2$, since $f(\emptyset) = 0$ and $f(\{2j-1, 2j\}) = 1$ for all j, $1 \leq j \leq n/2$. In fact, it is easy to see that $bs(f) = n/2$.

We now compute the sensitivity of f. We distinguish two cases according to the value of $f(H)$.

Case 1. $f(H) = 1$. Note that if $g_i(H) = 1$ for more than one value of i then the function value cannot be changed by altering only one input bit. If there is exactly one i such that $g_i(H) = 1$ then $f(H \oplus k) = 0$ precisely if $k \in \Delta_i$. This allows $\sqrt{n}$ choices of k.

Case 2. We have $f(H) = 0$. Then for each i, there exists at most one $k \in \Delta_i$ such that $g_i(H \oplus k) = 1$. This means $\leq \sqrt{n}$ choices of k to change the value of f. ∎

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