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8_3 in $PG(2, q)$

By

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To Prof. Dr. Günter Pickert on the occasion of his 70th birthday

1. Introduction. Let Π be a finite projective plane of order n (see e.g. Albert and Sandler [1] or Pickert [8]). We shall denote the line joining two points P and Q of Π by $[PQ]$; the point of intersection of two lines a, b will be written as (ab). This paper is concerned with embedding a certain configuration into $PG(2, q)$, the desarguesian projective plane of order q.

Here a *configuration* $\mathbf{K}$ is an ordered pair of m points and n lines with c lines on each point and with d points on each line. Such an object is often denoted as (m_c, n_d); in the special case $m = n$ (hence $c = d$), this is simplified to m_c. An interesting problem in the theory of finite projective planes is the study of configurations contained in such planes and of configurational propositions (see e.g. Skornyakov [11]). We will here study the case of configurations 8_3. One such configuration is $BAG(2, 3)$, the biaffine plane of order 3, which is obtained from the affine plane $AG(2, 3)$ by omitting a point P together with the lines incident with P. Let us first observe that this is the only 8_3 which is simultaneously a partial plane (i.e., no two points are on more than one common line); thus $BAG(2, 3)$ is the only candidate for embedding an 8_3 into any projective plane.

Lemma 1.1. *Let* $\mathbf{K}$ *be an* 8_3 *which is a partial plane. Then* $\mathbf{K}$ *is isomorphic to* $BAG(2, 3)$.

P r o o f. Given any line g of $\mathbf{K}$, there is a unique line g' disjoint from g (since each point of g is on exactly 3 lines). Choose a point P on g. Since P is on two other lines and since P cannot be joined twice to any other point, at least one line h through P intersects both g and g'. But then h' also intersects both g and g', and the lines g, g', h, h' form a quadrilateral. The remaining 4 lines are now uniquely determined: for instance, the third line through P cannot intersect g or h again and it cannot contain the point $(g'h')$; so it has to contain the points C and D (see Figure 1). Thus $\mathbf{K}$ is unique up to isomorphism and therefore isomorphic to $BAG(2, 3)$. □

In the remainder of this paper, we will denote $BAG(2, 3)$ by 8_3. It will be quite convenient to have another notation which allows the explicit description of the points and lines of an individual 8_3. We exemplify this for the 8_3 on the point set P, Q, R, S, A, B, C, D given in Figure 1: Its lines are the rows, columns and transversals of the punctured

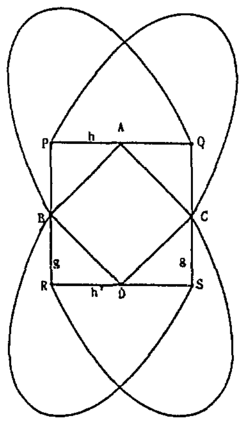

Figure 1

(3 × 3)-matrix

$$\begin{array}{ccc} P & A & Q \\ B & & C \\ R & D & S \end{array}$$

which is not surprising since the lines of an affine plane of order 3 on the point set of 8_3 together with a ninth point X are given by the rows, columns and transversals of the (3 × 3)-matrix

$$\begin{array}{ccc} P & A & Q \\ B & X & C \\ R & D & S \end{array}$$

(as is well-known). Thus our notation for 8_3 also shows how to construct the unique embedding of 8_3 into an affine plane of order 3.

Yet another way of writing 8_3 comes from the fact that $BAG(2, 3)$ (and more generally $BAG(2, q)$, see e.g. Jungnickel [6]) admits a cyclic Singer group and thus a representation by a relative difference set, e.g. by $\{0, 1, 3\}$ in the cyclic group $\mathbb{Z}_8$ of order 8. Thus an 8_3 on the point set $0, \ldots, 7$ is given by the following schema:

g_7	g_6	g_5	g_4	g_3	g_2	g_1	g_0
7	6	5	4	3	2	1	0
0	7	6	5	4	3	2	1
2	1	0	7	6	5	4	3

where the point i is on g_j if and only if $i - j \equiv 0, 1$ or $3 \bmod 8$. Note that here the unique line g_i' disjoint from g_i is g_{i+4}.

We remark in passing that the hypothesis that our 8_3 is a partial plane is indeed necessary to prove Lemma 1.1; for instance, the schema based on $\{0, 1, 2\}$ (in $\mathbb{Z}_8$) gives a configuration 8_3 which is not a partial plane.

We shall determine all planes $PG(2, q)$ containing 8_3, and in fact also the number of 8_3's they contain. Moreover, it turns out that any 8_3 contained in $PG(2, q)$ extends to an affine plane of order 3 contained in $PG(2, q)$. Thus our result implies the result of Ostrom and Sherk [7] who have shown that $PG(2, q)$ contains $AG(2, 3)$ if and only if q is a power of 3 or congruent to 1 modulo 3.

Finally we mention that 8_3 can also be taken as the base for a configurational proposition which might be stated as follows: If a projective plane contains 8 points and 7 lines forming on these points 7 lines of an 8_3, then it contains an eighth line inducing the missing 8th line of the 8_3. (Note that it does not matter which line is taken as the missing line, since 8_3 has a line-transitive group, as shown above.) According to Skornyakov [11], Rashevskii [9] has shown that the only planes satisfying this configurational proposition are $PG(2, 3)$ and $PG(2, 4)$. This motivates an explicit consideration of these two cases before turning to the general problem.

2. 8_3 in $PG(2, 3)$. Denote by Π the plane $PG(2, 3)$. A combinatorial schema for Π is based on the difference set $\{0, 1, 3, 9\}$ in $\mathbb{Z}_{13}$:

ℓ_{12}	ℓ_{11}	ℓ_{10}	ℓ_9	ℓ_8	ℓ_7	ℓ_6	ℓ_5	ℓ_4	ℓ_3	ℓ_2	ℓ_1	ℓ_0
1	2	3	4	5	6	7	8	9	10	11	12	0
2	3	4	5	6	7	8	9	10	11	12	0	1
4	5	6	7	8	9	10	11	12	0	1	2	3
10	11	12	0	1	2	3	4	5	6	7	8	9

It is to be observed, from the schema, that $P_i I \ell_j \Leftrightarrow i + j = 0, 1, 3, 9 \pmod{13}$.

Now, let any four collinear points, in the schema of Π, and any other point together with the lines incident with them, be neglected, the result is an 8_3-configuration. Hence the total number of resulting such configurations is $13 \cdot 9 = 117$.

To illustrate the method, upon neglecting the four colinear points incident with ℓ_{12} and any other point, say 3, together with the five lines incident with them, we obtain the configuration 8_3, which is described by the following schema

ℓ_9	ℓ_8	ℓ_7	ℓ_5	ℓ_4	ℓ_3	ℓ_2	ℓ_1
5	5	6	8	9	11	11	12
7	6	7	9	12	0	12	0
0	8	9	11	5	6	7	8

In the notation of section 1, we may write this 8_3 as

$$\begin{matrix} 5 & 7 & 0 \\ 12 & & 6 \\ 9 & 8 & 11 \end{matrix}$$

(cf. Figure 2). We thus have the following simple result:

Proposition 2.1. *$PG(2, 3)$ contains exactly* $117\ (= 13 \cdot 9)$ *configurations* 8_3.

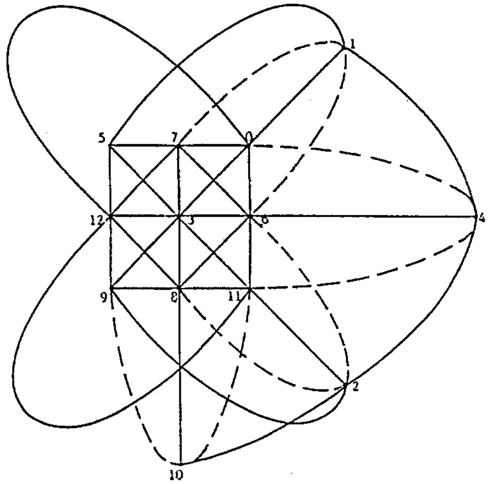

Figure 2

3. 8_3 in $PG(2, 4)$. The finite projective plane of order 4, denoted by Π, may be defined by the following schema:

ℓ_{20}	ℓ_{19}	ℓ_{18}	ℓ_{17}	ℓ_{16}	ℓ_{15}	ℓ_{14}	ℓ_{13}	ℓ_{12}	ℓ_{11}	ℓ_{10}	ℓ_9	ℓ_8	ℓ_7	ℓ_6	ℓ_5	ℓ_4	ℓ_3	ℓ_2	ℓ_1	ℓ_0
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	0
4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	0	1	2	3
5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	0	1	2	3	4
10	11	12	13	14	15	16	17	18	19	20	0	1	2	3	4	5	6	7	8	9
12	13	14	15	16	17	18	19	20	0	1	2	3	4	5	6	7	8	9	10	11

It is clear that $P_i I \ell_j \Leftrightarrow i + j \equiv 0, 3, 4, 9, 11 \pmod{21}$.

Now upon removing the five points 0, 3, 4, 9, 11 (incident with ℓ_0), wherever they appear, and also the five lines $\ell_0, \ell_3, \ell_4, \ell_9, \ell_{11}$ (incident with 0) we obtain the following schema for the symmetric net $S(4)$ of order 4:

ℓ_{20}	ℓ_{19}	ℓ_{18}	ℓ_{17}	ℓ_{16}	ℓ_{15}	ℓ_{14}	ℓ_{13}	ℓ_{12}	ℓ_{10}	ℓ_8	ℓ_7	ℓ_6	ℓ_5	ℓ_2	ℓ_1
1	2	6	7	5	6	7	8	12	14	13	14	15	16	19	20
5	5	7	8	8	10	10	12	13	15	16	17	18	19	1	2
10	6	12	13	14	15	16	17	18	20	17	18	19	20	2	8
12	13	14	15	16	17	18	19	20	1	1	2	5	6	7	10

It is clear that although we are using, in the above schema, only 16 of the residues modulo 21, we still have the rule that $P_i I \ell_j \Leftrightarrow i + j \equiv 0, 3, 4, 9, 11 \pmod{21}$.

Now, the configuration 16_4, defined by the last schema, may be divided into four tetrads of non-intersecting lines as follows:

$$\ell_{20}\ell_{17}\ell_7\ell_5,\ \ell_{19}\ell_{14}\ell_{13}\ell_{10},\ \ell_{16}\ell_{15}\ell_{12}\ell_2,\ \ell_{18}\ell_8\ell_6\ell_1.$$

A configuration 8_3 can be derived (in many ways) by choosing (in a proper way) two lines from each tetrad and discarding the points that appear only once. For this purpose, select any two points not both incident with any line of the configuration 16_4, say 17 and 20. Notice that the point 17 lies on lines ℓ_7, ℓ_{13}, ℓ_{15} and ℓ_8 (one in each tetrad), while the point 20 lies on lines $\ell_5, \ell_{10}, \ell_{12}$ and ℓ_1. From the first three tetrades, choose the six lines $\ell_7, \ell_5, \ell_{13}, \ell_{10}, \ell_{15}, \ell_{12}$ that contain 17 or 20; and from the fourth tetrad choose the two lines ℓ_{18}, ℓ_6 that contain neither 17 nor 20. We thus obtain the following schema:

ℓ_7	ℓ_5	ℓ_{13}	ℓ_{10}	ℓ_{15}	ℓ_{12}	ℓ_{18}	ℓ_6
14	16	8	14	6	12	6	15
17	19	12	15	10	13	7	18
18	20	17	20	15	18	12	19
2	6	19	1	17	20	14	5

Upon discarding the eight points 2, 16, 8, 1, 10, 13, 7, 5 that appear only once in this schema, we obtain the following simpler schema:

14	19	12	14	6	12	6	15
17	20	17	15	15	18	12	18
18	6	19	20	17	20	14	19

which represents the following 8_3:

14	12	6
17		20
18	15	19

Choosing instead the remaining lines: ℓ_{20}, ℓ_{17}, ℓ_{19}, ℓ_{14}, ℓ_{16}, ℓ_2, ℓ_8, ℓ_1 of the four tetrads, we obtain

1	7	2	7	5	1	13	2
5	8	5	10	8	2	16	8
10	13	13	16	16	7	1	10

which defines another 8_3 whose symbol is

1	5	10
16		2
13	7	8

Combining these two 8_3 configurations, as in Fig. 3, we obtain a cycle of four, each having one (simple) quadrangle in common with the next:

1	5	10	5	6	2	6	20	19	20	1	15
16		2	14		19	12		15	13		10
13	7	8	16	18	7	14	17	18	12	8	17

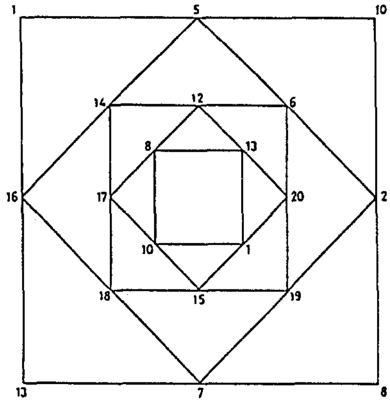

Figure 3

This means that we have a chain of four quadrangles 1 10 8 13, 20 15 17 12, 6 19 18 14 and 5 2 7 16, each is inscribed in the next, Fig. 3, resembling the Pappian chain of three triangles (Al-Dhahir [2]).

Thus we have shown:

Proposition 3.1. *The symmetric net $S(4)$ of order 4 may be partitioned into two disjoint configurations 8_3. Consequently, both $AG(2, 4)$ and $PG(2, 4)$ contain pairs of disjoint 8_3's.*

Of course, we may also describe $PG(2, 4)$ by using homogeneous coordinates over GF(4) (see e.g. Pickert [8]). We will write the elements of GF(4) as 0, 1, ω and ω^2, where $\omega^2 + \omega + 1 = 0$. After discarding the five points

$$(0, 1, 0),\ (1, 0, 1),\ (1, 1, 1),\ (1, \omega, 1),\ (1, \omega^2, 1),$$

which are incident with the line [1, 0, 1], we obtain the following two 8_3 configurations:

$(\omega^2, \omega, 1)$	$(0, 0, 1)$	$(\omega, 1, 1)$	$(0, \omega, 1)$	$(0, 1, \omega)$	$(0, 1, 1)$
$(\omega^2, 1, 1)$		$(1, 1, \omega^2)$	$(\omega, 0, 1)$		$(1, 0, \omega)$
$(1, 1, \omega)$	$(1, 0, 0)$	$(1, \omega, \omega^2)$	$(1, 1, 0)$	$(\omega, 1, 0)$	$(1, \omega, 0)$

Summarizing our results, we have proved

Theorem 3.2. *In $PG(2, 4)$, there exist:*

(1) *two disjoint configurations of the type 8_3; and*
(2) *a chain of four (simple) quadrangles each is inscribed in the next.*

In Section 5, we shall show that any 8_3 contained in $PG(2, 4)$ extends to an affine subplane of order 3 in $PG(2, 4)$; then it will be easy to see that the following holds:

Proposition 3.3. *$PG(2, 4)$ contains exactly 2520 configurations 8_3 and exactly 280 affine subplanes of order 3.*

4. 8_3 in $PG(n, 4)$. In this section we briefly consider the existence of pairwise disjoint 8_3's in projective and affine spaces over GF(4); this will be a simple application of Proposition 3.1. We first note:

Proposition 4.1. *$AG(n, 4)$ can be partitioned into 2^{2n-3} pairwise disjoint 8_3's.*

Proof. By Proposition 3.1, the assertion holds for $n = 2$. Now consider a parallel class of planes of $AG(n, 4)$ (where $n \geqq 3$); this is a partition of the points of $AG(n, 4)$ into 2^{2n-2} affine planes of order 4. But each of these affine planes can be partitioned into two disjoint 8_3's. □

Proposition 4.2. *Let g be a line of the projective space $\Pi = PG(n, 4)$. Then $\Pi \backslash g$ can be partitioned into $2^{2n-3} + 2^{2n-5} + \cdots + 2^3 + 2$ pairwise disjoint 8_3's.*

Proof. Again, the assertion holds for $n = 2$ by Proposition 3.1. Now let $n \geqq 3$ and choose a hyperplane H of Π. Then $\Pi \backslash H$ is isomorphic to $AG(n, 4)$ and may thus be partitioned into 2^{2n-3} configurations 8_3. Assume $g \subset H$; then the assertion follows by induction, observing that H is isomorphic to $PG(n-1, 4)$. □

5. 8_3 in Pappian planes. We now consider the general question of embedding 8_3 into the Pappian projective plane $PG(2, F)$ over some commutative field F. Let P, Q, R, S, A, B, C, D be 8 points of $PG(2, F)$ and assume that the lines of $PG(2, F)$ induce the 8_3 of Figure 1 on these points. Since the collineation group of $PG(2, F)$ is transitive on ordered quadrangles, we may assume $P = (1, 0, 0)$, $Q = (0, 1, 0)$, $R = (0, 0, 1)$ and $S = (1, 1, 1)$. Then the coordinates of A, B, C, D may be taken to be $A = (1, t, 0)$, $B = (1, 0, y)$, $C = (1, z, 1)$ and $D = (1, 1, x)$, where t, x, y, z are suitable elements of F. Since $(1, 0, 0)$, $(1, z, 1)$ and $(1, 1, x)$ are collinear, we have $\det \begin{pmatrix} 1 & 0 & 0 \\ 1 & z & 1 \\ 1 & 1 & x \end{pmatrix} = 0$, i.e. $zx = 1$; thus we may replace the coordinates of C by the scalar multiple $(x, zx, x) = (x, 1, x)$. Similarly, the collinearity of $(0, 1, 0)$, $(1, 0, y)$ and $(1, 1, x)$ implies $y = x$, and the collinearity of $(0, 0, 1)$,

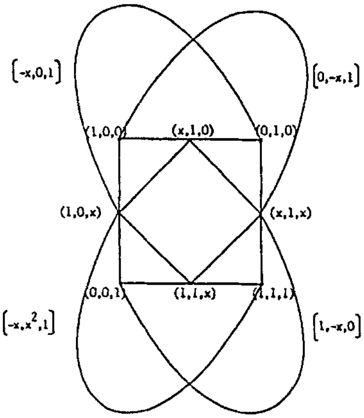

Figure 4

$(x, 1, x)$ and $(1, t, 0)$ shows $xt = 1$ and allows us to replace the coordinates of A by $(x, 1, 0)$. The coordinates are now as shown in Figure 4.

Finally, the points $(1, 1, 1)$, $(1, 0, x)$ and $(x, 1, 0)$ are collinear; this yields

$0 = \det \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & x \\ x & 1 & 0 \end{pmatrix}$; thus x satisfies

$$(*) \qquad x^2 - x + 1 = 0.$$

If char $F = 3$, $(*)$ can be written as $(x+1)^2 = 0$, and we get the unique solution $x = -1$. If char $F = 2$, $(*)$ reduces to $x^2 + x + 1 = 0$ and x has to be a primitive third root of unity. In all other cases, x has to be a primitive 6'th root of unity. Conversely, if x is chosen in this way, then $(*)$ is satisfied and

$$\begin{matrix} (1,0,0) & (x,1,0) & (0,1,0) \\ (1,0,x) & & (x,1,x) \\ (0,0,1) & (1,1,x) & (1,1,1) \end{matrix}$$

is an 8_3 embedded in $PG(2, F)$. Thus we have the following

Theorem 5.1. *Let F be a commutative field. Then the Pappian plane $PG(2, F)$ contains an 8_3 if and only if one of the following cases holds:*

(i) char $F = 3$;
(ii) char $F = 2$ *and F contains a primitive* 3rd *root of unity*;
(iii) *F contains a primitive sixth root of unity.*

Now any 8_3 can be embedded into $AG(2, 3)$; this poses the question whether or not this $AG(2, 3)$ is contained in $PG(2, F)$ if 8_3 itself is embedded into $PG(2, F)$. Thus consider the 8_3 of Figure 4, embedded into $PG(2, F)$. Now the lines $[(1, 0, 0)(1, 1, 1)] = [0, 1, -1]$ and $[(0, 1, 0)(0, 0, 1)] = [1, 0, 0]$ intersect in the point $(0, 1, 1)$. Thus the affine plane completing our 8_3 will be contained in $PG(2, F)$ if and only if the lines $[(1, 0, x)(x, 1, x)] = [-x, x^2 - x, 1]$ and $[(x, 1, 0)(1, 1, x)] = [x, -x^2, x - 1]$ contain $(0, 1, 1)$. But the condition for this just turns out to be equation $(*)$ and is therefore satisfied. We have shown:

Theorem 5.2. *Let F be a commutative field and assume that* **D** *is a configuration* 8_3 *contained in $PG(2, F)$. Then* **D** *extends to an affine subplane of order* 3 *of $PG(2, F)$ (for the case* char $F \neq 3$, *see Figure* 5).

For example, the two disjoint 8_3's within $PG(2, 4)$ given in section 3 extend to the two affine planes

$$\begin{matrix} 14 & 12 & 6 \\ 17 & 0 & 20 \\ 18 & 15 & 19 \end{matrix} \quad \text{and} \quad \begin{matrix} 1 & 5 & 10 \\ 16 & 0 & 2 \\ 13 & 7 & 8 \end{matrix}$$

intersecting in the point 0. We do not know whether or not $PG(2, 4)$ contains two disjoint affine subplanes of order 3.

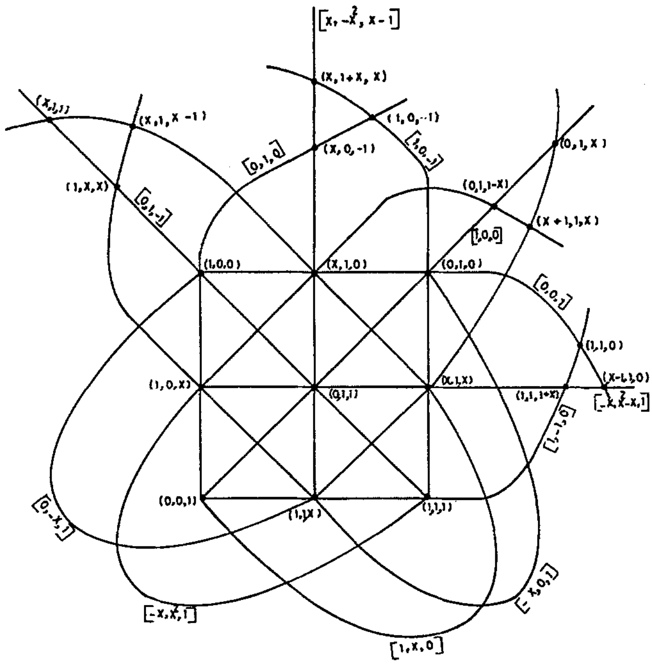

Figure 5

6. 8_3 in $PG(2, q)$. We now specialize the results of section 5 to the finite case. Since the multiplicative group of $F = \mathrm{GF}(q)$ is cyclic, we obtain the following:

Theorem 6.1. *$PG(2, q)$ contains 8_3 if and only if q is a power of 3 or 4 or q is congruent to 1 modulo 6. Any 8_3 contained in $PG(2, q)$ extends to an affine subplane of order 3 of $PG(2, q)$.*

Our proof in fact allows us to determine the number of 8_3's contained in $PG(2, q)$. For any quadrangle of $PG(2, q)$ gives rise to two 8_3's (for $q = 4^a$ or $q \equiv 1 \bmod 6$), since we have two choices for x in these cases. On the other hand, each 8_3 contains two quadrangles and thus is counted twice. Hence the number of 8_3's equals the number of quadrangles in $PG(2, q)$ and thus is $(q^2 + q + 1)(q^2 + q)\, q^2(q - 1)^2/24$, cf. Hirschfeld [5]. In the remaining case ($q = 3^a$), we have only one possibility for x (i.e. $x = -1$) and thus obtain only half as many 8_3's. Hence we have:

Proposition 6.2. *Then number of 8_3's contained in $PG(2, q)$ is*

$$(q^2 + q + 1)(q + 1)\, q^3(q - 1)^2 x,$$

where $x = 1/24$ for $q = 4^a$ or $q \equiv 1 \bmod 6$, $x = 1/48$ for $q = 3^a$ and $x = 0$ otherwise.

Since each 8_3 contained in $PG(2, q)$ extends to a unique $AG(2, 3)$ and since $AG(2, 3)$ contains 9 copies of 8_3, this also implies the following:

Corollary 6.3. *The number of affine subplanes of order* 3 *of* $PG(2, q)$ *is*

$$(q^2 + q + 1)(q + 1)\, q^3 (q - 1)^2 y,$$

where $y = 1/216$ *for* $q = 4^a$ *or* $q \equiv 1 \bmod 6$, $y = 1/532$ *for* $q = 3^a$ *and* $y = 0$ *otherwise.*

We finally mention that Rigby [10] has proved that the only planes $PG(2, q)$ containing an affine plane $AG(2, r)$ with $r \geqq 4$ are those for which q is a power of r. This still leaves the question, however, whether "large parts" of $AG(2, r)$ (for example, $BAG(2, r)$) can be contained in $PG(2, q)$ for other values of q. The result of Theorem 5.1 for $F = \mathbb{R}$ resp. $F = \mathbb{C}$ is already given by Coxeter [3] and Coxeter [4].

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