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The Maximal Rank Conjecture for Non-Special Curves in $\mathbb{P}^n$

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In this paper we conclude our study ([2, 3]) about the postulation of "general" curves embedded in a projective space by a non-special linear system. Recall that a curve $C \subset \mathbb{P}^n$ is said to be of maximal rank if for every $k \geqq 1$, the natural map of restriction $r_{C,n}(k)$: $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) \to H^0(C, \mathcal{O}_C(k))$ is surjective or injective. In this paper we prove the following result (over any algebraically closed base field).

Theorem. *Fix integers n, d, g with $n \geqq 5$, $g \geqq 0$, $d \geqq g+n$. Let X be a general curve of genus g and h: $X \to \mathbb{P}^n$ a general nondegenerate embedding with non-special hyperplane section,* $\deg h(X) = d$. *Then $h(X)$ has maximal rank.*

For $n=3$ and $n=4$ the corresponding result was proved respectively in [3] and [2]. In [2] we assumed that the base field has zero characteristic; however this assumption can be avoided quoting [11], Prop. 3 and Lemma 4, in the proof of [2], Lemma 1. Hence this paper, together with [2, 3], yields the so called "Maximal rank conjecture for non-special curves in $\mathbb{P}^n$, $n \geqq 3$". Since in the proof of the theorem we use induction on n, we need the main result of [2] (but not of [3]). As promised in the introduction of [2], here we use the skeleton of the proof (and often the notations) of the main theorem of [2]. The main difference with respect to [2, 4], is in §1 (intersection with a hyperplane). Furthermore we don't use any nilpotent.

We prove the theorem by induction. We try to construct by an inductive procedure called "la méthod d'Horace" (see [6, 9, 10]) a suitable reducible curve $Y \subset \mathbb{P}^n$, $\deg Y = d$, $p_a(Y) = g$, with good postulation. A theorem of Sernesi ([13]) and Hartshorne-Hirschowitz [7]) states that the curve Y can be deformed to a smooth curve $Z \subset \mathbb{P}^n$, $\deg Z = d$, $p_a(Z) = g$, with $h^1(Z, \mathcal{O}_Z(1)) = 0$. By semicontinuity, Z has good postulation.

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§ 0. Notations and Preliminaries

Definition. Let $Z^*(d, g; n)$ be the closure in Hilb $\mathbb{P}^n$ of the set of smooth irreducible curves $C \subset \mathbb{P}^n$ with $H^1(C, \mathcal{O}_C(1))=0$.

It is well-known that $Z^*(d, g; n)$ is irreducible. This fact will be used many times in the next sections without further mention.

For a curve C in $\mathbb{P}^n$, N_C is its normal bundle. A curve C is said to be k-secant to another curve D if it intersects D quasi-transversally and exactly at k points. The next lemma is due to Sernesi ([13]) and Hartshorne-Hirschowitz ([7]). It is the fundamental tool for this paper (as it was for [2] and [3]).

0.1 Lemma. *Take $Y \in Z^*(d, g; n)$. Denote by D a rational curve of degree $f \leqq n$ which spans a $\mathbb{P}^f$. Assume that D is k-secant to Y with $1 \leqq k \leqq f+1$. Then $Y \cup D \in Z^*(d+f, g+k-1; n)$. Furthermore if Y is a locally complete intersection with $h^1(Y, N_Y)=0$, then $h^1(Y \cup D, N_{Y \cup D})=0$.*

A map is said to be strictly surjective (or s-surjective) if it is surjective but not injective. Let E be a closed subscheme of the projective space V; $\mathcal{I}_{E,V}$ will denote its ideal sheaf. For all integers $k \geqq 1$, $r_{E,V}(k)$: $H^0(V, \mathcal{O}_V(k)) \to H^0(E, \mathcal{O}_E(k))$ is the restriction map. If $V=\mathbb{P}^m$, we write often $\mathcal{I}_{E,m}$, $r_{E,m}(k)$ instead of $\mathcal{I}_{E,V}$, $r_{E,V}(k)$. If $H=\mathbb{P}^{n-1}$, we write often $Z^*(d, g; H)$ instead of $Z^*(d, g; n-1)$.

For a real number x, $[x]$ denotes its integral part.

0.2 *Remark.* Let $S \subset \mathbb{P}^m$ be a general subset, $\#(S) \leqq m+3$. Then there is a smooth, rational normal curve $C \subset \mathbb{P}^m$ with $S \subset C$.

§ 1. Intersection with the Hyperplane

1.1 Definition. Let U, V be irreducible subvarieties of $\mathbb{P}^n$. The join, $U^0 V$, of U, V is the closure of the union of the lines $[x, y]$, $x \in U$, $y \in V$, $x \neq y$.

Note that $U^0 V$ is irreducible and $\dim(U^0 V) \leqq \dim(U)+\dim(V)+1$. By iteration one defines $V^{0i} := V^0 V^{0(i-1)}$. The following lemma is well-known (for ex. see [1]):

1.2 Lemma. *Let C be a nondegenerate, irreducible curve in $\mathbb{P}^n$. If U is an irreducible subvariety of $\mathbb{P}^n$, then* $\dim(U^0 C)=\min(n, \dim(U)+2)$.

Proof. If $\dim(U)<n$, since C is nondegenerate there exists $p \in C$ such that $p \notin \mathrm{Vert}(U) := \{x : x^0 U = U\}$ (note that $\mathrm{Vert}(U)$ is a linear subspace of U). Hence $\dim(C^0 U) \geqq \dim(P^0 U)=\dim(U)+1$. If $\dim(U^0 C)=\dim(U)+1$, then $p^0 U = C^0 U$. It follows that C is contained in $\mathrm{Vert}(U^0 C)$. Since C spans $\mathbb{P}^n$, $\mathrm{Vert}(U^0 C)=\mathbb{P}^n$, hence $U^0 C=\mathbb{P}^n$. □

1.3 Corollary. *Let $C \subset \mathbb{P}^n$ be a nondegenerate, irreducible curve, then* $\dim(C^{0(t+1)})=\min(n, 2t+1)$.

Let $H \subset \mathbb{P}^n$ be a hyperplane and $D \subset H$ be a closed subscheme of dimension at most one. Denote by C a nondegenerate curve in $\mathbb{P}^n$. For $h := [(n-2)/2]$ let $L_1, \ldots, L_h$ be h distinct lines, intersecting H transversally, not meeting D and satisfying the following incidence relations:

(a1) L_1 is 2-secant to C;

(a2) L_i, $i\geqq 2$, is 2-secant to $C\cup L_{i-1}$ with $L_i\cap C\neq\emptyset$, $L_{i-1}\cap L_i\neq\emptyset$. Also let $\{Y_j\}$, $1\leqq j\leqq t$, be t lines such that:

(b) for every j, $1\leqq j\leqq t$, $Y_j\cap C\neq\emptyset$ and Y_j is 2-secant to $X:=C\cup L_1\cup\ldots\cup L_h$. Set $Y:=L_1\cup\ldots\cup L_h\cup Y_1\cup\ldots\cup Y_t$.

1.4 Lemma. *With notations as above, assume that $r_{D\cup(Y\cap H),H}(k)$ is strictly surjective for a given $k>0$. Then we can deform $L_1,\ldots,L_h$, $Y_1,\ldots,Y_t$ to $L'_1,\ldots,L'_h$, $Y'_1,\ldots,Y'_t$, the L'_i, Y'_j satisfying* (a1), (a2) *and* (b), *and we can find a line A 2-secant to $X':=C\cup L'_1\cup\ldots\cup L'_h$ with $C\cap A\neq\emptyset$ and $r_{D\cup((Y'\cup A)\cap H),H}(k)$ surjective (Y' $:=L'_1\cup\ldots\cup L'_h\cup Y'_1\cup\ldots\cup Y'_t$).*

Proof. Let $S\subset H$ be a hypersurface of degree k containing $D\cup(Y\cap H)$. If $C^{02}\cap H\not\subset S$, we put $Y=Y'$ and take for A a generic 2-secant line to C. If $C^{02}\cap H\subset S$ but $(C^{03}\cap H)\not\subset S$ there exists $L'_1\subset C^{02}$ and $A\subset C^{03}$ such that: $A\cap L'_1\neq\emptyset$, $A\cap C\neq\emptyset$, and $A\cap H\not\subset S$. We deform L_1 to L'_1. Since for every line B, $(B^0 C)\cap S\neq\emptyset$, we can follow this deformation with a deformation $L'_2,\ldots,L'_h$, $Y'_1,\ldots,Y'_t$, of $L_2,\ldots,L_h$, $Y_1,\ldots,Y_t$, in such a way that the incidence relations (a1), (a2), (b), hold and such that $(D\cup(Y'\cap H))\subset S$. Then we have $h^0(H,\mathscr{I}_{D\cup((Y'\cup A)\cap H),H}(k))=h^0(H,\mathscr{I}_{D\cup(Y\cap H),H}(k))-1$ by semicontinuity and $r_{D\cup((Y'\cup A)\cap H),H}(k)$ is surjective.

If $(C^{03}\cap H)\subset S$, we go on this way. By 1.3 $\dim(C^{0(h+2)})=n$. Hence there is $s\leqq h$ such that $(C^{0(s+1)}\cap H)\subset S$ and $(C^{0(s+2)}\cap H)\not\subset S$. There exist lines $A\subset C^{0(s+2)}$, $L'_i\subset C^{0(i+1)}$, $1\leqq i\leqq s$, the L'_i's having the same incidence relations as the lines L_i and such that: $A\cap H\not\subset S$, $A\cap L'_s\neq\emptyset$, $A\cap C\neq\emptyset$. We deform $L_1,\ldots,L_s$ to $L'_1,\ldots,L'_s$. We follow this deformation with a deformation $L'_{s+1}\cup\ldots\cup L'_h\cup Y'_1\cup\ldots\cup Y'_t$ of $L_{s+1}\cup\ldots\cup L_h\cup Y_1\cup\ldots\cup Y_t$ in such a way that $D\cup(Y'\cap H)\subset S$ and (a1), (a2), (b) do hold. Then, as above, we are done. □

1.5 Lemma. *Fix integers d, g, n, s with $n\geqq 3$, $g\geqq 0$, $d\geqq g+n$, $s\geqq 1$, $g\geqq(s-n-3-(d-g-n))[n/2]$. Let S be a general subset of $\mathbb{P}^n$ with $\#(S)=s$. Then there exists a curve $X\in Z^*(d,g;n)$ with $S\subset X$ and $h^1(X,N_X)=0$.*

Proof. Take a general subset S' of $\mathbb{P}^n$, $\#(S')=\min(s,n+3)$. By 0.2 we may find a smooth, rational normal curve C in $\mathbb{P}^n$ with $S'\subset C$. Hence we may assume $s\geqq n+3$. If P_i, $1\leqq i\leqq d-g-n$, are $d-g-n$ general points, there are lines L_i, $1\leqq i\leqq d-g-n$, with $P_i\in L_i$ and L_i 1-secant to C. By 1.3, $C^{0(t+1)}=\mathbb{P}^n$, where $t=[n/2]$. Set $y=s-n-3-(d-g-n)$. Given any y general points $A_1,\ldots,A_y$ in $\mathbb{P}^n$, there are lines B_{ij}, $i=1,\ldots,t$, $j=1,\ldots,y$, with B_{1j} 2-secant to C and if $2\leqq i\leqq t$, B_{ij} intersecting both C and $B_{i-1,j}$ (but not $C\cap B_{i-1,j}$) with $A_j\in B_{tj}$, $j=1,\ldots,y$ (note that $B_{ij}\subset C^{0(i+1)}$). Now the union of C, $L_1,\ldots,L_{d-g-n}$ and B_{ij}, $1\leqq i\leqq t$, $1\leqq j\leqq y$, is a curve in $Z^*(d-(g-yt),yt;n)$ by 0.1; note that $g-yt\geqq 0$ by assumption. Adding further $(g-yt)$ general 2-secant lines to C, we get the curve in $Z^*(d,g;n)$ we were looking for.

1.6 Lemma. *Fix nonnegative integers d', g', n, e, d'', g'' with $0\leqq e\leqq d'-g'-n$, $d''\geqq g''+n-1$. Set $s=n+e$. Assume $g''\geqq(s-n-2-(d''-g''-n+1))[n/2]$. Let S be a general set of s points in a hyperplane H of $\mathbb{P}^n$.*

(a) *There existe $Y\in Z^*(d',g';n)$ through S and $D\in Z^*(d'',g'';H)$ through S. For general such Y and D we have $Y\cup D\in Z^*(d'+d'',g'+g''+s-1;n)$.*

(b) *Set* $B=Y\cap(H\setminus S)$. *Assume that* $r_{D,H}(k)$ *is surjective for some* k. *Then we may assume that* $h^0(H,\mathscr{I}_{D\cup B,H}(k))=\max(0,h^0(H,\mathscr{I}_{D,H}(k))-\#(B))$.

Proof. (a) The existence of D' with $h^1(D,N_{D'})=0$ and passing through S follows from 1.5. Fix any such D' and $S'\subset S$ with $\#(S')=n$. Take C in $Z^*(d'-e,g';n)$ through S' with $h^1(C,N_C)=0$. Then take the union E of e lines 1-secant to C and such that $E\cap H=S\setminus S'$. By 0.1 $h^1(D'\cup C\cup E,N_{D'\cup C\cup E})=0$ and $D'\cup C\cup E\in Z^*(d'+d'',\ g'+g''+s-1;\ n)$. We may deform D', $C\cup E$ and S to general D, Y, S preserving the incidence relations.

(b) The last part follows from 1.4. □

1.7 Lemma. *Let* $H\subset\mathbb{P}^n$ *be a hyperplane,* $k\geqq 1$ *an integer,* $C\subset\mathbb{P}^n$, $D\subset H$ *reduced subschemes; assume that no component of* C *is contained in* H.

(a) *If* $r_{C,n}(k-1)$ *and* $r_{D\cup(H\cap C),H}(k)$ *are injective, then* $r_{C\cup D}(k)$ *is injective.*

(b) *Assume that* $r_{C,n}(k-1)$ *and* $r_{D\cup(H\cap C),H}(k)$ *are surjective. Then*

$$h^0(\mathbb{P}^n,\mathscr{I}_{C\cup D,n}(k))\leqq h^0(\mathbb{P}^n,\mathscr{I}_{C,n}(k-1))+h^0(H,\mathscr{I}_{D\cup(C\cap H),H}(k)).$$

Proof. Take $f\in\operatorname{Ker}(r_{C\cup D,n}(k))$. Since $f|H$ vanishes on $D\cup(C\cap H)$, f is divided by the equation z of H. Since f/z vanishes on C, $f=0$. (b) Take general subsets $A\subset\mathbb{P}^n\setminus H$, $B\subset H$ with $\#(A)=h^0(\mathbb{P}^n,\mathscr{I}_{C,n}(k-1))$, $\#(B)=h^0(H,\mathscr{I}_{D\cup(C\cap H),H}(k))$. Then apply (a) to $C\cup A$ and $D\cup B$. □

§ 2. Basic Inductive Statements

From now on in this paper we fix integers d, g, n with $d\geqq g+n$, $g\geqq 0$, $n\geqq 5$. By [2] and induction we assume the theorem in $\mathbb{P}^{n-1}$.

2.1 Definition. The critical value $v(t,s,n)$, $t\geqq s+n$, $s\geqq 0$, is defined by

$$v(t,s,n)=\min\{k\geqq 1\colon h^0(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(k))\geqq kt-s+1\}.$$

We set $j:=v(d,g,n)$.

2.2 Definition. We define integers $g(k,n)$, $f(k,n)$ for $k\geqq 2$ by:

$$k(g(k,n)+n)-g(k,n)+1+f(k,n)=\binom{n+k}{n},\qquad 0\leqq f(k,n)\leqq k-2. \tag{1}$$

Set $g(1,n):=0$, $f(1,n):=0$.

2.3 Definition. We set $r=\max\{k\colon g(k,n)\leqq g\}$.

2.4 Definition. For $r\leqq k\leqq j$ we define integers $d(k)$, $h(k)$ by:

$$k\,d(k)-g+1+h(k)=\binom{n+k}{n},\qquad 0\leqq h(k)\leqq k-1. \tag{2}$$

From the definitions we immediately get:

2.5 Lemma. (i) *If* $t \leqq t'$, *then* $v(t, s, n) \leqq v(t', s, n)$.

(ii) $g(u, n) + n = \max\{t \geqq n: v(t, t-n, n) = u\}$; $d(k) = \max\{t \geqq g + n: v(t, g, n) = k\}$.

(iii) *We have* $r \leqq j$ *with equality if and only if* $d = g(r, n) + n$, $g = g(r, n)$.

Now we introduce the basic inductive statements:

2.6 $H(k)$, $k \geqq 1$: A general $C \in Z^*(g(k, n)+n, g(k, n)-f(k, n); n)$ has $r_{C,n}(k)$ bijective.

This definition makes sense because $g(k, n) \geqq f(k, n)$ (see 4.3).

2.7 $R(k)$, $k \geqq r+1$: There exists (X, Z, T) such that:

(1) $X = Z \cup T$, $Z \cap T = \varnothing$, $r_{X,n}(k)$ is bijective;

(2) $Z \in Z^*(d(k)-h(k), g; n)$ and T is the union of $h(k)$ disjoint lines.

We will use $R(k)$ only when it makes sense, i.e. only when $d(k)-h(k) \geqq g+n$ (see 3.3, 2.5(ii)); if $k \geqq r+2$, there is no problem by 4.6.

2.8 $R'(r+1)$, *if* $g - h(r+1) \geqq 0$: There exists Y in $Z^*(d(r+1), g-h(r+1); n)$ with $r_{Y,n}(r+1)$ bijective.

§ 3. Proof of the Basic Inductive Statements

In this section we prove the statements $H(k)$, $k \geqq 1$, $R(k)$, $k \geqq r+1$, $R'(r+1)$, modulo some numerical lemmas whose proof is postponed to Sect. 4. These numerical lemmas will also be used in Sect. 5 (proof of the theorem).

Recall also that, by [2] and induction, we may assume the theorem for $n-1$, $n \geqq 5$.

3.1 Lemma. *For* $k \geqq 1$, $H(k)$ *holds.*

Proof. $H(1)$ is clear and $H(2)$ was proved in [4], Prop. 1.1.

Assume $k \geqq 3$ and that $H(k-1)$ is true.

(i) First suppose: $n - f(k, n) + f(k-1, n) \geqq 1$.

Set $x := g(k, n) - g(k-1, n)$. By 4.1, 4.3 we have $x \geqq n-1$. By the theorem for $n-1$ there exists $d \in Z^*(x, x-n+1; H)$, H a hyperplane in $\mathbb{P}^n$, with $r_{D,H}(k)$ of maximal rank. Note that

$$\binom{n-1+k}{n-1} - (kx - (x-n+1)+1) = g(k-1, n) + f(k, n) - f(k-1, n) > 0 \quad \text{(see 4.3)}.$$

Hence $r_{D,H}(k)$ is s-surjective. Let $S \subset H$ be a set of $s := n - f(k, n) + f(k-1, n)$ general points. Since $g(k, n) - g(k-1, n) \geqq n-1 + (f(k-1, n) - f(k, n) - 2)[(n-1)/2]$ (see 4.2) and $n + f(k-1, n) \geqq s$, we may assume by 1.5, 1.6 that there exists $Y \in Z^*(g(k-1, n)+n, g(k-1, n) - f(k-1, n); n)$ with $r_{Y,n}(k-1)$ bijective and such that Y and D intersect (quasi-transversally) exactly at S and with $X := Y \cup D \in Z^*(g(k, n)+n, g(k, n)-f(k, n); n)$. By 1.7 we get that $r_{X,n}(k)$ is bijective.

(ii) Now assume $n - f(k, n) + f(k-1, n) \leqq 0$.

This time we take for D an element of $Z^*(x, z; n-1)$, x as above and

$z := g(k,n) - g(k-1,n) + f(k-1,n) - f(k,n)$. Note that $x \geqq z + n - 1$ and that $f(k,n) \leqq k-2$. By 4.1 we have $z \geqq 0$. We have

$$\binom{n-1+k}{n-1} - kx + z - 1 = n + g(k-1,n) - 1.$$

hence $r_{D,H}(k)$ is s-surjective. We conclude as above (but with $\#(S)=1$). □

3.2 Lemma. *If $r \geqq 2$ and $g - h(r+1) \geqq g(r,n) - f(r,n)$ (\$), then $R'(r+1)$ is true.*

Proof. Since $g(r,n) \geqq f(r,n)$ by 4.3, $R'(r+1)$ is well-defined.

Set $x := d(r+1) - g(r,n) - n$, $y := \min(x+1-n, g - h(r+1) - g(r,n) + f(r,n))$ and $s := 1 + g - h(r+1) - g(r,n) + f(r,n) - y$. If $y = x+1-n$, we have $y \geqq 0$, because $d(r+1) - g(r,n) \geqq 2n$ (see 4.5(b)). So in any case $y \geqq 0$ and $x \geqq y + n - 1$. Hence, by the theorem for $n-1$, there is a hyperplane H in $\mathbb{P}^n$ and $D \in Z^*(x, y; H)$ with $r_{D,H}(r+1)$ of maximal rank. We have

$$\binom{n+r}{n-1} - (r+1)x + y - 1 = g(r,n) + n - s.$$

Hence $r_{D,H}(r+1)$ is strictly surjective. Let $S \subset H$ be a general set of s points. Note that $s \geqq 1$ (assumption (\$) and definition of y). To apply 1.5 we have to check that $y \geqq (s - x + y - 3)[(n-1)/2]$. But if $s = 1$, there is nothing to check. Hence we may assume $y = x + 1 - n$. In this case we see that $s \leqq n + f(r,n)$ and the inequality follows from 4.5(a). Similarly since $s \leqq n + f(r,n)$, we can apply 1.6 with $Y \in Z^*(g(r,n)+n, g(r,n) - f(r,n); n)$. By 3.1, we may assume that Y satisfies $H(r)$. By 1.7, 1.4 and semicontinuity, $X := Y \cup D$ satisfies $R'(r+1)$. □

3.3 Lemma. *If $g - h(r+1) < g(r,n) - f(r,n)$, $r \geqq 2$, then $R(r+1)$ is true.*

Proof. By the assumption and 4.4, $d(r+1) - h(r+1) \geqq g + n$, hence $R(r+1)$ makes sense. Let $H \subset \mathbb{P}^n$ be a hyperplane and $C \subset H$ be a general smooth curve of degree $c := d(r+1) - h(r+1) - g(r,n) - n$ and genus $g - g(r,n) + f(r,n) \leqq r-1$. By 4.4 $c \geqq n - 1 + (g - g(r,n) + f(r,n))$. Denote by T the general union of $h(r+1)$ disjoint lines such that $T \cap C = \varnothing$. Finally set $D = T \cup C$. We will see in 3.4 that $r_{D,H}(r+1)$ is strictly surjective. We have:

$$h^0(\mathcal{O}_H(r+1)) - h^0(\mathcal{O}_D(r+1)) = 2g(r,n) + n - g - f(r,n) - 1 > 0$$

(by the hypothesis).

Let $S \subset H$ be a general set of $s := 1 + g - g(r,n) + f(r,n)$ points. By 2.2, $s \geqq 1$. By the hypothesis, $s \leqq h(r+1)$. Hence $0 \geqq (s - n - 2 - (c - n + 1))$ and we can apply 1.5 to C and S. Now take $Y \in Z^*(g(r,n)+n, g(r,n) - f(r,n); n)$ satisfying $H(r)$. Since $g(r,n) - f(r,n) \geqq (s - n - f(r,n))[n/2]$ (see 4.3) by 1.5 we may assume $C \cap Y = S$. By 0.1 we may assume that $Z := C \cup Y$ belongs to $Z^*(d(r+1) - h(r+1), g; n)$. Finally by 1.7, 1.4 and semi-continuity, we get that $r_{Z \cup T, n}(r+1)$ has maximal rank. □

3.4 Sublemma. *The map $r_{D,H}(r+1)$ is strictly surjective.*

Proof. Let f be the maximal integer such that

$$rf+1-(g-g(r,n)+f(r,n)) \leqq \binom{n-1+r}{n-1}.$$

Note that $f \geqq n+r-2$ for $n \geqq 5$, $r \geqq 1$. Set $u=\min(c,f)$. By the theorem for $n-1$ there is $E \in Z^*(u, g-g(r,n)+f(r,n); H)$ such that $r_{E,H}(r)$ is surjective. Fix a hyperplane V of H. In V consider the union B of a general rational curve F, $\deg(F)=c-u$, F intersecting E quasi-transversally and only at a point, and $h(r+1)$ general disjoint lines.

Note that if $r \geqq 1$ and $n \geqq 5$ we have

$$r(r+2)+(n-2)(r+1)+1 \leqq \binom{n-1+r}{n-2}.$$

Hence by [8] if $n=5$, by [5] if $n \geqq 6$, $r_{B,V}(r+1)$ is surjective. By 0.1, 1.7, we may assume the surjectivity of $r_{V \cup B,H}(r+1)$. By semicontinuity $r_{D \cup T,H}(r+1)$ is surjective. By counting dimensions, it is also strictly surjective. □

3.5 Lemma. *$R'(r+1)$ implies $R(r+2)$.*

Proof. In a hyperplane H we take for Z a general rational curve of degree $x := d(r+2)-d(r+1)-h(r+2)$ and for T the union of $h(r+2)$ general lines. By [5] $D := Z \cup T$ is of maximal rank. We have:

$$\binom{n+r+1}{n-1}-(r+2)\,x-1-(r+3)\,h(r+2)=d(r+1)-h(r+1)-1>0 \quad \text{(by 4.5, 4.3)}.$$

Hence $r_{D,H}(r+2)$ is strictly surjective. Then we take a general set S of $s := h(r+1)+1$ points in H. Since $x+1 \geqq s$ (4.4) we may assume $S \subset Z$. Now let Y be a general element of $Z^*(d(r+1), g-h(r+1); n)$ (hence satisfying $R'(r+1)$). Since $d(r+1) \geqq g+n$ (2.5), by 1.6 we may assume $S \subset Y$. By 0.1 (recall $x+1 \geqq s$) and 1.7, 1.4 and semicontinuity, $X := Z \cup Y$ $Z^*(d(r+2)-h(r+2), g; n)$ and $r_{X \cup T,n}(r+2)$ has maximal rank. □

3.6 Lemma. *For $k \geqq r+1$, $R(k)$ implies $R(k+1)$.*

Proof. (i) First assume $h(k+1) \geqq h(k)$.

In a hyperplane H we take a general rational curve Z of degree $x := d(k+1)-d(k)-h(k+1)+h(k)$. By 4.4 $x \geqq n-1$. Let $T \subset H$ be the general union of $h(k+1)-h(k)$ lines. By [5] $D := Z \cup T$ has maximal rank. We have:

$$\binom{n+k}{n-1}-h^0(\mathcal{O}_D(k+1))=d(k)-1>0.$$

By $R(k)$ there is $X := Y \cup T'$ with $r_{X,n}(k)$ bijective. We may assume that Z and Y intersect at one point and that $Z \cup Y$ is smoothable (0.1). We conclude with 1.7, 1.4 and semicontinuity.

(ii) Now assume: $h(k) > h(k+1)$.

This time we take $D \subset H$, a general rational curve of degree $x := d(k+1) - d(k)$. By 4.4, $x \geqq n-1$. We have:

$$\binom{n+k}{n-1} - h^0(\mathcal{O}_D(k+1)) = d(k) - 1 + h(k+1) - h(k) \tag{3}$$

which according for instance to 4.4 is strictly positive. Hence $r_{D,H}(k+1)$ is strictly positive. As above, $R(k)$ gives $X := Y \cup T'$. We would like that D and Y intersect at one point and that D meets $h(k) - h(k+1)$ lines of T'. This is possible if we can impose $s' := 1 + h(k) - h(k+1)$ general points to D. According to 1.5 this is possible, because by 4.4 $d(k+1) \geqq d(k) + h(k) - h(k+1) - 2$. Now $D \cup X$ is smoothable (0.1). We conclude with 1.7, 1.4, and semicontinuity. □

§ 4. Numerical Lemmas

4.1 Lemma. *For $n \geqq 5$ and $k \geqq 3$ we have $g(k, n) - g(k-1, n) \geqq n + k - 2$.*

Proof. From the definition (1) of $g(k, n)$, $g(k-1, n)$, we get:

$$(k-2)(g(k, n) - g(k-1, n)) = \binom{n-1+k}{n-1} - g(k, n) + f(k-1, n) - f(k, n). \tag{4}$$

Set $F_n(k) = \binom{n+k}{n} - (k-1) \cdot \binom{n-1+k}{n-1} - n - 1 + (k-1)(k-2)(n+k-2)$ and $G_n(k) = \binom{n-2+k}{n-2} - 2n - 3k + 7$; hence $F_n(k-1) - F_n(k) = (k-2)\, G_n(k)$.

From the definition (1) of $g(k, n)$ we obtain

$$g(k, n) = \left[\left(\binom{n+k}{n} - kn - 1 - f(k, n)\right) \Big/ (k-1)\right]. \tag{5}$$

Assume $g(k, n) - g(k-1, n) \leqq n + k - 3$. By (4) and (5) to obtain a contradiction, it is sufficient to check that $F_n(k) < 0$. We easily see that $G_n(k+1) \geqq G_n(k)$, $k \geqq 2$, $n \geqq 5$, and that $G_n(2) > 0$, $F_n(3) < 0$, $n \geqq 5$. □

4.2 Lemma. *We have $g(k, n) - g(k-1, n) \geqq n - 1 + (k-5)(n-1)/2$ if $k \geqq 5$, $n \geqq 5$.*

Proof. Set

$$F(n, k) = \binom{n+k-1}{n-1} - \binom{n+k-1}{n} \Big/ (k-2) - n - (k-2) - k(n-1) - k(k-5)(n-1)/2.$$

By (1), (4), (5), it is sufficient to check that $F(n, k) \geqq 0$ if $n \geqq 5$, $k \geqq 5$. Set $F'(n, k) := F(n+1, k) - F(n, k)$ and $F''(n, k) := F'(n+1, k) - F'(n, k)$.

Since $F''(n, k) = \binom{n+k-1}{n+1} - \binom{n+k-1}{n+2} \Big/ (k-2) \geqq 0$ if $n \geqq 5$, $k \geqq 3$, it is sufficient to check that $F(5,k)\geqq 0$ and $F'(5,k)\geqq 0$ for every $k\geqq 5$. This is left to the reader. □

By the definition (1) of $g(k,n)$ the following lemma follows immediately.

4.3 Lemma. *We have* $g(k,n)\geqq k\cdot n/2+(k-2)$ *for all* $k\geqq 2$, $n\geqq 5$.

4.4 Lemma. *For every* $k\geqq r+1$, $r\geqq 1$, *we have* $d(k)-d(k-1)\geqq n+2k-2$, *except in the following cases:* $k=3$, $n=5$; $k=2$, $n=5$ *or* $n=6$.

Proof. By the definition (2) of $d(k)$, $d(k-1)$, we find:

$$k(d(k)-d(k-1))+d(k-1)+h(k)-h(k-1)=\binom{n+k-1}{n-1}. \tag{6}$$

First assume $k\geqq 3$. Set

$$G(n,k)=\binom{n+k-1}{n-1}-\binom{n+k-1}{n}\Big/(k-2)-kn-2k^2-3k+1.$$

Since $d(k-1)\leqq g(k-1)+n$, by (5), (6), it is sufficient to check when $G(n,k)\geqq 0$. Set $G'(n,k)=G(n+1,k)-G(n,k)$ and $G''(n,k)=G'(n+1,k)-G'(n,k)$. Note that $G''(n,k)\geqq 0$ for every $k\geqq 3$, $n\geqq 5$. It is easy to check that $G'(5,k)\geqq 0$ for every $k\geqq 3$ and that $G(5,k)\geqq 0$ for every $k\geqq 6$. Furthermore $G(7,3)\geqq 0$. Since $d(2)\leqq g(2,n)+n$, $d(1)=n$, it is easy to check the thesis for $k=3$, $n=6$; $k=4$, $n=6$; $k=2$, $n\geqq 7$. □

4.5 Lemma. (a) *We have* $d(r+1)-g(r,n)\geqq 2n+(r-4)(n-1)/2$ *if* $r\geqq 4$, $n\geqq 5$.
(b) *We have* $d(r+1)-g(r,n)\geqq 2n$ *if* $r\geqq 1$, $n\geqq 5$.

Proof. From the definitions (1) and (2) we obtain:

$$(r+1)(d(r+1)-g(r,n)-n)+2g(r,n)+n-g+h(r+1)-f(r,n)=\binom{n+r}{n-1}. \tag{7}$$

(a) Set

$$M(n,r)=\binom{n+r}{n-1}-\binom{n+r}{n}\Big/(r-1)-n-r-2n(r+1)-(r+1)(r-4)(n-1)/2.$$

By (5) and the inequality: $g(r,n)\leqq g$, it is sufficient to check when $M(n,r)\geqq 0$. Set $M'(n,r):=M(n+1,r)-M(n,r)$ and $M''(n,r):=M'(n+1,r)-M'(n,r)$. Since $M''(n,r)\geqq 0$ for $n\geqq 5$, $r\geqq 2$, it is sufficient to check (left to the reader) that $M(5,r)\geqq 0$ and $M'(5,r)\geqq 0$ for every $r\geqq 4$.

(b) If $r\geqq 4$, this is covered by part (a); the remaining cases have to be checked by hands. Assume for instance $r=1$; it is sufficient to check that $2(2n)+1\leqq(n+2)(n+1)/2$ if $n\geqq 5$. □

4.6 Lemma. *We have* $d(r+1)-g(r,n)\geqq n-1+2r$ *if* $n\geqq 5$, $r\geqq 2$.

Proof. Set $Z(n,r)=\binom{n+r}{n-1}-\binom{n+r}{n}-(r-1)-n-r-(2r-1)(r+1)$. Again it is sufficient to check when $Z(n, r)\geqq 0$. Set $Z'(n, r)=Z(n+1, r)-Z(n, r)$ and $Z''(n, r)=Z'(n+1, r)-Z'(n, r)$. $Z''(n, r)\geqq 0$ for every $n\geqq 5$, $r\geqq 2$, $Z'(n, r)\geqq 0$ for every $r\geqq 2$, $Z(5, r)\geqq 0$ if $r\geqq 3$, $Z(6, 2)\geqq 0$; if $n=5$, $r=2$, we have $g(2, 5)=10$, $g\geqq g(2, 5)$, hence $d(3)\geqq 21$. □

§ 5. Proof of the Theorem

We fix $n\geqq 5$, $g\geqq 0$, $d\geqq g+n$, and we set $j=v(d, g, n)$ (see 2.1). Since $Z^*(d, g; n)$ is irreducible, it is sufficient to find X, Y in $Z^*(d, g; n)$ such that $r_{X,n}(j)$ is surjective and $r_{Y,n}(j-1)$ is injective (semicontinuity and Castelnuovo-Mumford's lemma ([12], p. 99)). If $j\leqq 2$ see [4], Prop. 1.1. Hence we assume $j\geqq 3$. We distinguish several cases.

5.1 Lemma. *If $j=r$ the general curve X in $Z^*(d, g; n)$ has $r_{X,n}(j)$ surjective.*

Proof. According to 2.5(iii) we have $g=g(r, n)$ and $d=g(r, n)+n$. We show that there exist $X\in Z^*(g(r, n)+n, g(r, n); n)$ and a set P of $f(r, n)$ points such that $r_{X\cup P,n}(r)$ is bijective. For this we repeat the proof of "$H(r-1)$ implies $H(r)$, $n-f(r, n)+f(r-1, n)\geqq 1$" (see 3.1), with two minor modifications. To get the correct genus we put $\#(S)=n+f(r-1, n)$ (instead of $n+f(r-1, n)-f(r, n)$). To have the right number of conditions in H, we add $P\subset H$, a set of $f(r, n)$ general points. We have only to check that $g(r, n)-g(r-1, n)\geqq (f(r-1, n)-2)[(n-1)/2]$, to apply 1.5. This follows from 4.2. Note also that $\deg(Y)-p_a(Y)\geqq s$, where Y satisfies $H(r-1)$. Hence we can apply 1.6, 1.4 and conclude by 1.7. □

5.2 Lemma. *If $j=r$ the general curve X in $Z^*(d, g; n)$ has $r_{X,n}(r-1)$ injective.*

Proof. If $r=3$ the injectivity is contained in [4]. Hence we may assume $r\geqq 4$. To prove the lemma it is sufficient to construct J in $Z^*(g(r-1, n)+n+1, g(r-1, n)+1; n)$ with $r_{J,n}(r-1)$ injective. Indeed, then we may take as X the union of J and $g(r, n)-g(r-1, n)-1$ 2-secant lines to J (0.1). To construct J we modify the proof of "$H(r-2)$ implies $H(r-1)$" (see 3.1).

Set $x=g(r-1, n)-g(r-2, n)+1$ and $s=f(r-2, n)+n$. By 4.1 $x\geqq n-1$. Hence for general $D\in Z^*(x, x-n+1, H)$, H a hyperplane, $r_{D,H}(r-1)$ is of maximal rank. We have: $h^0(\mathcal{O}_H(r-1))-h^0(\mathcal{O}_D(r-1))=g(r-2, n)-r+1+f(r-1, n)-f(r-2, n)>0$ (by 4.3). Since $g(r-1, n)-g(r-2, n)\geqq n-2+(f(r-2, n)-2)[(n-1)/2]$ (by 4.2), we may apply 1.5 and assume that D contains a general set S of s points in H. By 1.6(a) there is Y in $Z^*(g(r-2, n)+n, g(r-2, n)-f(r-2, n); n)$ satisfying $H(r-2)$ and such that $Y\cap D=S$, $Y\cup D\in Z^*(g(r-1, n)+n+1, g(r-1, n)+1; n)$. Since $\deg(Y)-s\geqq h^0(\mathcal{O}_H(r-1))-h^0(\mathcal{O}_D(r-1))$, we conclude with 1.7, 1.6(b). □

5.3 Lemma. *If $j=r+1$ the general curve X in $Z^*(d, g; n)$ has $r_{X,n}(j)$ surjective.*

Proof. (i) First assume $d=d(r+1)$.

We have $d(r+1)\geqq g(r, n)+2n-1$ by 4.4. Note that $g\geqq g(r, n)$ by the definition of r. Set $x=d-g(r, n)-n$, $y=\min(x+1-n, g-g(r, n)+f(r, n))$, $s=g-g(r, n)+f(r, n)-y+1$. We have $y\geqq 0$, $x\geqq y+n-1$ and $1\leqq s\leqq n+f(r, n)$($\$\$$). Let H be a hyperplane. By the theorem for $n-1$, for a general $D\in Z^*(x, y; n-1)$, $r_{D,H}(r+1)$

is of maximal rank. We have:

$$\binom{n+r}{n-1}-h^0(\mathcal{O}_D(r+1))=g(r,n)+n-s+h(r+1)>0.$$

We show the existence of $C\in Z^*(d(r+1),g;n)$ and of a set P of $h(r+1)$ points such that $r_{C\cup P,n}(r+1)$ is bijective. For this we repeat the proof of 3.2 with D, s defined as above and $Y\in Z^*(g(r,n)+n,g(r,n)-f(r,n);n)$. Furthermore, to have the right number of conditions, we add $h(r+1)$ general points P in H. We have to verify the hypothesis of 1.5 for D and s. However we may assume $y=x+1-n$ (otherwise $s=1$). Then the condition is:

$$d(r+1)-g(r,n)-2n+1\geqq(g-d(r+1)+n+f(r,n)-3)[(n-1)/2].$$

Since $d(r+1)\geqq g+n$, this follows from 4.5(a). Finally, since $\deg(Y)-p_a(Y)\geqq s$ (by ($\$\$$)), we conclude with 1.6 and 1.7.

(ii) Now assume $d<d(r+1)$.

Set $c=d-g(r,n)-n$, $v=\min(c+1-n,g-g(r,n))$, $u=\max(v,0)$ and $s=1+g-g(r,n)-n$. Take $D\in Z^*(c,u;H)$ and $Y\in Z^*(g(r,n)+n,g(r,n);n)$ with $r_{Y,n}(r)$ surjective. The existence of Y was shown in the proof of 5.1. If $u=v=g-g(r,n)$, then $s=1$ and we conclude with 1.6, 1.7. If $u=v=c+1-n$, then $s\leqq n$ and we conclude with 1.5, 1.6, 1.7. Finally $s\leqq c+1$ if $c\leqq n-1$. Then D is a rational normal curve in $\mathbb{P}^c$. We conclude with 0.1, 1.6(b), 1.7. □

5.4 Lemma. *If $j=r+1$, the general curve X in $Z^*(d,g;n)$ has $r_{X,n}(r)$ injective.*

Proof. The lemma follows from the existence of $T\in Z^*(g(r,n)+n+1,g(r,n)+e;n)$ ($e=0$ if $g=g(r,n)$, $e=1$ otherwise) with $r_{T,n}(r)$ injective. If $e=1$ the existence of T was shown in the proof of 5.2. A slight modification of that proof also yields the case $e=0$. We have just to check the corresponding condition for 1.5 which is:

$$g(r,n)-g(r-1,n)\geqq n-2+(f(r-1,n)-2)[(n-1)/2].$$

See 4.2. □

5.5 Lemma. *If $j\geqq r+2$, the general $T\in Z^*(d,g;n)$ has $r_{T,n}(j-1)$ injective.*

Proof. The lemma follows from the existence of $X\in Z^*(d(j-1)+1,g;n)$ with $r_{X,n}(j-1)$ injective (note that $d\geqq d(j-1)+1$, see 2.5).

(i) Assume $R(j-2)$ holds (which is always the case if $j\geqq r+4$ by 3.4, 3.5). We apply 1.6(b), 1.7 with: $D\subset H$ a rational curve of degree $x:=d(j-1)-d(j-2)+1$; $Y=Z\cup T$ given by $R(j-2)$ ($Z\in Z^*(d(j-2)-h(j-2),g;n)$, T the union of $h(j-2)$ disjoint lines). We require that D intersects Z and each line of T at one point. This is possible since $x+3\geqq h(j-2)+1$ (see 4.4). Clearly $Y\cup D$ is smoothable. Set $m:=h^0(\mathcal{O}_H(j-1))-h^0(\mathcal{O}_D(j-1))=d(j-2)+h(j-1)-h(j-2)-j$. By 4.4 or 4.3, 4.5, we have $m>0$. Since $\deg(Z)\geqq m+1$, $r_{Y\cup D,n}(j-1)$ is injective by 1.6(b), 1.7.

(ii) Now assume $j=r+3$ and that $R'(r+1)$ holds.

In a similar way we take $Y\in Z^*(d(r+1),g-h(r+1);n)$ satisfying $R'(r+1)$, a rational curve $D\subset H$, $\deg(D)=x:=d(r+2)-d(r+1)+1$. Since $x\geqq h(r+1)$ (4.4), by 0.1 we may assume that D and Y intersects quasi-transversally in $s:=h(r+1)+1$ points, and that $D\cup Y\in Z^*(d(r+2)+1, g; n)$. We conclude with 1.6(b), 1.7.

(iii) Finally assume $j=r+2$.

Again the proof is similar with $Y\in Z^*(g(r, n)+n, g(r, n)-f(r, n); n)$ satisfying $H(r)$, $D\in Z^*(x, y; H)$, $x:=d(r+1)+1-g(r, n)-n$, $y:=\min(x-n+1, g-g(r, n)+f(r, n))$ and $s:=g-g(r, n)+f(r, n)+1-y$. The condition $x\geqq n-1$ is equivalent to $d(r+1)-g(r, n)\geqq 2n-2$ (see 4.5(b)). We always have $1\leqq s\leqq n+f(r, n)-1$, hence $\deg(Y)-p_a(Y)\geqq s$. To use 1.5 for D and s we may assume $y=x+n-1$ (otherwise $s=1$). Since $d(r+1)\geqq g+n$ (2.5) it is enough to show that $d(r+1)+2-g(r, n)-2n\geqq(f(r, n)-2)[(n-1)/2]$ (see 4.5). □

5.6 Lemma. *If $j\geqq r+2$ and $d-d(j-1)\geqq h(j-1)$, the general T in $Z^*(d, g; n)$ has $r_{T,n}(j)$ surjective.*

Proof. (i) Assume $R(j-1)$ holds.

We take $Y\cup T$ satisfying $R(j-1)$, $D\subset H$ rational of degree $x:=d-d(j-1)$ which intersects Y and each line of T at one point. This is possible since $x\geqq h(j-1)$ by assumption. Hence $A=D\cup T\cup Y$ is connected, $p_a(A)=g$, and by 1.6(b), 1.7, $r_{A,n}(j)$ is surjective.

(ii) In a similar way if $j=r+2$ and $R'(r+1)$ holds, we take Y satisfying $R'(r+1)$, $D\subset H$ rational of degree $x=d-d(r+1)$ and $s:=h(r+1)+1$. By [5] we may assume D of maximal rank. We have $\deg(Y)-p_a(Y)\geqq s$ and $x+3\geqq s$. We conclude as usual with 1.6(b) and 1.7. □

5.7 Lemma. *If $j\geqq r+2$ and $d-d(j-1)<h(j-1)$, the general curve T in $Z^*(d, g; n)$ has $r_{T,n}(j)$ surjective.*

Proof. (i) Assume that $R(j-2)$ holds.

Then as in 5.5(i) we construct $X\in Z^*(d(j-1), g; n)$ with $r_{X,n}(j-1)$ surjective (for this we have to check: $d(j-1)-d(j-2)\geqq h(j-2)$ (see 4.4)). Then we take $D\subset H$ rational of degree $d-d(j-1)$, of maximal rank and intersecting X at one point.

(ii) Assume $j=r+3$ and that $R'(j-2)$ holds.

As in 5.5(ii) we construct $X\in Z^*(d(j-1), g; n)$ with $r_{X,n}(j-1)$ surjective (we have to check that $d(r+2)-d(r+1)\geqq h(r+1)$: 4.4). Once we have X, we conclude as above.

(iii) Assume $j=r+2$.

We construct $X\in Z^*(d(j-1), g; n)$ with $r_{X,n}(j-1)$ surjective. For this we repeat the proof of 5.5(iii) but with $x:=d(r+1)-g(r, n)-n$. The arithmetical conditions now are: $d(r+1)-g(r, n)\geqq 2n-1$ (4.5(b)) and $d(r+1)+1-g(r, n)-2n\geqq(f(r, n)-2)[(n-1)/2]$ (4.5(a)). Note that we have $1\leqq s\leqq n+f(r, n)$ and hence $\deg(Y)-p_a(Y)\geqq s$. □

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