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ON THE BOUNDARY INTEGRAL EQUATIONS FOR THE CRACK OPENING DISPLACEMENT OF FLAT CRACKS

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The boundary integral equations for the crack opening displacement in acoustic and elastic scattering problems are discussed in the case of flat cracks by means of the Fourier analysis technique.The pseudo-differential nature of the hypersingular integral operators is shown and their symbols explicited. It is then proved that the variational problems assocaited with these BIE are well-posed in a Sobolev functional framework which is closely linked with the elastic energy. A decomposition of the vector integral equation in the elastic case into scalar integral equations is obtained as a by-product of the variational formulation.

0. INTRODUCTION

An effective method for crack detection in materials is the nondestructive testing by means of ultrasound. It consists of deducing the presence and characteristics of cracks from the analysis of the wave diffraction pattern. Mathematically, this *inverse problem* still remains one of the most challenging applied problems. Studies of the direct problem of the elastic wave scattering by cracks are then extensively pursued with the purpose of acquiring more useful information. For the case of a penny-shaped crack, a fairly complete account of the state of the art in 1983 was given by Martin & Wickham [22]. A very recent work is that of Budreck and Achenbach [6]. The main tool in these studies is the boundary integral equation (BIE) for the crack opening displacement (COD). However, this BIE suffers from a double disavantage of being a first kind Fredholm integral equation and of having a hyper-singular kernel. For this reason, almost all the references in [22] focus on the techniques of transforming this BIE to a more tractable second kind integral equation, or to a Neumann's serie problem. Some techniques of regularization are recently investigated to calculate the hyper-singular integral. See e.g. [6], Bui *et al.*. [7], Hirose & Niwa [17], Nishimura & Kobayashi [28].

Another path was opened by Nedelec who showed the good functional properties of the first kind BIE and how to deal with the hyper singular character of the kernel. His results for the Laplace equation ([25], [26], [27]) were extended by others for the Helmholtz equation [16], the biharmonic equation [13], the electromagnetic or elastic waves scattering problems [3], [2], ... Numerical experimentations were performed, giving accurate and stable results in many different situations including crack calculus [10], transient acoustic scattering problems [12] or electromagnetic scattering by single or gratings of antennas [4], [5]...

On the other hand, Wendland and his co-workers have made important contributions to the analysis of the discretizations of first kind integral equations, including scalar and vectorial problems. See e.g.[11, [18], [29] and [30].

An important fact which should be stressed on is that although compactness prevents first kind integral operators from being continuous and invertible from a Banach space into itself, there is nothing to prevent them from being continuous and invertible form a Banach space to an other one ! A key idea of Nedelec in proving that such is really the case for the integral operator associated with the Laplace equation [27] is the use of a variational formulation for the integral equation. For the Neumann's problems, the same idea proves particularly efficient in that it permits moreover an elegant treatment of the hyper-singular kernel. See e.g. [25] and [16].

Naturally, the variational treatment cannot make the integral equations solvable when they are not ! This is the case of the BIE in scattering problems, when the square of the frequency is an eigen value of an interior problem.

This paper deals with the BIE in the scattering problems by a flat crack of arbitrary shape, for arbitrary harmonic incident waves. Using a Fourier transform with respect to the variables of the cracks plane, the pseudo-differential nature of the integral operators is shown, and theirs full symbols explicited. From this, a coerciveness estimate (thus, more precise than a Garding inequality) is obtained for the associated bilinear form, and this allows us to prove the solvability of the equivalent variational problem. This is done first for scalar (or anti-plane) waves in part I, where the essential ideas are discussed. In part II, the more complicated calculations for the (vector) general elastic waves are presented, following the same path.

In [24] , the authors advocate for the definition of the hypersingular integrals by the Hadamard finite part which 'do not involve the tangential derivatives' of the integrand functions.However, this absence is, in our view, only *apparent* .The modern theory of pseudo-differential operators, cf. Chazarain & Piriou [8], clarifies this point by using the Fourier integral method. The framework of Sobolev spaces appears natural in the context of this theory. This is also the point of view adopted in this paper.

I. THE SCALAR PROBLEM

1. The scattering problem and its associated BIE

If the incident wave impinging on the crack is an anti-plane shear wave, so is the scattered wave and the whole problem is scalar, identical to the acoustic scattering problem in fluids. Thus, with a convenient scaling of the measure units, and suppressing the harmonic factor $e^{i\omega t}$, we have the following problem

$$
(P)\begin{cases}
\Delta^* u = (\omega^2+\Delta)u = 0 \quad \text{in } \Omega = \mathbb{R}^3\setminus\overline{\Gamma} & (1.1)\\
\dfrac{\partial u}{\partial r} - i\omega u = 0\left(\dfrac{1}{r}\right) \text{ when } r = |x| \to +\infty & (1.2)\\
\dfrac{\partial u}{\partial n} = f \qquad \text{on } \Gamma & (1.3)
\end{cases}
$$

where Γ is a bounded smooth domain of the plane $\{ x_3 = 0 \}$ and Ω the surrounding elastic medium. Actually, the crack is better represented by two coplanar (traction-free) surfaces that are infinitesimally close : the upper (Γ_+) and lower (Γ_-) faces of Γ.

The known function f is the opposite normal trace of the incident wave on Γ. The radiation condition (1.2) expresses the fact that the scattered wave is outgoing. Finally, ω is a given real, positive frequency.

This problem is well studied. In particular, we know (see e.g. Colton & Kress [9]) that any solution u of (1.1 - 1.2) (for brevity, such solution will be called an outgoing wave) can be represented by the layer potentials on Γ :

$$u(x) = \int_\Gamma G(x,y)\left[\frac{\partial u}{\partial n}(y)\right] dy - \int_\Gamma \frac{\partial}{\partial n_y} G(x,y)\,[\,u(y)\,]\, dy \tag{1.4}$$

where

$$G(x,y) = \frac{e^{i\omega|x-y|}}{4\pi|x-y|} \tag{1.5}$$

is the fundamental solution of Δ^*, and [g] designates the jump of a function across Γ :

$$[\,g(y)\,] = \lim_{\substack{y'\to y\\(y'_3<0)}} g(y') - \lim_{\substack{y'\to y\\(y'_3>0)}} g(y') = g_-(y) - g_+(y)$$

Now, since the condition (1.3) implies that $\left[\frac{\partial u}{\partial n}\right] = 0$ for the solution of problem (P), this solution is then completely determined by its COD $\varphi = [u]$:

$$u(x) = -\int_\Gamma \frac{\partial}{\partial n_y}\left(\frac{e^{i\omega|x-y|}}{4\pi|x-y|}\right)\varphi(y)dy \quad (x \in \Omega) \tag{1.6}$$

We will see in section 2 a new proof for formula (1.6) in the particular case of a flat crack, using the Fourier technique.
Taking the normal derivative of (1.6) and comparing this to (1.3), we obtain the classical BIE for the COD of u :

$$D\varphi(x) = f(x) \qquad (x \in \Gamma) \tag{1.7}$$

where

$$D\varphi(x) := \frac{-\partial}{\partial n_x}\int_\Gamma \frac{\partial}{\partial n_y}\left(\frac{e^{i\omega|x-y|}}{4\pi|x-y|}\right)\varphi(y)dy \tag{1.8}$$

However, since

$$\frac{\partial^2}{\partial n_x\,\partial n_y}\left(\frac{e^{i\omega|x-y|}}{|x-y|}\right) \equiv 0\left(\frac{1}{|x-y|^3}\right) \text{ when } (x \to y) \;,$$

the operator D has an integral kernel which is hyper-singular, and must be defined carefully.
The following equivalent definition of D comes from the well-known properties of the double layer potential, and for the above reason, is preferable to (1.8) :

$$D\varphi = \frac{\partial u}{\partial n} \qquad \text{on } \Gamma \tag{1.9}$$

where u is an outgoing wave with the following jump condition across Γ:

$$\begin{cases} [\,u\,] = u_- - u_+ = \varphi \\ [\frac{\partial u}{\partial n}] = 0 \end{cases} \tag{1.10}$$

we will designate by (Q) the problem (1.1, 1.2 and 1.10).
It should be noted that, since u is analytic in Ω, the condition (1.10) is actually equivalent to :

$$\begin{cases} [\,u\,](x) = \varphi_0(x) \\ [\frac{\partial u}{\partial n}](x) = 0 \qquad \text{for } x \in \{x_3 = 0\} \end{cases} \tag{1.11}$$

where φ_0 is the extension of φ by 0 on $\{x_3 = 0\}\backslash\overline{\Gamma}$.

In the next section, using a partial Fourier transform with respect to the variables of the crack plane, we will show that D has a simple expression in terms of the Fourier variables. It follows from this expression two main consequences in our point of view :
First, it will be obvious that the right way to deal with the BIE (1.7) is the variational method ; and secondly, we will be able to prove, with this method, the solvability of D for all ω. This is the essential difference between our case here and the case when the BIE (1.7) is considered on a closed surface Γ.

2. The Fourier expression of D

We first prove a very simple "limiting absorption principle" for the solution of problem (Q).
Let us denote by $x' = (x_1, x_2)$ the spatial variable in the crack plane, and $\xi' = (\xi_1, \xi_2)$ its Fourier dual variable. The radiation condition (1.2), joined to the analyticity of u in Ω allows us to consider the partial Fourier transform with respect to x' :

$$\hat{u}(\xi', x_3) = F_{x'}\, u(x', x_3)$$

for $x_3 \neq 0$. From (1.1) $\hat{u}$ is then solution of the ordinary differential equation :

$$(-|\xi'|^2 + \omega^2)\, \hat{u}(\xi', x_3) + \frac{d^2}{dx_3^2}\hat{u}(\xi', x_3) = 0 \text{ in } \{x_3 > 0\} \cup \{x_3 < 0\} \tag{2.1}$$

And the jump condition (1.11) is transformed into :

$$\begin{cases} [\hat{u}](\xi', 0) = \hat{u}(\xi', 0-) - \hat{u}(\xi', 0+) = \widehat{\varphi_0}(\xi') \\ \left[\dfrac{\partial \hat{u}}{\partial n}(\xi', 0) = 0\right] \end{cases} \tag{2.2}$$

The solution of problem (Q) (the uniqueness of which is as usual assured by the Rellich theorem) is then given in terms of its Fourier transform in the following lemma :
LEMMA 1. *For* $\varphi_0 \in H^{1/2}(R^2)$ *, the solution u of problem (Q) is in* $H^1{}_{loc}\Omega)$ *and is given by :*

$$\hat{u}(\xi', x_3) = \begin{cases} -\frac{1}{2}\hat{\varphi}_0(\xi') \exp(ix_3 Z(\xi', \omega)) & \text{for } x_3 > 0 \\ \frac{1}{2}\hat{\varphi}_0(\xi') \exp(-ix_3 Z(\xi', \omega)) & \text{for } x_3 < 0 \end{cases} \tag{2.3}$$

where the symbol Z is:

$$Z(\xi', \omega) = \begin{cases} \sqrt{\omega^2 - |\xi'|^2} & \text{if } |\xi'| < \omega \\ i\sqrt{|\xi'|^2 - \omega^2} & \text{if } |\xi'| \geq \omega \end{cases} \tag{2.4}$$

PROOF : a) Let ε be a positive real and $\omega_\varepsilon = \omega + i\varepsilon$. Then, the solution of equation (2.1) that does not blow up when $|x_3| \to +\infty$ is :

$$\hat{u}_\varepsilon(\xi', x_3) = \begin{cases} \hat{u}_\varepsilon(\xi', 0_+) \exp(ix_3 Z_\varepsilon) & (x_3 > 0) \\ \hat{u}_\varepsilon(\xi', 0_-) \exp(-ix_3 Z_\varepsilon) & (x_3 < 0) \end{cases} \tag{2.5}$$

where Z_ε is the square root of $\omega_\varepsilon^2 - |\xi'|^2$ with strictly positive imaginary part.
Imposing on $\hat{u}_\varepsilon$ the same jump condition (2.2) as for $\hat{u}$, we get :

$$\hat{u}_\varepsilon(\xi', x_3) = \begin{cases} -\frac{1}{2}\hat{\varphi}_0(\xi') \exp(ix_3 Z_\varepsilon) & (x_3 > 0) \\ \frac{1}{2}\hat{\varphi}_0(\xi') \exp(-ix_3 Z_\varepsilon) & (x_3 < 0) \end{cases} \tag{2.6}$$

This is clearly an integrable function of x_3, and its Fourier transform with respect to this variable is obtained easily :

$$\hat{u}_\varepsilon(\xi', \xi_3) = \frac{i\xi_3 \hat{\varphi}_0(\xi')}{\xi_3^2 - Z_\varepsilon^2} = \frac{i\xi_3 \hat{\varphi}_0(\xi')}{|\xi|^2 - \omega_\varepsilon^2}$$

with $\xi = (\xi', \xi_3)$.
By the following correspondence in Fourier transform in R^3 :

$$\frac{1}{|\xi|^2 - \omega_\varepsilon^2} \leftrightarrow \frac{e^{i\omega_\varepsilon |x|}}{4\pi|x|} \tag{2.7}$$

one gets :

$$u_\varepsilon(x) = \int_{\{y_3=0\}} -\frac{\partial}{\partial y_3}\left(\frac{e^{i\omega_\varepsilon |x-y|}}{4\pi|x-y|}\right) \varphi_0(y)\, dy \tag{2.8}$$

The integration is actually on Γ because of the support of φ_0.
b) Suppose now $\varphi_0 \in H^{1/2}(R^2)$, and let us prove that u_ε has a limit in the sense of $H^1_{loc}(\Omega)$ when $\varepsilon \to 0$.
Since u_ε is analytic in Ω, in particular for $x \in \{ x_3 = 0 \} \setminus \Gamma$, it suffices to prove that $\| u_\varepsilon \|_{H^1(V)}$ is bounded independently of ε for $V = V_{\pm\alpha}$ where $V_{\pm\alpha} = \{ x ; x_3 \in]0, \pm\alpha[\}$, $\alpha > 0$ is given . By (2.6) and the Parseval formula, one gets :

$$\|u_\varepsilon\|^2_{H^1(V_\alpha)} = \frac{1}{4}\int_{IR^2} |\hat{\varphi}_0(\xi')|^2 \; (1 + |\xi'|^2 + |Z_\varepsilon|^2) \; \frac{1 - e^{-2\alpha \operatorname{Im} Z_\varepsilon}}{2 \operatorname{Im} Z_\varepsilon} \, d\xi'$$

Now, we can remark that :

(i) In $\{ |\xi'| \leq \omega \}$, $\operatorname{Im} Z_\varepsilon \to 0$ when $\varepsilon \to 0$, but we can always overestimate the ratio $\frac{(1 - e\text{-}2a \operatorname{Im} Ze)}{2 \operatorname{Im} Ze}$ by α,

(ii) In $\{ |\xi'| > \omega \}$, $\operatorname{Im} Z_\varepsilon \to \sqrt{|\xi'|^2 - \omega^2}$.

So that, there is no difficulty to obtain the estimate :

$$(1 + |\xi'|^2 + |Z_\varepsilon|^2) \frac{1 - e^{-2\alpha \operatorname{Im} Z_\varepsilon}}{2 \operatorname{Im} Z_\varepsilon} \leq C(\alpha,\omega)\,(1 + |\xi'|^2)^{1/2}$$

where the constant is independent of $|\xi'|$, and of ε in $]0,1[$. The boundness of $\| u_\varepsilon \|_{H^1(V_\alpha)}$ follows from this estimate.

c) The lemma is totally proved from these part a/ and b/, and from : (i) The classical results on elliptic equations ; (ii) The trace theorems in $H^1_{loc}(\Omega_+)$ and $H^1_{loc}(\Omega_-)$ $(\Omega_\pm = \Omega \cap R^3_\pm)$.

Note that the correspondence (2.7) is valid for $\varepsilon = 0$, so is the representation (2.8) in this case. Thus, one gets again (1.6), by Fourier technique, as claimed in section 1 ♦

Remarks 1.

a) The result in Lemma 1 (essentially formula (2.3)) is not new, but may be it is worth setting here, for convenient use later, and also for the simple limiting absorption argument in its proof. Naturally, this simplicity is due to the geometry of our problem.

b) More importantly, il should be noted that, it is *not* sufficient to suppose that the COD φ is in $H^{1/2}(\Gamma)$ to obtain the validity of the lemma. The extension by 0 out of Γ is actually not continuous from $H^{1/2}(\Gamma)$ to $H^{1/2}(R^2)$. We refer to the treatise of Lions Magenes [21] for all of the results on Sobolev spaces used in this paper. In particular, we recall that the relevant fact concerning the above question is that the subspace $H^{1/2}_{\Gamma}(IR^2)$, of the elements of $H^{1/2}(R^2)$ with support in Γ, can be identified to the space

$$H^{1/2}_{00}(\Gamma) = \{ \varphi \in H^{1/2}(\Gamma) \,;\, \rho^{-1/2}\varphi \in L^2(\Gamma) \} \tag{2.9}$$

where ρ is a positive, regular function on Γ, equivalent to the distance of x to the boundary $\partial\Gamma$ in a vicinity of this boundary. The space $H^{1/2}_{00}(\Gamma)$ is algebraically strictly included in $H^{1/2}(\Gamma)$, and its topology defined by the norm

$$\|\varphi\|_{H_{00}^{1/2}(\Gamma)} = \{ \|\varphi\|^2_{H^{1/2}(\Gamma)} + \| \rho^{-1/2}\varphi \|^2_{H^0(\Gamma)} \}^{1/2} , \tag{2.10}$$

is strictly finer than the topology induced by that of $H^{1/2}(\Gamma)$.

A simple consequence of (2.9), which is worth noting, is that if $\varphi \in H_{00}^{1/2}(\Gamma)$ is a continuous function on Γ, than it must vanish on $\partial\Gamma$. Then (2.9) can be interpreted as the general "*edge - condition*" to be imposed on φ when dealing with problem (Q) or BIE (1.7). The restricted edge condition "$\varphi = 0$ on $\partial\Gamma$" was prescribed by Jones in a work of 1956, solving the scattering problem by a penny-shaped crack.

We can now precise the operator D in this functional framework :

THEOREM 1. *The operator D defined by (1.8) is a continuous operator from* $H_{00}^{1/2}(\Gamma)$ *into* $(H_{00}^{1/2}(\Gamma))'$ *, and can be expressed by :*

$$D\varphi = (R_\Gamma \circ T \circ P_\Gamma)\, \varphi \tag{2.11}$$

where P_Γ *is the extension operator by 0 out of* Γ, *in* R^2 *;*

R_Γ *is the restriction operator on* Γ*;*

and T *is the pseudo-differential operator on* R^2 *defined by :*

$$\widehat{T\psi}(\xi') = -\frac{i}{2} Z(\xi',\omega)\, \hat{\psi}(\xi') \tag{2.12}$$

where the symbol Z is as in (2.4).

PROOF : From (1.9), $D\varphi$ is the restriction on Γ of $\frac{\partial u}{\partial x_3}(x',0)$, where u is the solution of problem (Q). And from lemma 1,

$$\frac{\partial \hat{u}}{\partial x_3}(\xi',0) = -\frac{i}{2}\hat{\varphi}_0(\xi')\, Z(\xi',\omega)$$

where $\varphi_0 = P_\Gamma \varphi$.

The conclusion follows, recalling that R_Γ is a continuous operator from $H^{-1/2}(\mathbb{R}^2)$ onto $(H_{00}^{+1/2}(\Gamma))'$.

We note that $(H_{00}^{+1/2}(\Gamma))'$ contains strictly $H^{-1/2}(\Gamma)$, however , even if in the BIE (1.7), f is given in $H^{-1/2}(\Gamma)$, one cannot drop the edge-condition (2.9) for φ. This should be kept in mind when discretising (1.7).

3. The variational solution of the BIE (1.7)

We present in this section the main result of the paper : that the BIE (1.7) is well-posed in an appropriated functional framework, for all frequency ω. The key of it is the following

THEOREM 2. *The sesqui-linear form*

$$b(\varphi,\psi) := < D\varphi, \psi >_\Gamma \tag{3.1}$$

defined by the duality brackets of $H_{00}^{1/2}(\Gamma)$, *can be written in Fourier variables as :*

$$b(\varphi,\psi) = \frac{-i}{2.(2\pi)^2}\int_{R^2} Z(\xi,\omega)\,\hat{\varphi}_0(\xi)\overline{\hat{\Psi}_0}(\xi)\,d\xi \tag{3.2}$$

and satisfies the following coerciveness estimate :

$$|b(\varphi,\varphi)| \geq \frac{C}{(\omega^2+1)^{1/2}}\,\|\varphi\|^2_{1/2,\omega} \qquad \forall\ \varphi \in H_{00}^{1/2}(\Gamma) \tag{3.3}$$

In (3.2), $\varphi_0 = P_\Gamma\,\varphi$ as defined in th. 1. And in (3.3), we have adopted the equivalent norm in $H_{00}^{1/2}(\Gamma)$ defined by :

$$\|\varphi\|^2_{1/2,\omega} = \int_{IR^2} (1+|\xi|^2+\omega^2)^{1/2}\,|\hat{\varphi}_0|^2\,d\xi \tag{3.4}$$

Finally, the constant C in (3.3) is independent of ω.

PROOF : a) To prove (3.2), we first remark that for $v \in (H_{00}^{1/2}(\Gamma))'$ and $\psi \in H_{00}^{1/2}(\Gamma)$, then

$$< v, \psi >_\Gamma \;=\; < \tilde{v}, P_\Gamma\,\psi >_{IR^2} \tag{3.5}$$

where $\tilde{v}$ is any extension of v in $H^{-1/2}(R^2)$ and the brackets in the right-hand side of (3.5) design duality between $H^{-1/2}(R^2)$ and $H^{1/2}(R^2)$. See e.g. [11]. Now, it suffices to apply this with $v = D\varphi \quad \varphi \in H_{00}^{1/2}(\Gamma)$. By th. 1, the extension of v can be chosen as $T_0\,P_\Gamma\,\varphi$, so that :

$$< D\varphi\,,\psi >_\Gamma = \;< T\varphi_0\,,\psi_0 >_\Gamma$$

and one gets (3.2) by the Parseval formula.

b) We proceed now to prove the estimate (3.4).

First, we note that since the support $\overline{\Gamma}$ of φ_0 is bounded, the following inequalities are obvious :

$$|\hat{\varphi}_0(\xi')|^2 \leq C\,\|\varphi_0\|^2_{L^2(IR^2)} \leq \frac{C}{(\omega^2+1)^{1/2}}\,\|\varphi_0\|^2_{1/2,\omega} \quad \forall\ \xi' \in IR^2 \tag{3.6}$$

where c depends only on Γ.

Next, let us decompose the integral $J = \|\varphi_0\|^2_{1/2,\,\omega}$ as follows :

$$\int_{IR^2_\xi} = \int_{\{0<|\xi|<\beta\omega\}} + \int_{\{\beta\omega<|\xi|<\omega\}} + \int_{\{\omega<|\xi|<\alpha\omega\}} + \int_{\{|\xi|>\alpha\omega\}}$$
$$:= J_1 + J_2 + J_3 + J_4$$

where the reals α, β with $\beta < 1 < \alpha$ will be chosen later.

The same decomposition will be applied to the integral

$$I = \int_{IR^2} |Z| |\hat{\varphi}_0(\xi)|^2 d\xi = I_1 + I_2 + I_3 + I_4$$

since Z is either a positive real, or a purely imaginary positive number, I is merely $8\pi^2 \ |b|$.

We will compare J_k to I_k.

c) Let us begin with J_3 and J_4.

Using (3.6) and the Cauchy-Schwarz inequality, one gets :

$$J_3^2 \le \frac{C}{(\omega^2+1)^{1/2}} \|\varphi_0\|^2_{1/2,\omega} \left(\int_{\{\omega<|\xi|<\alpha\omega\}} \frac{1+\omega^2+|\xi|^2}{(|\xi|^2-\omega^2)^{1/2}} d\xi \right) \left(\int_{\{\omega<|\xi|<\alpha\omega\}} (|\xi|^2-\omega^2)^{1/2} |\hat{\varphi}_0(\xi)|^2 d\xi \right)$$

Now, using polar coordinates in the plane, one gets :

$$\int_{\{\omega<|\xi|<\alpha\omega\}} \frac{1+\omega^2+|\xi|^2}{(|\xi|^2-\omega^2)^{1/2}} d\xi = 2\pi \int_\omega^{\alpha\omega} \frac{1+\omega^2+r^2}{(r^2-\omega^2)^{1/2}} r dr$$

$$= 2\pi\, \omega \sqrt{\alpha^2-1} \left(\frac{\alpha^2\omega^2}{3} + \frac{5}{3}\omega^2 + 1\right)$$

So that, with the choice

$$\omega\sqrt{\alpha^2-1} = 1, \text{ i.e. } \alpha = \frac{\sqrt{1+\omega^2}}{\omega^2},$$

the following inequality is proved :

$$J_3^2 \le c_3 (1+\omega^2)^{1/2} \|\varphi_0\|^2_{1/2,\omega} I_3 .$$

On $\{ |\xi| > \alpha\omega \}$ we have on the other hand :

$$\frac{1+\omega^2+|\xi|^2}{|\xi|^2-\omega^2} \le \frac{(\alpha^2+1)\omega^2+1}{\omega^2(\alpha^2-1)} ,$$

then

$$J_4^2 \le \int_{\{|\xi|>\alpha\omega\}} \left(\frac{1+\omega^2+|\xi|^2}{|\xi|^2-\omega^2}\right)^{1/2} (1+|\omega|^2+|\xi|^2)^{1/2} |\hat{\varphi}_0|^2 d\xi \int_{|\xi|>\alpha\omega} (|\xi|^2-\omega^2)^{1/2} |\hat{\varphi}_0|^2 d\xi$$

$$\leq \frac{[(\alpha^2 + 1)\omega^2 + 1]^{1/2}}{\omega(\alpha^2 - 1)^{1/2}} \| \varphi_0 \|^2_{1/2,\omega} I_4$$

And the same choice of α as above yields the estimate

$$J_4^2 \leq c_4 \, (1 + \omega^2)^{1/2} \| \varphi_0 \|^2_{1/2,\omega} I_4$$

d) The same calculus can be applied to J_1 and J_2, and the choice of

$$\beta = \frac{\sqrt{\omega^2 - 1}}{\omega} \text{ if } \omega > 1 \text{ and } \beta = 0 \text{ if } \omega \leq 1$$

(which means that in the last case, the integrals J_1 and I_1 are evacuated) yields the same type of estimates as for J_3 and J_4 :

$$J_k^2 \leq c_k \, (1 + \omega^2)^{1/2} \|\varphi_0\|^2_{1/2,\omega} I_k \tag{3.7}$$

Now,

$$\|\varphi_0\|^4_{1/2,\omega} = J^2 = (\sum_{k=1}^{4} J_k)^2 \leq 4 \sum_{1}^{4} J_k^2 \, ,$$

and (3.3) follows from (3.7) and the definition of I ♦

The solvability of the BIE (1.7) is now an immediat corollary of th. 1 and 2 :

COROLLARY : *For* $f \in (H_{00}^{1/2}(\Gamma))'$, *there exists an unique solution* $\varphi \in (H_{00}^{1/2}(\Gamma)$ *of eq. (1.7), which is also the unique solution of the variational problem*

$$\begin{cases} \text{To find } \varphi \in H_{00}^{1/2}(\Gamma) \text{ such that} \\ < D\varphi \, , \psi >_\Gamma = < f \, , \psi >_\Gamma \quad \forall \, \psi \in H_{00}^{1/2}(\Gamma) \end{cases} \tag{3.8}$$

Remark 2 .

Theorem 2 and this corollary make clear a fundamental difference between the problems of wave scattering by a bulky objet and by a crack. In the first case, the usual boundary integral equations for the problem are all solvable *except for* a sequence of spurious frequencies ω. Such a frequency is the square root of an eigenvalue of an interior laplacian problem. Naturally, there is no interior problem in the case of a crack. However, from this absence of the interior problems, it was not clear that the quadratic form $b(\varphi,\varphi)$ must be coercive.

4. Some further remarks

We conclude this part I with two remarks, on the hyper-singularity of D and on the connexion of the form b to the wave energy.

a) It was proved by Hamdi [16] , in the case of the BIE $D\varphi = f$ on a closed surface, following an original idea of [25] , that the form $b(\varphi, \psi)$ can be written with only weakly singular integrals.

In our case, this can be found again from the Fourier expression (3.2). For a convenient use in part II, we set out here this stricking observation of Bamberger [2] . The beginning point is the following correspondence of Fourier transform in R^2 :

$$-\frac{1}{2i\,Z(\xi,\omega)} \leftrightarrow \frac{e^{i\omega|x|}}{4\pi|x|} \tag{4.1}$$

Now, we can write

$$-\frac{iZ}{2} = \frac{1}{2iZ}(\omega^2 - |\xi|^2)$$

On the other hand,

$$|\xi|^2\,\widehat{\varphi_0}(\xi)\,\overline{\widehat{\psi_0}}\,(\xi) = (i\xi_2\,\widehat{\varphi_0})\,\overline{(i\xi_2\,\widehat{\psi_0})} + (-i\xi_1\,\widehat{\varphi_0})\,\overline{(-i\xi_1\widehat{\psi_0})}$$
$$= \widehat{\text{curl}\,\varphi_0}\,.\,\overline{\widehat{\text{curl}\,\psi_0}}$$

where $\text{curl}\,\varphi_0 = (\frac{\partial\varphi_0}{\partial x_2}\,,\,-\frac{\partial\varphi_0}{\partial x_1})$ is the rotational vector of a scalar function in R^2.
From these calculations, it follows that

$$b(\varphi,\psi) = \int\int_{IR^2 x\, IR^2} \frac{e^{i\omega|x-y|}}{4\pi|x-y|}(\text{curl}\,\varphi_0(x).\text{curl}\,\overline{\psi_0}(y) - \omega^2\varphi_0(x)\overline{\psi_0}(y))dxdy$$

$$= \int\int_{\Gamma\, x\, \Gamma} \frac{e^{i\omega|x-y|}}{4\pi|x-y|}(\text{curl}_\Gamma\,\varphi(x).\text{curl}\,\overline{\psi}(y) - \omega^2\varphi(x)\overline{\psi}(y))dxdy \tag{4.2}$$

This is the formula of Hamdi, where the hyper-singularity of the kernel in operator D is now transferred to the functions φ and ψ as derivatives.

With these all properties of the form b, the good way to discretize the BIE (1.7) is clear : in our view, it should be done by a finite element method an Γ, with P_1 - elements null on the boundary $\partial\Gamma$, (to respect the edge-condition (2.9)). We refer to [26] for the analysis of these boundary - finite element method.

b) Now, let us say some words on the energy question when dealing with the boundary integral equation method. We start with the classical Green formula :

$$\int_\Gamma D\varphi(x)\overline{\varphi}(x)d\sigma(x) = \int_{\Omega_R} (|\nabla u|^2 - |\omega u|^2)dx - \int_{\{|x|=R\}} \frac{\partial u}{\partial n}\,\overline{u}\,d\sigma \tag{4.3}$$

where u is the double layer potential (1.6), and Ω_R is $\Omega \cap \{ |x| < R \}$ with R sufficiently large. We recall also that the radiation condition (1.2) is equivalent to the following behavior of the scattered wave u at infinity :

$$\left\{ \begin{aligned} & u(x) = \frac{e^{i\omega r}}{r} A_0(\Theta)\left(1 + 0\left(\frac{1}{r}\right)\right) \\ & \frac{\partial u}{\partial r} = \frac{e^{i\omega r}}{r} i\omega A_0(\Theta)\left(1 + 0\left(\frac{1}{r}\right)\right) \end{aligned} \right. \tag{4.4}$$

($r = |x| \to +\infty$ and $\Theta = \frac{x}{r}$). The function A_0 is usually called the far field pattern of u. Reporting (4.4) in the last integral of (4.3), one gets :

$$\left\{ \begin{aligned} & \operatorname{Re} b(\varphi,\varphi) = \lim_{(R\to\infty)} \int_{\Omega_R} (|\nabla u|^2 - |\omega u|^2)\, dx \\ & -\operatorname{Im} b(\varphi,\varphi) = \omega \| A_0 \|^2_{L^2(S_2)} \end{aligned} \right. \tag{4.5}$$

Formula (4.5) links the imaginary part of the quadratic form b to the energy of u which is radiated to the infinity. This was used in [15] to explain why, in the discretization of the variational problem (3.8) (in the case of an obstacle of arbitrary shape), one can always solve the finite dimensional linear equation by a Cholesky decomposition of the matrix.

On the other hand, the real part of b can be interpreted as the difference of the potential and the kinetic energy of u. In our case, by (3.2) we have

$$\left\{ \begin{aligned} & \operatorname{Re} b(\varphi,\varphi) = \frac{1}{2} \int_{\{|\xi| > \omega\}} \sqrt{|\xi|^2 - \omega^2}\, |\hat{\varphi}_0(\xi)|^2\, d\xi \\ & -\operatorname{Im} b(\varphi,\varphi) = \frac{1}{2} \int_{\{|\xi| < \omega\}} \sqrt{\omega^2 - |\xi|^2}\, |\hat{\varphi}_0(\xi)|^2\, d\xi \end{aligned} \right.$$

Thus, Re b(φ,φ) is also positive contrarily to the case of a closed surface Γ. Actually, the coerciveness of b is no longer true in this general case, because of the existence of spurious frequencies.

We note also that the connexion between b (φ,φ) and the energy of u is more striking for the time - dependent waves. Indeed, in this case, the causality of waves allows us to extend the domain of the frequency variable to the half complex plan $\{\operatorname{Im}\omega > 0\}$, and then one can show that the last integral in (4.3) tends to zero when $R \to +\infty$, such that

$$\operatorname{Re} \langle D\varphi, i\omega\varphi \rangle = \int (|\nabla u|^2 + |i\omega u|^2)\, dx \tag{4.6}$$

and we have in the right hand side of (4.6) the usual energy of the acoustic wave $v=e^{iwt}u(x)$:

$$E(v) = \int (|\nabla u|^2+|\dot{u}|^2)dx = e^{-2\mathrm{Im}\omega t}\int (|\nabla u|^2+|i\omega u|^2)dx\ .$$

II - THE ELASTIC SCATTERING PROBLEM

5. The problem and its associated BIE

The geometric notations are that of part I.

The space IR^3 is now an isotropic, homogeneous elastic medium with the Lamé constants γ and μ. And we want to solve the problem of the elastic scattering by the crack Γ. With the convention of summation for the repeated indices, the equations are :

$$\sigma_{ij,j}(u) + \rho\,\omega^2 u_i = 0 \quad \text{in } \Omega\,,\ 1 \le i \le 3 \qquad (5.1)$$

$$\sigma_{i3} \equiv \sigma(u).n = g_i \quad \text{on } \Gamma\,,\ 1 \le i \le 3 \qquad (5.2)$$

where u is the displacement vector $u = (u_1, u_2, u_3)$ and $\sigma(u)$ the strain tensor :

$$\begin{cases} \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \\ \sigma_{ij} = \lambda\, \varepsilon_{kk}\, \delta_{ij} + 2\mu\, \varepsilon_{ij} \end{cases} \qquad (5.3)$$

$\omega > 0$ is the pulsation of the incident wave u^I and the vector function g in (5.2) is $-(\sigma(u^I).n)$, a known quantity.

We must add to the equations (5.1, 5.2) an outgoing radiation condition that fixes the decrease of the scattered wave u at infinity :

$$\begin{cases} |u| = 0\left(\frac{1}{r}\right) \\ |\sigma(u).n_r - iTu| = 0\left(\frac{1}{r^2}\right) \qquad r = |x| \to +\infty \end{cases} \qquad (5.4)$$

where T is the operator defined on the sphere of radius r by :

$$Tu = (\lambda + 2\mu)\, k_p\, (u.n)n + \mu\, k_s\, (u - (u.n)n) \qquad (5.5)$$

$(n = n_r = \frac{x}{r}\ )$.

In (5.5), the pressure and shearing wave numbers k_p, k_s are defined as usual :

$$\begin{cases} k_p = \dfrac{\omega}{c_p} = \omega\sqrt{\dfrac{\rho}{\lambda + 2\mu}} \\ k_s = \dfrac{\omega}{c_s} = \omega\sqrt{\dfrac{\rho}{\mu}} \end{cases} \tag{5.6}$$

The existence and uniqueness of the solution of problem (5.1, 5.2, 5.4) can be found in the Kupradze's book [20] . Here, as in the part I we are interested in the solution of the BIE for the crack opening displacement of u.
From the divergence theorem and the condition (5.2), any scattered wave can be represented by the well-known elastic double layer potential with the COD $\varphi = [u]$ as density :

$$u_j(x) = - \int_\Gamma \Sigma^i_{jk}\, n_k\, \varphi_i(y)\, d\sigma(y) \quad \left(x \in \Omega\,,\ 1 \leq j \leq 3\right) \tag{5.7}$$

where the tensor Σ is the stress Green tensor, which is related to the fundamental tensor G of the displacement equation (5.1) by :

$$\begin{aligned} \Sigma^i_{jk} &= \sigma_{jk}(G^i)(x-y) \\ &= (\lambda\, \delta_{jk}\, \delta_{lm} + \mu(\delta_{jl}\, \delta_{km} + \delta_{jm}\, \delta_{kl}))\, G^i_{l,m}(x-y) \\ &= \lambda\, G^i_{l,l}(x-y)\, \delta_{jk} + \mu(G^i_{j,k}(x-y) + G^i_{k,j}(x-y)) \end{aligned} \tag{5.8}$$

The fundamental tensor G is given in [20] :

$$\begin{cases} G^i_j(x) = \dfrac{1}{\mu}\left(\dfrac{1}{4\pi\,|x|}\, e^{ik_s|x|}\right)\delta_{ij} \\ \quad + \dfrac{1}{\rho\omega^2}\, \dfrac{\partial^2}{\partial x_i\, \partial_{x_j}}\left(\dfrac{e^{ik_s|x|} - e^{ik_p|x|}}{4\pi\,|x|}\right) \end{cases} \tag{5.9}$$

There are no confusion between the index i and the symbol $i = \sqrt{-1}$ in (5.9).
Now, as in the case of the scalar waves, the surface traction on Γ of the double layer potential (5.7) is an hyper-singular integral of φ. Let us denote by $D\varphi$ this integral :

$$D\varphi = \sigma(u)|_\Gamma \,.\, n \text{ with } u \text{ by } (5.7) \tag{5.10}$$

The scattering problem (5.1 - 5.2 - 5.4) is then equivalent to the vectorial BIE :

$$D\varphi = g \quad \text{on } \Gamma \tag{5.11}$$

As indicated in the introduction, our purpose is to prove the well-posedness of this BIE in a convenient functional framework, using a variational approach. We follow the same steps as in the scalar case, using the Fourier calculations of Bamberger [2].

6. The Fourier expression of the integral operator D

The following equivalent definition of D is obtained from the well-known properties of the elastic double layer potential :

$$D\varphi = \sigma(u).n|_\Gamma \tag{6.1}$$

where u is solution of (5.1), (5.4) and the following jump conditions on Γ :

$$\begin{cases}[\sigma(u).n] = 0 \\ [u] = \varphi\end{cases} \tag{6.2}$$

Let us denote again by (Q) the problem (5.1 - 5.4 - and 6.2).

By a partial Fourier transform with respect to $x' = (x_1, x_2)$, we will obtain the solution of (Q) by a limiting absorption argument, and then obtain the Fourier expression of D as in part I.

First, the transform of eq. (5.1) is :

$$A_2 \frac{\partial^2 \hat{u}}{\partial x_3^2}(\xi', x_3) + i A_1 \frac{\partial \hat{u}}{\partial x_3} + A_0 \hat{u} = 0 \tag{6.3}$$

where $\xi' = (\xi_1, \xi_2)$ is the dual variable of x' and

$$A_2 = \begin{bmatrix} c_s^2 & 0 & 0 \\ 0 & c_s^2 & 0 \\ 0 & 0 & c_p^2 \end{bmatrix} \tag{6.4.i}$$

$$A_1 = (c_p^2 - c_s^2) \begin{bmatrix} 0 & 0 & \xi_1 \\ 0 & 0 & \xi_2 \\ \xi_1 & \xi_2 & 0 \end{bmatrix} \tag{6.4.ii}$$

$$A_0 = (\omega^2 - c_s^2 |\xi|^2) I - (c_p^2 - c_s^2)\, \xi.\xi^T \tag{6.4.iii}$$

in (6.4iii) and henceforth, except in some precised places, we will write $\xi = (\xi_1, \xi_2, 0)^T$ and identify ξ with ξ'.

Replacing ω by $\omega + i\varepsilon$ $(\varepsilon > 0)$ in (6.4.iii), the solution of (6.2) in the half-space $\{x_3 > 0\}$, that does not explode when $x_3 \to +\infty$, can be obtained by the plane waves method (see [2]) :

$$\widehat{u}_\varepsilon(\xi,x_3) = \begin{bmatrix} -\xi_2 & \xi_1 Z_s^\varepsilon & \xi_1 \\ \xi_1 & \xi_2 Z_s^\varepsilon & \xi_2 \\ 0 & -|\xi|^2 & Z_p^\varepsilon \end{bmatrix} \begin{bmatrix} \widetilde{u}_1 & e^{ix_3 Z_s^\varepsilon} \\ \widetilde{u}_2 & e^{ix_3 Z_s^\varepsilon} \\ \widetilde{u}_3 & e^{ix_3 Z_p^\varepsilon} \end{bmatrix} \tag{6.5}$$

where $\widetilde{u}_1$ are constant, and the symbols z_s^ε, z_s^ε are defined by

$$\begin{cases} (Z_s^\varepsilon)^2 = \dfrac{(\omega + i\varepsilon)^2}{c_s^2} - |\xi|^2 \\ (Z_p^\varepsilon)^2 = \dfrac{(\omega + i\varepsilon)^2}{c_s^2} - |\xi|^2 \\ \operatorname{Im} Z_s^\varepsilon > 0 \quad , \quad \operatorname{Im} Z_p^\varepsilon > 0 \end{cases}$$

when $\varepsilon \to 0$, their limits are

$$Z_S = \begin{cases} \sqrt{k_S^2 - |\xi|^2} & \text{on } \{|\xi| < k_S\} \\ i\sqrt{|\xi|^2 - k_p^2} & \text{on } \{|\xi| \geq k_S\} \end{cases} \tag{6.6}$$

$$Z_p = \begin{cases} \sqrt{k_p^2 - |\xi|^2} & \text{on } \{|\xi| < k_p\} \\ i\sqrt{|\xi|^2 - k_p^2} & \text{on } \{|\xi| \geq k_p\} \end{cases} \tag{6.7}$$

Denote by M^ε the matrix in (6.5), the boundary value of $\hat{u}_\varepsilon$ is then $\widehat{u}_\varepsilon(\xi,0_+) = M^\varepsilon \widetilde{u}$, and we can write (6.5) as :

$$\widehat{u}_\varepsilon(\xi,x_3) = M^\varepsilon . \begin{bmatrix} e^{ix_3 z_s^\varepsilon} & 0 & 0 \\ 0 & e^{ix_3 z_s^\varepsilon} & 0 \\ 0 & 0 & e^{ix_3 z_p^\varepsilon} \end{bmatrix} (M^\varepsilon)^{-1}\, \widehat{u}_\varepsilon(\xi,0_+)$$

After some calculations, one gets :

$$\widehat{u}_\varepsilon(\xi,x_3) = E_\varepsilon(\xi,x_3)\, \widehat{u}_\varepsilon(\xi,0_+) \tag{6.8}$$

with

$$E_\varepsilon(\xi,x_3) = e_p^\varepsilon I + \frac{1}{P_\varepsilon(\xi)} (e_p^\varepsilon - e_s^\varepsilon)\, \xi_p^\varepsilon . (e_s^\varepsilon)^T \tag{6.9}$$

where

$$\begin{cases} e^{\varepsilon}_{\alpha} = e^{ix_3 Z^{\varepsilon}_{\alpha}} & \alpha = S, P \\ \xi^{\varepsilon}_{\alpha} = (\xi_1, \xi_2, Z^{\varepsilon}_{\alpha})^T & \alpha = S, P \\ P_{\varepsilon}(\xi) = Z^{\varepsilon}_s Z^{\varepsilon}_p + |\xi|^2 \end{cases} \tag{6.10}$$

we can now prove the following "limiting absorption principle" :

LEMMA 2 . *If* $u_\varepsilon(.\,, 0_+) = u_+ \in (H^{1/2}(\mathbb{R}^2))^3$ *the solution* u_ε *of (6.8) is in* $(H^1(\mathbb{R}^3_+))^3$*, and tends in the sense of* $(H^1_{loc}(\mathbb{R}^3_+))^3$*, when* $\varepsilon \to 0$*, to the function* u *which satisfies (5.1 , 5.4) in* $\mathbb{R}^3_+ = \{x_3 > 0\}$ *and the boundary condition* $u(., 0_+) = u_+$.
In Fourier variables with respect to (x_1, x_2)*, u is given by*

$$\hat{u}(\xi, x_3) = E(\xi, x_3)\, \hat{u}_+(\xi) \quad (x_3 > 0) \tag{6.11}$$

where E is as in (6.9) with ε *null and dropped.*

PROOF : To simplify the notations, we will drop the ε in all the below calculations. To avoid confusion, if suffices to remember that ω has now a positive imaginary part.
Consider the following orthonormal basis of $\mathbb{C}^3$:

$$d_1 = \frac{1}{|\xi|}\left(-\xi_2, \xi_1, 0\right)^T$$

$$d_2 = \frac{1}{\left(|\xi|^2 + |Z_p|^2\right)^{1/2}}\, \xi_p = \frac{\xi_p}{|\xi_p|}$$

$$d_3 = \frac{1}{|\xi|\left(|\xi|^2 + |Z_p|^2\right)^{1/2}}\, (\xi_1 \overline{Z}_p, \xi_2 \overline{Z}_p, -|\xi|^2)^T$$

Let $\hat{u}_+ = f_j\, d_j$ be the decomposition of $\hat{u}_+$ in this basis. From (6.7) and (6.8), one gets :

$$|\hat{u}(\xi, x_3)|^2 \leq |f_1|\, e^{-2 x_3 \operatorname{Im} Z_s} + |f_3|^2 e^{-2 x_3 \operatorname{Im} Z_s} + |f_2|^2 e^{-2 x_3 \operatorname{Im} Z_p}$$

$$+ |f_3|^2 \frac{|\overline{Z}_p - z_s|^2}{\left||\xi|^2 + z_p z_s\right|^2}\, |e^{i x_3 z_p} - e^{i x_3 z_s}|^2 \tag{6.12}$$

An analogous inequality can be easily derived for $\left|\frac{\partial \hat{u}}{\partial x_3}\right|^2$, and the lemma follows by similar calculations as in lemma 1. The only different term is the last term in (6.12). After integration for x_3, from 0 to x > 0 this can be handled with by an asymptotic development when $|\xi| \to +\infty$.
The formula (6.11) is clear ♦

We can now come back to the case of real (positive) ω. In the same manner as in th. 1, we have the following expression of D :

THEOREM 3. *The operator D defined by (6.1) can be written as :*

$$D\varphi = (R_\Gamma \circ T \circ P_\Gamma)\,\varphi \tag{6.13}$$

where R_Γ and P_Γ are defined in th.1, and T the pseudo-differential operator of order 1 on $\mathbb{R}^2$ *defined by :*

$$\widehat{T\psi}(\xi) = T(\xi)\,\hat{\psi}(\xi) \tag{6.14}$$

$$T(\xi) = \frac{-i\mu}{2(\omega^2/c_s^2)} \begin{bmatrix} \frac{P_R(\xi)}{Z_s} - \frac{\xi_2^2 P_c(\xi)}{Z_s} & \xi_1\xi_2 \frac{P_c(\xi)}{Z_s} & 0 \\ \xi_1\xi_2\frac{P_c(\xi)}{Z_s} & \frac{P_R(\xi)}{Z_s} - \frac{\xi_1^2 P_c(\xi)}{Z_s} & 0 \\ 0 & 0 & \frac{P_R(\xi)}{Z_p} \end{bmatrix} \tag{6.15}$$

where

$$P_c(\xi) = |\xi|^2 - 3Z_s^2 + 4Z_s Z_p \tag{6.16}$$

$$P_R(\xi) = \left(|\xi|^2 - Z_s^2\right)^2 + 4|\xi|^2 Z_s Z_p \tag{6.17}$$

In particular, D is a continuous operator from $\left(H_{00}^{1/2}(\Gamma)\right)^3$ *into* $\left(H_{00}^{1/2}(\Gamma)'\right)^3$.

PROOF : From (6.1) one gets after some easy but rather lengthy calculations :

$$\widehat{\sigma_+^3}(\xi) = \widehat{\sigma(u).n}(\xi, 0_+) = H^+(\xi)\,\hat{u}_+(\xi) \tag{6.18}$$

where

$$H_+(\xi) = \frac{i\mu}{P(\xi)} \begin{bmatrix} Z_p k_s^2 + \xi_2^2(Z_s - Z_p) & -\xi_1\xi_2(Z_s - Z_p) & \xi_1(k_s^2 - 2Z_s(Z_s - Z_p)) \\ \xi_1\xi_2(Z_s - Z_p & Z_p k_s^2 + \xi_1^2(Z_s - Z_p) & \xi_2(k_s^2 - 2Z_s(Z_s - Z_p)) \\ -\xi_1(k_s^2 - 2Z_s(Z_s - Z_p)) & -\xi_2(k_s^2 - 2Z_s(Z_s - Z_p)) & Z_s k_s^2 \end{bmatrix} \tag{6.19}$$

Similar formulas can be obtained for $x_3 < 0$ and the traces on $\{x_3 = 0_-\}$. Then, (6.13) results from the boundary conditions (6.2).

All these calculus are in Bamberger's work [2]. The reason of (6.13) is as in th.1.

Now, there are no more difficulty to check that each term in the matrix T is a symbol of order 1 in ξ , regular for $|\xi| > k_s$ ♦

Remarks 3.
a) It can be verified that

$$\det(H^+) = (i\mu)^3 \frac{P_R(\xi)\, k_s}{P(\xi)}$$

where P_R , given in (6.17), is the Rayleigh function, which involves in the research of surface in the half-space { $x_3 > 0$ } with the free surface boundary condition $\sigma .n = 0$. The formula (6.18) gives then another way to obtain the Rayleigh waves.
b) On the other hand,

$$\det T(\xi) = \left(\frac{-i\mu}{2}\right)^3 \frac{P_R^2(\xi)}{Z_p(\xi)\, k_s^4}$$

Thus, we find again that if $\left|\xi\right| = k_R = \frac{\omega}{c_R}$ where c_R is the Rayleigh wave velocity, the matrix $T(\xi)$ becomes singular. This expected result, however, is not contradictory to the coerciveness of operator D in $\left(H_{00}^{1/2}(\Gamma)\right)^3$, because a function of this space cannot be a Dirac distribution on $\left\{\left|\xi\right| = \frac{\omega}{c_R}\right\}$.
c/ The pseudo-differential character of the integral operator presented here is not new (see [8] for a general presentation of the theory). In the literature on integral equations, Wendland others used to prove Garding inequality for some first kind integral operators with this method. See the raferences of [30]. Stephan did the same in a crack problem of static elasticity in [29]. The main difference between our theorems 1 and 3 with the classical results is that we have in formulas (2.11) and (6.13) the *full* symbols (not only the principal ones) of the interested operators. Of course, this is due to the geometry of our problem. Meister and Speck in a similar context (a half- plan crack problem, [24]) used also this full symbol $T(\xi)$ for a Wiener-Hopf solution of the problem.

7. The variational solution of the BIE (5.11)
Like the theorem 2 in the scalar case, we have the following :

THEOREM 4. *The sesqui-linear form*

$$b(\varphi,\psi) := \langle D\varphi , \psi \rangle_\Gamma \tag{7.1}$$

can be written in Fourier variables as

$$b(\varphi,\psi) = \frac{1}{(2\pi)^2}\int_{IR^2}\left(T(\xi)\,\widehat{\varphi}_0(\xi)\,,\,\widehat{\psi}_0(\xi)\right)d\xi \qquad (7.2)$$

where (,) designates the scalar product in C^3. The following coerciveness estimate is satisfied for all $\varphi \in \left(H_{00}^{1/2}(\Gamma)\right)^3$:

$$|b(\varphi,\varphi)| \geq C\,\|\varphi\|^2_{(H_{00}^{1/2}(\Gamma))^3} \qquad (7.3)$$

where the constant C is dependent of ω.

PROOF : The equality (7.2) results from (6.13) in the same manner as it was shown for ((3.2) in th.2.
Now, let us write the matrix $T(\xi)$ as :

$$T(\xi) = \frac{-i\mu}{2k_s^2}\left\{\frac{P_R(\xi)}{Z_s\,Z_p}\begin{pmatrix} Z_p & 0 & 0 \\ 0 & Z_p & 0 \\ 0 & 0 & Z_s \end{pmatrix} - \frac{P_c(\xi)}{Z_s}\begin{pmatrix} \xi_2^2 & -\xi_1\xi_2 & 0 \\ -\xi_1\xi_2 & \xi_1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right\}$$

From this, it is immediat that the following orthonormal basis of C^3 is actually composed of eigenvectors of T :

$$v_1 = \frac{1}{|\xi|}\begin{pmatrix} -\xi_2 \\ \xi_1 \\ 0 \end{pmatrix} \quad v_2 = \frac{1}{|\xi|}\begin{pmatrix} \xi_1 \\ \xi_2 \\ 0 \end{pmatrix} \quad v_3 = \frac{1}{|\xi|}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad (7.4)$$

More precisely, designing by ν the constant $\left(-\frac{i\mu}{k_s^2}\right)$, we have :

$$\left\{\begin{array}{l} T(\xi)\,v_1 = \nu\,k_s^2\,Z_s(\xi)\,v_1 \\ T(\xi)\,v_2 = \nu\,(P_R(\xi)/Z_s(\xi))\,v_2 \\ T(\xi)\,v_3 = \nu\,(P_R(\xi)/Z_p(\xi))\,v_3 \end{array}\right\} \qquad (7.5)$$

Then, if $\widehat{\varphi}_0 = \widehat{\eta}_i\,v_i$ is the decomposition of $\widehat{\varphi}_0$ on the basis (v_i), and according to (6.5), (6.6), (6.17), we can decompose $b(\varphi,\varphi)$ into the following parts :

$$b(\varphi,\varphi) = \nu(I_1 + I_2 + i\,I_3 + i\,I_4 + i\,I_5)$$

with

$$I_1 = \int_{\{|\xi| \le k_p\}} \left\{ k_s^2 \sqrt{k_s^2 - |\xi|^2}\, |\hat{\eta}_1(\xi)|^2 + \frac{P_R(\xi)}{\sqrt{k_s^2 - |\xi|^2}} |\hat{\eta}_2(\xi)|^2 + \frac{P_R(\xi)}{\sqrt{k_p^2 - |\xi|^2}} |\hat{\eta}_3(\xi)|^2 \right\} d\xi$$

$$I_2 = \int_{\{k_p < |\xi| \le k_s\}} \left\{ k_s^2 \sqrt{k_s^2 - |\xi|^2}\, |\hat{\eta}_1(\xi)|^2 + \frac{\left(2|\xi|^2 - k_s^2\right)^2}{\sqrt{k_s^2 - |\xi|^2}} |\hat{\eta}_2(\xi)|^2 + 4|\xi|^2 \sqrt{k_s^2 - |\xi|^2}\, |\hat{\eta}_3(\xi)|^2 \right\} d\xi$$

$$I_3 = \int_{\{k_p < |\xi| \le k_s\}} \left\{ 4|\xi|^2 \sqrt{|\xi|^2 - k_p^2}\, |\hat{\eta}_2(\xi)|^2 - \frac{\left(2|\xi|^2 - k_s^2\right)^2}{\sqrt{|\xi|^2 - k_p^2}} |\hat{\eta}_3(\xi)|^2 \right\} d\xi$$

$$I_4 = \int_{\{k_s < |\xi| \le k_R\}} \left\{ k_s^2 \sqrt{|\xi|^2 - k_s^2}\, |\hat{\eta}_1(\xi)|^2 - \frac{P_R(\xi)}{\sqrt{|\xi|^2 - k_s^2}} |\hat{\eta}_2(\xi)|^2 - \frac{P_R(\xi)}{\sqrt{|\xi|^2 - k_p^2}} |\hat{\eta}_3(\xi)|^2 \right\} d\xi$$

and finally, I_5 is the integral on $\{|\xi| > k_R\}$ of the same (real) function as in I_4. The distinction between these two last integrals is that on $\{|\xi| > k_R = \frac{\omega}{c_R}\}$ the Rayleigh function is negative, so that the integral function in I_5 is sum of three positive functions. Actually, we recall (cf.[1]), that P_R can be factorised by

$$P_R(\xi) = (|\xi|^2 - k_R^2)\, Q_R(|\xi|) \tag{7.6}$$

where Q_R is a $\mathcal{C}^\infty$ function of $|\xi|$, which satisfies the following estimates :

$$\begin{cases} 0 < m_* \le |Q_R(|\xi|)| \le m^* < +\infty \\ Q_R(|\xi|) = -2\dfrac{\lambda + \mu}{\lambda + 2\mu} k_s^2 \left(1 + 0\left(\dfrac{1}{|\xi|^2}\right)\right) \quad (|\xi| \to +\infty) \end{cases} \tag{7.7}$$

From this, and from (6.17), it is clear that Q_R is real, non null in $\{|\xi| > k_s\}$.
Now, the difference with the scalar case is that the integrals I_3 and I_4 contribute to Im B with non definite signs, and then prevent us from obtaining a direct lower bound of $|b|$. So, we proceed to prove (7.3) with a contradiction argument.

Suppose that (7.3) is not true. Then there exists a sequence (φ^m) in $\left(H_{00}^{1/2}(\Gamma)\right)^3$ such that

$$\begin{cases} \|\varphi^m\|^2_{\left(H_{00}^{1/2}(\Gamma)\right)^3} = 1 \\ b_m = b(\varphi^m, \varphi^m) \to 0 \quad m \to +\infty \end{cases} \tag{7.8}$$

From the real part of b_m, it follows that

$$\lim_{m\to\infty} I_1^m = \lim_{m\to\infty} I_2^m = 0$$

On the other hand, by the orthogonality of (v_i), we have

$$|\hat{\eta}^m(\xi)| = |\hat{\varphi}_0^m(\xi)| \quad (\forall \xi \neq 0),$$

such that (3.6) is satisfied by $\hat{\eta}(\xi)$ and consequently

$$\lim_{m\to\infty} \|\hat{\eta}^m\|_{(L^2(|\xi|<k_s))^3} = \lim \|\hat{\varphi}_0^m\|_{(L^2(|\xi|<k_s))^3} = 0 \tag{7.9}$$

Now, $H_{00}^{1/2}(\Gamma)$ is imbedded compactly into $L^2(\Gamma)$, we can choose from (φ^m) a subsequence, also denoted by (φ^m), which converges to $(\hat{\varphi}_0)$ in $L^2(\Gamma)$. Then (7.8) implies that $\hat{\varphi}_0(\xi) = 0$ for almost all ξ in $\{|\xi| < k_s\}$. But as the Fourier transform of a function of compact support, $\hat{\varphi}_0$ is an analytic function of ξ. It is then identically null. It follows immediatly that the subsequences (I_3^m) and (I_4^m) tend also to zero, and the same is true for (I_5^m).

Applying to these integrals the same calculations as in the proof of theorem 2, we get finally

$$\lim_{m\to\infty} \int_{\{|\xi|>k_R\}} (1+|\xi|^2)^{1/2} |\hat{\eta}^m(\xi)|^2 \, d\xi = 0$$

and the contradiction with the assumption (7.8) is obtained.

The theorem is proved ♦

COROLLARY :*For all* $g \in \left[\left(H_{00}^{1/2}(\Gamma)\right)'\right]^3$, *the vectorial BIE (5.11) has an unique solution* $\varphi \in \left[\left(H_{00}^{1/2}(\Gamma)\right)\right]^3$, *which is also unique solution of the variational problem :*

$$b(\varphi,\psi) = \langle g,\psi\rangle \quad \forall\, \psi \in \left[\left(H_{00}^{1/2}(\Gamma)\right)\right]^3 \tag{7.10}$$

We conclude now this section with some remarks about the variational problem (7.1). First, repeating the arguments of section 4, it is easy to show that the bilinear form $b(\varphi, \psi)$ can be written with integrals of only weakly singular kernels.
Moreover, it is clear from the expression of $T(\xi)$ that problem (7.1) is actually decoupled into two smaller and independent variational problems, with respectively the tangential and normal parts of φ as unknowns. Remarkably enough, the problem for the tangential part $\varphi' = (\varphi_1 , \varphi_2)$ turns out to be itself decoupled into two independant problems with the unknowns η_1, η_2 which Fourier transforms are the components of $\widehat{\varphi'}$ on the basis (v_1, v_2).
Indeed, from the above discussions, it is easy to verify that $\hat{\eta}_1, \hat{\eta}_2$ are resp. solution of the following variational equations:

$$\int_{R^2} Z_S \widehat{\eta_1} \cdot \overline{\widehat{\psi_1}} d\xi = \int_{R^2} \widehat{\gamma_1} \cdot \overline{\widehat{\psi_1}} d\xi \qquad \forall \psi_1 \tag{7.11}$$

and

$$\int_{R^2} \frac{P_R}{Z_S} \widehat{\eta_2} \cdot \overline{\widehat{\psi_2}} d\xi = \int_{R^2} \widehat{\gamma_2} \cdot \overline{\widehat{\psi_2}} d\xi \qquad \forall \psi_2 \tag{7.12}$$

where γ_1 and γ_2 are functions of the data g, defined by

$$\begin{cases} \Delta \gamma_1 = \operatorname{curl} g' & \text{on } \Gamma \\ \gamma_1 = 0 & \text{on } \partial\Gamma \end{cases} \tag{7.13}$$

and

$$\begin{cases} \Delta \gamma_2 = \operatorname{div} g' & \text{on } \Gamma \\ \gamma_2 = 0 & \text{on } \partial\Gamma \end{cases} \tag{7.14}$$

with $g'=(g_1,g_2)$.
Once the problems (7.11-7.14) solved, we recover φ_1, φ_2 by

$$\begin{cases} \varphi_1 = \operatorname{curl} \eta' \\ \varphi_2 = \operatorname{div} \eta' \end{cases} \tag{7.15}$$

Finally, we would like to point out that our Fourier method is clearly applicable to the two-dimensional scattering problem by a rectilinear crack.

CONCLUSIONS

We have shown in this paper that the natural BIE for the COD of a flat crack of arbitrary shape is actually well-posed in suitable functional spaces. Moreover, a variational treatment of the involved BIE, not only permits to prove this result but also reveals to be an efficient method to circumvent the hyper-singularity of its kernel.

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