Integr Equat Oper Th 15 (1992) 0378-620X/92/030378-11\$1.50+0.20/0

INTERPOLATION PROBLEMS, EXTENSIONS OF SYMMETRIC OPERATORS AND REPRODUCING KERNEL SPACES II

(Missing Section 3)

Daniel Alpay, Piet Bruinsma, Aad Dijksma, Henk de Snoo

By an oversight on the part of the authors this section was not included in the paper previously published in Integral Equations Operator Theory, volume 14/4 (1991), 466–500.

3. Interpolation via reproducing kernels

The theory of reproducing kernel spaces of holomorphic functions given in Sections 1 and 2 will now be applied to the interpolation problem (IP). This approach to solve interpolation problems was initiated by Dym in [Dy1,2]. As mentioned in the Introduction, the method developed in this paper has several points of contact with Dym's work.

We build from the data of the interpolation problem (IP) the following finite dimensional, resolvent invariant space $\mathfrak{M}$ of rational functions:

$$\mathfrak{M} = \{ ({W \atop V})(Z-\ell)^{-1}c \mid c \in \mathbb{C}^r \},$$

where r, the $n\times r$ matrices V,W and the $r\times r$ matrix Z are all associated with the data of (IP) and defined in the Introduction. In order to apply the results of Section 2 to our interpolation problem, we connect the notations used there, cf. (2.3)–(2.6), with the ones used in the formulation of (IP) by setting the $2n\times 1$ vectors c_{jq} equal to

$$c_{jq} = \begin{pmatrix} W_{jq} \\ V_{jq} \end{pmatrix}, \quad V_{jq} = \mathcal{K}_j^{(q)}(w_j)^*/q!, \quad W_{jq} = \mathcal{L}_j^{(q)}(w_j)^*/q!, \quad 1 \le j \le m, \ 0 \le q \le r_j,$$

so that C is the $2n\times r$ matrix

$$C = (c_{10} \vdots c_{11} \vdots \ldots \vdots c_{1r_1} \vdots \ \ldots \vdots c_{m0} \vdots \ldots \vdots c_{mr_m}) = ({W \atop V}).$$

Thus the linear space $\mathfrak{M}$ can be written as

$$\mathfrak{M} = \text{l.s.}\{ F_{jq}(\ell) \mid j = 1,2,\ldots,m, \ q = 0,1,\ldots,r_j \},$$

where $F_{jq}(\ell)$, the q-th element of the j-th chain, is given by

$$(3.1) \quad F_{jq}(\ell) = \sum_{h=0}^{q} (\ell - \bar{w}_j)^{-q-1+h} c_{jh}.$$

From the formula

$$\mathcal{F}(\ell) = \big(F_{10}(\ell):F_{11}(\ell):\ldots:F_{1r_1}(\ell):\ldots:F_{m0}(\ell):\ldots:F_{mr_m}(\ell) \big) = -C(Z-\ell)^{-1}$$

it easily follows that $\mathfrak{M}$ is resolvent invariant. Except when explicitly stated otherwise, we fix in this section the signature matrix J and set it equal to

$$J = \begin{pmatrix} 0 & -iI_n \\ iI_n & 0 \end{pmatrix}.$$

Note that now the Lyapunov equation (0.6) can be rewritten as

$$\mathbb{P}Z - Z^*\mathbb{P} = -C^*(iJ)C,$$

which agrees with (2.7).

The space $\mathfrak{M}$ can be used to provide a test to check whether a Nevanlinna pair $(M(\ell),N(\ell))$ is or is not a solution of the interpolation problem (IP). To formulate this test we associate with each ordered pair $(M(\ell),N(\ell))$ of matrix functions satisfying (2.14) the linear mapping $\tau = \tau_{M,N}$ from the space $\mathfrak{M}$ to some linear space of functions defined by

$$(3.2) \quad (\tau F)(\ell) = (-M^{\#}(\ell) : N^{\#}(\ell))F(\ell). \quad F \in \mathfrak{M},$$

cf. (2.17). If τ maps $\mathfrak{M}$ into the space $\mathfrak{L}(M,N)$, then the same is true for each ordered pair equivalent to $(M(\ell),N(\ell))$ and we shall say that τ associated with the Nevanlinna pair $(M(\ell),N(\ell))$ takes $\mathfrak{M}$ into $\mathfrak{L}(M,N)$, cf. the remarks after Theorem 2.5. The test is stated in the first part of the next theorem. For the definition of the $r \times r$ matrix $\mathbb{P}_{M,N}$ where $(M(\ell),N(\ell)) \in \mathbf{N}^n$, appearing in the second part of the theorem, we refer to the Introduction, formula (0.8).

THEOREM 3.1. *(i) The Nevanlinna pair $(M(\ell),N(\ell))$ is a solution of the interpolation problem* (IP) *if and only if the mapping τ takes $\mathfrak{M}$ into $\mathfrak{L}(M,N)$. (ii) If the Nevanlinna pair $(M(\ell),N(\ell))$ is a solution of the interpolation problem* (IP), *then*

$$(3.3) \quad [\tau\mathcal{F},\tau\mathcal{F}]_{\mathfrak{L}(M,N)} = \mathbb{P}_{M,N},$$

i.e., in terms of the matrix components $[\tau F_{jq},\tau F_{ip}]_{\mathfrak{L}(M,N)} = (\mathbb{P}_{M,N})^{pq}_{ij}$.

To prove the theorem we need the following technical lemma, which we also apply in the proof of Theorem 4.1 in Section 4.

LEMMA 3.2. *Let $(M(\ell),N(\ell))$ be an ordered pair of holomorphic matrix functions on $\mathbb{C}\backslash\mathbb{R}$ that satisfies* (2.14) *and is a solution of the interpolation problem* (IP). *Then there exist unique vectors $e_{jq} \in \mathbb{C}^n$, $j=1,2,\ldots,m$, $q=0,1,\ldots,r_j$, such that the coefficients c_{jh} of $F_{jq}(\ell)$ in* (3.1) *are given by*

$$(3.4) \quad c_{jh} = \sum_{t=0}^{h} \Big((D_\ell)^{h-t} \begin{pmatrix} N(\ell) \\ M(\ell) \end{pmatrix} \Big) \Big|_{\ell=\bar{w}_j} e_{jt}, \qquad j=1,2,\ldots,m, \ h=0,1,\ldots,r_j,$$

and then we have

(3.5) $\quad (\tau F_{jq})(\ell) = \sum_{t=0}^{q} \left((D_{\bar{\lambda}})^{q-t} \mathsf{L}_{M,N}(\ell,\lambda) \right) |_{\lambda=w_j} e_{jt}.$

If, in addition, the ordered pair satisfies the relation $\ell M(\ell)+N(\ell)=I$, then the vectors e_{jh} are given by

(3.6) $\quad e_{jt} = (D_{\bar{\lambda}})^{t} (\lambda \mathcal{K}_j(\lambda) + \mathcal{L}_j(\lambda))^{*} |_{\lambda=w_j}$

and hence,

(3.7) $\quad (\tau F_{jq})(\ell) = (D_{\bar{\lambda}})^{q} \left(\mathsf{L}_{M,N}(\ell,\lambda)(\lambda \mathcal{K}_j(\lambda) + \mathcal{L}_j(\lambda))^{*} \right) |_{\lambda=w_j}.$

Proof. For any $N=0,1,\ldots$ we associate with the $2n\times n$ matrix function $\mathcal{X}(\ell) = \begin{pmatrix} N(\ell) \\ M(\ell) \end{pmatrix}$ the two $(N+1)\times(N+1)$ block matrix functions $A_N(\ell)$ and $B_N(\ell)$, whose ij-th blocks are the $n\times 2n$, $2n\times n$ matrix functions, defined by

$$(A_N(\ell))_{ij} = ((D_\ell)^{i-j} \mathcal{X}^{\#}(\ell))J, \quad i\ge j, \qquad (A_N(\ell))_{ij}=0, \quad i<j.$$

$$(B_N(\ell))_{ij} = (D_\ell)^{i-j} \mathcal{X}(\ell), \qquad i\ge j, \qquad (B_N(\ell))_{ij}=0, \quad i<j,$$

respectively, where we recall the notation $(D_\ell)^j = (1/j!)(d/d\ell)^j$, $j\ge 0$. The symmetry condition in (2.14) can be rewritten as $\mathcal{X}^{\#}(\ell)J\mathcal{X}(\ell)=0$, $\ell\in\mathbb{C}\backslash\mathbb{R}$, and after differentiating this identity k times, we obtain

$$\sum_{j=0}^{k} \left((D_\ell)^{k-j} \mathcal{X}^{\#}(\ell)\right) J\, (D_\ell)^j \mathcal{X}(\ell) = 0, \qquad k=0,1,\ldots,N,$$

which implies that $A_N(\ell)B_N(\ell)=0$, $\ell\in\mathbb{C}\backslash\mathbb{R}$. It follows that $\Re(B_N(\ell))\subset\nu(A_N(\ell))$, $\ell\in\mathbb{C}\backslash\mathbb{R}$. To show that

(3.8) $\quad \nu(A_N(\ell)) = \Re(B_N(\ell)), \qquad \ell\in\mathbb{C}\backslash\mathbb{R},$

we use a simple dimension argument. By the nondegeneracy condition in (2.14), $\operatorname{rank}\mathcal{X}(\ell) = \operatorname{rank}\mathcal{X}(\ell)^* = n$, from which one gets that the $(N+1)n\times 2(N+1)n$ matrix $A_N(\ell)$ and the $2(N+1)n\times(N+1)n$ matrix $B_N(\ell)$ have full rank $(N+1)n$. Hence,

$$\dim\nu(A_N(\ell)) = 2(N+1)n - \dim\Re(A_N(\ell)) = (N+1)n = \dim\Re(B_N(\ell))$$

and now (3.8) easily follows. Next we rewrite the interpolation problem (IP) at w_i as

$$\sum_{k=0}^{p} (D_\ell)^k ((\mathcal{L}_i(\ell) : \mathcal{K}_i(\ell))J((D_\ell)^{p-k}\mathcal{X}(\ell))|_{\ell=w_i} = 0, \quad 0\le p\le r_i,$$

and by taking adjoints we obtain

$$A_{r_i}(w_i)\begin{pmatrix} c_{i0} \\ \vdots \\ c_{ir_i} \end{pmatrix} = \left(\sum_{k=0}^{p} (((D_\ell)^{p-k}\mathcal{X}(\ell))|_{\ell=w_i})^* J\, c_{ik} \right)_{p=0}^{r_i} = 0.$$

It follows from (3.8) and the fact that $B_{r_i}(\bar{w}_i)$ has full rank, that there exist unique vectors $e_{ip}\in\mathbb{C}^n$, $i=1,2,\dots,m$, $p=0,1,\dots,r_i$, such that

$$\begin{pmatrix} c_{i0} \\ \vdots \\ c_{ir_i} \end{pmatrix} = B_{r_i}(\bar{w}_i) \begin{pmatrix} e_{i0} \\ \vdots \\ e_{ir_i} \end{pmatrix}$$

or, equivalently, such that c_{ih} is as in (3.4). Substituting (3.4) into formula (3.1) we find that

$$F_{jq}(\ell) = \sum_{t=0}^{q} \left((D_{\bar\lambda})^{q-t} (\ell-\bar\lambda)^{-1} \mathcal{X}(\bar\lambda) \right) |_{\lambda=w_j} e_{jt}$$

and hence,

$$(\tau F_{jq})(\ell) = (-\mathcal{M}^\sharp(\ell) : \mathcal{N}^\sharp(\ell)) F_{jq}(\ell) = i\mathcal{X}^\sharp(\ell) J F_{jq}(\ell) =$$

$$= i \sum_{t=0}^{q} \left((D_{\bar\lambda})^{q-t} (\ell-\bar\lambda)^{-1} \mathcal{X}^\sharp(\ell) J \mathcal{X}(\bar\lambda) \right) |_{\lambda=w_j} e_{jt}.$$

Since $\mathsf{L}_{\mathcal{M},\mathcal{N}}(\ell,\lambda) = (\ell-\bar\lambda)^{-1} \mathcal{X}^\sharp(\ell)(iJ)\mathcal{X}(\bar\lambda)$, this proves (3.5). Formula (3.6) follows from the following calculations:

$$(D_\ell)^t (\ell \mathcal{K}_j(\ell) + \mathcal{L}_j(\ell))|_{\ell=w_j} = (D_\ell)^t ((\mathcal{L}_j(\ell) : \mathcal{K}_j(\ell)) \begin{pmatrix} I \\ \ell \end{pmatrix})|_{\ell=w_j} = c_{jt}^* \begin{pmatrix} I \\ w_j \end{pmatrix} + c_{j,t-1}^* \begin{pmatrix} 0 \\ I \end{pmatrix} =$$

$$= \sum_{r=0}^{t} e_{jr}^* \left((D_\ell)^{t-r} \mathcal{X}(\ell)|_{\ell=\bar{w}_j} \right)^* \begin{pmatrix} I \\ w_j \end{pmatrix} + \sum_{r=0}^{t-1} e_{jr}^* \left((D_\ell)^{t-1-r} \mathcal{X}(\ell)|_{\ell=\bar{w}_j} \right)^* \begin{pmatrix} 0 \\ I \end{pmatrix} =$$

$$= \sum_{r=0}^{t} e_{jr}^* \left((D_\ell)^{t-r} (I:\ell)\mathcal{X}(\ell)|_{\ell=\bar{w}_j} \right)^* = e_{jt}^*.$$

It is only in this last equality that we have used the condition that $(I:\ell)\mathcal{X}(\ell)=I$. Finally, (3.7) follows from formula (3.5) by substituting for e_{jt} the righthand side of formula (3.6). This completes the proof of the lemma.

Proof of Theorem 3.1. In order to prove the theorem we consider representatives $\mathcal{M}(\ell),\mathcal{N}(\ell)$ of the Nevanlinna pair that are holomorphic matrix functions in $\mathbb{C}\backslash\mathbb{R}$. We first establish the sufficiency in (i). Suppose that the map τ takes $\mathfrak{M}$ into $\mathfrak{L}(\mathcal{M},\mathcal{N})$. Then by Proposition 2.4 the function $(\tau F_{jq})(\ell)$ is holomorphic on $\mathbb{C}\backslash\mathbb{R}$. Since

(3.9) $$(\tau F_{jq})(\ell) = (-\mathcal{M}^\sharp(\ell) : \mathcal{N}^\sharp(\ell)) F_{jq}(\ell) =$$

$$= (\ell - \bar{w}_j)^{-q-1} \left(\mathcal{N}^\sharp(\ell) \sum_{h=0}^{q} (\ell-\bar{w}_j)^h V_{jh} - \mathcal{M}^\sharp(\ell) \sum_{h=0}^{q} (\ell-\bar{w}_j)^h W_{jh} \right)$$

for $j=1,2,\dots,m$, $q=0,1,\dots,r_j$, we find by integrating both sides over a small circle around $\ell=\bar{w}_j$ and using Cauchy's formula, that

$$0 = (D_\ell)^q \left(\mathcal{N}^\sharp(\ell)\mathcal{K}_j^\sharp(\ell) - \mathcal{M}^\sharp(\ell)\mathcal{L}_j^\sharp(\ell) \right) |_{\ell=\bar{w}_j} =$$

$$= \left((D_\ell)^q \left(\mathcal{K}_j(\ell)\mathcal{N}(\ell) - \mathcal{L}_j(\ell)\mathcal{M}(\ell) \right) |_{\ell=w_j} \right)^*, \quad q=0,1,\ldots,r_j.$$

It follows that $\mathcal{K}_j(\ell)\mathcal{N}(\ell)-\mathcal{L}_j(\ell)\mathcal{M}(\ell)$ has a zero of order r_j+1 at $\bar{w}_j$, which shows that the pair $(\mathcal{M}(\ell),\mathcal{N}(\ell))$ is a solution of the interpolation problem (IP). If conversely, the pair $(\mathcal{M}(\ell),\mathcal{N}(\ell))$ is a solution of (IP) then by Lemma 3.2 and Proposition 1.1 we have that $\tau F_{jq}\in\mathfrak{L}(\mathcal{M},\mathcal{N})$. Since the F_{jq}'s form a basis for $\mathfrak{M}$, this proves the necessity in (*i*). We now come to the proof of (*ii*). If the representing pair $(\mathcal{M}(\ell),\mathcal{N}(\ell))$ has the stronger property that $\ell\mathcal{M}(\ell)+\mathcal{N}(\ell)=I$, $\ell\in\mathbb{C}\backslash\mathbb{R}$, then (3.3) can easily be deduced from (3.7), the reproducing property of the kernel $\mathsf{L}_{\mathcal{M},\mathcal{N}}(\ell,\lambda)$ and the definition of $\mathbb{P}_{\mathcal{M},\mathcal{N}}$. The general case follows from the observation that both sides of this equality are independent of the chosen representative of the Nevanlinna pair $(\mathcal{M}(\ell),\mathcal{N}(\ell))$. This completes the proof of the theorem.

Part (*i*) of Theorem 3.1 can be considered as a weak version of Theorem 2.5(*i*). The spaces in Theorem 2.5 have a Hilbert space inner product and the mapping τ there is a contraction. In Theorem 3.1(*i*) the space $\mathfrak{M}$ has not been provided with a Hilbert space structure and the test to determine whether or not a Nevanlinna pair is a solution of the interpolation problem (IP) only involves the range of the mapping τ. An inner product on $\mathfrak{M}$ is suggested by part (*ii*) of Theorem 3.1, namely the one which makes τ an isometry between Hilbert spaces. If $(\mathcal{M}(\ell),\mathcal{N}(\ell))\in\mathbf{N}^n$ is a solution of (IP) then we know from the Appendix of Part I that $\mathbb{P}=\mathbb{P}_{\mathcal{M},\mathcal{N}}$ is a Pick matrix, i.e., a hermitian solution of the Lyapunov equation (cf. the proof of Proposition 1.7 of Part I, but the reader should note that this Appendix contains an independent study of the Lyapunov equation and does not rely on the extension theory in part I). If moreover, $\mathbb{P}$ is positive, then there exists a uniquely determined inner product on $\mathfrak{M}$ which makes τ an isometry between $\mathfrak{M}$ and $\mathfrak{L}(\mathcal{M},\mathcal{N})$. In this case, $\mathcal{F}(\ell)$ is a basis of $\mathfrak{M}$ and $\mathbb{P}$ is the Gram matrix associated with $\mathcal{F}(\ell)$ and therefore a positive solution of the Lyapunov equation. Now, a converse also holds and by turning things around, that is, by starting with a positive solution $\mathbb{P}$ of the Lyapunov equation we obtain a parametrization of all solutions $(\mathcal{M}(\ell),\mathcal{N}(\ell))\in\mathbf{N}^n$ of the interpolation problem (IP) that satisfy the condition $\mathbb{P}_{\mathcal{M},\mathcal{N}}=\mathbb{P}$.

THEOREM 3.3. *Let* $\mathbb{P}$ *be a positive solution of the Lyapunov equation* (0.6) *and let the space* $\mathfrak{M}$ *be endowed with an inner product* $[.,.]_{\mathfrak{M}}$ *such that* $\mathbb{P}$ *is the Gram matrix associated with the basis* $\mathcal{F}(\ell)$, *i.e.,* $\mathbb{P}=[\mathcal{F},\mathcal{F}]_{\mathfrak{M}}$. *Then:*

(*i*) $\mathfrak{M}=\mathfrak{H}(\Theta)$, *where the J inner function* $\Theta(\ell)$ *is given by*

$$(3.10)\quad \Theta(\ell)=I+\begin{pmatrix} W \\ V \end{pmatrix}(Z-\ell)^{-1}\mathbb{P}^{-1}(W^*:V^*)\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

(*ii*) $\Theta(\ell)$ *has the property that the formula*

$$(3.11)\quad \begin{pmatrix} N(\ell) \\ M(\ell) \end{pmatrix} = \Theta(\ell) \begin{pmatrix} \mathcal{B}(\ell) \\ \mathcal{A}(\ell) \end{pmatrix}$$

establishes a one to one correspondence between all the solutions $(M(\ell),N(\ell))\in\mathbf{N}^n$ *of the interpolation problem* (IP) *satisfying the supplementary condition* $\mathbb{P}_{M,N}=\mathbb{P}$ *and all Nevanlinna pairs* $(\mathcal{A}(\ell),\mathcal{B}(\ell))$.

A J inner function $\Theta(\ell)$ with the property mentioned in part (ii) of Theorem 3.3 is called a solution matrix for the interpolation problem (IP) associated with the positive Pick matrix $\mathbb{P}$. Note that the solution matrix $\Theta(\ell)$ defined by (3.10) coincides with the solution matrix $U_\infty(\ell)$ given in Part I, Theorem 3.7. We shall come back to this in Section 4. We recall that if the interpolation points w_i are such that $w_i\neq\bar{w}_j$ for all indices i,j, then the Lyapunov equation has a unique solution $\mathbb{P}$ and it automatically coincides with $\mathbb{P}_{M,N}$. In this case all solutions of the problem (IP) satisfy the relation $\mathbb{P}_{M,N}=\mathbb{P}$. Only in case some interpolation points occur in conjugate pairs, this relation implies that the Nevanlinna pair $(M(\ell),N(\ell))$ satisfies certain additional interpolation requirements at these points.

Proof of Theorem 3.3. Part (i) follows from Theorem 2.2, and the representation (3.10) of $\Theta(\ell)$ comes from formula (2.8). To prove (ii), first assume that $(M(\ell),N(\ell))\in\mathbf{N}^n$ is a solution of the interpolation problem (IP) with $\mathbb{P}_{M,N}=\mathbb{P}$. Then by Theorem 3.1 the mapping τ is an isometry from $\mathfrak{M}=\mathfrak{H}(\Theta)$ into $\mathfrak{L}(M,N)$ and by Theorem 2.5(i) (3.11) is valid for some Nevanlinna pair $(\mathcal{A}(\ell),\mathcal{B}(\ell))$. Since $\Theta(\ell)$ is J inner this pair is uniquely determined. For the proof in the other direction, assume that $(M(\ell),N(\ell))$ is given by (3.11) for some $(\mathcal{A}(\ell),\mathcal{B}(\ell))\in\mathbf{N}^n$. Then $(M(\ell),N(\ell))\in\mathbf{N}^n$ and again by Theorem 2.5(i) $\tau=\tau_{M,N}$ maps $\mathfrak{M}=\mathfrak{H}(\Theta)$ into $\mathfrak{L}(M,N)$ and is a contraction. On account of Theorem 3.1(i) the former conclusion implies that $(M(\ell),N(\ell))$ is a solution of the interpolation problem (IP), whereas the latter implies that $\mathbb{P}_{M,N}\leq\mathbb{P}$. Since both members of this inequality are solutions of the Lyapunov equation and $\sigma(Z)\cap\mathbb{R}=\emptyset$, the diagonal entries of $\mathbb{P}_{M,N}$ and $\mathbb{P}$ are the same. Hence the matrix $\mathbb{P}-\mathbb{P}_{M,N}$ is nonnegative and has a zero main diagonal which implies that $\mathbb{P}_{M,N}=\mathbb{P}$. This completes the proof of the theorem.

We have seen in the proof of Theorem 3.3(ii) that the mapping τ is an isometry. Therefore on account of Theorem 2.5(ii) the following result holds, to which we return in Section 4.

COROLLARY 3.4. *If* $\Theta(\ell)$ *is given by* (3.10) *and* (3.11) *is valid, then the space* $\mathfrak{L}(M,N)$ *has the orthogonal decomposition*

$$\mathfrak{L}(M,N)=(-M^\#:N^\#)\mathfrak{H}(\Theta)\oplus\mathfrak{L}(\mathcal{A},\mathcal{B}).$$

Up till now we have been concerned with the solvability of the interpolation problem (IP) when the corresponding Lyapunov equation possesses positive solutions. For this case

Theorem 3.3 provides for an enumeration of all solutions of the interpolation problem over the Nevanlinna class $\mathbf{N}^n$. If it is given that the corresponding Lyapunov equation possesses a nonnegative solution, then with the techniques we have developed in this paper we can prove, see Theorem 3.5 below, that the interpolation problem (IP) also has a solution. In this case it is also possible to obtain an enumeration of all solutions, see [Br].

THEOREM 3.5. *The interpolation problem* (IP) *has a solution if and only if the Lyapunov equation* (0.6) *has a nonnegative solution* $\mathbb{P}\geq 0$.

To prove this theorem we again state and prove a proposition concerning perturbed Lyapunov equations. Since this result may be of independent interest, we provide a complete proof. In the proposition the matrix J is an arbitrary signature matrix.

PROPOSITION 3.6. *Let Z be as in the Introduction, and suppose that the pair (C,Z) is observable. Assume that the Lyapunov equation*

$$\mathbb{P}Z-Z^*\mathbb{P}=-C^*(iJ)C$$

has a nonnegative solution $\mathbb{P}$. Then there exists a sequence of matrices $C(\varepsilon)$, $\varepsilon>0$, such that (i) $C(\varepsilon)\to C$ as $\varepsilon\to 0$, and (ii) the corresponding Lyapunov equations

$$\mathbb{P}Z-Z^*\mathbb{P}=-C^*(\varepsilon)(iJ)C(\varepsilon),$$

possess positive solutions $\mathbb{P}(\varepsilon)>0$ such that $\mathbb{P}(\varepsilon)\to\mathbb{P}$ as $\varepsilon\to 0$.

Proof. Without loss of generality we may and shall assume that $Z=\operatorname{diag}(Z_+,Z_-)$, where the spectrum $\sigma(Z_+)\subset\mathbb{C}^+$ and the spectrum $\sigma(Z_-)\subset\mathbb{C}^-$. We decompose the matrix C as $C=(C_+:C_-)$ such that the number of columns of $C_\pm$ is equal to the size of $Z_\pm$. Let $U(\varepsilon)$, $\varepsilon>0$ and small, be a family of invertible matrices, such that

$$U(\varepsilon)JU(\varepsilon)^*<J,\qquad U(\varepsilon)\to I \text{ as } \varepsilon\to 0.$$

For example, if J is represented as $J=\mathcal{E}^*\operatorname{diag}(I_p,-I_q)\mathcal{E}$, where $\mathcal{E}$ is a constant unitary matrix, take

$$U(\varepsilon)=(1+\varepsilon^2)^{-\frac{1}{2}}\mathcal{E}^*\operatorname{diag}((1-\varepsilon)I_p,(1+\varepsilon)I_q)\mathcal{E}.$$

Next we define $C_\pm(\varepsilon)$ by

$$C_+(\varepsilon)=JU(\varepsilon)^{-1}JC_+,\qquad C_-(\varepsilon)=U(\varepsilon)^*C_-.$$

Clearly $C_\pm(\varepsilon)\to C_\pm$ as $\varepsilon\to 0$. We consider the equation

$$\mathbb{P}Z-Z^*\mathbb{P}=-i(C_+(\varepsilon):C_-(\varepsilon))^*J(C_+(\varepsilon):C_-(\varepsilon))=-iC^*JC-i\operatorname{diag}(\Delta_+(\varepsilon),\Delta_-(\varepsilon)),\tag{3.12}$$

where we have put

$$\Delta_+(\varepsilon)=C_+^* JU(\varepsilon)^{-*}(J-U(\varepsilon)^*JU(\varepsilon))U(\varepsilon)^{-1}JC_+,\quad \Delta_-(\varepsilon)=C_-^*(U(\varepsilon)JU(\varepsilon)^*-J)C_-.$$

To obtain a solution of the equation (3.12), it suffices to look for solutions of the equations

(3.13) $\mathbb{P}_+(\varepsilon)Z_+-Z_+^*\mathbb{P}_+(\varepsilon)=-i\Delta_+(\varepsilon)$,

(3.14) $\mathbb{P}_-(\varepsilon)Z_--Z_-^*\mathbb{P}_-(\varepsilon)=-i\Delta_-(\varepsilon)$.

For, if $\mathbb{P}_+(\varepsilon)$ is a solution of (3.13) and $\mathbb{P}_-(\varepsilon)$ one of (3.14) then $\mathbb{P}(\varepsilon)=\mathbb{P}+\mathrm{diag}(\mathbb{P}_+(\varepsilon),\mathbb{P}_-(\varepsilon))$ is a solution of (3.12). As $\sigma(Z_+)\subset\mathbb{C}^+$ and $\sigma(Z_-)\subset\mathbb{C}^-$ these equations have unique, and hence, hermitian solutions $\mathbb{P}_+(\varepsilon)$ and $\mathbb{P}_-(\varepsilon)$, respectively. Since the observability of (C,Z) implies that of (C_+,Z_+) and of (C_-,Z_-), we may apply Theorem 3.3 of [G] to each of the equations (3.13) and (3.14) and conclude that the spectra $\sigma(\mathbb{P}_\pm(\varepsilon))$ are contained in the right half plane $\{z\in\mathbb{C}\,|\,\mathrm{Re}z>0\}$ and hence that $\mathbb{P}_\pm(\varepsilon)>0$, $\varepsilon>0$. In the Appendix of Part I it is shown that the entries of the solutions $\mathbb{P}_\pm(\varepsilon)$ can be expressed as linear combinations of the entries of $\Delta_\pm(\varepsilon)$, cf. the proof of Proposition A.1 in Part I. Since $\Delta_\pm(\varepsilon)\to 0$, we see that $\mathbb{P}_\pm(\varepsilon)\to 0$, as $\varepsilon\to 0$. From these results for $\mathbb{P}_\pm(\varepsilon)$ it now easily follows that $\mathbb{P}(\varepsilon)=\mathbb{P}+\mathrm{diag}(\mathbb{P}_+(\varepsilon),\mathbb{P}_-(\varepsilon))$ is a positive solution of the given Lyapunov equation with the property that $\mathbb{P}(\varepsilon)\to\mathbb{P}$, as $\varepsilon\to 0$, which proves the proposition.

From the proof of Proposition 3.6 and the construction of $\mathbb{P}(\varepsilon)$ it follows that the rate of convergence in which $\mathbb{P}(\varepsilon)$ tends to $\mathbb{P}$ is at least equal to the rate of convergence in which $C(\varepsilon)$ tends to C as $\varepsilon\to 0$.

Proof of Theorem 3.5. If the Nevanlinna pair $(M(\ell),N(\ell))$ is a solution of the interpolation problem (IP), then by (3.3) $\mathbb{P}_{M,N}\geq 0$, since $\mathfrak{L}(M,N)$ is a Hilbert space. As already observed before, $\mathbb{P}_{M,N}$ satisfies the Lyapunov equation and thus the Lyapunov equation has a nonnegative solution. To prove the converse, we assume that the Lyapunov equation (0.6) has a nonnegative solution $\mathbb{P}$. At the interpolation points w_j, $j=1,2,\ldots,m$, we perturb the interpolation data V and W in such a way that the perturbed interpolation data $V(\varepsilon)$ and $W(\varepsilon)$, $\varepsilon>0$, give rise to a perturbed Lyapunov equation

$$\mathbb{P}Z-Z^*\mathbb{P}=V(\varepsilon)^*W(\varepsilon)-W(\varepsilon)^*V(\varepsilon),$$

which for every $\varepsilon>0$ posseses a positive solution $\mathbb{P}(\varepsilon)$, and that $V(\varepsilon)$, $W(\varepsilon)$ and $\mathbb{P}(\varepsilon)$ have the limiting behaviour

$$W(\varepsilon)\to W,\ V(\varepsilon)\to V,\ \mathbb{P}(\varepsilon)\to\mathbb{P},\ \text{as } \varepsilon\to 0.$$

By Proposition 3.6 such a perturbation is possible. Indeed, without loss of generality we may assume that the data satisfy the following rank condition:

For each $j=1,2,\ldots,m$ the set $\{\begin{pmatrix}V_{i0}\\W_{i0}\end{pmatrix}\,|\,w_i=w_j\}$ is linearly independent in $\mathbb{C}^{2n}$.

which implies that the pair (C,Z) is observable. That this may be assumed without loss of generality can be seen as follows. If the rank condition is not satisfied for some j, we reorder the equations in the interpolation problem (IP) such that only the first s equations, say, involve the interpolation point w_j and such that $r_1 \geq r_2 \geq \ldots \geq r_s$. Let t be the first index such that the set

$$\left\{ \begin{pmatrix} V_{i0} \\ W_{i0} \end{pmatrix} \mid i = 1,2,\ldots,t \right\}$$

is linearly dependent. Then in the equations

$$(\mathcal{K}_t(\ell)\mathcal{N}(\ell))^{(p)}|_{\ell=w_j} = (\mathcal{L}_t(\ell)\mathcal{M}(\ell))^{(p)}|_{\ell=w_j},\ 0 \leq p \leq r_t,$$

the terms involving the highest derivatives of $\mathcal{M}(\ell)$ and $\mathcal{N}(\ell)$ can be eliminated and then the resulting equations involve derivatives of at most order r_t-1. In this way the number of equations is reduced by at least one without changing the appearance of the interpolation problem, i.e., the new equations are also of the form (IP). If necessary one can start all over again to obtain another reduction of the number of equations. Since there are only finitely many equations, this reduction procedure must terminate and this occurs precisely when the rank condition is met.

With the perturbed data we build the space $\mathfrak{M}(\varepsilon)$ with basis $\mathcal{F}_\varepsilon(\ell) = -\begin{pmatrix} W(\varepsilon) \\ V(\varepsilon) \end{pmatrix}(Z-\ell)^{-1}$ and we endow $\mathfrak{M}(\varepsilon)$ with the inner product determined by $[\mathcal{F}_\varepsilon,\mathcal{F}_\varepsilon] = \mathbb{P}(\varepsilon)$. Then, by Theorem 3.3, there exists a J unitary rational function $\Theta_\varepsilon(\ell)$, such that $\mathfrak{M}(\varepsilon) = \mathfrak{H}(\Theta_\varepsilon)$ and such that for each Nevanlinna pair $(\mathcal{A}(\ell),\mathcal{B}(\ell))$ the pair $(\mathcal{M}_\varepsilon(\ell),\mathcal{N}_\varepsilon(\ell))$ defined by

$$\begin{pmatrix} \mathcal{N}_\varepsilon(\ell) \\ \mathcal{M}_\varepsilon(\ell) \end{pmatrix} = \Theta_\varepsilon(\ell)\begin{pmatrix} \mathcal{B}(\ell) \\ \mathcal{A}(\ell) \end{pmatrix} \tag{3.15}$$

is a solution of the perturbed interpolation problem $(\mathrm{IP})_\varepsilon$, i.e., (IP) with V and W replaced by $V(\varepsilon)$ and $W(\varepsilon)$. Fix a parameter $(\mathcal{A}(\ell),\mathcal{B}(\ell)) \in \mathbf{N}^n$ and define the Nevanlinna pair $(\mathcal{M}_\varepsilon(\ell),\mathcal{N}_\varepsilon(\ell))$ by (3.15). Let the ordered pair $(\hat{\mathcal{M}}_\varepsilon(\ell),\hat{\mathcal{N}}_\varepsilon(\ell))$ be the representative of $(\mathcal{M}_\varepsilon(\ell),\mathcal{N}_\varepsilon(\ell))$ with the property that $\ell\hat{\mathcal{M}}_\varepsilon(\ell)+\hat{\mathcal{N}}_\varepsilon(\ell) = I$. Then on account of (2.16) $-\hat{\mathcal{M}}_\varepsilon(\ell)$ is a Nevanlinna function and hence holomorphic on $\mathbb{C}\backslash\mathbb{R}$, and

$$\|\hat{\mathcal{M}}_\varepsilon(\ell)\| \leq 1/|\mathrm{Im}\ell|,\quad \ell \in \mathbb{C}\backslash\mathbb{R}.$$

By Vitali's theorem it follows that, for some sequence $\varepsilon(n) > 0$ with $\varepsilon(n) \to 0$ as $n \to \infty$, the limits $\lim_{n\to\infty} \hat{\mathcal{M}}_{\varepsilon(n)}(\ell)$ and $\lim_{n\to\infty} \hat{\mathcal{N}}_{\varepsilon(n)}(\ell)$ exist uniformly on compact subsets on $\mathbb{C}\backslash\mathbb{R}$. We write

$$\mathcal{M}(\ell) = \lim_{n\to\infty} \hat{\mathcal{M}}_{\varepsilon(n)}(\ell),\qquad \mathcal{N}(\ell) = \lim_{n\to\infty} \hat{\mathcal{N}}_{\varepsilon(n)}(\ell).$$

The matrix functions $\mathcal{M}(\ell)$ and $\mathcal{N}(\ell)$ are holomorphic on $\mathbb{C}\backslash\mathbb{R}$ and it is easy to verify that the pair $(\mathcal{M}(\ell),\mathcal{N}(\ell))$ is a representative of a Nevanlinna pair with the property that $\ell\mathcal{M}(\ell)+\mathcal{N}(\ell) = I$. Moreover, for all $p \in \mathbb{N}$ we have that

$$M^{(p)}(\ell)=\lim_{n\to\infty}\hat{M}^{(p)}_{\varepsilon(n)}(\ell),\qquad N^{(p)}(\ell)=\lim_{n\to\infty}\hat{N}^{(p)}_{\varepsilon(n)}(\ell)$$

and therefore, since for each $\varepsilon>0$ the pair $(\hat{M}_\varepsilon(\ell),\hat{N}_\varepsilon(\ell))$ solves the perturbed interpolation problem $(\mathrm{IP})_\varepsilon$, the Nevanlinna pair $(M(\ell),N(\ell))$ is a solution of the interpolation problem (IP). This completes the proof of the theorem.

If $(M(\ell),N(\ell))\in\mathbf{N}^n$ is a function, then $\det M(\ell)\not\equiv 0$ on $\mathbb{C}^+$ and on $\mathbb{C}^-$, $Q(\ell)=N(\ell)M(\ell)^{-1}$ is a Nevanlinna function, and the pair $(M(\ell),N(\ell))$ is a solution of the interpolation problem (IP) if and only if $Q(\ell)$ satisfies

$$\mathcal{K}_i(\ell)Q(\ell)-\mathcal{L}_i(\ell)=o((\ell-w_i)^{r_i}),\quad i=1,2,\ldots,m.$$

Similarly, if $(M(\ell),N(\ell))\in\mathbf{N}^n$ is the inverse of a function, then $\det N(\ell)\not\equiv 0$ on $\mathbb{C}^+$ and on $\mathbb{C}^-$, $P(\ell)=N(\ell)M(\ell)^{-1}$ is a Nevanlinna function, and the pair $(M(\ell),N(\ell))$ is a solution of the interpolation problem (IP) if and only if $P(\ell)$ satisfies

$$\mathcal{K}_i(\ell)-\mathcal{L}_i(\ell)P(\ell)=o((\ell-w_i)^{r_i}),\quad i=1,2,\ldots,m.$$

In part I, Corollary 3.5, we formulated necessary and sufficient conditions which ensure that all solutions of (IP) are functions or inverses of functions. In the proof we gave there we made use of the correspondence between the selfadjoint extensions of a symmetric relation associated with the data of the problem (IP) and its solutions. Now we repeat the corollary and give another proof based on the method developed in the present part of the paper.

COROLLARY 3.7. *Suppose that the Lyapunov equation* (0.6) *has a nonnegative solution. Then* (*i*) *all solutions of the interpolation problem* (IP) *are functions if and only if* $\operatorname{rank}V=n$, *and* (*ii*) *all solutions of the interpolation problem* (IP) *are inverses of functions if and only if* $\operatorname{rank}W=n$.

If $\mathbb{P}>0$ the proof is a straightforward consequence of Proposition A.6 and Theorem A.9 in the Appendix. We show this for item (*ii*). In (3.11) $N(\ell)$ is invertible for all $(\mathcal{A}(\ell),\mathcal{B}(\ell))\in\mathbf{N}^n$ if and only if

$$\mathrm{l.s.}\{(I:0)f(\mu)\mid f\in\mathfrak{H}(\Theta)\}=\mathbb{C}^n$$

for one (and hence for all) $\mu\in\Omega(\Theta)\backslash\mathbb{R}$. Since

$$\mathcal{F}(\ell)=-\begin{pmatrix}W\\V\end{pmatrix}(Z-\ell)^{-1}$$

is a basis of $\mathfrak{H}(\Theta)$ and thus $(I:0)\mathcal{F}(\ell)=-W(Z-\ell)^{-1}$, we see that $N(\ell)$ is invertible if and only if $\operatorname{rank}W=n$.

Proof of Corollary 3.7. (*i*) Assume that $(M(\ell),N(\ell))\in\mathbf{N}^n$ is a solution of the interpolation problem (IP) and that it is not a function. Denote by $(\hat{M}(\ell),\hat{N}(\ell))$ the representative of the Nevanlinna pair $(M(\ell),N(\ell))$ which has the property that $\hat{M}(\ell)+\ell\hat{N}(\ell)=I$.

Then there exists a vector $c\in\mathbb{C}^n$, $c\neq 0$, such that $\hat{M}(\ell)c=0$ or, equivalently, $\ell\hat{N}(\ell)c=c$ for all $\ell\in\mathbb{C}\backslash\mathbb{R}$, and it follows from formula (3.9) (with $M(\ell),N(\ell)$ replaced by $\hat{M}(\ell),\hat{N}(\ell)$, respectively) that

$$\ell c^*(\tau F_{jq})(\ell)=\ell c^*\sum_{h=0}^{q}(\ell-\bar{w}_j)^{h-q-1}\hat{N}^{\#}(\ell)V_{jh}=$$

$$=c^*\sum_{h=0}^{q}(\ell-\bar{w}_j)^{h-q-1}V_{jh},\quad j=1,2,\dots,m,\quad q=0,1,\dots,r_j.$$

The lefthand side is holomorphic on $\mathbb{C}\backslash\mathbb{R}$ and hence $c^*V_{jh}=0$ for all indices $j=1,2,\dots,m$, $h=0,1,\dots,r_j$, in other words $c^*V=0$. Thus we have shown that if $\operatorname{rank}V=n$ then all solutions of the interpolation problem (IP) are functions. To prove the converse. we assume that there exists a vector $c\in\mathbb{C}^n$, $c\neq 0$, such that $c^*V=0$. Let $w\in\mathbb{C}\backslash\mathbb{R}$ be a point which does not coincide with any of the interpolation points of (IP) or their complex conjugates. Consider the new interpolation problem $(\mathrm{IP})_0$, which consists of finding all solutions $(M(\ell),N(\ell))$ of (IP) which also satisfy the condition $c^*M(w)=0$. We denote the matrices for this new problem that correspond to the matrices V, W and Z introduced in connection with the interpolation problem (IP) by V_0, W_0 and Z_0, respectively. Then $V_0=(V:0)$, $W_0=(W:c)$ and $Z_0=\operatorname{diag}(Z,w)$ and it follows that if $\mathbb{P}$ is a nonnegative solution of the Lyapunov equation (0.6) associated with (IP), then $\mathbb{P}_0=\operatorname{diag}(\mathbb{P},0)$ is a nonnegative solution of the Lyapunov equation associated with the interpolation problem $(\mathrm{IP})_0$. By Theorem 3.5 the interpolation problem $(\mathrm{IP})_0$ has at least one solution $(M(\ell),N(\ell))$, say. This Nevanlinna pair is also a solution of the problem (IP) but is not a function, since $c^*\hat{M}(w)=0$ and hence $c^*\hat{M}(\ell)=0$ for all $\ell\in\mathbb{C}\backslash\mathbb{R}$. Thus we have shown that if all solutions are functions of the interpolation problem (IP) then $\operatorname{rank}V=n$. (*ii*) This part follows from (*i*) by taking advantage of the symmetry of the problem (IP): interchange $\mathcal{K}(\ell)$ and $-\mathcal{L}(\ell)$, and $N(\ell)$ and $-M(\ell)$ and use the fact that $(M(\ell),N(\ell))\in\mathbf{N}^n$ if and only if $(N(\ell),-M(\ell))\in\mathbf{N}^n$, cf. the proof of Part I, Corollary 3.5 (*ii*).

D. Alpay
Department of Theoretical Mathematics
The Weizmann Institute of Science
Rehovot 76100
Israel

Present address:
Department of Mathematics
Ben-Gurion University of the Negev
Beersheva
Israel

P. Bruinsma, A. Dijksma, H.S.V. de Snoo
Department of Mathematics
University of Groningen
Postbox 800
9700 AV Groningen
The Netherlands

MSC 1991, Primary 47A57; Secondary, 47B25,46E22