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The formal Teichmüller space for stable Mumford curves

By

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The purpose of this paper is to construct the formal Teichmüller space $\hat{T}_g$. $\hat{T}_g$ is a formal scheme which is a moduli space for uniformized stable Mumford curves.

A (non-singular) Mumford curve over a complete local ring $\mathcal{O}$ is a stable curve C over $\mathcal{O}$ (in the sense of [2]) with non-singular generic and totally degenerated special fibre. Mumford showed in [10] that such curves can be uniformized by an action of the free non-commutative group F_g on $\mathbb{P}^1$. This can easily be generalized to any stable curves C with totally degenerated special fibre (stable Mumford curves) by embedding C into a nonsingular deformation. Instead of the action of F_g on $\mathbb{P}^1$ one gets an action of F_g on a tree of projective lines, a so-called F_g-tree (see [9]). The formal Teichmüller space thus can be thought of as a formal neighbourhood of the subspace corresponding to totally degenerated curves in the moduli space $B^{F_g}_{\dot{F}_g}$ of F_g-trees as constructed in [9].

Unfortunately, $B^{F_g}_{\dot{F}_g}$ is only a pro-scheme, not a scheme, so we have to work in a different way:

F_g-trees are classified by the set of all cross-ratios of the attracting fixed points of any four elements of $\dot{F}_g = \{\text{primitive elements of } F_g\}$, thus $B^{F_g}_{\dot{F}_g}$ is naturally embedded in $\mathbb{P}^V_1$, $V = \{(v_1, \ldots, v_4) \mid v_i \neq v_j \, \forall i \neq j, \, v_i \in \dot{F}_g\}$. By covering $\mathbb{P}^V_1$ by copies of $\mathbb{A}^V_1$ we get a covering of $B^{F_g}_{\dot{F}_g}$ by affine schemes $U_c = \operatorname{Spec} A_c$, $c \in C = \{\text{maps } V \to \{\pm 1\}\}$. Let $Y_c = \operatorname{Spec} \bar{A}_c$ be the subspaces of U_c corresponding to totally degenerated curves, and $\hat{Y}_c = \operatorname{Spf} \hat{A}_c$ the completion of U_c along Y_c.

We then have to glue the formal schemes $\hat{Y}_c$ over "their intersections" $\hat{Y}_{c,d} = (U_c \cap U_d$ completed along $Y_c \cap Y_d)$.

The key point in this paper is to show that this is possible, i.e. that the maps $\hat{Y}_{c,d} \to \hat{Y}_c$ are open immersions. This is done as follows:

After introducing the basic objects and notions in Section 1 we show in Section 2 – Section 4 that $\bar{A}_c$ is a finitely generated $\mathbb{Z}$-Algebra (Theorem 4.7): In Section 2 a tree T is constructed corresponding to a point of Y_c, and it is shown that F_g acts on T and T/F_g is finite. In Section 3 it is shown that $\bar{A}_c$ is essentially of finite type over $\mathbb{Z}$, and this fact combined with the results of Section 2 is used to get a finite covering of each Y_c by schemes $Y_{c,\varepsilon}$, for which it is possible to show that they are of finite type over $\mathbb{Z}$ (Section 4).

In Section 5 it is shown that S_c/S_c^2 is finitely generated, where S_c is the ideal sheaf of Y_c in U_c. This (together with the results of Section 2–Section 4) yields the existence of the formal Teichmüller space $\hat{T}_g$ (Theorem 5.3). $\hat{T}_g$ then is a moduli space for

$$\{\textit{stable Mumford curves} + \textit{basis of the fundamental group}\}/\textit{Inn}\, F_g$$

(Theorem 5.10).

Finally in Section 6 the formal Teichmüller space is related to the rigid analytic Teichmüller space $\mathcal{T}_g$ (see [4], [7], [11]) through the fact that $\mathcal{T}_g$ is an open subspace of the rigid analytic space $\hat{T}_g^{an}$ associated with $\hat{T}_g$.

1. Basic concepts. Denote by F_g the free non-commutative group of rank g, let $\dot{F}_g$ be the subset of primitive elements (i.e. $\dot{F}_g = \{\gamma \in F_g \mid \gamma \neq \delta^n \forall \delta \in F_g,\ n \geqq 2\}$) and $V := \{v = (v_1, v_2, v_3, v_4) \mid v_i \in \dot{F}_g,\ v_i \neq v_j \forall i \neq j\}$.

Note that $\operatorname{Aut} F_g$ acts on $\dot{F}_g$ and hence on V. Let F_g act as inner automorphisms. Let $\varepsilon = \{\varepsilon_1, \ldots, \varepsilon_g\}$ be a base of F_g. Then each $\gamma \in F_g$ has a unique representation as a reduced word in $\varepsilon_{\pm 1}, \ldots, \varepsilon_{\pm g}$, where we define $\varepsilon_{-i} := \varepsilon_i^{-1}$.

1.1. Definition. Let $\gamma \in F_g$.
$1(\gamma) := 1_\varepsilon(\gamma) :=$ length of $\gamma :=$ number of letters in the reduced word associated with γ.
$\operatorname{st}(\gamma) := \operatorname{st}_\varepsilon(\gamma) :=$ first letter in the reduced word associated with γ. If $\gamma = \alpha\beta \in F_g$, $\operatorname{st}(\beta) \neq \operatorname{st}(\alpha^{-1})$, we write $\gamma = \alpha \cdot \beta$.

For later use we proof some Lemma's on F_g:

1.2. Lemma. $\gamma \in \dot{F}_g, \gamma = \alpha \cdot \beta \cdot \alpha^{-1}$, *$\beta$ cyclic reduced (i.e.* $\operatorname{st}(\beta) \neq \operatorname{st}(\beta^{-1})$), $\mu \in F_g$. *Then* $\operatorname{st}(\mu\gamma\mu^{-1}) \neq \operatorname{st}(\mu) \Rightarrow \beta = \beta_1 \cdot \beta_2, \mu^{-1} = \alpha \cdot \beta^n \cdot \beta_1, n \geqq 0$, *or* $\alpha = \mu^{-1} \cdot \alpha'$.

Proof.
(i) $\alpha = \mathrm{id}$, i.e. $\gamma = \beta$. Then we can find β_1 (possibly $= \mathrm{id}$), $\beta_2, \eta \in F_g$ s. th. $\beta = \beta_1 \cdot \beta_2$, $\mu = \eta \cdot \beta_1^{-1}$, $\operatorname{st}(\eta^{-}1) \neq \operatorname{st}(\beta_2)$ if $\beta_2 \neq \mathrm{id}$.
(a) $\beta_2 \neq \mathrm{id}$. Then $\mu\beta\mu^{-1} = \eta\beta_1^{-1}\beta_1\beta_2\beta_1\eta^{-1} = \eta\beta_1\eta^{-1} = \eta \cdot \beta_2 \cdot \beta_1 \cdot \eta^{-1} \cdot \operatorname{st}(\mu\beta\mu^{-1}) \neq \operatorname{st}(\mu)$ $\Rightarrow \eta = \mathrm{id}$.
(b) $\beta_2 = \mathrm{id}$. Then $\mu\beta\mu^{-1} = \eta\beta\eta^{-1}$, so by induction on $1(\eta)$ and using (a) we find $\operatorname{st}(\mu\beta\mu^{-1}) \neq \operatorname{st}(\mu) \Rightarrow \mu^{-1} = \beta^n \cdot \beta_1', \beta = \beta_1' \cdot \beta_2'$.
(ii) $\alpha \neq \mathrm{id}$. Let $\lambda = \mu\alpha$. Then $\mu\gamma\mu^{-1} = \lambda\beta\lambda^{-1}$.
(a) $\operatorname{st}(\lambda) = \operatorname{st}(\mu)$. Then $\operatorname{st}(\mu\gamma\mu^{-1}) \neq \operatorname{st}(\mu) \Rightarrow \operatorname{st}(\lambda) \Rightarrow_{(i)} \lambda^{-1} = \beta^n \cdot \beta_1 \Rightarrow \mu^{-1} = \alpha \cdot \beta^n \cdot \beta_1$.
(b) $\operatorname{st}(\lambda) \neq \operatorname{st}(\mu)$: Then $\alpha = \mu^{-1} \cdot \alpha'$. □

1.3. Lemma. *Let $\alpha, \beta, \gamma \in \dot{F}_g$ be pairwise distinct. Then there exists a unique $\mu \in F_g$ s.th.* $\#\{\operatorname{st}(\mu\alpha\mu^{-1}), \operatorname{st}(\mu\beta\mu^{-1}), \operatorname{st}(\mu\gamma\mu^{-1})\} = 3$.

Proof. Uniqueness follows directly from Lemma 1.2. We proof the existence of μ by induction on $1(\alpha, \beta, \gamma) := 1(\alpha) + 1(\beta) + 1(\gamma)$:

For $1(\alpha, \beta, \gamma) = 3$ there is nothing to prove. Let now $1(\alpha, \beta, \gamma) = n$, and suppose the lemma is true for all $1 < n$.

Conjugation with the greatest common starting sequence of α, β, γ leads (without increasing length) to w.l.o.g. $\operatorname{st}(\alpha) = \operatorname{st}(\beta) \neq \operatorname{st}(\gamma)$ or all three different.

If α and β are not cyclic reduced, a suitable conjugation decreases length. So let β be cyclic reduced.

Let $\alpha = \eta \cdot \alpha' \cdot \eta^{-1}$, α' cyclic reduced.

(i) $\beta = \eta \cdot \beta'$. Then conjugation by η leads to α and β cyclic reduced.

(a) $\alpha = \zeta \cdot \alpha'$, $\beta = \zeta \cdot \beta' \neq \mathrm{id}$, $\# \{\mathrm{st}(\alpha'), \mathrm{st}(\beta'), \mathrm{st}(\zeta')\} = 3$. Then $\mu = \zeta^{-1}$.

(b) $\beta = \alpha^k \cdot \zeta \cdot \beta'$, $\alpha = \zeta \cdot \alpha'$, $\# \{\mathrm{st}(\alpha'), \mathrm{st}(\beta'), \mathrm{st}(\zeta^{-1})\} = 3$. Then $\mu = \zeta^{-1} \alpha^{-k}$.

(ii) $\eta = \beta^k \cdot \beta' \cdot \eta'$, $\beta = \beta' \cdot \beta''$, $\# \{\mathrm{st}(\eta'), \mathrm{st}(\beta''), \mathrm{st}(\beta'^{-1})\} = 3$. Then $\mu = (\beta^k \beta')^{-1}$. □

1.4. Lemma. $\alpha, \beta \in \dot{F}_g$, $\alpha \neq \beta$. *Then there exist only finitely many* $\mu \in F_g$ *s. th.* $\mathrm{st}(\mu^{-1} \alpha \mu) \neq \mathrm{st}(\mu^{-1})$, $\mathrm{st}(\mu^{-1} \beta \mu) \neq \mathrm{st}(\mu^{-1})$.

Proof. Since there can be only finitely many μ's with $\alpha = \mu \cdot \alpha' \cdot \mu^{-1}$ or $\beta = \mu \cdot \beta' \cdot \mu^{-1}$, suppose we had infinitely many $\mu = \alpha' \cdot \zeta^n \cdot \zeta_1 = \beta' \cdot \eta^m \cdot \eta_1$, with $\alpha = \alpha' \cdot \zeta_1 \cdot \zeta_2 \cdot \alpha'^{-1}$, $\beta = \beta' \cdot \eta_1 \cdot \eta_2 \cdot \beta'^{-1}$. Then there would also exist infinitely many such μ's with ζ_1, η_1 fixed. Let μ_0 be one of them, and define for each $\mu\, v := \mu_0^{-1} \mu = \zeta_1^{-1} \zeta^{n-n_0} \zeta_1 = \eta_1^{-1} \cdot \eta^{m-m_0} \cdot \eta_1$, or $\zeta_1^{-1} \zeta^k \zeta_1 = \eta_1^{-1} \eta^l \eta_1$ with $k > 0$.

Let $x = \zeta_1^{-1} \zeta \zeta_1 = \zeta_2 \zeta_1$, $y = (\eta_2 \eta_1)^{\mathrm{sign}\, l}$, $r := |l|$.

Then $x^k = y^r$, x, y cyclic reduced and primitive. A simple calculation shows that this implies $x = y$, so $\zeta_2 \cdot \zeta_1 = (\eta_2 \cdot \eta_1)^{\pm 1}$.

(i) $\zeta_2 \cdot \zeta_1 = (\eta_2 \cdot \eta_1)^{-1}$. Then $v = (\eta_2 \cdot \eta_1)^{n-n_0} = (\eta_2 \cdot \eta_1)^{m-m_0} = (\zeta_2 \cdot \zeta_1)^{m_0-m} \Rightarrow n + m = n_0 + m_0$, so the number of such v's is finite.

(ii) $\zeta_2 \cdot \zeta_1 = \eta_2 \cdot \eta_1 =: w$, $\mu = \alpha' \cdot \zeta_1 \cdot w^n = \beta' \cdot \eta_1 \cdot w^m \cdot$ w.l.o.g. $k = n - m \geqq 0$.
Then $\beta = \beta' \eta_1 \eta_2 \beta'^{-1} = \beta' \eta_1 w (\beta' \eta_1)^{-1} = \alpha' \zeta_1 w^k w (\alpha' \zeta_1 w^k)^{-1} = \alpha' \zeta_1 w \xi_1'^{-1} \alpha'^{-1} = \alpha$. □

Next we want to introduce some rings and their spectra, which are the building blocks for the spaces we want to construct:

1.5. Definition.

(i) $A^* := \mathbb{Z}[\lambda_v, \lambda_v^{-1} \mid v \in V]/_{I^*}$, where I^* is the ideal generated by

(a) the kernel of the map $\mathbb{Z}[\lambda_v, \lambda_v^{-1} \mid v \in V] \to \mathbb{Z}[x_\gamma, (x_\gamma - x_\delta)^{-1} \mid \gamma, \delta \in \dot{F}_g, \gamma \neq \delta]$ which sends λ_v to the cross-ratio $(x_{v_1} - x_{v_3})(x_{v_1} - x_{v_4})^{-1}(x_{v_2} - x_{v_3})^{-1}(x_{v_2} - x_{v_4})$ (this kernel we want to call the "cross-ratio relations") and

(b) all $\lambda_{\gamma \cdot v} - \lambda_v$, $v \in V$, $\gamma \in F_g$ (the "F_g-invariance relations").

(ii) Let $c, d: V \to \{\pm 1\}$ be any maps. Then $A_c :=$ subring of A^* generated by all $\lambda_v^{c(v)}$, $A_{c,d} :=$ subring of A^* generated by all $\lambda_v^{c(v)}, \lambda_v^{d(v)}$.

(iii) $\forall c, d: V \to \{\pm 1\}$ let T_c, $T_{c,d}$ be the ideal in $A_c, A_{c,d}$, generated by all λ_v, $v = (\gamma \alpha \gamma^{-1}, \alpha, \gamma, \gamma^{-1})$ ($= A_c$ if $c(v) = -1$). Let $\bar{A}_c = A_{c/T_c}$, $\bar{A}_{c,d} := A_{c,d/T_{c,d}}$.

(iv) Denote by $\hat{A}_c, \hat{A}_{c,d}$ the completion of $A_c, A_{c,d}$ w.r.t. the T-adic topology.

(v) $Y_c := \mathrm{Spec}\, \bar{A}_c$, $Y_{c,d} := \mathrm{Spec}\, \bar{A}_{c,d}$
$U_c := \mathrm{Spec}\, A_c$, $U_{c,d} := \mathrm{Spec}\, A_{c,d}$
$\hat{Y}_c := \mathrm{Spf}\, \hat{A}_c$, $\hat{Y}_{c,d} := \mathrm{Spf}\, \hat{A}_{c,d}$

1.6. Remark. The typical cross-ratio relations are

(i) $\lambda_{v_1, v_2, v_3, v_4} = \lambda_{v_2, v_1, v_3, v_4}^{-1}$

(ii) $\lambda_{v_1, v_2, v_3, v_4} = 1 - \lambda_{v_1, v_3, v_2, v_4}$

(iii) $\lambda_{v_1, v_2, v_3, v_4} = \lambda_{v_1, v_5, v_3, v_4} \cdot \lambda_{v_2, v_5, v_3, v_4}^{-1}$.

1.7. Definition. $t_\gamma := \lambda_{\gamma \alpha \gamma^{-1}, \alpha, \gamma, \gamma^{-1}} \in A^*$ (which does obviously not depend on α) is called the multiplier of γ.

$$\left.\begin{array}{ll} u_{\gamma, \alpha, \beta} := \lambda_{\alpha, \beta, \gamma, \gamma^{-1}} \in A^* & u_{ijk}^{(\varepsilon)} := u_{\varepsilon_i, \varepsilon_j, \varepsilon_k} \\ v_{\gamma, \alpha, \beta} := \lambda_{\gamma \alpha \gamma^{-1}, \beta, \gamma, \gamma^{-1}} \in A^* & v_{ijk}^{(\varepsilon)} := v_{\varepsilon_i, \varepsilon_j, \varepsilon_k} \end{array}\right\} \text{ for any base } \varepsilon.$$

1.8. Lemma. *In A^* we have*

(i) $(1-t_\alpha)\lambda_{\alpha\delta\alpha^{-1},\beta,\gamma,\delta} = \lambda_{\alpha,\beta,\gamma,\delta} - t_\alpha \cdot \lambda_{\alpha^{-1},\beta,\gamma,\delta}$

(ii) $(t_\alpha \lambda_{\alpha,\beta,\gamma,\delta} - \lambda_{\alpha^{-1},\beta,\gamma,\delta}) \cdot \lambda_{\alpha\gamma\alpha^{-1},\beta,\gamma,\delta} = \lambda_{\alpha,\beta,\gamma,\delta}\lambda_{\alpha^{-1},\beta,\gamma,\delta}(1-t_\alpha)$

or

(i)′ $(1-t_\alpha)\lambda_{\alpha\delta\alpha^{-1},\beta,\gamma,\delta} = (1 - t_\alpha \lambda_{\alpha^{-1},\alpha,\gamma,\delta}) \cdot \lambda_{\alpha,\beta,\gamma,\delta}$

(ii)′ $(1 - t_\alpha \lambda_{\alpha,\alpha^{-1},\gamma,\delta}) \cdot \lambda_{\alpha\gamma\alpha^{-1},\beta,\gamma,\delta} = (1-t_\alpha) \cdot \lambda_{\alpha,\beta,\gamma,\delta}$

Proof.

(i)′ $(1 - t_\alpha \lambda_{\alpha^{-1},\alpha,\gamma,\delta})\lambda_{\alpha,\beta,\gamma,\delta} = (1 - \lambda_{\alpha\delta\alpha^{-1},\delta,\alpha,\alpha^{-1}} \cdot \lambda^{-1}_{\gamma,\delta,\alpha,\alpha^{-1}}) \cdot \lambda_{\alpha,\beta,\gamma,\delta} = (1 - \lambda_{\alpha\delta\alpha^{-1},\gamma,\alpha,\alpha^{-1}}) \cdot \lambda_{\alpha,\beta,\gamma,\delta} =$ $\lambda_{\alpha\delta\alpha^{-1},\alpha,\gamma,\alpha^{-1}} \cdot \lambda^{-1}_{\alpha\delta\alpha^{-1},\alpha,\gamma,\delta} \cdot \lambda_{\alpha\delta\alpha^{-1},\beta,\gamma,\delta} = \lambda_{\alpha\delta\alpha^{-1},\alpha,\delta,\alpha^{-1}} \cdot \lambda_{\alpha\delta\alpha^{-1},\beta,\gamma,\delta} \cdot = (1-t_\alpha)\lambda_{\alpha\delta\alpha^{-1},\beta,\gamma,\delta}.$

(ii)′ is proved in the same way, and (i), (ii) are easy consequences of (i)′, (ii)′. □

2. The tree associated to a point of Y_c.

2.1. Definition. Let $\varepsilon = \{\varepsilon_1, \ldots, \varepsilon_g\}$ be a basis of F_g.

$$C_\varepsilon := \{c: V \to \{\pm 1\} \mid c(v_1, v_2, v_3, v_4) = c(v_1, v_2, v_4, v_3) \forall v \in V \text{ with } \mathrm{st}(v_1) = \mathrm{st}(v_2),\ \#\{\mathrm{st}(v_2), \mathrm{st}(v_3), \mathrm{st}(v_4)\} = 3\}$$

$$\dot{C}_\varepsilon := \{c \in C_\varepsilon \mid \bar{A}_c \neq 0\}$$

$$\dot{C} := \bigcup_{\substack{\varepsilon \text{ basis} \\ \text{of } F_g}} \dot{C}_\varepsilon$$

In this paragraph we fix ε, $c \in \dot{C}_\varepsilon$ and a k-valued point of $\bar{A}_c$ (k any field). By λ_v we always mean the value of λ_v in this point ($\lambda_v \in \mathbb{P}_1(k)$), if nothing is explicitly specified.

2.2. Lemma. *Let M be any subset of $\dot{F}_g$, $S(M) := \{(\alpha_1, \alpha_2, \alpha_3) \in M^3 \mid \alpha_i \neq \alpha_j \forall i \neq j\}$. Then $R := \{((\alpha_1, \alpha_2, \alpha_3), (\beta_1, \beta_2, \beta_3)) \in S(M) \times S(M) \mid \lambda_{\alpha_i,\alpha_j,\beta_k,\beta_l} \neq 1$ whenever $\#\{\alpha_i, \alpha_j, \beta_k, \beta_l\} = 4\}$ is an equivalence relation on $S(M)$.*

Proof.

(i) Reflexivity: obvious

(ii) Symmetry: obvious

(iii) Transivity: Take $(\alpha), (\beta), (\gamma) \in S(M)$, $((\alpha), (\beta)) \in R$, $((\beta), (\gamma)) \in R$ and suppose $((\alpha), (\gamma)) \notin R$.
Then w.l.o.g. $\lambda_{\alpha_1,\alpha_2,\gamma_1,\gamma_2} = 1$, hence $\lambda_{\alpha_1,\gamma_1,\alpha_2,\gamma_2} = 0$.
In A^* we have $\lambda_{\beta_i,\gamma_1,\alpha_2,\gamma_2} = \lambda_{\alpha_1,\gamma_1,\alpha_2,\gamma_2} \cdot \lambda_{\beta_i,\alpha_1,\alpha_2,\gamma_2}$, hence $\forall i: \lambda_{\beta_i,\gamma_1,\alpha_2,\gamma_2} = 0 \vee \lambda_{\beta_i,\alpha_1,\gamma_2,\alpha_2} = 0$.
$\Rightarrow$ w.l.o.g. $\lambda_{\beta_1,\gamma_1,\alpha_2,\gamma_2} = \lambda_{\beta_2,\gamma_1,\alpha_2,\gamma_2} = 0$
$\Rightarrow \lambda_{\beta_1,\alpha_2,\gamma_1,\gamma_2} = \lambda_{\beta_2,\alpha_2,\gamma_1,\gamma_2} = 1 \Rightarrow \lambda_{\beta_1,\beta_2,\gamma_1,\gamma_1} = 1 \Rightarrow ((\beta), (\gamma)) \notin R$. □

2.3. Remark. $((\alpha, \gamma, \delta), (\beta, \gamma, \delta)) \in R \Leftrightarrow \lambda_{\alpha,\beta,\gamma,\delta} \in k^*$.

2.4. Definition.

(i) $T_0(M) := S(M)/R$, the equivalence classes are denoted by $[\alpha_1, \alpha_2, \alpha_3]$.

(ii) $T_1(M) := \{(P, Q) \in T_0(M) \times T_0(M) \mid \exists \alpha, \beta, \gamma, \delta \in M \text{ s.th. } P = [\alpha, \gamma, \delta], Q = [\beta, \gamma, \delta], \lambda_{\alpha,\beta,\gamma,\delta} = 0, \forall \varepsilon \in M: \lambda_{\alpha,\varepsilon,\gamma,\delta} \neq 0 \vee \lambda_{\varepsilon,\beta,\gamma,\delta} \neq 0\}$

(iii) $\mathscr{A}: T_1(M) \to T_0(M)$, $\mathscr{A}(P, Q) := P$
$\overline{}: T_1(M) \to T_0(M)$, $\overline{(P, Q)} := (Q, P)$

(iv) We denote by $T(M) := (T_0(M), T_1(M), \mathscr{A}, {}^-)$ the graph given by the data in (i), (ii), (iii), $T_0(M)$ being the vertices, $T_1(M)$ the edges.

2.5. Remark. F_g acts on $T := T(\dot{F}_g)$ by

$$\gamma \cdot [\alpha_1, \alpha_2, \alpha_3] := [\gamma\alpha_1\gamma^{-1}, \gamma\alpha_2\gamma^{-1}, \gamma\alpha_3\gamma^{-1}], \gamma \cdot (P, Q) := (\gamma \cdot P, \gamma \cdot Q).$$

Denote the quotient T/F_g by G.

Proof. F_g acts on $T_0(\dot{F}_g)$ as an easy consequence of the F_g-invariance relations: F_g acts on $S(\dot{F}_g)$, and equivalence is preserved since

$$\lambda_{\gamma\alpha_i\gamma^{-1}, \gamma\alpha_j\gamma^{-1}, \gamma\beta_k\gamma^{-1}, \gamma\beta_l\gamma^{-1}} = \lambda_{\alpha_i, \alpha_j, \beta_k, \beta_l}.$$

From the definition of the action of F_g on T_1 it is clear that $\mathscr{A}$ and $-$ are F_g-equivariant.

2.6. Lemma. *Given $P, Q \in T_0(M)$, $P \neq Q$, there exist $\alpha, \beta, \gamma, \delta \in M$ s.th. $P = [\alpha, \gamma, \delta]$, $Q = [\beta, \gamma, \delta]$, $\lambda_{\alpha,\beta,\gamma,\delta} = 0$.*

Proof. $P = [\alpha_1, \alpha_2, \alpha_3]$, $Q = [\beta_1, \beta_2, \beta_3]$, w.l.o.g. $\lambda_{\alpha_1,\alpha_2,\beta_1,\beta_2} = 1$, $\lambda_{\alpha_3,\beta_1,\alpha_1,\alpha_2} \neq \infty$, $\lambda_{\beta_3,\alpha_1,\beta_1,\beta_2} \neq \infty$.

(i) $\lambda_{\alpha_3,\beta_1,\alpha_1,\alpha_2} = 0$. Then $P = [\alpha_3, \alpha_1, \beta_1]$ since $1 = \lambda_{\alpha_3,\alpha_1,\beta_1,\alpha_2} = \lambda_{\beta_1,\alpha_2,\alpha_3,\alpha_1}$

(ii) $\lambda_{\alpha_3,\beta_1,\alpha_1,\alpha_2} \neq 0$. Then $P = [\beta_1, \alpha_1, \alpha_2]$ since $\lambda_{\alpha_3,\beta_1,\alpha_1,\alpha_2} \in k^*$. The same holds for Q, so take $\gamma = \alpha_1$, $\delta = \beta_1$.

2.7. Definition. $\gamma, \delta \in M$, $\gamma \neq \delta$.
$(\gamma, \delta) := \{[\alpha, \gamma, \delta] \in T_0(M) \mid \alpha \neq \gamma, \alpha \neq \delta\}$ together with the ordering $[\alpha, \gamma, \delta] < [\beta, \gamma, \delta]$ $\Leftrightarrow \lambda_{\alpha,\beta,\gamma,\delta} = 0$ is called the axis from γ to δ.

2.8. Proposition. *$T(M)$ is connected.*

Proof. $P, Q \in T_0(M)$, $P \neq Q$. Choose an axis $1 = (\gamma, \delta)$ with $P, Q \in 1$, $P < Q$.

Claim. *$W := \{R \in (\gamma, \delta) \mid P < R < Q\}$ is finite.*

Proof. $P = [\alpha, \gamma, \delta]$, $Q = [\beta, \gamma, \delta]$, $\lambda_{\alpha,\beta,\gamma,\delta} = 0$.
We may assume $M = \dot{F}_g$ since $T_0(M) \subset T_0(\dot{F}_g)$ and $W = W(\dot{F}_g) \cap T_0(M)$. For any $\mu \in F_g$, the map μ: $T_0(\dot{F}_g) \to T_0(\dot{F}_g)$ induces a bijection $W \cong \mu W = \{R \in (\mu\gamma^{-1}\mu, \mu\mu\delta\mu^{-1}) \mid \mu \cdot P < R < \mu \cdot Q\}$, hence we may assume $\# \{\mathrm{st}(\beta), \mathrm{st}(\gamma), \mathrm{st}(\delta)\} = 3$ by Lemma 1.3. Suppose $\# W = \infty$. Then $W = \{[\sigma_i, \gamma, \delta] \mid i \in \mathbb{Z}\}$ with $\lambda_{\alpha,\sigma_i,\gamma,\delta} = \lambda_{\sigma_i,\sigma_j,\gamma,\delta} = \lambda_{\sigma_j,\beta,\gamma,\delta} = 0\ \forall i < j$. Each σ_i uniquely determines $\mu_i \in F_g$ by Lemma 1.3 s.th. $\# \{\mathrm{st}(\mu_i\sigma_i\mu_i^{-1}), \mathrm{st}(\mu_i\gamma\mu_i^{-1}), \mathrm{st}(\mu_i\delta\mu_i^{-1})\} = 3$.
Note that $\lambda_{\sigma_i,\beta,\gamma,\delta} = 0$ implies $\lambda_{\mu_i\gamma\mu_i^{-1}, \mu_i\beta\mu_i^{-1}, \mu_i\sigma_i\mu_i^{-1}, \mu_i\delta\mu_i^{-1}} = \lambda_{\gamma,\beta,\sigma_i,\delta} = 0$, hence $\mathrm{st}(\mu_i) = \mathrm{st}(\mu_i\delta\mu_i^{-1})$ for $\mu_i \neq \mathrm{id}$ because $\mathrm{st}(\mu_i\delta\mu_i^{-1}) \neq \mathrm{st}(\mu_i)$ would imply $\mathrm{st}(\mu_i\gamma\mu^{-1}) = \mathrm{st}(\mu_i) = \mathrm{st}(\mu_i\beta\mu_i^{-1})$ in contradiction to $c \in \dot{C}_\varepsilon$. But then $\mathrm{st}(\mu_i\gamma\mu_i^{-1}) \neq \mathrm{st}(\mu_i)$ by the definition of μ_i. $\lambda_{\alpha,\sigma_i,\gamma,\delta} = 0 \Rightarrow \lambda_{\alpha,\delta,\gamma,\sigma_i} = 0$ $\Rightarrow \mathrm{st}(\mu_i\alpha\mu_i^{-1} \neq \mathrm{st}(\mu_i\delta\mu_i^{-1}) = \mathrm{st}(\mu_i)$ because $c \in \dot{C}_\varepsilon$. But by Lemma 1.4. there can only exist finitely many such μ_i's, so we have a μ s.th. $\mu_i = \mu$ for infinitely many σ_i. $I := \{i \in \mathbb{Z} \mid \mu_i = \mu\}$. For i, j in I, $i < j$ we have $0 = \lambda_{\sigma_i,\sigma_j,\gamma,\delta} = \lambda_{\mu\sigma_i\sigma^{-1}, \mu\sigma_j\mu^{-1}, \mu\gamma\mu^{-1}, \mu\delta\mu^{-1}}$, so $c \in \dot{C}_\varepsilon \Rightarrow \mathrm{st}(\mu\sigma_j\mu^{-1}) \neq \mathrm{st}(\mu\sigma_i\mu^{-1})$.
So $\# I \leqq \# \{\mathrm{st}(\mu\sigma_i\mu^{-1}) \mid i \in I\} < \infty$.
Since W is finite, we can write $W = \{R_1, \ldots, R_n\}$, $P =: R_0 < R_1 < \ldots < R_n < R_{n+1} := Q$. But then the definition of $T_1(M)$ yields $(R_i, R_{i+1}) \in T_1(M)$, so we have found a path joining P with Q. □

2.9. Remark. The proof of 2.8. also shows that if $P, Q \in (\gamma, \delta)$ then there exists a path in (γ, δ) joining P and Q.

For a finite $M \subset \dot{F}_g$ we have an alternative way to associate a graph with a k-valued point of Y_c:

Let B_M be the moduli-scheme of stable M-pointed trees of projective lines (s. [5]). Projective coordinates on B_M are given by the λ_v, $v_i \in M$, relations between them are the cross-ratio relations. Hence any k-valued point of Y_c uniquely determines a k-valued point of B_M, i.e. a stable M-pointed tree of projective lines. Let $T'(M)$ be its intersection graph, which is a tree.

2.10. Proposition. *$T(M) = T'(M)$ for each finite $M \subset \dot{F}_g$.*

Proof. Any element of $T_0'(M)$ is the "median" $[\alpha, \beta, \gamma]'$ of three marked points. A simple calculation shows $[\alpha, \beta, \gamma]' = [\alpha', \beta', \gamma']' \Leftrightarrow [\alpha, \beta, \gamma] = [\alpha', \beta', \gamma']$, so $T_0(M) = T_0'(M)$. We have $T_1'(M) = \{(P, Q) \in T_0'(M) \times T_0'(M) \mid L_P \cap L_Q \neq \emptyset\}$ where L_P, L_Q are the components of the tree of proj. lines corresponding to P, Q. $(P, Q) \in T_1'(M) \Rightarrow P = [\alpha, \gamma, \delta]'$, $Q = [\beta, \gamma, \delta]'$ (the tree is stable, so any end component has a marked point on it), $\lambda_{\alpha,\beta,\gamma,\delta} = 0$ $(P \neq Q)$, for any $\varepsilon \in M$ we have $\lambda_{\alpha,\varepsilon,\gamma,\delta} \neq 0$ or $\lambda_{\varepsilon,\beta,\gamma,\delta} \neq 0$ (because otherwise $[\varepsilon, \gamma, \delta]$ would correspond to a component "between" L_P and L_Q and then $L_P \cap L_Q = \emptyset$) $\Rightarrow (P, Q) \in T_0(M)$.

$(P, Q) \in T_0(M) \Rightarrow P = [\alpha, \gamma, \delta]$, $Q = [\beta, \gamma, \delta]$, $\lambda_{\alpha,\beta,\gamma,\delta} = 0$, for any ε $\lambda_{\alpha,\varepsilon,\gamma,\delta} \neq 0$ or $\lambda_{\varepsilon,\beta,\gamma,\delta} \neq 0$: P, Q are on the path between the component with the marked point "γ" and the component with the marked point "δ", and there is no component between them $\Rightarrow (P, Q) \in T_1'(M)$ because $T'(M)$ is a tree.

Obviously $\mathscr{A}$ and $-$ are the same maps in both graphs. □

2.11. Corollary. *$M \subset \dot{F}_g$ finite $\Rightarrow T(M)$ tree.*

2.12. Proposition. *$T(M)$ is a tree for all subsets M of $\dot{F}_g$.*

Proof. We have to show $T(M)$ is simply connected.

Suppose $w = (P_0, \ldots, P_n)$ simply closed path in $T(M)$ (i.e. $(P_i, P_{i+1}) \in T_1(M)$, $P_0 = P_n$, $P_i \neq P_j \forall i < j, i \neq 0$ or $j \neq n$), $P_i = [\alpha_i, \beta_i, \gamma_i]$. Choose $N \subset M$, N finite, $\alpha_i, \beta_i, \gamma_i \in N \forall i$. Then $P_i \in T_0(N)$, $(P_i, P_{i+1}) \in T_1(N)$, $P_0 = P_n$, $P_i \neq P_j \forall i < j$, $i \neq 0$ or $j \neq n$, i.e. w is a simply closed path in $T(N)$, which contradicts 2.11. □

2.13. Proposition. *The action of F_g on T is free and $G = T/F_g$ is a finite graph.*

Proof.

(i) Since F_g is a free group, the action on T is free if it is fixed point free. Suppose there exist $\gamma \in F_g$, $P \in T_0$ s.th. $\gamma \neq$ id., $\gamma \cdot P = P$, $\gamma = \alpha^n$ for some $\alpha \in \dot{F}_g$, $n > 0$. Let $Q := \pi_\alpha(P) :=$ uniquely determined vertex in (α, α^{-1}) with (path from P to Q) $\cap (\alpha, \alpha^{-1}) = Q$ (called the projection of P onto (α, α^{-1})).

We have $\pi_\alpha(\gamma \cdot P) = \gamma \cdot \pi_\alpha(P)$ because $\gamma \cdot (\alpha, \alpha^{-1}) = (\alpha, \alpha^{-1})$, (path joining $\gamma \cdot R$ with $\gamma \cdot R'$) $= \gamma \cdot$ (path joining R with R'), so $\gamma \cdot Q = Q$. Let $Q = [\beta, \alpha, \alpha^{-1}]$, then $\gamma \cdot Q = [\gamma \beta \gamma^{-1}, \gamma \alpha \gamma^{-1}, \gamma \alpha^{-1} \gamma^{-1}] = [\alpha^n \beta \alpha^{-n}, \alpha, \alpha^{-1}]$ with

$$\begin{aligned}\lambda_{\alpha^n \beta \alpha^{-n}, \beta, \alpha, \alpha^{-1}} &= \lambda_{\alpha^n \beta \alpha^{-n}, \alpha^{n-1} \beta \alpha^{1-n}, \alpha, \alpha^{-1}} \cdot \lambda_{\alpha^{n-1} \beta \alpha^{1-n}, \alpha^{n-2} \beta \alpha^{2-n}, \alpha, \alpha^{-1}} \cdot \cdots \cdot \lambda_{\alpha \beta \alpha^{-1}, \beta, \alpha, \alpha^{-1}} \\ &= t_\alpha \cdot t_\alpha \cdot \cdots \cdot t_\alpha = t_\alpha^n = 0\end{aligned}$$

which contradicts $\gamma \cdot Q = Q$. So F_g acts freely on T.

(ii) Let $F_g \cdot [\alpha', \beta', \gamma'] \in G_0$. By Lemma 1.3. we find $\alpha, \beta, \gamma \in F_g$ with $\#\{\mathrm{st}(\alpha), \mathrm{st}(\beta), \mathrm{st}(\gamma)\} = 3$, and $F_g \cdot [\alpha', \beta', \gamma'] = F_g \cdot [\alpha, \beta, \gamma]$. $\mathrm{st}(\alpha) = \varepsilon_i \Rightarrow [\alpha, \beta, \gamma] = [\varepsilon_i, \beta, \gamma]$, $\mathrm{st}(\beta) = \varepsilon_j \Rightarrow [\varepsilon_i, \beta, \gamma] = [\varepsilon_i, \varepsilon_j, \gamma]$, $\mathrm{st}(\gamma) = \varepsilon_k \Rightarrow [\varepsilon_i, \varepsilon_j, \gamma] = [\varepsilon_i, \varepsilon_j, \varepsilon_k]$ (use 2.3. and $c \in \dot{C}_\varepsilon$). This shows that the map $\{[\varepsilon_i, \varepsilon_j, \varepsilon_k] \mid i \neq j \neq k, i \neq k\} \to G_0$, $[\varepsilon_i, \varepsilon_j, \varepsilon_k] \to F_g \cdot [\varepsilon_i, \varepsilon_j, \varepsilon_k]$ is surjective, hence G_0 finite.

(iii) T is the universal covering of G, F_g the group of cover transformations $\Rightarrow F_g$ is the fundamental group of $G \Rightarrow$ the cyclomatic number of G is g.
But a graph with a finite number of vertices and finite cyclomatic number can only have finitely many edges. □

2.14. Corollary. *T is a locally finite tree.*

2.15. Definition.

(i) $P, Q \in T_0$. $d(P, Q) := \min\{n \mid \exists$ path in T joining P and Q with length $n\}$ is a metric on T.

(ii) A basis $w = w_1, \ldots, w_g$ of F_g is called *Schottky-basis* in a point of Y_c iff $v_{ijk}^{(w)} = 0 \ \forall\, i, j, k \in \{\pm 1, \ldots, \pm g\}, j, k \neq i$ in this point.

(iii) A basis of F_g is called a geometric basis for the action of F_g on T if it can be constructed by the following process (given by Bass and Serre, s. [13]):
Let H be a lifting of a maximal subtree of G to a subtree of T, let $l_1, \ldots, l_g$ be liftings of the remaining edges of G with $\mathscr{A}(l_i) \in H_0$, $\mathscr{A}(\bar{l}_i) \notin H_0$.
Then there exist uniquely determined $w_1, \ldots, w_g \in F_g$ s.th. $w_i(\mathscr{A}(\bar{l}_i)) \in H_0$. An easy calculation shows that $w_1, \ldots, w_g$ form a basis of F_g.

2.16. Proposition. *For each point of Y_c, $c \in \dot{C}_\varepsilon$, there exists a Schottky-base of F_g. In fact: Every geometric basis of F_g for the action on T is a Schottky-basis.*

Proof. Let $w_1, \ldots, w_h$ be a geometric basis, H the corresponding lifting of the maximal subtree, l_i liftings of the free edges.

(i) Claim. $\forall i \in \{1, \ldots, g\}$ *we have* $(w_i, w_i^{-1}) \cap H_0 \neq \emptyset$.
Proof. Suppose the contrary. Then $d((w_i, w_i^{-1}), H_0) := \min\{d(P, Q) \mid P \in (w_i, w_i^{-1}), Q \in H_0\} \geqq 1 \Rightarrow d(w_i \cdot P, H_0) \geqq 3 \,\forall P \in H_0$ (because w_i acts as a translation on (w_i, w_i^{-1})). But $d(w_i \cdot \mathscr{A}(l_i), H_0) = 1$.

(ii) Claim. $\mathscr{A}(l_i), \mathscr{A}(\bar{l}_i) \in (w_i, w_i^{-1}) \,\forall i$
Proof. Suppose $P \in \{\mathscr{A}(l_i), \mathscr{A}(\bar{l}_i)\}$, $P \notin (w_i, w_i^{-1})$. Let Q be the projection of P onto (w_i, w_i^{-1}). From the facts that $d(P, H_0) \leqq 1$, $(w_i, w_i^{-1}) \cap H_0 \neq \emptyset$, H and T are trees we conclude $Q \in H_0$. But then $d(w_i \cdot P, H_0) \geqq d(w_i \cdot P, w_i \cdot Q) + d(w_i \cdot Q, H_0) \geqq 1 + 1 = 2$ in contradiction to the definition of l_i.

(iii)
$$l_i \neq l_j,\ l_i \neq \bar{l}_j \,\forall i \neq j \Rightarrow_{(i),(ii)} (w_i, w_i^{-1}) \cap (w_j, w_j^{-1}) \subset H_0$$

Claim. $[w_j, w_i, w_i^{-1}] \in H_0$.

Proof.
a) $(w_i, w_i^{-1}) \cap (w_j, w_j^{-1}) = \emptyset$.
Then $[w_j, w_i, w_i^{-1}] =$ (projection of $\mathscr{A}(l_j)$ onto $(w_i, w_i^{-1})) \in H_0$
b) $(w_i, w_i^{-1}) \cap (w_j, w_j^{-1}) \neq \emptyset$. Then $[w_j, w_i, w_i^{-1}] \in (w_i, w_i^{-1}) \cap (w_j, w_j^{-1}) \subset H_0$.

(iv) $u_{ijk}^{(w)} \neq \infty \Rightarrow v_{ijk}^{(w)} = 0$
$u_{ijk}^{(w)} = \infty \Rightarrow v_{ijk}^{(w)} = 0$. Then
$[w_k, w_i, w_i^{-1}], [w_j, w_i, w_i^{-1}] \in H_0$
$\Rightarrow \mathscr{A}(l_i) \geqq [w_j, w_i, w_i^{-1}] > [w_k, w_i, w_i^{-1}] > w_i \cdot \mathscr{A}(l_i) \geqq w_i \cdot [w_j, w_i, w_i^{-1}]$
$= [w_i, w_j, w_i^{-1}, w_i, w_i^{-1}]$ in (w_i, w_i^{-1})
$\Rightarrow v_{ijk}^{(w)} = 0$. □

3. The rings $\bar{A}_c$. Fix a basis ε and a map $c \in \dot{C}_\varepsilon$.

Let B_1 be the subring of $\bar{A}_c$ generated by all $\lambda_{\gamma,\varepsilon_i,\varepsilon_j,\varepsilon_k} \in \bar{A}_c$ (i.e. $c(\gamma, \varepsilon_i, \varepsilon_j, \varepsilon_k) = 1$ or ($c(\gamma, \varepsilon_i, \varepsilon_j, \varepsilon_k) = -1$ and $\lambda^{-1}_{\gamma,\varepsilon_i,\varepsilon_j,\varepsilon_k}$ unit)).

3.1. Lemma. *$\bar{A}_c$ is generated as $\mathbb{Z}$-Algebra by all $f \in \bar{A}_c$ with $f \in B_1$ or $f^{-1} \in B_1$ ($\Rightarrow \bar{A}_c$ is essentially of finite type over B_1).*

Proof.
(i) Let B the subring of $\bar{A}_c$ generated by all $f \in \bar{A}_c$ with $f \in B_1$ or $f^{-1} \in B_1$. We have to show: $\lambda_v \in \bar{A}_c \Rightarrow \lambda_v \in B$. By Lemma 1.3. we know $\lambda_v = \lambda_{\alpha_1,\alpha_2,\alpha_3,\alpha_4} \in \bar{A}_c$ with $\# \{\mathrm{st}(\alpha_2), \mathrm{st}(\alpha_3), \mathrm{st}(\alpha_4)\} = 3$.
(ii) x unit in $\bar{A}_c$, $x^{-1} \in B \Rightarrow x^{-1} = P(f_1, \ldots, f_n, g_1, \ldots, g_m)$ with $f_i \in B_1, g_i^{-1} \in B_1, g_i \in \bar{A}_c$, $P \in \mathbb{Z}[x_1, \ldots, x_{n+m}]$. Define
$y := \prod g_i^{-\deg_{x_{n+i}} P} \in B_1$.
Then $x^{-1} y \in B_1$, $x^{-1} y$ is a unit in $\bar{A}_c \Rightarrow xy^{-1} \cdot y \in B$.
(iii)
$$\mathrm{st}(\alpha_1) = \mathrm{st}(\alpha_4) \Rightarrow \lambda_v^{-1} \in \bar{A}_c,\ \lambda_v \text{ unit in } \bar{A}_c$$
$$\mathrm{st}(\alpha_1) = \mathrm{st}(\alpha_3) \Rightarrow \lambda_v = 1 - \lambda_{\alpha_1,\alpha_3,\alpha_2,\alpha_4},\ \lambda_{\alpha_1,\alpha_3,\alpha_2,\alpha_4} \in \bar{A}_c.$$
So we may assume $\mathrm{st}(\alpha_1) \neq \mathrm{st}(\alpha_3), \mathrm{st}(\alpha_4)$.
(iv) $\lambda_{\varepsilon_i,\alpha_2,\alpha_3,\alpha_4} = \lambda_{\alpha_1,\alpha_2,\alpha_3,\alpha_4} \cdot \lambda_{\varepsilon_i,\alpha_1,\alpha_3,\alpha_4} \in \bar{A}_c$ if $\mathrm{st}(\alpha_1) = \varepsilon_i$, hence we may assume $\alpha_1 = \varepsilon_i$.
(v) $\mathrm{st}(\alpha_2) = \varepsilon_j$, $\mathrm{st}(\alpha_3) = \varepsilon_k$, $\mathrm{st}(\alpha_4) = \varepsilon_1$
a) $i \neq j$:
$$\begin{aligned}\lambda_{\varepsilon_i,\alpha_2,\alpha_3,\alpha_4} &= \lambda_{\alpha_3,\alpha_4,\varepsilon_i,\alpha_2}\\ &= \lambda^{-1}_{\varepsilon_k,\alpha_3,\varepsilon_i,\alpha_2} \cdot \lambda_{\varepsilon_k,\alpha_4,\varepsilon_i,\alpha_2}\\ &= (1 - \lambda_{\alpha_4,\alpha_2,\varepsilon_k,\varepsilon_i})(1 - \lambda_{\alpha_3,\alpha_2,\varepsilon_k,\varepsilon_i})^{-1}\\ &= (1 - \lambda_{\alpha_4,\varepsilon_j,\varepsilon_k,\varepsilon_i} \lambda^{-1}_{\alpha_2,\varepsilon_j,\varepsilon_k,\varepsilon_i})(1 - \lambda_{\alpha_3,\varepsilon_j,\varepsilon_k,\varepsilon_i} \cdot \lambda^{-1}_{\alpha_2,\varepsilon_j,\varepsilon_k,\varepsilon_i})^{-1}\\ &= uv^{-1},\ u \in B,\ v \in B,\ v^{-1} \in \bar{A}_c \overset{(ii)}{\Rightarrow} uv^{-1} \in B\end{aligned}$$
b) $i = j$. Choose $m \neq j, k, l$
$$\begin{aligned}\lambda_{\varepsilon_i,\alpha_2,\alpha_3,\alpha_4} &= \lambda_{\varepsilon_m,\alpha_2,\varepsilon_k,\alpha_3} \cdot \lambda_{\varepsilon_m,\alpha_2,\varepsilon_k,\alpha_4} \cdot \lambda_{\varepsilon_i,\varepsilon_m,\alpha_3,\alpha_4}\\ &= (1 - \lambda_{\alpha_2,\varepsilon_j,\varepsilon_m,\varepsilon_k} \lambda_{\alpha_3,\varepsilon_j,\varepsilon_k,\varepsilon_m})^{-1} (1 - \lambda_{\alpha_2,\varepsilon_j,\varepsilon_m,\varepsilon_k} \cdot \lambda_{\alpha_4,\varepsilon_j,\varepsilon_k,\varepsilon_m}) \cdot \lambda_{\varepsilon_i,\varepsilon_m,\alpha_3,\alpha_4}\\ &= u^{-1} v \text{ with } u, v \in B,\ u^{-1} \in \bar{A}_c\\ &\Rightarrow u^{-1} v \in B.\end{aligned}$$

3.2 Lemma. *Let B_2 be the subring of B_1 generated by all*
$$\lambda_{\gamma\varepsilon_i\gamma^{-1},\varepsilon_j,\varepsilon_k,\varepsilon_l} \in \bar{A}_c,\ \# \{j, k, l\} = 3.$$
Then $B_1 = B_2$.

Proof. $\lambda_{\gamma,\varepsilon_j,\varepsilon_k,\varepsilon_l} \in B_1$
(i) $z := \lambda_{\gamma^{-1},\gamma,\varepsilon_k,\varepsilon_l} \in \bar{A}_c$. Then by Lemma 1.8.(i)′ we know
$$\lambda_{\gamma,\varepsilon_j,\varepsilon_k,\varepsilon_l} = \lambda_{\gamma\varepsilon_l\gamma^{-1},\varepsilon_j,\varepsilon_k,\varepsilon_l} \in B_2$$
(ii) $z \notin \bar{A}_c \Rightarrow \lambda_{\gamma,\gamma^{-1},\varepsilon_k,\varepsilon_l} \in \bar{A}_c \Rightarrow \lambda_{\gamma,\varepsilon_j,\varepsilon_k,\varepsilon_l} = \lambda_{\gamma\varepsilon_k\gamma^{-1},\varepsilon_j,\varepsilon_k,\varepsilon_l} \in B_1$ again by Lemma 1.8. (i)′ □

3.3. Lemma. *Let B_3 be the subring of B_2 generated by all $\lambda_{\varepsilon_i, \varepsilon_j, \varepsilon_k, \varepsilon_l} \in \bar{A}_c$ and all v_{ijk}. Then B_2 is essentially of finite type over B_3.*

Proof. Let B_4 be the subring of B_2 generated by all $f \in B_2$ with $f \in B_3$ or $f^{-1} \in B_3$. We have to show: $\lambda = \lambda_{\gamma\varepsilon_i\gamma^{-1}, \varepsilon_j, \varepsilon_k, \varepsilon_l} \in \bar{A}_c \Rightarrow \lambda \in B_4$.

We do induction on $l(\gamma)$:

(i) $l(\gamma) = 1$:

(a) $\gamma = \varepsilon_i^{\pm 1}$. Then there is nothing to prove.

(b) $\gamma = \varepsilon_r$, $r \neq \pm i$, $r \neq l$ (otherwise λ is a unit, exchange k and l), $r \neq k$ (look at $1 - \lambda$).

(b_1) $r \neq j$: $\lambda = \lambda_{\varepsilon_r, \varepsilon_j, \varepsilon_k, \varepsilon_l} \cdot \lambda_{\varepsilon_r\varepsilon_i\varepsilon_r^{-1}, \varepsilon_r, \varepsilon_k, \varepsilon_l}$, and the second factor is a unit, hence $\lambda_{\varepsilon_r, \varepsilon_j, \varepsilon_k, \varepsilon_l} \in \bar{A}_c$. So w.l.o.g.:

(b_2) $r = j$: If $k = -j$ or $l = -j$, then $\lambda^{\pm 1} = 1 - v_{j,i,?}$, which is a unit in B_4. If $k \neq -j$, $l \neq -j$, then $\lambda = (1 - v_{j,i,k})(1 - v_{j,i,l})^{-1} \in B_4$.

(ii) $l(\gamma) = n + 1$, and assume the Lemma is proved for $l(\gamma) \leqq n$. w.l.o.g. $\mathrm{st}(\gamma^{-1}) \neq \varepsilon_i^{\pm 1}$, $\mathrm{st}(\gamma) = \varepsilon_r \neq \varepsilon_k, \varepsilon_l$. As in (i) we may assume $r = j$ and $k, l \neq -j$.

$$
\begin{aligned}
\lambda &= \lambda_{\gamma\varepsilon_i\gamma^{-1}, \varepsilon_j, \varepsilon_k, \varepsilon_j^{-1}} \cdot \lambda^{-1}_{\gamma\varepsilon_i\gamma^{-1}, \varepsilon_l, \varepsilon_j^{-1}} \\
&= (1 - \lambda_{\varepsilon_j^m \beta\varepsilon_i\beta^{-1}\varepsilon_j^{-m}, \varepsilon_k, \varepsilon_j, \varepsilon_j^{-1}})(1 - \lambda_{\varepsilon_j^m \beta\varepsilon_i\beta^{-1}\varepsilon_j^{-m}, \varepsilon_l, \varepsilon_j, \varepsilon_j^{-1}})^{-1} \\
&= (1 - t_j^m \lambda_{\beta\varepsilon_i\beta^{-1}, \varepsilon_k, \varepsilon_j, \varepsilon_j^{-1}})(1 - t_j^m \lambda_{\beta\varepsilon_i\beta^{-1}, \varepsilon_l, \varepsilon_j, \varepsilon_j^{-1}})^{-1} \\
&= (1 - t_j^{m-1} v_{j,s,k} \lambda_{\beta\varepsilon_i\beta^{-1}, \varepsilon_s, \varepsilon_j, \varepsilon_j^{-1}})(1 - t_j^{m-1} v_{j,s,l} \lambda_{\beta\varepsilon_i\beta^{-1}, \varepsilon_s, \varepsilon_j, \varepsilon_j^{-1}})^{-1} \\
&= u v^{-1}
\end{aligned}
$$

with $\varepsilon_s = \mathrm{st}(\beta) \neq \varepsilon_j^{\pm 1}$, hence $u, v \in B_2$, $u, v \in B_4$ by the induction assumption and $v^{-1} \in B_2$ because $v = \lambda_{\gamma\varepsilon_i\gamma^{-1}, \varepsilon_j, \varepsilon_l, \varepsilon_j^{-1}}$ is a unit in $\bar{A}_c$ ($\mathrm{st}(\gamma\varepsilon_i\gamma^{-1}) = \varepsilon_j$), so $v^{-1} \in B_4$ and then $\lambda = uv^{-1} \in B_4$. □

3.4. Proposition. *$\bar{A}_c$ is essentially of finite type over $\mathbb{Z}$.*

Proof. This follows immediately from Lemma 3.1. to 3.3. using Proposition 6.3.15 in [3], Chapter 0. (B. ess. of. fin. type over A, C ess. of fin. type over $B \Rightarrow C$ ess. of fin. type over A).

3.5. Corollary. *$\bar{A}_c$ is noetherian.*

3.6. Proposition. *If $c \in \dot{C}$ then*

$$Y_c = \bigcup_{s=1}^{n=n(c)} Y_{c, \varepsilon^{(s)}}$$

with: $\varepsilon^{(s)}$ basis of F_g

$$Y_{c, \varepsilon^{(s)}} := \{ y \in Y_c \mid v_{i,j,k}^{(s)}(y) = 0 \, \forall i, j, k, j, k \neq \pm i \}.$$

Proof. Let $\mathfrak{p}$ be a minimal prime ideal of $\bar{A}_c$, $K = \mathrm{Quot}(\bar{A}_c)$, $\bar{A}_c \to K$ the corresponding k-valued point of Y_c. By Proposition 2.16. there exists a Schottky-basis ε, i.e. a basis of F_g with $v_{ijk} \in \mathfrak{p}$, hence $\mathrm{Spec}\, \bar{A}_{c/\mathfrak{p}} \subset Y_{c,\varepsilon}$. $\bar{A}_c$ is a noetherian ring (3.15), so the number of minimal prime ideals is finite. □

4. Schottky-domains. Throughout this paragraph we fix $c \in \dot{C}_w$ and a basis ε of F_g. We define

$$\bar{A}_{c, \varepsilon} := \bar{A}_{c/(v_{ijk}^{\varepsilon} \mid \text{all } i, j, k)}.$$

4.1. Lemma. *$j, k, l \in \{\pm 1, \dots, \pm g\}$, $\# \{j, k, l\} = 3$, $\gamma = \varepsilon_j^r \cdot \beta$, $\gamma \varepsilon_i \gamma^{-1} = \varepsilon_j^r \cdot \beta \cdot \varepsilon_i \cdot \beta^{-1} \varepsilon_j^{-r}$. Then*

$$\lambda_{\gamma\varepsilon_i\gamma^{-1}, \varepsilon_j, \varepsilon_k, \varepsilon_l} = 1 \quad \text{in } \bar{A}_{c,\varepsilon}.$$

Proof. We do induction on $n = l(\beta)$.

(i) $n = 0: \gamma = \varepsilon_j^r$, $i \neq \pm j$.
We may assume $k, l \neq -j$.

$$\lambda_{\varepsilon_j^r \varepsilon_i \varepsilon_j^{-r}, \varepsilon_j, \varepsilon_k, \varepsilon_l} = (1 - t_j^{r-1} v_{jik})(1 - t_j^{r-1} v_{jil})^{-1} = 1$$

(ii) $n \to n+1: \gamma = \varepsilon_j^r \cdot \beta$, $\beta = \varepsilon_m^s \cdot \delta$, $l(\delta) \leqq n$, $m \neq \pm j$.
Then $\lambda_{\beta\varepsilon_i\beta^{-1}, \varepsilon_m, \varepsilon_j, \varepsilon_j^{-1}} = 1$, and so
$\lambda_{\gamma\varepsilon_i\gamma^{-1}, \varepsilon_j, \varepsilon_k, \varepsilon_l}$
$= (1 - t_j^{r-1} v_{j,m,k} \cdot \lambda_{\beta\varepsilon_i\beta^{-1}, \varepsilon_m, \varepsilon_j, \varepsilon_j^{-1}}) \cdot (1 - t_j^{r-1} v_{j,m,l} \cdot \lambda_{\beta\varepsilon_i\beta^{-1}, \varepsilon_m, \varepsilon_j, \varepsilon_j^{-1}})^{-1} = 1.$ □

4.2. Lemma. *$j, k, l \in \{\pm 1, \dots, \pm g\}$, $\# \{j, k, l\} = 3$, $\mathrm{st}_\varepsilon(\gamma) = \varepsilon_j$. Then $\lambda_{\gamma, \varepsilon_j, \varepsilon_k, \varepsilon_l} = 1$ in $\bar{A}_{c,\varepsilon}$.*

Proof. Lemma 1.8. implies $\lambda_{\gamma, \varepsilon_j, \varepsilon_k, \varepsilon_l} = \lambda_{\gamma\varepsilon_m\gamma^{-1}, \varepsilon_j, \varepsilon_k, \varepsilon_l}$ with $m = k$ or $m = 1$ (s. proof of Lemma 3.3.), and Lemma 4.1. implies $\lambda_{\gamma\varepsilon_m\gamma^{-1}, \varepsilon_j, \varepsilon_k, \varepsilon_l} = 1$. □

4.3. Lemma. *$i, j \in \{\pm 1, \dots, \pm g\}$, $i \neq j$, $\mathrm{st}_\varepsilon(\gamma_1) = \varepsilon_i$ or $\mathrm{st}_\varepsilon(\gamma_2) = \varepsilon_j$, $\mathrm{st}_\varepsilon(\gamma_1) \neq \varepsilon_j$ and $\mathrm{st}_\varepsilon(\gamma_2) \neq \varepsilon_i$. Then $\lambda_{\gamma_1, \varepsilon_i, \gamma_2, \varepsilon_j} = 1$.*

Proof.

(i) $\mathrm{st}_\varepsilon(\gamma_1) = \varepsilon_i$, $\mathrm{st}_\varepsilon(\gamma_2) \neq \varepsilon_i, \varepsilon_j$, $\mathrm{st}_\varepsilon(\gamma_2) = \varepsilon_k$.
Then

$$\begin{aligned}\lambda_{\gamma_1, \varepsilon_i, \gamma_2, \varepsilon_j} &= 1 - \lambda_{\gamma_1, \gamma_2, \varepsilon_i, \varepsilon_j} = 1 - \lambda_{\gamma_1, \varepsilon_k, \varepsilon_i, \varepsilon_j} \cdot \lambda_{\gamma_2, \varepsilon_k, \varepsilon_i, \varepsilon_j} \\ &= 1 - (1 - \lambda_{\gamma_1, \varepsilon_i, \varepsilon_k, \varepsilon_j}) \lambda_{\gamma_2, \varepsilon_k, \varepsilon_i, \varepsilon_j}^{-1} = 1 - (1 - 1) \cdot 1^{-1} = 1\end{aligned}$$

(ii) $\mathrm{st}_\varepsilon(\gamma_1) = \varepsilon_i$, $\mathrm{st}_\varepsilon(\gamma_2) = \varepsilon_j$. Choose $k \neq i, j$. Then

$$\lambda_{\gamma_1, \varepsilon_i, \gamma_2, \varepsilon_j} = 1 - (1 - \lambda_{\gamma_1, \varepsilon_i, \varepsilon_k, \varepsilon_j}) \cdot (1 - \lambda_{\gamma_2, \varepsilon_j, \varepsilon_k, \varepsilon_j}) = 1$$

in $\bar{A}_{c,\varepsilon}$. □

4.4. Proposition. *$\gamma_1, \dots, \gamma_4 \in F_g$, $\mathrm{st}_\varepsilon(\gamma_i) = \varepsilon_{k_i}$, $\# \{k_i \mid i = 1, \dots, 4\} \geqq 3$. Then $\lambda_{\gamma_1, \dots, \gamma_4} = \lambda_{\varepsilon_{k_1}, \dots, \varepsilon_{k_4}}$ in $A_{c,\varepsilon}$, where we define*

$$\lambda_{\varepsilon_{k_1}, \dots, \varepsilon_{k_4}} := \begin{cases} 0 & \text{if } k_1 = k_3 \text{ or } k_2 = k_4 \\ 1 & \text{if } k_1 = k_2 \text{ or } k_3 = k_4 \\ \infty & \text{if } k_1 = k_4 \text{ or } k_2 = k_3. \end{cases}$$

Proof. w.l.o.g. $\# \{k_i \mid i \geqq 2\} = 3$.

(i) $\gamma_3 = \varepsilon_{k_3}$, $\gamma_4 = \varepsilon_{k_4}$:
$\lambda_{\gamma_1, \gamma_2, \varepsilon_{k_3}, \varepsilon_{k_4}} = \lambda_{\gamma_1, \varepsilon_{k_2}, \varepsilon_{k_3}, \varepsilon_{k_4}} \cdot \lambda_{\gamma_2, \varepsilon_{k_2}, \varepsilon_{k_4}, \varepsilon_{k_3}} = \lambda_{\varepsilon_{k_1}, \dots, \varepsilon_{k_4}} \cdot \lambda_{\gamma_1, \varepsilon_{k_2}, \varepsilon_{k_3}, \varepsilon_{k_4}} \cdot \lambda_{\gamma_2, \varepsilon_{k_2}, \varepsilon_{k_3}, \varepsilon_{k_4}}$ in A^* and the proposition follows from Lemma 4.3.

(ii) $\varepsilon_{k_1} = \varepsilon_{k_2}$:

$\lambda_{\gamma_1,\gamma_2,\gamma_3,\gamma_4} = (1 - \lambda_{\varepsilon_{k_2},\varepsilon_{k_4},\gamma_1,\gamma_3}) \cdot \lambda^{-1}_{\gamma_1,\varepsilon_{k_2},\gamma_4,\varepsilon_{k_4}} \cdot \lambda^{-1}_{\gamma_2,\varepsilon_{k_2},\gamma_3,\varepsilon_{k_4}} \cdot \lambda_{\gamma_2,\varepsilon_{k_2},\gamma_4,\varepsilon_{k_4}}$ in A^*

$\underset{\text{Lemma 4.3}}{\Rightarrow} \lambda_{\gamma_1,\ldots,\gamma_4} = 1 - \lambda_{\varepsilon_{k_2},\varepsilon_{k_4},\gamma_1,\gamma_3}$ in $\bar{A}_{c,\varepsilon}$.

$\underset{(i)}{\Rightarrow} \lambda_{\gamma_1,\ldots,\gamma_4} = \lambda_{\varepsilon_{k_1},\ldots,\varepsilon_{k_4}}$.

(iii) $\varepsilon_{k_1} \neq \varepsilon_{k_2}$: $\lambda_{\gamma_1,\ldots,\gamma_4} = \lambda_{\gamma_1,\varepsilon_{k_2},\gamma_3,\gamma_4} \cdot \lambda^{-1}_{\gamma_1,\varepsilon_{k_2},\gamma_3,\gamma_4}$ in A^*

$\underset{(ii)}{\Rightarrow} \lambda_{\gamma_1,\ldots,\gamma_4} = \lambda_{\gamma_1,\varepsilon_{k_2},\gamma_3,\gamma_4}$ in $\bar{A}_{c,\varepsilon}$

$\underset{(ii)}{\Rightarrow} \lambda_{\gamma_1,\ldots,\gamma_4} = \lambda_{\gamma_1,\varepsilon_{k_2},\gamma_3,\varepsilon_{k_3}}$ in $\bar{A}_{c,\varepsilon}$

$\underset{(ii)}{\Rightarrow} \lambda_{\gamma_1,\ldots,\gamma_4} = \lambda_{\varepsilon_{k_1},\ldots,\varepsilon_{k_4}}$ in $\bar{A}_{c,\varepsilon}$ □

4.5. Corollary. *$\bar{A}_{c,\varepsilon}$ is of finite type over $\mathbb{Z}$, in fact: $\bar{A}_{c,\varepsilon}$ is generated by all $\lambda_v^{c(v)}$ with $v \in \{\varepsilon_{\pm 1}, \ldots, \varepsilon_{\pm g}\}^4 \cap V$.*

Proof. $v = (v_1, \ldots, v_4) \in V$. There exists $\mu \in F_g$ s.th. $\#\{\mathrm{st}(\mu v_i \mu^{-1}) \mid i \geq 2\} = 3$. $\gamma_i := \mu v_i \mu^{-1}$, $i = 1, \ldots, 4$, $\mathrm{st}_\varepsilon(\mu v_i \mu^{-1}) = \varepsilon_{k_i}$. Then $\lambda_v^{c(v)} = \lambda^{c(v)}_{\gamma_1,\ldots,\gamma_4} = \lambda^{c(v)}_{\varepsilon_{k_1},\ldots,\varepsilon_{k_4}}$. □

4.6. Lemma. *Let B be a noetherian ring, $A \to B$ a ringhomomorphism s.th. for each minimal prime ideal $\mathfrak{p}$ of B the homomorphism $A \to B/\mathfrak{p}$ is of finite type. Then B is of finite type over A.*

Proof. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the minimal prime ideals of B, let $x_j^{(i)}$ be liftings of the generators of $B/\mathfrak{p}$ over A, and let $t_j^{(i)}$ be generators of $\mathfrak{p}_i$, $s_1, \ldots, s_r$ generators of $\sqrt{(0)}$ as B-modules. Then the subring C of B, generated over A by all $x_j^{(i)}, t_j^{(i)}, s_j$ is of finite type over A.

Claim. $B = C$.

Proof. $f \in B \Rightarrow \exists f_1 \in C, \lambda_j^{(1)} \in B$ s.th. $f = f_1 + \sum \lambda_j^{(1)} t_j^{(1)}$ because $C \to B/\mathfrak{p}_1$ is surjective. To each $\lambda_j^{(1)}$ exist $\lambda_{jk}^{(2)} \in B$ and $\alpha_j^{(1)} \in C$ s.th. $\lambda_j^{(1)} = \alpha_j^{(1)} + \sum \lambda_{jk}^{(2)} t_k^{(2)}$. Continuing this we get

$$f = g + h, g \in C, h \in \bigcap_{i=1}^{n} \mathfrak{p}_i = \sqrt{(0)}.$$

By the same procedure we get

$$h = h'_l + h''_l, h'_l \in C, h''_l \in \sqrt{(0)}^{\,l}$$

for any l. Since B is noetherian, $\sqrt{(0)}^{\,l} = (0)$ for some l, hence $h \in C$ and finally $f \in C$.

4.7. Theorem. *$c \in \mathring{C} \Rightarrow \bar{A}_c$ is a finitely generated $\mathbb{Z}$-algebra.*

Proof. $\bar{A}_c$ is noetherian by 3.5., and to each mimimal prime $\mathfrak{p}$ of $\bar{A}_c$ there exists a base ε s.th. $\bar{A}_{c/\mathfrak{p}} \cong \bar{A}_{c,\varepsilon/\mathfrak{p}} \cap \bar{A}_{c,\varepsilon}$ by 3.6. $\bar{A}_{c,\varepsilon}$ is finite type over $\mathbb{Z}$ (Cor. 4.5.), so $\bar{A}_{c/\mathfrak{p}}$ is of finite type over $\mathbb{Z}$ for each minimal prime. Then $\bar{A}_c$ is a finitely generated $\mathbb{Z}$-algebra by Lemma 4.6. □

5. The spaces $\bar{T}_g$ and $\hat{T}_g$.

5.1. Lemma. *Let $c \in \dot{C}$ and ε be a basis of F_g, $\mathfrak{p}$ the kernel of the map $A_c \to \bar{A}_{c,\varepsilon}$. Then $\mathfrak{p}/\mathfrak{p}^2$ is a finitely generated A_c-module.*

Proof. $\mathfrak{p}/\mathfrak{p}^2$ is generated by all $v_{ijk}^{(\varepsilon)}$ (finitely many) and all t_γ. Let $\gamma = \alpha \cdot \beta$ reduced in the basis ε, and assume $\mathrm{st}_\varepsilon(\beta^{-1}) \neq \mathrm{st}_\varepsilon(\alpha)$, i.e. γ cyclic reduced. Take $\delta \in F_g$. s.th. $\mathrm{st}_\varepsilon(\delta) \neq (\mathrm{st}_\varepsilon \alpha^{-1})$, $\mathrm{st}_\varepsilon(\beta)$ and $\mathrm{st}_\varepsilon(\delta^{-1}) \neq \mathrm{st}_\varepsilon(\beta^{-1})$.

Then

$$\begin{aligned} t_\gamma &= \lambda_{\gamma\beta^{-1}\delta\gamma^{-1},\beta^{-1}\delta,\gamma,\gamma^{-1}} \\ &= \lambda_{\alpha\cdot\delta\cdot\beta^{-1}\cdot\alpha^{-1},\delta\cdot\beta^{-1},\alpha\cdot\beta,\beta^{-1}\cdot\alpha^{-1}} \cdot \lambda_{\delta\cdot\beta^{-1},\beta^{-1}\cdot\delta,\alpha\cdot\beta,\beta^{-1}\cdot\alpha^{-1}} \\ &= uv \in \mathfrak{p}^2 \end{aligned}$$

because

$$\mathrm{st}_\varepsilon(\alpha \cdot \delta \cdot \beta^{-1} \cdot \alpha^{-1}) = \mathrm{st}_\varepsilon(\alpha \cdot \beta) \Rightarrow u \in \mathfrak{p}$$

and

$$\mathrm{st}_\varepsilon(\beta^{-1} \cdot \delta) = \mathrm{st}_\varepsilon(\beta^{-1} \cdot \alpha^{-1}) \Rightarrow v \in \mathfrak{p}\,.$$

Hence $\mathfrak{p}/\mathfrak{p}^2$ is generated by all $v_{ijk}^{(\varepsilon)}$ and all t_γ with $l(\gamma) \leq 1$.

5.2. Proposition. *$A_c/T_c \cdot \sqrt{T_c}$ is noetherian.*

Proof.

$$R := A_c/T_c^2, \quad T := \ker(R \to \bar{A}_c),\ T^2 = 0\,.$$

(i) Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be liftings of the minimal primes of $\bar{A}_c$ to R ($\bar{A}_c$ noetherian), $I_\varepsilon := \ker(R \to \bar{A}_{c,\varepsilon})$. $I_\varepsilon/I_\varepsilon^2$ is a finitely generated $\bar{A}_c$-module, hence noetherian, so we know that R/I_ε^2 is noetherian. But to each $\mathfrak{p}_i$ there exists a basis ε s.th. $I_\varepsilon \subset \mathfrak{p}_i$, hence $R/\mathfrak{p}_i^2$ noetherian.

(ii) Choose finitely generated ideals $\mathfrak{a}_i$ in R s.th. $\mathfrak{p}_i = \mathfrak{a}_i + T$ (note that $\mathfrak{p}_i/T$ is finitely generated). Then $\mathfrak{p}_i^2 = \mathfrak{a}_i^2 + \mathfrak{a}_i T$, and $\mathfrak{p}_i^2/\mathfrak{p}_i T = \mathfrak{a}_i^2/\mathfrak{a}_i T$ is finitely generated, hence noetherian. But then $R/\mathfrak{p}_i^2$ noetherian implies $R/\mathfrak{p}_i T$ noetherian.

(iii) Choose finitely generated ideals T_i in R s.th. $T = T_i + \alpha_i T$. Then there exists a finitely generated ideal $\tilde{T}$ in R s.th. $T = \tilde{T} + \left(\prod_i \mathfrak{a}_i\right) T$. Obviously $\prod_i \mathfrak{a}_i \subset \sqrt{T}$ hence $T = \tilde{T} + T \cdot \sqrt{T}$. Now $T/T \cdot \sqrt{T}$ is noetherian, because it is finitely generated over $R/\sqrt{T}$, a noetherian ring. From this fact we conclude that $A_c/T_c \cdot \sqrt{T_c} = R/T \cdot \sqrt{T}$ is noetherian.

5.3. Theorem.

(i) *The rings $\hat{A}_c$, $\hat{A}_{c,d}$ are noetherian and adic.*

(ii) *The morphisms $\mathrm{Spf}\,\hat{A}_{c,d} \to \mathrm{Spf}\,\hat{A}_c$ are of finite type.*

(iii) *The morphisms $\mathrm{Spf}\,\hat{A}_{c,d} \to \mathrm{Spf}\,\hat{A}_c$ are open immersions.*

Proof.

(α) To each pair $c, d \in C$ with $\hat{A}_{c,d} \neq 0$ we can find $e \in \dot{C}$ s.th. $\hat{A}_{c,d} = \hat{A}_e$.

(β) $S_c := \sqrt{T_c}$. Then A_c/S_c is noetherian and S_c/S_c^2 is finitely generated (5.2). Then by [3], 0.7.2.5 and 0.7.2.7 $\hat{A}_c$ is noetherian and adic.

(γ) The morphism $\mathrm{Spf}\,\hat{A}_{c,d} \to \mathrm{Spf}\,\hat{A}_c$ is adic, $\mathrm{Spec}\,\bar{A}_{c,d} \to \mathrm{Spec}\,\bar{A}_c$ is of finite type, hence by [3], 10.13.1 the morphism is of finite type.

(δ) The underlying topological spaces of $\mathrm{Spf}\,\hat{A}_c$ and $\mathrm{Spec}\,\bar{A}_c$ are the same for all $c \in \dot{C}$. The maps $\bar{A}_c \to \bar{A}_{c,d}$ are localizations, and they are of finite type. Hence the maps $Y_{c,d} \to Y_c$ are open immersions, and top $(\mathrm{Spf}\,\hat{A}_{c,d})$ is an open subset of top $(\mathrm{Spf}\,\hat{A}_c)$. Let Z be the multiplicative system in A_c generated by all $\lambda_v^{c(v)}$ with $d(v) \neq c(v)$. Then obviously $\hat{A}_{c,d} = \hat{A}_c\{Z^{-1}\}$ (strictly convergent power series, terminology of EGA), and the stalks of the structure sheaves of $\mathrm{Spf}\,\hat{A}_c\{Z^{-1}\}$ and $\mathrm{Spf}\,\hat{A}_c$ are the same. So $\mathrm{Spf}\,\hat{A}_{c,d} \to \mathrm{Spf}\,\hat{A}_c$ gives an isomorphism of $\mathrm{Spf}\,\hat{A}_{c,d}$ with an open formal subscheme of $\mathrm{Spf}\,\hat{A}_c$. □

5.4. Definition.
(i) The formal scheme $\hat{T}_g$ obtained by glueing all the Spf $\hat{A}_c$ on the "overlaps" Spf $\hat{A}_{c,d}$ is called the (formal) Teichmüller space for degenerating curves.
(ii) The scheme $\bar{T}_g$ obtained by glueing all the Y_c's over the $Y_{c,d}$'s is called the Teichmüller space for totally degenerate curves.
(iii) $\psi_g := \mathrm{Aut}\, F_g/\mathrm{Inn}\, F_g =: \mathrm{Out}\, F_g$ is called Teichmüller modular group.

5.5. Remark. $\hat{T}_{g,\mathrm{red}} = \bar{T}_{g,\mathrm{red}}$.

5.6. Proposition. *$\hat{T}_g$ is separated, locally noetherian and a formal* $\mathrm{Spf}\,\mathbb{Z}[\![t_\gamma \mid \gamma \in \dot{F}_g]\!]$-*scheme locally of finite type.*

Proof. We only have to proof separatedness:
The morphism $\hat{T}_g \to \mathrm{Spf}\,\mathbb{Z}[\![t_\gamma \mid \gamma \in \dot{F}_g]$ is inductive limit of the sequence

$$(\hat{T}_g, \mathcal{O}_{\hat{T}_g}/J^{n+1}) \to \mathrm{Spf}\,(\mathbb{Z}[\![t_\gamma \mid \gamma \in F_g]\!]/J^{n+1}),$$

where J denotes the ideal (– sheaf) generated by all t_γ. By [3], 10.15.2 we have to show that $\bar{T}_g$ is separarated over $\mathbb{Z}$. Using [3], 5.3.6 it is enough to know that $\bar{A}_{c,d}$ is generated by $\bar{A}_c$ and $\bar{A}_d\, \forall\, c, d \in \dot{C}$. But this is obvious.

5.7. Remark. The group ψ_g acts on $\hat{T}_g$ by $\lambda_{\alpha(v)}^{(c\cdot\alpha^{-1})(\alpha(v))} \to \lambda_v^{c(v)}$ for $\alpha \in \psi_g$. This action induces isomorphisms of the trees corresponding to x and $\alpha(x)$, $x \in \bar{T}_g$.

We want now to establish the connection to moduli theory:
Let A be a complete noetherian local ring with maximal ideal $\mathfrak{m}$ and quotient field K. Let $C \to \mathrm{Spec}\, A$ be a stable curve s.th. $C_s := C \times \mathrm{Spec}\, A/\mathfrak{m}$ is totally degenerated and $C_\eta := C \times \mathrm{Spec}\, K$ nonsingular.
The completion $\hat{C}$ of C can be uniformized by a flat Schottky group $\Gamma \subset \mathrm{PGL}(2, K)$, see [10]. Fix a basis of Γ, or equivalently an isomorphism $\tau: F_g \to \Gamma$, and let $\tilde{\lambda}_v$ be the cross-ratio of $\tau(v_1), \ldots, \tau(v_4)$ for $v \in V$. Then Γ flat means $\tilde{\lambda}_v \in A$ or $\tilde{\lambda}_v^{-1} \in A\; \forall\, v \in V$. Note that Γ is unique up to conjugation in $\mathrm{PGL}(2, K)$, thus the collection of $\tilde{\lambda}_v$ is unique up to outer automorphisms of F_g.

5.8. Lemma. *In the situation above, there exists a basis $\varepsilon_1, \ldots, \varepsilon_g$ of F_g s.th. $\tilde{\lambda}_v \in A \backslash \mathfrak{m}$ for all $v \in V$ with $\#\{\mathrm{st}_\varepsilon(v_2), \ldots, \mathrm{st}_\varepsilon(v_4)\} = 3$ and $\mathrm{st}_\varepsilon(v_1) = \mathrm{st}_\varepsilon(v_2)$.*

Proof. $C \times \mathrm{Spec}\, K$ nonsingular implies that there exists a complete noetherian valuation ring $\mathcal{O}$ and a continuous homomorphism $A \to \mathcal{O}$ s.th. $C \times \mathrm{Spec}\, \mathcal{O}$ is generically nonsingular. But if the image of $\tilde{\lambda}_v$ is in $\mathcal{O} - \mathfrak{m}_{\mathcal{O}}$, then $\tilde{\lambda}_v$ is in $A - \mathfrak{m}$. Thus we may assume that A is a valuation ring. But then K is a complete ultrametric valued field and we can use results of rigid analysis: Γ has a Schottky basis $w_1, \ldots, w_g$, and this means that there are $2g$ disjoint disks $C_{\pm 1}, \ldots, C_{\pm g}$ in $\mathbb{P}_K^1$ s.th. the attracting fixed point of γ is in $C_{\pm 1}$ if $\mathrm{st}_w(\gamma) = \pm w_i$, see [6]. Let $\varepsilon_i := \tau^{-1}(w_i)$, then its easy to see that $|\tilde{\lambda}_v| = 1$ if v satisfies the conditions of the Lemma. □

5.9. Proposition. *Let A be a complete noetherian local ring with maximal ideal $\mathfrak{m}$, $k := A/\mathfrak{m}$, $K := \mathrm{Quot}\, A$. Let C be a stable curve over $\mathrm{Spec}\, A$ with $C_s := C \times \mathrm{Spec}\, k$ totally degenerated and $C_\eta := C \times \mathrm{Spec}\, K$ nonsingular. Let $\varepsilon_1 \cdots \varepsilon_g$ be a basis of the uniformizing Schottky-group Γ and $\tilde{\lambda}_v$ the corresponding cross-ratios. Then: There exists a unique morphism $\varphi: \mathrm{Spf}\, A \to \hat{T}_g$ s.th. $\tilde{\lambda}_v = \varphi^* \lambda_v$.*

Proof.
I. Existence: By Lemma 5.8. we can choose $c: V \to \{\pm 1\}$ s.th. $c \in C_\varepsilon$ and $\tilde{\lambda}_v^{c(v)} \in A \; \forall v \in V$. Let $\Psi_1: \mathbb{Z}[\lambda_v, \lambda_v^{-1} \mid v \in V] \to K$ be the homomorphism sending λ_v to $\tilde{\lambda}_v$. Since $\tilde{\lambda}_v$ are cross-ratios of points in $\mathbb{P}_K^1$ and $\tilde{\lambda}_{\gamma \cdot v} = \tilde{\lambda}_v \; \forall \gamma \in \operatorname{Inn} F_g$, we have $\Psi_1(I^*) = 0$, and Ψ_1 induces $\psi_2: A^* \to K$.
Because $\tilde{\lambda}_v^{c(v)} \in A \; \forall v \in V$, Ψ_2 induces $\Psi_3: A_c \to A$.
$\Psi_3(t_\gamma) = \tilde{\lambda}_{\gamma\alpha\gamma^{-1}, \alpha, \gamma, \gamma^{-1}} =: \tilde{t}_\gamma$, the multiplier of $\tau(\gamma)$ and $\tilde{t}_\gamma \in \mathfrak{m}$ (all $\tau(\gamma)$, $\gamma \neq \mathrm{id}$, are hyperbolic), thus $T_c \subset \varphi^{-1}(\mathfrak{m})$ because T_c is generated by all t_γ. Hence Ψ_3 is continuous and induces $\hat{\Psi}_3: \hat{A}_c \to A$, which in turn gives $\varphi: \operatorname{Spf} A \to \hat{T}_g$ because $c \in \dot{C}_\varepsilon \subset \dot{C}$. Obviously $\varphi^* \lambda_v = \tilde{\lambda}_v$.

II. Uniqeness. Let φ_1, φ_2 be two such morphisms. Then φ_1 be induced by $\Psi_1: \hat{A}_c \to A$, φ_2 by $\Psi_2: \hat{A}_d \to A$. But then $\tilde{\lambda}_v^{c(v)}, \tilde{\lambda}_v^{d(v)} \in A$, and there exists $\Psi_3: \hat{A}_{c,d} \to A$ s.th. φ_1, φ_2 factor over Ψ_3. But this means $\varphi_1 = \varphi_2$. □

Now let $\mathscr{CLNR}$ be the category of complete noetherian local rings, let $\mathscr{S}: \mathscr{CLNR} \to$ sets be the functor

$$\mathscr{S}(A) := \left\{ (C, (\varepsilon_1, \ldots, \varepsilon_g)) : \begin{array}{l} C \text{ stable curve over } A,\ C_s \text{ totally degenerated,} \\ (\varepsilon_1, \ldots, \varepsilon_g) \text{ basis of the fundamental group of } C_s \end{array} \right\} / \operatorname{Inn} F_g .$$

Let $\hat{T}_g^0$ and $\hat{T}_g^{00}$ be the open subschemes of $\hat{T}_g$ with

$$\operatorname{top}(\hat{T}_g^0) = \{x \in \operatorname{top}(\hat{T}_g) : \lambda_v \neq 0 \text{ in } \mathcal{O}_{\hat{T}_g, x} \; \forall v\}$$
$$\operatorname{top}(\hat{T}_g^{00}) = \{x \in \operatorname{top}(\hat{T}_g^0) : \widehat{\mathcal{O}_{\hat{T}_g, x}} \text{ regular}\}$$

$h_{\hat{T}_g^{00}}, h_{\hat{T}_g^0}, h_{\hat{T}_g}$ the point functors.

5.10 Theorem. *There exists a morphism of functors* $\Phi: \mathscr{S} \to h_{\hat{T}_g}$ *with*
(i) $\Phi(A)$ *injective.*
(ii) $h_{\hat{T}_g^{00}}(A) \subseteqq \operatorname{Im} \Phi(A) \subseteqq h_{\hat{T}_g^0}(A)$.
$\forall$ A *in* $\mathscr{CLNR}$.

Proof.
(i) Let A be as in 5.9, $(C, (\varepsilon_1, \ldots, \varepsilon_g)) \in \mathscr{S}(A)$. Let $\mathscr{X} \to \mathscr{M}$ be the universal deformation of C_s (see [2]). There exists a unique morphism $\psi: \operatorname{Spf} A \to \mathscr{M}$ s.th. $\hat{C} = \mathscr{X} \times_{\mathscr{M}} \operatorname{Spf} A$ and $C_S \xrightarrow{\sim} \mathscr{X}_s$. Then $\mathscr{X}_\eta$ is nonsingular, and $(\varepsilon_1, \ldots, \varepsilon_g)$ determines a basis of the uniformizing group. By 5.9 we find a unique morphism $\varphi: \mathscr{M} \to \hat{T}_g$.
Define $\Phi(A)(C, (\varepsilon_1, \ldots, \varepsilon_g)) := \varphi \circ \psi$. Obviously this is well-defined and functorial, and the uniqueness of φ and ψ gives injectivity.
(ii) $(f: \operatorname{Spf} A \to \hat{T}_g) \in \operatorname{Im} \Phi(A)$ factors through $\varphi: \mathscr{M} \to \hat{T}_g$ with $\varphi^* \lambda_v = \tilde{\lambda}_v \neq 0 \; \forall v$ as in 5.9. Thus $f \in h_{\hat{T}_g^0}(A)$.
(iii) $f \in h_{\hat{T}_g^{00}}(A)$ factors through $\varphi: \operatorname{Spf} \widehat{\mathcal{O}_{\hat{T}_g, x}} \to \hat{T}_g^{00}$, and $\varphi^* \lambda_v \neq 0 \; \forall v$. Then there exists a flat Schottky-group $\Gamma \subset \mathrm{PGL}(2, \operatorname{Quot} \widehat{\mathcal{O}_{\hat{T}_g, x}})$ with cross-ratios $\varphi^* \lambda_v$ with respect to some basis $(\varepsilon_1, \ldots, \varepsilon_g)$. Applying Mumfords construction ([10]) to Γ we obtain a curve $\tilde{C} \to \operatorname{Spec} \mathcal{O}_{\hat{T}_g, x}$, and by pullback $C \to \operatorname{Spec} A$. Then $f = \Phi(A)(C, (\varepsilon_1, \ldots, \varepsilon_g))$. □

5.11 Remark. One can construct (replacing $\dot{F}_g$ by $\dot{F}_g \cup \{z\}$ and repeating the whole construction) a formal scheme $\mathscr{Z}_g \to \hat{T}_g$ together with an action of F_g on $\mathscr{Z}_g$. The fibres of $\mathscr{Z}_g$ are open formal subschemes of "trees of projective lines" (see [9]), and F_g acts partially by translation of the components and the stabilizer groups of the components act as Schottky-groups.

The closed fibre $\mathscr{X}_g$ is a tree of projective lines, and the intersection graph is the tree described in Section 2.

$\mathscr{X}_g/F_g \to \hat{T}_g$ is a family of Mumford curves, and $\mathscr{X}_g \to \hat{T}_g$ should make $\hat{T}_g$ into a fine moduli space.

However there are some technical difficulties in the construction, and I will carry it out in a subsequent paper.

6. Rigid analytic aspects. In this paragraph we construct a rigid analytic space $\hat{T}_g^{an}$ associated with $\hat{T}_g$ and show that the rigid analytic Teichmüller space $\mathscr{T}_g$ for nonsingular curves (see [4], [7], [11]) can be embedded into $\hat{T}_g^{an}$ as an open analytic subspace. In order to limit the length of this section (which is more like an appendix to the rest of the paper) we don't give proofs in full detail. For a definition and properties of rigid analytic spaces we refer to [1]. Let $\mathcal{O}$ be a complete valuation ring, m its maximal ideal, k its quotient field (which is assumed to be algebraically closed) and $\bar{k} = \mathcal{O}/m$ its residue field. If π is a nonzero element of m, then $\pi\mathcal{O}$ is an ideal of definition for the topology of $\mathcal{O}$.

6.1. Definition.

(i) $R := \mathbb{Z}[\![t_\gamma \mid \gamma \in F_g]\!]$ with the $(\sum t_\gamma R)$-adic topology

(ii) $c, d \in C$, $0 \neq \pi, \varrho \in m$:

$$\tilde{\mathscr{A}}_{c,\pi,d,\varrho} := \hat{A}_{c,d} \hat{\bigotimes_R} \mathcal{O}\left\{\frac{t_\gamma}{\pi}, \frac{t_\gamma}{\varrho} \mid \gamma \in F_g\right\}$$

where $\{\ \}$ denotes strictly convergent power series

$$\mathscr{A}_{c,\pi,d,\varrho} := \tilde{\mathscr{A}}_{c,\pi,d,\varrho} \otimes k$$

$$\mathscr{A}^0_{c,\pi,c,\varrho} := \operatorname{im}(\tilde{\mathscr{A}}_{c,\pi,\varrho} \to \mathscr{A}_{c,\pi,d,\varrho}).$$

For $c = d$ and $\pi = \varrho$ we get $\tilde{\mathscr{A}}_{c,\pi}, \mathscr{A}_{c,\pi}, \mathscr{A}^0_{c,\pi}$.

6.2. Remark. The topologies on the $\mathcal{O}$-algebras in 6.1. are the ones induced by $\mathcal{O}$.

6.3. Proposition.

(i) $\operatorname{Spf} \mathscr{A}^0_{c,\pi,d,\varrho} \to \operatorname{Spf} \mathcal{O}$ *is of finite type.*

(ii) $\mathscr{A}_{c,\pi,d,\varrho}$ *is a k-affinoid algebra.*

Proof.

(i) By [3], 10.13.1 we have to show that $\mathscr{A}^0_{c,\pi}/\pi\mathscr{A}^0_{c,\pi}$ is a finitely generated $\mathcal{O}/\pi\mathcal{O}$-algebra. But since $\hat{T}_c$ is finitely generated ($\hat{A}_c$ is noetherian), there exists a surjective homomorphism

$$\bar{A}_c \otimes_{\mathbb{Z}} \mathcal{O}/\pi[z_1, \ldots, z_n] \to A^0_{c,\pi}/\pi\mathscr{A}^0_{c,\pi}$$

(the z_i are mapped to $1 \otimes \frac{\mu_i}{\pi}$, μ_i generators of $\hat{T}_c/\hat{T}_c^2$). But $\bar{A}_c$ is finitely generated.

(ii) is a consequence of (i), see [12]. □

6.4. Proposition. *The obvious homomorphism* $\mathscr{A}_{c,\pi} \xrightarrow{\eta} \mathscr{A}_{c,\pi,d,\varrho}$ *identifies* $\operatorname{Sp} \mathscr{A}_{c,\pi,d,\varrho}$ *with the (open) affinoid subdomain*

$$U_{c,\pi,d,\varrho} := \{x \in \operatorname{Sp} A_{c,\pi} \mid |t_\gamma(x)| \leqq |\varrho|,\ \forall \gamma \in F_g, |\lambda_v^{c(v)}(x)| \geqq 1\ \ \forall c(v) \neq d(v)\}$$

of $\operatorname{Sp} \mathscr{A}_{c,\pi} =: U_{c,\pi}$.

Proof. We have to show that η represents all affinoid morphisms $\mathrm{Sp}(\phi)\colon \mathrm{Sp}\, C \to \mathrm{Sp}\, \mathscr{A}_{c,\pi}$ with image in $U_{c,\pi,d,\varrho}$:

Let ϕ be such a morphism. Then $\|\varphi(t_\gamma)\| \leqq |\varrho|$, $\|\phi(\lambda_v^{c(v)})\| \geqq 1$ if $c(v) \neq d(v)$. Then we find

$$\phi_1\colon A_c \to C^0,\ \phi_2\colon \mathcal{O}\left\{\frac{t_\gamma}{\pi} \,\middle|\, \gamma \in F_g\right\} \to C^0 \ (C^0 = \{f \in C \mid \|f\| \leqq 1\})$$

s.th.

$$\begin{array}{ccccc} A_c & & & & \mathcal{O}\left\{\frac{t_\gamma}{\pi} \,\middle|\, \gamma \in F_g\right\} \\ & \searrow & & \swarrow & \\ {\scriptstyle \phi_1}\downarrow & & \mathscr{A}_{c,\pi} & & \downarrow{\scriptstyle \phi_2} \\ & & \downarrow{\scriptstyle \phi} & & \\ C^0 & & & & C^0 \\ & \searrow & & \swarrow & \\ & & C & & \end{array}$$

commutes.

Then we find (uniquely determined) continuous extensions

$$\psi_1\colon A_{c,d} \to C^0 \quad \text{and} \quad \psi_2\colon \mathcal{O}\left\{\frac{t_\gamma}{\pi}, \frac{t_\gamma}{\varrho} \,\middle|\, \gamma \in F_g\right\} \to C^0,$$

and they give a homomorphism $\psi\colon \mathscr{A}_{c,\pi,d,\varrho} \to C$ with $\psi \circ \eta = \phi$. Obviously ψ is uniquely determined by ϕ.

6.5. Definition. $\hat{T}_g^{an}$:= rigid k-analytic space obtained by glueing all $U_{c,\pi}$ over $U_{c,\pi,d,\varrho}$, $\forall c \in \dot{C}$, $0 \neq \pi \in m$.

6.6. Remark. $\mathrm{Aut}\, F_g$ acts on $\hat{T}_g^{an}$ by

$$\alpha\colon \mathscr{A}_{c\pi} \to \mathscr{A}_{c\circ\alpha,\pi}$$
$$\lambda_v^{c(v)} \to \lambda_{\alpha(v)}^{(c\circ\alpha^{-1})(\alpha(v))}$$

for any $\alpha \in \mathrm{Aut}\, F_g$. $\mathrm{Inn}\, F_g$ acts trivial, so there is an action of $\mathrm{Out}\, F_g$ on $\hat{T}_g$.

Let now $\mathscr{T}_g$ be the rigid analytic Teichmüller space for nonsingular curves. For the following facts about $\mathscr{T}_g$ see [11]. It is a fine moduli space for
$\{(\gamma_1, \ldots, \gamma_g) \mid \gamma_i \in \mathrm{PGL}(2, k), \langle \gamma_1, \ldots, \gamma_g \rangle$
= subgroup of $\mathrm{PGL}(2,k)$ generated by $\gamma_1, \ldots, \gamma_g$ is a Schottky group of rank $g\}/\mathrm{PGL}(2,k)$.
Let $\varepsilon_1, \ldots, \varepsilon_g$ be a basis of F_g. Then $\tau(\varepsilon_i)(\zeta) := \gamma_i$, where ζ = conjugation class of $(\gamma_1, \ldots, \gamma_g)$, γ_1 has fixed points $0, \infty$ and γ_2^{-1} has attracting fixed point 1, defines an injective group-homomorphism $\tau\colon F_g \to \mathrm{Aut}_{\mathscr{T}_g}(\mathbb{P}^1 \times \mathscr{T}_g)$ with image Γ, the "universal" Schottky-group over $\mathscr{T}_g$. Over each affinoid subdomain $\mathrm{Sp}\, B \subset \mathscr{T}_g$, $\gamma \in \Gamma$ is represented by $M_\gamma \in \mathrm{GL}(2, B)$. We can take

$$M_\gamma = \begin{pmatrix} x_\gamma - t_\gamma x_{\gamma^{-1}} & x_\gamma x_{\gamma^{-1}}(t_\gamma - 1) \\ 1 - t_\gamma & t_\gamma x_\gamma - x_{\gamma^{-1}} \end{pmatrix}$$

where $t_\gamma, x_\gamma \in B$ are the multiplier and the attracting fixed points of γ, i.e.

$$\|t_\gamma\| < 1,\ \frac{\gamma(z) - x_\gamma}{\gamma(z) - x_{\gamma^{-1}}} = t_\gamma \frac{z - x_\gamma}{z - x_{\gamma^{-1}}}\ \forall z \in \mathbb{P}^1.$$

For $v \in V$ define

$$\delta_v := \frac{x_{\alpha_1} - x_{\alpha_3}}{x_{\alpha_2} - x_{\alpha_4}} : \frac{x_{\alpha_2} - x_{\alpha_3}}{x_{\alpha_2} - x_{\alpha_4}} \in \mathcal{O}(\mathcal{T}_g), \quad \alpha_i = \tau(v_i).$$

Then obviously the δ_v satisfy the cross-ratio relations, and for $\beta \in F_g$ we have $\delta_{\beta v \beta^{-1}} = \delta_v$. The group $\operatorname{Aut} F_g$ acts on $\mathcal{T}_g$ by $\operatorname{Aut} F_g \ni \alpha: (\gamma_1, \ldots, \gamma_g) \to (\alpha(\gamma_1), \ldots, \alpha(\gamma_g))$, and $\alpha(\delta_v) = \delta_{\alpha(v)}$. $B_g := \{(\gamma_1, \ldots, \gamma_g) \mid \gamma_1, \ldots, \gamma_g$ are a Schottky-basis for $\langle \gamma_1, \ldots, \gamma_g \rangle\}$ is an admissible open subset of $\mathcal{T}_g$, and

$$\mathcal{T}_g = \bigcup_{\alpha \in \operatorname{Aut} F_g} \alpha(B_g)$$

is an admissible covering. It is described by

$$B_g = \{\zeta \in \mathcal{T}_g \,\big|\, |v_{ijk}(\zeta)| := |\delta_{\varepsilon_i \varepsilon_j \varepsilon_i^{-1}, \varepsilon_k, \varepsilon_i, \varepsilon_i^{-1}}(\zeta)| < 1\}$$

and, using the embedding

$$\mathcal{T}_g \to k^{3g-3}$$

$$(\gamma_1, \ldots, \gamma_g) \to (t_{\gamma_i}, x_{\gamma_i}, x_{\gamma_i^{-1}})$$

by

$$B_g = \left\{(t_i, x_i, x_{-i}) \,|\, 0 < |t_i| < 1, \left| t_i \cdot \frac{x_j - x_i}{x_j - x_{-i}} : \frac{x_k - x_i}{x_k - x_{-i}} \right| < 1, x_i \neq x_j \right\}.$$

Let

$$B_{g,c,\pi,n} := \{\zeta \in B_g \,\big|\, |\pi^n| \leqq |t_i|, |\pi^n| \leqq |\delta_v| \;\forall v \in V_\varepsilon, |t_\gamma| \leqq |\pi| \;\forall \gamma \in F_g, |\delta_v^{c(v)}| \leqq 1 \;\forall v \in V\}$$

$$\forall c \in \dot{C}_\varepsilon, \; 0 \neq \pi \in m, \; n \in \mathbb{N}, \; V_\varepsilon := \{v \in V \mid v_i \in \{\varepsilon_1^{\pm 1}, \ldots, \varepsilon_g^{\pm 1}\}\}$$

$B_{g,c,\pi,n}$ is affinoid, and is an affinoid subdomain of k^{3g-3}. ($B_{g,c,\pi,n}$ can be defined by finitely many inequalities, see e.g. [6]). The k-algebra homomorphism $\mathcal{A}_{c,\pi} \to \mathcal{O}(B_{g,c,\pi,n})$ given by $\lambda_v^{c(v)} \to \delta_v^{c(v)}$ defines a morphism of k-analytic spaces $B_{g,c,\pi,n} \to U_{c,\pi}$. Let

$$V_{c,\pi,n} := \{x \in U_{c,\pi} \,\big|\, |t_{\varepsilon_i}(x)| \geqq |\pi|^n, |\lambda_v| \geqq |\pi|^n, \;\forall v \in V, |v_{ijk}^{(\varepsilon)}(x)| \leqq |\pi|\}.$$

6.7. Lemma.

(i) λ_v *is a unit in* $\mathcal{O}(V_{c,\pi,n}) \;\forall v \in V$

(ii) $\lambda_v = \dfrac{x_{v_1} - x_{v_3}}{x_{v_1} - x_{v_4}} : \dfrac{x_{v_2} - x_{v_3}}{x_{v_2} - x_{v_4}}$ *for* $x_{v_i} := \lambda_{v_i, \varepsilon_2^{-1}, \varepsilon_1, \varepsilon_1^{-1}}$.

Proof. Obvious.

6.8. Lemma.

$$\tilde{M}_\gamma := \begin{pmatrix} x_\gamma - t_\gamma^n x_{\gamma^{-1}} & x_\gamma x_{\gamma^{-1}} (t_\gamma^n - 1) \\ 1 - t^n & x_\gamma t_\gamma^n - x_{\gamma^{-1}} \end{pmatrix} \in \mathrm{GL}(2, \mathcal{O}(V_{c,\pi,n}))$$

if $\gamma = \alpha^n, \alpha \in \dot{F}_g, \alpha \neq \varepsilon_1^{\pm 1}, n > 0$

$$\tilde{M}_{\varepsilon_1^n} = \begin{pmatrix} t_{\varepsilon_1}^n & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(2, (V_{c,\pi,n})).$$

Then $\tilde{M}_\gamma = u_\gamma \cdot \tilde{M}_{\varepsilon_{i_1}} \dots \tilde{M}_{\varepsilon_{i_n}}$ *if* $\gamma = \varepsilon_{i_1} \dots \varepsilon_{i_n}$, *and* u_γ *is a unit in* $\mathcal{O}(V_{c,\pi,n})$.

Proof. The matrices act on $\mathbb{P}^1_{\mathcal{O}(V_{c,\pi,n})}$, so they act on sections $\operatorname{Spec} \mathcal{O}(V_{c,\pi,n}) \to \mathbb{P}^1$. One easily finds $\tilde{M}_\gamma(x_\alpha) = x_{\gamma\alpha\gamma^{-1}} \forall \gamma, \alpha \in F_g$. Thus

$$\begin{aligned} \tilde{M}_\gamma(x_\alpha) &= x_{\gamma\alpha\gamma^{-1}} = x_{\varepsilon_{i_n}\dots\varepsilon_{i_n}\alpha\varepsilon_{i_n}^{-1}\dots\varepsilon_{i_1}^{-1}} \\ &= \tilde{M}_{\varepsilon_{i_1}} \dots \tilde{M}_{\varepsilon_{i_n}}(x_\alpha) \; \forall \alpha \in F_g \end{aligned}$$

especially for $x_{\varepsilon_1} = 0$, $x_{\varepsilon_1^{-1}} = \infty$, $x_{\varepsilon_2^{-1}} = 1$.

So $M_{\bar{\gamma}}^{-1} \tilde{M}_{\varepsilon_{i_n}} \dots \tilde{M}_{\varepsilon_{i_n}}$ acts trivial on these *sections*, and this implies 6.8. □

6.9. Proposition. *The map* $j_0 : B_{g,c,\pi,n} \to U_{c,\pi}$ *induces an isomorphism* $B_{g,c,\pi,n} \xrightarrow{\sim} V_{c,\pi,n}$.

Proof. The morphism

$$\begin{aligned} &V_{c,\pi,n} \to k^{3g-3} \\ &(x) \to (t_{\varepsilon_i}(x),\; x_{\varepsilon_i}(x), x_{\varepsilon_i^{-1}}(x)) \end{aligned}$$

has it's image in $B_{g,c,\pi,n}$, so it factors over $B_{g,c,\pi,n}$. Let $\psi : \mathcal{O}(B_{g,c,\pi,n}) \to \mathcal{O}(V_{c,\pi,n})$ be the corresponding algebra homomorphism. The morphism $B_{g,c,\pi,n} \to U_{c,\pi}$ has it's image in $V_{c,\pi,n}$. Let $\phi : \mathcal{O}(V_{c,\pi,n}) \to \mathcal{O}(B_{g,c,\pi,n})$ be the corresponding algebra-homomorphism. Obviously $\phi \circ \psi = \mathrm{id}$. Let $\Phi = \psi \circ \phi$. If $\tilde{M}_\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}$ then $\Phi(\tilde{M}_{\varepsilon_i}) = \tilde{M}_{\varepsilon_i}$, and

$$\Phi(\tilde{M}_\gamma) = \Phi(\tilde{M}_{\varepsilon_{i_1}}) \dots \Phi(\tilde{M}_{\varepsilon_{i_n}}) = \tilde{M}_{\varepsilon_{i_1}} \dots \tilde{M}_{\varepsilon_{i_n}} = u_\gamma^{-1} \tilde{M}_\gamma$$

thus

$$\Phi(a_\gamma) = u_\gamma^{-1} a_\gamma, \quad \Phi(b_\gamma) = \dots.$$

Define $\mu_\gamma := \dfrac{a_\gamma d_\gamma - b_\gamma c_\gamma}{(a_\gamma + d_\gamma)^2} = \dfrac{t_\gamma}{1 + t_\gamma^2}$. Then $\phi(\mu_\gamma) = \mu_\gamma$ and $\Phi(t_\gamma) = t_\gamma$ because $\|t_\gamma\| < 1$. Now one can easily see that $\Phi(\lambda_v) = \lambda_v \forall v \in V$, hence $\Phi = \mathrm{id}$ because $V_{c,\pi,n}$ is a rational subdomain of $U_{c,\pi}$. □

The affinoid domains $\alpha(B_{g,c\pi,n})$ form an admissible covering of $\mathcal{T}_g$. Using the action of $\operatorname{Aut} F_g$ on $\mathcal{T}_g$ and $\hat{T}_g^{an}$ we obtain open immersions $j_\alpha = \alpha \circ j_0 \circ \alpha^{-1} : \alpha(B_{g,c,\pi,n}) \to \hat{T}_g^{an}$. Obviously $j_\alpha = j_\beta$ on $\alpha(B_{g,c\pi,n}) \cap \beta(B_{g,c\pi,n})$, so we can glue all these maps to get an open immersion $j : \mathcal{T}_g \to \hat{T}_g^{an}$.

Concluding we have

6.10. Theorem. *There exists a natural open embedding* $j : \mathcal{T}_g \to \hat{T}_g^{an}$ *with image*

$$j(\mathcal{T}_g) = \{x \in \hat{T}_g^{an} \mid \lambda_v(x) \neq 0 \; \forall v \in V\}.$$

Proof. $x \in \hat{T}_g^{an} \Rightarrow x \in \operatorname{Sp} \mathcal{A}_{c,\pi}$ for $c \in \dot{C}_{\alpha(\varepsilon)}, \pi \in m$.

Let $w^{(1)}, \dots, w^{(n)}$ be bases of F_g s.th. $\operatorname{Spec} \bar{A}_c = \bigcup_i \operatorname{Spec} \bar{A}_{c, w^{(i)}}$ x defines a continuous homomorphism $\phi_x : \hat{A}_c \to \mathcal{O}$, hence a $\bar{k}$-valued point $\bar{x}$ of $\operatorname{Sp} \bar{A}_c$. Thus there exists an index $l \in \{1, \dots, n\}$ s.th. $\bar{v}_{ijk}^{(w_l)}(\bar{x}) = 0 \forall i, j, k$, i.e. $\phi_x(v_{ijk}^{(w_l)}) \in m \forall i, j, k$ or

$$|v_{ijk}^{(w_l)}(x)| < 1 \; \forall i, j, k.$$

Let $|\varrho| = \max(\{|v_{ijk}^{(w_l)}(x)|, i, j, k\} \cup \{|\pi|\}, p \in m$.

Then for

$$\#\{\mathrm{st}_{w^{(l)}}(v_i) \mid i = 1, \ldots, 4\} = 3,\ \mathrm{st}_{w^{(l)}}(v_3) = \mathrm{st}_{w^{(l)}}(v_4)$$

we have

$$|\lambda_v(x) - 1| \leqq \max(\{|t_\gamma(x)|,\ \gamma \in F_g\} \cup \{|v_{ijk}^{(w_l)}(x)|,\ i, j, k\}) \leqq |\varrho|$$

thus $|\lambda_v(x)| = 1$ and we assume $c \in \dot{C}_{w^{(l)}}$.

Next we may assume $w^{(l)} = \varepsilon$ because $j(\mathscr{T}_g)$ and $\{x \in \hat{T}_g^{an} \mid \lambda_v(x) \neq 0 \,\forall v\}$ are $\mathrm{Aut}\, F_g$ invariant.

If now $\lambda_v(x) \neq 0 \,\forall v \in V$ we can find $n \in \mathbb{N}$ s.th. $x \in V_{c,\varrho,n}$, thus $x \in j(\mathscr{T}_g)$. On the other hand clearly $x \in j(\mathscr{T}_g)$ implies $\lambda_v(x) \neq 0 \,\forall v \in V$.

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