BOUNDARY INTEGRAL EQUATIONS FOR SCREEN PROBLEMS IN $\mathbb{R}^3$

Ernst P. Stephan

Here we present a new solution procedure for Helmholtz
and Laplacian Neumann screen or Dirichlet screen problems in $\mathbb{R}^3$
via boundary integral equations of the first kind having as
unknown the jump of the field or of its normal derivative,
respectively, across the screen S. Under the assumption of
local finite energy we show the equivalence of the integral
equations and the original boundary value problems. Via the
Wiener-Hopf method in the halfspace, localization and the calcu-
lus of pseudodifferential operators we derive existence, unique-
ness and regularity results for the solution of our boundary
integral equations together with its explicit behavior near the
edge of the screen. We give Galerkin schemes based on our inte-
gral equations on S and obtain high convergence rates by using
special singular elements besides regular splines as test and
trial functions.

1. INTRODUCTION

This paper presents a solution procedure for both the
<u>Dirichlet</u> and the <u>Neumann screen</u> problems for the scalar
Helmholtz equation (with small wave number k) via boundary
integral equations on the screen surface S. The problems under
consideration are the following ones. <u>For given g or h on S
find u in</u> $\Omega_S := \mathbb{R}^3 \setminus \bar{S}$ <u>satisfying</u>

$$(\Delta + k^2)u = 0 \ \underline{\text{in}}\ \Omega_S \qquad (1.1)$$

$$\left. \begin{array}{l} u = g\ \underline{\text{on}}\ S\ (\underline{\text{Dirichlet}})\ \underline{\text{or}} \\[2mm] \dfrac{\partial u}{\partial n} = h\ \underline{\text{on}}\ S\ (\underline{\text{Neumann}})\ \underline{\text{and}} \end{array} \right\} \qquad (1.2)$$

$$\frac{\partial u}{\partial r} - iku = o(r^{-1})\ \underline{\text{for}}\ k \neq 0\ \underline{\text{or}}\ u = O(r^{-1})\ \underline{\text{for}}\ k = 0 \qquad (1.3)$$

$$\underline{\text{as}}\ r := |x| \to \infty.$$

Such problems appear in scattering of acoustic fields u by
obstacles of different hardness (see [8]): The Dirichlet

condition represents a soft screen whereas the Neumann condi-
tion represents a hard screen. For k = 0 the above Dirichlet
problem describes the electrostatic field u of an electrified
screen and one looks for the charge density (see [1]).

        We make the general assumption:

        S is a bounded, simply connected, orientable
        smooth, open surface in $\mathbb{R}^3$ with a smooth   (1.4)
        boundary curve γ which does not intersect
        itself (see Fig. 1).

Our solution procedure is to derive boundary integral equations
of the first kind on S for the jump of the normal derivative
$[\frac{\partial u}{\partial n}]$ across the screen S in the Dirichlet case and for the jump
of the field [u] across S in the Neumann case, respectively.
In Section 2 we reduce the Dirichlet problem by use of Green's
formula to the weakly singular boundary integral equation

$$V_S[\frac{\partial u}{\partial n}](x) := \frac{1}{2\pi} \int_S \frac{e^{ik|x-y|}}{|x-y|} [\frac{\partial u}{\partial n}](y) dS_y = 2g(x), \qquad x \in S \qquad (1.5)$$

Similarly, the Neumann screen problem is reduced to the hyper-
singular boundary integral equation

$$D_S[u](x) := \frac{1}{2\pi} \int_S \frac{\partial^2}{\partial n_x \partial n_y} \frac{e^{ik|x-y|}}{|x-y|} [u](y) dS_y = -2h(x), \qquad (1.6)$$
$$x \in S$$

Both integral equations (1.5), (1.6) have been already sug-
gested by physical arguments in [8]. Although the integral
operators in (1.5), (1.6) possess quite different properties in
the classical theory of integral equations, they both belong to
the larger class of pseudodifferential operators. The solva-
bility of the equations (1.5) and (1.6) hinges on the fact that
V_S and D_S are strongly elliptic, that is they satisfy a Gårding
inequality in suitable Sobolev spaces (Lemma 2.8). Therefore,
V_S, D_S are Fredholm operators of index zero and, thus, the
injectivity of the operators implies their bijectivity. Since
the integral equation (1.5) is equivalent to the Dirichlet prob-
lem (1.1)-(1.3) and the integral equation (1.6) to the cor-
responding Neumann problem, respectively, our integral equa-
tions have no eigensolutions for Im k $\geq$ 0. Following [16] the
analysis of the integral equations using the calculus of

pseudodifferential operators and Wiener-Hopf technique provides
the explicit edge behavior of the unknown densities $[\frac{\partial u}{\partial n}]$ and
[u] (Theorem 2.9). In case of the Dirichlet problem we obtain
that the jump in the normal derivative of the acoustic field
behaves near the edge like $\rho^{-1/2}$ even for C^{∞} data where ρ
denotes the Euclidean distance to the edge γ of the screen S.
This is in agreement with the "edge condition" in physics (see
[8]). Therefore, in general, the solution of (1.5) is not
continuous for continuous data g contrary to Hayashi's claim
[5], [6]. Both integral equations (1.5) and (1.6) are derived
under the only assumption that the acoustic field u in (1.1)-
(1.3) has local finite energy, i.e., $u \in H^1_{loc}$ near S -- which
is the physically relevant property. For the Neumann problem
we obtain the explicit edge behavior for the solution of (1.6)
which improves the results by Durand [3].

        The knowledge of the explicit edge behavior of the
solutions of (1.5) and (1.6) is important for an effective
numerical scheme (based on our integral equations) in order to
diminish the pollution effect caused by the edge singularity.
Solving (1.5), (1.6) approximately with regular splines we
obtain only low convergence rates in the energy norm for the
boundary element Galerkin solutions (Theorem 3.1). This is in
accordance to the standard variational finite element method
for (1.1)-(1.3) in the domain. In order to improve the conver-
gence rate we incorporate in the Galerkin scheme as test and
trial functions special singular elements which simulate the
edge behavior of the exact solutions of the integral equations.
With this augmentation method we obtain higher convergence
rates (Theorem 3.2) as in the two-dimensional case [19].

        2. BOUNDARY INTEGRAL EQUATIONS
        Before we derive the integral equations (1.5) and
(1.6) we reformulate the original screen problems: <u>Let</u>
$g \in H^{1/2}(S)$ ($h \in H^{-1/2}(S)$) <u>be given, we look for</u> $u \in L_S$ <u>satis-</u>
<u>fying the Dirichlet (Neumann) boundary condition in</u> (1.2) <u>where</u>
$$L_S = \{u \in H^1_{loc}(\Omega_S) : (\Delta+k^2)u = 0 \text{ in } \Omega_S, u \text{ satisfies (1.3)}\} \quad (2.1)$$

For completeness, let us introduce the Sobolev spaces $H^S(\Omega)$, $H^S(\Gamma)$, and $H^S(S)$, $s \in \mathbb{R}$, where Ω is a bounded domain with a smooth boundary Γ and a smooth boundary piece $S \subset \Gamma$. We recall from [9], [11] the function spaces used which are incorporated with their natural norms:

$$H^S(\Omega) = \{u|_\Omega : u \in H^S(\mathbb{R}^3)\} \quad (s \in \mathbb{R})$$

$$H^S(\Gamma) = \begin{cases} \{u|_\Gamma : u \in H^{s+\frac{1}{2}}(\mathbb{R}^3) & (s > 0) \\ L^2(\Gamma) & (s = 0) \\ H^{-S}(\Gamma))' \text{ (dual space)} & (s < 0) \end{cases}$$

$$H^S(S) = \{u|_S : u \in H^S(\Gamma)\} \quad (s \geq 0)$$

$$\widetilde{H}^S(S) = \{u \in H^S(\Gamma) : \text{supp } u \subset \bar{S}\}, \quad H^S(S) = H^S(\Gamma)/\widetilde{H}^S(\Gamma \backslash \bar{S})$$

As in the two-dimensional case (see [19]) we prove uniqueness of the screen Dirichlet and the screen Neumann problem (1.1)-(1.3) for Im $k \geq 0$ by transforming them into appropriate transmission problems.

LEMMA 2.1. *For* Im $k \geq 0$ *the homogeneous screen Dirichlet (Neumann) problem has at most the trivial solution in* L_S.

For the proof and further investigation we extend S to an arbitrary smooth, simply connected, closed, orientable manifold (surface) ∂G_1 enclosing a bounded domain G_1 with boundary ∂G_1 (see Fig. 1). Let $\frac{\partial}{\partial n}$ denote the exterior normal derivative to ∂G_1. Let [v] denote the jump $v_- - v_+$ where the subscript + (-) means the limit from $\mathbb{R}^3 \backslash \bar{G}_1$ (from G_1) to ∂G_1. Furthermore, let B denote a sufficiently large ball with radius R including $\bar{G}_1$ and let $G_2 := B \cap (\mathbb{R}^3 \backslash \bar{G}_1)$ and ∂B denote the boundary of B.
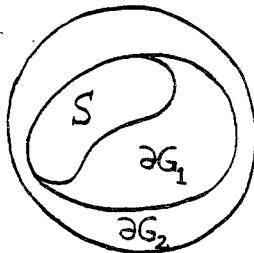

Fig. 1.

PROOF.   Let us consider first the homogeneous Dirichlet problem; thus $u \in L_S$ with $u|_S = g = 0$.   Then $u \in H^1(G_1) \cup H^1_{loc}(\mathbb{R}^3 \setminus \bar{G}_1)$ solves the following transmission problem:

$$u = u_1 \text{ in } G_1 \text{ with } \Delta u_1 + k^2 u_1 = 0 \text{ in } G_1$$

$$u = u_2 \text{ in } \mathbb{R}^3 \setminus \bar{G}_1 \text{ with } \Delta u_2 + k^2 u_2 = 0 \text{ in } \mathbb{R}^3 \setminus \bar{G}_1$$

satisfying (1.3) and

$$u_1 = u_2 \text{ on } \partial G_1 \text{ and } \frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} \text{ on } \partial G_1 \setminus \bar{S}. \qquad (2.2)$$

Application of the Green's formula in G_1 and G_2 then yields with the transmission conditions in (2.2) by eliminating the integral over S

$$\int_{\partial B} \frac{\partial u_2}{\partial n} \bar{u}_2 ds = -k^2 \int_{G_1} |u_1|^2 dx - k^2 \int_{G_2} |u_2|^2 dx$$

$$+ \int_{G_1} |\nabla u_1|^2 dx + \int_{G_2} |\nabla u_2|^2 dx \qquad (2.3)$$

Note that (2.3) holds with the traces of u_j since for $u \in L_S$ we also have $\Delta u_j \in L^2(G_j)$, $j = 1,2$.   For Im $k > 0$ or $k = 0$, the left hand side in (2.3) tends to zero for $R \to \infty$.   Hence, both imaginary and real parts of the right hand side in (2.3) vanish which implies $u_1 \equiv 0$ and $u_2 \equiv 0$.   For Im $k = 0$, $k > 0$, we use the radiation condition (1.3) and then take the imaginary part of (2.3) obtaining

$$k \int_{\partial B} |u_2|^2 ds + o(1) = 0.$$

This gives with Rellich's theorem [23, Theorem 4.2] $u_2 \equiv 0$ and with (2.2) also $u_1 \equiv 0$.

The uniqueness for the Neumann screen problem follows by the same arguments if one considers (2.2) with the new transmission conditions

$$\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} \text{ on } \partial G_1 \quad \text{and} \quad u_1 = u_2 \text{ on } \partial G_1 \setminus \bar{S}. \qquad \square$$

For the boundary integral equations on S we shall need some properties of the traces of the solution $u \in L_S$ of (1.1)-(1.3).   As in the two-dimensional case (see [19]) we

obtain for the screen Dirichlet (Neumann) problem the fol-
lowing result on the jump of the normal derivative (of the
trace) of the weak solution.

LEMMA 2.2.   *Let* $u \in H^1_{loc}(\Omega_S)$ *be a weak solution of
the screen problem* (1.1)-(1.3). *Then for the Dirichlet problem
we have*

$$[\frac{\partial u}{\partial n}]\big|_S \in \widetilde{H}^{-1/2}(S) = (H^{1/2}(S))' \qquad (2.4)$$

and for the Neumann problem we have

$$[u]\big|_S \in \widetilde{H}^{1/2}(S). \qquad (2.5)$$

PROOF.   Since $(\Delta+k^2)u = 0$ in Ω_S and $B \subset \Omega_S := \mathbb{R}^3 \setminus \bar{S}$
from Green's formula

$$\int_{G_j} v \cdot \Delta u \; dx + \int_{G_j} \nabla v \cdot \nabla u \; dx = (-1)^j \; \langle v, \frac{\partial u}{\partial n} \rangle_{L^2(\partial G_j)} \qquad (j = 1,2)$$

we have $v \in H^{1/2}(\partial G_j)$ and $\frac{\partial u}{\partial n} \in H^{-1/2}(\partial G_j)$ by duality. Here we
can take v with supp v $\subset\subset$ B such that the integral along ∂B
vanishes.

Hence the jump $[\frac{\partial u}{\partial n}]$ belongs to $H^{-1/2}(\partial G_1)$. By assump-
tion $u \in H^1_{loc}(\Omega_S)$, therefore away from S the relation
$(\Delta+k^2)u = 0$ implies $u \in H^3_{loc}(\Omega_S)$, and repeating this argument
implies even $u \in C^\infty$ away from S. Thus $[\frac{\partial u}{\partial n}] = 0$ on $\partial G_1 \setminus \bar{S}$.
This together with $[\frac{\partial u}{\partial n}]\big|_S \in H^{-1/2}(S)$ yields (2.4) because

$$\widetilde{H}^{-1/2}(S) = \{w \in H^{-1/2}(\partial G_1): \text{supp } w \subset \bar{S}\} = \overline{C_0^\infty(S)}^{H^{-1/2}(\partial G_1)}$$

The assertion (2.5) follows by similar arguments with

$$\widetilde{H}^{1/2}(S) = \overline{C_0^\infty(S)}^{H^{1/2}(\partial G_1)} . \qquad \qquad \square$$

The above properties of the Cauchy data enable us to
derive boundary integral equations to solve (1.1)-(1.3) even
for weak solutions $u \in L_S$.

Before we give our solution procedure let us briefly
recall the idea of layer potentials by introducing the funda-
mental solution ϕ of $(\Delta+k^2)u = 0$ in G_j (see Fig. 1).

$$\phi(z,\zeta) := \begin{cases} -\dfrac{1}{4\pi|z-\zeta|} & \text{for } k = 0 \\[2ex] \dfrac{e^{ik|z-\zeta|}}{-4\pi|z-\zeta|} & \text{for } k \neq 0 \end{cases} \qquad (2.6)$$

DEFINITION 2.3. Let $u \in C^\infty(\Gamma)$, Γ a bounded closed C^∞-surface. Then

$$V_{G_j} u(z) := -2 \int_\Gamma \phi(z,\zeta) u(\zeta) dS_\zeta \qquad (z \in G_j) \qquad (2.7)$$

$$K_{G_j} u(z) := -2 \int_\Gamma u(\zeta) \frac{\partial}{\partial n_\zeta} \phi(z,\zeta) dS_\zeta$$

The same definition of the potential of the simple and the double layer is valid for arbitrary distributions u on Γ since for $z \notin \Gamma$ the above kernels are C^∞ functions on Γ.

These potentials give the following representation formula for the solution of $(\Delta+k^2)u = 0$ in G_j.

LEMMA 2.4. For $u \in H^1_{loc}(G_j)$ $(j = 1,2)$ with Cauchy data $v := u|_{\partial G_j}$, $\psi := \frac{\partial u}{\partial n}|_{\partial G_j}$ and for $z \in G_j$ we have

$$u(z) = (-1)^j \frac{1}{2} (K_{G_j} v(z) - V_{G_j} \psi(z)) \qquad (2.8)$$

PROOF. This representation formula is well-known for smooth boundaries ∂G_j and smooth layers v and ψ. Since the operators in (2.7) have C^∞ kernels, formula (2.8) remains valid for the Cauchy data of $u \in H^1_{loc}(G_j)$, $j = 1,2$ (compare Fig. 1).□

Next, we give a representation of the weak solution of (1.1)-(1.3) via the operator of the single layer potential for the Dirichlet problem and via the operator of the normal derivative of the double layer potential in case of the Neumann problem.

Let us first consider the Dirichlet problem and proceed as in [18]. Using Green's formula we shall end up with an integral equation of the first kind for the jump of the normal derivative of the solution (1.1)-(1.3) on the open surface piece S. When we use the geometrical situation of Fig. 1 application of the representation formula (2.8) yields for $x \in G_1$

$$u(x) = -\frac{1}{2} (K_{G_1} u(x) - V_{G_1} \frac{\partial u}{\partial n}(x)),$$

$$0 = -\frac{1}{2} (K_{G_2} u(x) - V_{G_2} \frac{\partial u}{\partial n}(x)), \qquad (2.9)$$

where K_{G_j}, V_{G_j}, $j = 1,2$, are defined according to (2.7).
Note that $u|_{\partial G_j} \in H^{1/2}(\partial G_j)$ and $\frac{\partial u}{\partial n}\big|_{\partial G_j} \in H^{-1/2}(\partial G_j)$, $j = 1,2$.
Hence addition of (2.9) yields with the outer boundary
$\partial B = \{y \in \mathbb{R}^3 : |y| = R\}$ and the fundamental solution ϕ in (2.6)

$$u(x) = \int\limits_{|y|=R} u(y) \frac{\partial}{\partial n_y} \phi(x,y)\, dS_y - \int\limits_{|y|=R} \frac{\partial u}{\partial n}(y)\phi(x,y)\, dS_y$$
$$- \int\limits_{S} [\frac{\partial u}{\partial n}](y)\,\phi(x,y)\, dS_y,$$

since $[\frac{\partial u}{\partial n}]\big|_{\partial G_j \setminus \bar{S}} = 0$ according to Lemma 2.2.

For $x \to S$ the trace theorem yields together with the
boundary condition $u|_S = g$ the relation

$$g(x) = \int\limits_{|y|=R} \{u(y) \frac{\partial}{\partial n_y} \phi(x,y) - \frac{\partial u}{\partial n}(y)\phi(x,y)\}\, dS_y$$

$$- \int\limits_{S} [\frac{\partial u}{\partial n}](y)\,\phi(x,y)\, dS_y. \qquad (2.10)$$

Since the radiation condition (1.3) holds for u and ϕ, the
integral in (2.10) over $|y| = R$ vanishes as $R \to \infty$ and therefore
(2.10) becomes for $x \in S$

$$g(x) = -\int\limits_{S} [\frac{\partial u}{\partial n}](y)\,\phi(x,y)\, dS_y \qquad (2.11)$$

Now, let $[\frac{\partial u}{\partial n}] \in \widetilde{H}^{-1/2}(S)$ (compare Lemma 2.2) be a solution of
(2.11). Then the potential of the single layer

$$u(x) = -\int\limits_{S} [\frac{\partial u}{\partial n}](y)\,\phi(x,y)\, dS_y \qquad (2.12)$$

is well defined for every $x \notin S$ since $\phi(x,y) \in H^{1/2}(S)$. The
same holds for any derivative $(\frac{\partial}{\partial x})^\ell (\frac{\partial}{\partial y})^m \phi(x,y)\big|_S$. Now the
single layer potential V in (2.12) is a ψdo of order $-3/2$ as a
mapping from functions on the boundary ∂G_1 into functions on
the bounded domain G_1 and on its complement $G_2 = \mathbb{R}^3 \setminus \bar{G}_1$ [4, §8].
Thus for $\psi^* \in H^{-1/2}(\partial G_1)$ we have $V\psi^* \in H^1(G_1) \cup H^1_{loc}(G_2)$ where
$\psi^* = 0$ on $\partial G_1 \setminus \bar{S}$ and $\psi^* = [\frac{\partial u}{\partial n}]$ on S.

For $x \notin S$ we can interchange differentiation and inte-
gration in (2.12) obtaining

$$(\Delta + k^2)u = 0 \text{ in } \Omega_S = \mathbb{R}^3 \setminus \bar{S}.$$

For $|x| \geq R$ the potential (2.12) is C^∞ and satisfies the

radiation condition (1.3). Hence the potential u in (2.12)
belongs to $H^1_{loc}(\Omega_S)$.

The above results we sum up as follows:

THEOREM 2.5. $u \in H^1_{loc}(\Omega_S)$ *is the solution of the
screen Dirichlet problem* (1.1)-(1.3) *if and only if the jump*
$[\frac{\partial u}{\partial n}]|_S \in \tilde{H}^{-1/2}(S)$ *is the solution of the weakly singular inte-
gral equation*

$$V_S[\frac{\partial u}{\partial n}](x) := -2 \int_S [\frac{\partial u}{\partial n}](y)\, \phi(x,y)\, dS_y = 2g(x) \qquad (2.13)$$

for $x \in S$ *with* ϕ *as in* (2.6) *and given* $g \in H^{1/2}(S)$.

Next we consider the screen Neumann problem. Taking
in (2.9) the normal derivative we obtain for $x \in G_1$

$$\frac{\partial u}{\partial n}(x) = \int_{|y|=R} \{u(y)\, \frac{\partial}{\partial n_x}\, \frac{\partial}{\partial n_y}\, \phi(x,y)\, - \frac{\partial u}{\partial n}(y)\, \frac{\partial}{\partial n_x}\, \phi(x,y)\}\, dS_y$$

$$- \int_S [u](y)\, \frac{\partial^2}{\partial n_x \partial n_y}\, \phi(x,y)\, dS_y$$

since $[u]|_{\partial G_j \setminus \bar{S}} = 0$. Since ϕ and its derivatives satisfy (1.3)
we obtain for $x \to S$ and $R \to \infty$ together with $\frac{\partial u}{\partial n}|_S = h$ the rela-
tion

$$h(x) = -\int_S [u](y)\, \frac{\partial^2}{\partial n_x \partial n_y}\, \phi(x,y)\, dS_y \qquad (2.14)$$

Conversely, let $[u] \in \tilde{H}^{1/2}(S)$ be a solution of (2.14). Then
for $x \in \Omega_S =: \mathbb{R}^3 \setminus \bar{S}$ the potential of the double layer

$$u(x) = -\int_S [u](y)\, \frac{\partial}{\partial n_y}\, \phi(x,y)\, dS_y \qquad (2.15)$$

is well defined since $\frac{\partial}{\partial n_y}\, \phi(x,y) \in H^{-1/2}(S)$ together with all
derivatives with respect to x and y. The double layer poten-
tial K in (2.15) is a pseudodifferential operator of order -1/2
as a mapping from functions on the boundary ∂G_1 into functions
on the domains G_1 and $G_2 = \mathbb{R}^3 \setminus \bar{G_1}$ [4, §8]. Thus for
$v^* \in H^{1/2}(\partial G_1)$ we have $Kv^* \in H^1(G_1) \cup H^1_{loc}(G_2)$ where $v^* = 0$
on $\partial G_1 \setminus \bar{S}$ and $v^* = [u]$ on S.

For $x \notin S$ we can interchange differentiation and
integration obtaining $(\Delta+k^2)u = 0$ in Ω_S and for $|x| = R \to \infty$ the
potential (2.15) satisfies the radiation condition (1.3).

Summing up, we have the following equivalence:

THEOREM 2.6. $u \in H^1_{loc}(\Omega_S)$ *is the solution of the screen Neumann problem* (1.1)-(1.3) *if and only if the jump* $[u]|_S \in \widetilde{H}^{1/2}(S)$ *is the solution of the hypersingular integral equation*

$$D_S[u](x) := 2 \int_S [u](y) \frac{\partial^2}{\partial n_x \partial n_y} \phi(x,y) dS_y = -2h(x) \qquad (2.16)$$

for $x \in S$ *with* ϕ *as in* (2.6) *and given* $h \in H^{-1/2}(S)$.

We note that for the derivation of the integral equations (2.13) and (2.16) we only assumed local finite energy of the solution u of (1.1)-(1.3), i.e., $u \in H^1_{loc}(\Omega_S)$. We need no additional regularity assumptions. Using the calculus of pseudodifferential operators we derive the following existence results for the solutions of (2.13) and (2.16), respectively.

THEOREM 2.7. *Let* Im $k \geq 0$. *Then there holds:*
(i) *For given* $g \in H^{1/2}(S)$ *there exists exactly one solution* $\psi \in \widetilde{H}^{-1/2}(S)$, $\psi := [\frac{\partial u}{\partial n}]|_S$, *of the integral equation* (2.13).
(ii) *For given* $h \in H^{-1/2}(S)$ *there exists exactly one solution* $v \in \widetilde{H}^{1/2}(S)$, $v = [u]|_S$, *of the integral equation* (2.16).

The proof of Theorem 2.7 is based on the following lemma showing the coerciveness of the operators V_S and D_S in the form of a Gårding inequality in the appropriate trace spaces.

LEMMA 2.8. (i) *The mappings* $V_S: \widetilde{H}^s(S) \to H^{s+1}(S)$ *and* $D_S: \widetilde{H}^s(S) \to H^{s-1}(S)$ *are continuous for any real number s.*
(ii) *There exist constants* $\gamma_i > 0$ (i = 1,2) *and an operator* C_1 *from* $\widetilde{H}^{-1/2}(S)$ *into* $H^{\frac{1}{2}+\varepsilon}(S)$ *and an operator* C_2 *from* $\widetilde{H}^{1/2}(S)$ *into* $H^{-\frac{1}{2}+\varepsilon}$ *for some* $\varepsilon > 0$ *such that for all* $\psi \in \widetilde{H}^{-1/2}(S)$ *and* $v \in \widetilde{H}^{1/2}(S)$

$$< (V_S+C_1)\psi, \psi >_{L^2(S)} \geq \gamma_1 \|\psi\|^2_{\widetilde{H}^{-1/2}(S)} \qquad (2.17)$$

$$< (D_S+C_2)v, v >_{L^2(S)} \geq \gamma_2 \|v\|^2_{\widetilde{H}^{1/2}(S)} \qquad (2.18)$$

PROOF. For any $\psi \in \widetilde{H}^s(S)$ the extension ψ^* by zero on $\Gamma \backslash \bar{S}$ belongs to $H^s(\Gamma)$ with Γ being the closed smooth bounded manifold ∂G_1 in Fig. 1. Thus, the assertion (i) follows since

V and D are continuous mappings from $H^s(\Gamma)$ into $H^{s+1}(\Gamma)$ or into $H^{s-1}(\Gamma)$, respectively, because V and D are pseudodifferential operators of order minus one and plus one respectively (see [2], [10], [15]). The Gårding inequalities (2.17), (2.18) for V_S and D_S are consequences of the strong ellipticity of V and D, since their respective principal symbols are

$$\sigma(V)(\xi) = |\xi|^{-1}, \quad \sigma(D)(\xi) = |\xi| \tag{2.19}$$

where $|\xi| = \sqrt{\xi_1^2 + \xi_2^2} \neq 0$. By use of a partition of unity the screen S is locally mapped to the halfspace $\mathbb{R}_+^2$ and therefore (2.17), (2.18) follow from (2.19) by standard arguments (see [16]).

The first expression in (2.19) is easily obtained by applying the Fourier transformation to the kernel ϕ in (2.6) yielding the Fourier transformed kernel (see [12])

$$\hat{\phi}(\xi) = (|\xi|^2 - k^2)^{-1/2}$$

having the asymptotic expansion

$$(|\xi|^2 - k^2)^{-1/2} \sim |\xi|^{-1}\{1 + 2k^2|\xi|^{-2} + \frac{3}{8}k^4|\xi|^{-4} + \ldots\}$$
$$\text{for } |\xi| > |k| \tag{2.20}$$

This expansion into homogeneous functions of decreasing degree in the Fourier transformed variable ξ shows that the operator of the single layer potential is a pseudodifferential operator of order -1 (see also [16]).   □

PROOF OF THEOREM 2.7. From (2.17) and (2.18) follows that V and D are Fredholm operators of index zero from $\tilde{H}^{-1/2}(S)$ into $H^{1/2}(S)$ and from $\tilde{H}^{1/2}(S)$ into $H^{-1/2}(S)$, respectively. On the other hand, the integral equations (2.13) and (2.16) are equivalent to the screen Dirichlet problem and the screen Neumann problem, respectively. Due to Lemma 2.1 both problems do have no eigensolutions. Hence the above mappings V_S and D_S are injective and therefore bijective.   □

We now come to the point of our main concern -- the singularity of the densities of the integral equations (2.13), (2.16) near the edge γ of the screen S.

The following results give the asymptotic behavior of

the exact solution $[\frac{\partial u}{\partial n}]$ of the integral equation (2.13) and $[u]$ of (2.16) near the edge γ. The analysis here follows the procedure in [16] by (i) mapping locally S onto $\mathbb{R}_+^2$, (ii) applying the Wiener-Hopf technique in the halfspace $\mathbb{R}_+^2$ and (iii) patching together the local results.

THEOREM 2.9. (i) Let $g \in H^{3/2+\sigma}(S)$ be given. Then the solution of the integral equation (2.13) has the form

$$[\frac{\partial u}{\partial n}] = \beta(s)\rho^{-1/2}\chi(\rho) + \psi_r \text{ on } S \qquad (2.21)$$

with $\beta \in H^{\frac{1}{2}+\sigma}(\gamma)$, $\psi_r \in \tilde{H}^{\frac{1}{2}+\sigma'}(S)$, $0 < \sigma' < \sigma < 1/2$.
(ii) Let $h \in H^{\frac{1}{2}+\sigma}(S)$ be given. Then the solution of the integral equation (2.16) has the form

$$[u] = \alpha(s)\rho^{1/2}\chi(\rho) + v_r \text{ on } S \qquad (2.22)$$

with $\alpha \in H^{\frac{1}{2}+\sigma}(\gamma)$, $v_r \in L^2(I;H^{\frac{1}{2}+\sigma}(\gamma)) \cap \tilde{H}^{3/2+\sigma'}(I;L^2(\gamma))$, $0 < \sigma' < \sigma < 1/2$, where S is identified with $I \times \gamma$, $I = [0,1]$.

(Here s denotes the parameter of arclength of γ, ρ corresponds to the Euclidean distance to γ, χ is a C^∞ cut-off function with $\chi \equiv 1$ for $|\rho| < 1/2$ and $\chi \equiv 0$ for $|\rho| > 1$.)

PROOF. As a first step to prove the decompositions (2.21), (2.22) we discuss the halfspace case where $\Gamma = \partial G_1$ (compare Fig. 1) coincides with the plane $x_3 = 0$ and S coincides with $\mathbb{R}_+^2$ given by $x_3 = 0$ and $x_2 > 0$ such that its boundary γ becomes $\mathbb{R}^1$.

Following the ideas in [16], [17] the key of our analysis is to consider instead of (2.13), (2.16) the following equations which can be solved by the Wiener-Hopf technique:

$$p_+\hat{V}\psi_+ = 2g, \quad p_+\hat{D}_+v_+ = -2h \text{ on } \mathbb{R}_+^2 \qquad (2.23)$$

for given data

$$g \in H^{3/2+\sigma}(\mathbb{R}_+^2), \quad h \in H^{\frac{1}{2}+\sigma}(\mathbb{R}_+^2), \quad |\sigma| < 1/2 \qquad (2.24)$$

Here $\hat{V}(\hat{D})$ is the pseudodifferential operator with symbol $(\xi_2 + (|\xi_1| + 1)^2)^{(\mp)\frac{1}{2}}$, respectively, where ξ_1, ξ_2 are the dual variables to x_1, x_2 of the Fourier transformation. p_+ denotes the restriction to the halfspace $\mathbb{R}_+^2$. The operators $\hat{V}$ and $\hat{D}$ have the suitable form to perform the Wiener-Hopf technique

according to [4, §7]. In [16], [17] we derive via the factori-
zations $\hat{V} = \hat{V}_+\hat{V}_- = \hat{V}_-\hat{V}_+$ and $\hat{D} = \hat{D}_+\hat{D}_- = \hat{D}_-\hat{D}_+$ the solutions of
(2.23) as

$$\psi_+ = 2\hat{V}_+^{-1}p_+\hat{V}_-^{-1}\ell g$$

and

$$v_+ = -2\hat{D}_+^{-1}p_+\hat{D}_-^{-1}\ell h$$

where $\hat{V}_{(\overset{+}{-})}$ has the symbol $(\xi_{2(\overset{+}{-})}i(|\xi_1| + 1))^{-1/2}$ and $\hat{D}_{(\overset{+}{-})}$ has
the symbol $(\xi_{2(\overset{+}{-})}i(|\xi_1| + 1))^{\frac{1}{2}}$.

Under the assumption (2.24) we can improve the above
decompositions to

$$\psi_+ = \psi_s + \psi_r, \quad \begin{cases} \psi_s = 2\hat{V}_+^{-1}\hat{\Lambda}_+^{-1}ip'\hat{V}_-^{-1}\ell g \\ \psi_r = \hat{V}_+^{-1}\hat{\Lambda}_+^{-1}p_+\hat{\Lambda}_+\hat{V}_-^{-1}\ell g \end{cases} \quad (2.25)$$

$$v_+ = v_s + v_r, \quad \begin{cases} v_s = -2\hat{D}_+^{-1}\hat{\Lambda}_+^{-1}ip'\hat{D}_-^{-1}\ell h \\ v_r = -2\hat{D}_+^{-1}\hat{\Lambda}_+^{-1}p_+\hat{\Lambda}_+\hat{D}_-^{-1}\ell h \end{cases} \quad (2.26)$$

where p' denotes the restriction to $\mathbb{R}^1$ and $\hat{\Lambda}_{(\overset{+}{-})}$ denotes the
pseudodifferential operator with symbol $(\xi_{2(\overset{+}{-})}i(|\xi_1| + 1))$.

From (2.25), (2.26) follows $\psi_r \in \tilde{H}^{\frac{1}{2}+\sigma}(\mathbb{R}^2_+)$,
$v_r \in \tilde{H}^{3/2+\sigma}(\mathbb{R}^2_+)$. Since both $\hat{V}_+^{-1}\hat{\Lambda}_+^{-1}$ and $\hat{D}_+^{-1}\hat{\Lambda}_+^{-1}$ do not have
the transmission property, ψ_s and v_s do not belong to those
spaces: ψ_s and v_s are less regular, namely $\psi_s \in \tilde{H}^{-1/2}(S)$,
$v_s \in \tilde{H}^{1/2}(S)$.

As shown in [16], [17] with application of Fourier
transformation we deduce from (2.25), (2.26)

$$\psi_s(x) = b(x_1,x_2)\theta^+x_2^{-1/2}, \quad v_s(x) = a(x_1,x_2)\theta^+x_2^{1/2} \quad (2.27)$$

with the Heaviside function $\theta^+(x_2) = 1$ for $x_2 > 0$ and
$\theta^+(x_2) = 0$ for $x_2 < 0$. Here

$$b(\xi_1,x_2) = c\exp[-x_2(1 + |\xi_1|)]\tilde{\beta}(\xi_1)$$
$$a(\xi_1,x_2) = c\exp[-x_2(1 + |\xi_1|)]\tilde{\alpha}(\xi_1) \quad (2.28)$$

where

$$\beta := 2p'\hat{V}_-^{-1}\ell g, \quad \alpha := -2p'\hat{D}_-^{-1}\ell h, \quad (2.29)$$

and c denotes a generic constant.

We observe that the singular terms in (2.27) are <u>not</u> tensor products in x_1 and x_2. We derive in [16], [17] the claimed decompositions (2.21) and (2.22) by rewriting the singular terms, for example,

$$\widetilde{\psi}_s(\xi_1,x_2) = x_{2+}^{-1/2}\widetilde{\beta}(\xi_1) + r(\xi_1,x_2)$$

$$r(\xi_1,x_2) = x_{2+}^{-1/2}[b(\xi_1,x_2) - \widetilde{\beta}(\xi_1)]$$

and studying the regularity of the remainders in anisotropic Sobolev spaces. Applying Theorem 1.4.4 in [16] we obtain the claimed decompositions (2.21)-(2.22) for the halfspace case $S = \mathbb{R}_+^2$.

In order to prove those decompositions for a smooth, bounded, open surface S we proceed as follows: First, we observe V and $\widehat{V}$ have the same principal symbols. Therefore, they differ by a pseudodifferential operator C_1 of order $-1-\varepsilon$ for some $\varepsilon > 0$. Similarly $C_2 = D-\widehat{D}$ is a pseudodifferential operator of order $1-\varepsilon$ for some $\varepsilon > 0$. Hence we can rewrite the integral equations (2.13), (2.16) as Riesz-Schauder equations

$$[I + \widehat{V}^{-1}(V-\widehat{V})]\psi = 2\widehat{V}^{-1}g$$
$$[I + \widehat{D}^{-1}(D-\widehat{D})]v = -2\widehat{D}^{-1}h \tag{2.30}$$

Note that due to Rellich's embedding theorem the operators $V^{-1}(V-\widehat{V})$ and $D^{-1}(D-\widehat{D})$ are compact perturbations of the identity on S. The detailed analysis in [16] shows that the decompositions (2.21), (2.22) are not altered by such compact perturbations. Therefore, using standard localization techniques and patching the local results together completes the proof of Theorem 2.9.   ⬜

3. A BOUNDARY ELEMENT GALERKIN METHOD

In this section we solve the boundary integral equations (2.13), (2.16) in finite dimensional subspaces S_h^1, S_h^2 of the Sobolev spaces $\widetilde{H}^{-1/2}(S)$ and $\widetilde{H}^{1/2}(S)$, respectively. For conformity of our boundary element method, we assume that the families of finite element subspaces S_h^1, S_h^2 satisfy for integers t,k

$$S_h^1 = S_h^{t-1,k-1}(S) \subset H^{k-1}(S) \subset \tilde{H}^{-1/2}(S),$$

$$S_h^2 = S_h^{t,k}(S) \subset H^k(S) \cap \tilde{H}^1(S) \subset \tilde{H}^{1/2}(S), \quad t > k \geq 1 \tag{3.1}$$

In agreement with our general assumption (1.4) on S
we can assume that S is given by local representations such
that regular partitions in the parameter domains are mapped
onto a corresponding partition of S. On the partitions in the
parameter domains we use a regular (t,k)-system, called $S_h^{t,k}$,
of finite elements. Then the local representation of S trans-
plants these finite element functions onto S. In their coordi-
nates the finite elements appear as simple functions over the
parameter domains. The parameters in $S_h^{t,k}$ have the following
meanings: h, $0 < h \leq h_0$, is the mesh size of the partition of S,
for example h stands for the longest side of a triangle in a uni-
form triangulation; t-1 is the degree of piecewise polynomials
constituting the corresponding finite element, k describes the
conformity $S_h^{t,k} \subset H^k(S)$.

Now the <u>Galerkin procedures</u> to (2.13) and (2.16) read
as follows:

<u>For the Dirichlet screen problem find</u> $\psi_h \in S_h^{t-1,k-1}(S)$ <u>such that</u>

$$\langle V_S \psi_h, \underset{\sim}{\phi} \rangle_{L^2(S)} = \langle 2g, \underset{\sim}{\phi} \rangle_{L^2(S)} \tag{3.2}$$

<u>for all</u> $\underset{\sim}{\phi} \in S^{t-1,k-1}$ <u>with t,k as in</u> (3.1).

<u>For the Ñeumann screen problem find</u> $v_h \in S_h^{t,k}(S)$ <u>such that</u>

$$\langle D_S v_h, w_h \rangle_{L^2(S)} = -\langle 2h, w_h \rangle_{L^2(S)} \tag{3.3}$$

<u>for all</u> $w_h \in S_h^{t,k}(S)$ <u>with t,k as in</u> (3.1).

The solvability of the above Galerkin equations and
the convergence of the procedures are based on the Gårding
inequalities (2.17), (2.18) and the uniqueness of the integral
equations (2.13), (2.16). Application of the general results
on the Galerkin procedure for strongly elliptic pseudodifferen-
tial equations ([7], [18], [21], [22]) yields the following
theorem.

THEOREM 3.1. *(i) There exists a meshwidth* $h_0 > 0$
such that the Galerkin equations (3.2), (3.3) *are uniquely*

solvable for any h, $0 < h \leq h_0$.

(ii) For decreasing meshsize h → 0 *the Galerkin solutions*
$\psi_h \in S_h^{t-1,k-1}(S)$ *and* $v_h \in S_h^{t,k}(S)$ *converge to the exact solu-*
tions of (2.13), (2.16), $\psi \in \tilde{H}^{-1/2}(S)$ *and* $v \in \tilde{H}^{1/2}(S)$.
Furthermore, we have the quasi-optimal asymptotic error esti-
mates

$$\| \psi - \psi_h \|_{-1/2,S} \leq c \inf_{\underset{\sim}{\chi} \in S_h^{t-1,k-1}} \| \psi - \chi \|_{-1/2,S} \tag{3.4}$$

$$\| v - v_h \|_{1/2,S} \leq c \inf_{\underset{\sim}{y} \in S_h^{t,k}} \| v - y \|_{1/2,S} \tag{3.5}$$

where $\| \ \|_{r,S}$ *denotes the norm in* $\tilde{H}^r(S)$ *and the constant* c *is*
independent of ψ, ψ_h, v, v_h *and* h.

Due to Theorem 2.9, the exact solutions ψ,v of
$V_S \psi = 2g$ and of $D_S v = -2h$ behave like $\rho^{-1/2}$ and $\rho^{1/2}$, respec-
tively, where ρ is the distance to the boundary γ of the screen
S, i.e., ψ is not square integrable on S. Since $\rho^{1/2} \in H^{1-\epsilon}$
for some $\epsilon > 0$ the estimates (3.4), (3.5) give at most conver-
gence of order $\frac{1}{2} - \epsilon$. As in the two-dimensional case [19], we
can improve the asymptotic rate of convergence by using so-
called singular elements as test and trial functions in the
Galerkin procedures (3.2) and (3.3). Thus we augment the stan-
dard finite element spaces $S_h^{t,k}$ by special global singular
elements according to the form of the exact solution given in
Theorem 2.9. This gives the augmented finite element spaces
$z_h^{1/2}(S)$, $z_h^{3/2}(S)$ on S:

$$z_h^{1/2}(S) := \{ \psi = \underset{\sim}{\psi}_r + \underset{\sim}{\beta} \rho^{-1/2} \chi : \underset{\sim}{\beta} \in S_h^{t',\ell}(\gamma), \underset{\sim}{\psi}_r \in \overset{\circ}{S}_h^{t-1,k-1}(S) \}$$
$$\tag{3.6}$$
$$z_h^{3/2}(S) := \{ \underset{\sim}{v} = \underset{\sim}{v}_r + \underset{\sim}{\alpha} \rho^{1/2} \chi : \underset{\sim}{\alpha} \in S_h^{t',\ell}(\gamma), \underset{\sim}{v}_r \in \overset{\circ}{S}_h^{t,k}(S) \}$$

where $\underset{\sim}{\alpha}, \underset{\sim}{\beta} \in S_h^{t',\ell}(\gamma) \subset H^1(\gamma); \underset{\sim}{\psi}_r \in \overset{\circ}{S}_h^{t-1,k-1}(S) \subset H^{k-1}(S) \cap \tilde{H}^1(S);$

$$\underset{\sim}{v}_r \in \overset{\circ}{S}_h^{t,k}(S) \subset H^k(S) \cap \tilde{H}^2(S). \tag{3.7}$$

with $t' > \ell \geq 1$; $t',\ell \in \mathbb{N}$ and t,k as in (3.1).

The inclusions (3.7) yield $t \geq 3$, $k \geq 2$, i.e., the
simplest conform elements are continuous piecewise linear, one-

dimensional elements $\underset{\sim}{\alpha}, \underset{\sim}{\beta}$ on γ, piecewise linear elements[+] $\underset{\sim}{\psi}_r$ on
S with $\underset{\sim}{\psi}_r \in C^0(S)$ and piecewise quadratics $\underset{\sim}{v}_r$ on S with
$\underset{\sim}{v}_r \in C^1(S)$ satisfying on γ

$$\underset{\sim}{v}_r = \frac{\partial}{\partial x_1} \underset{\sim}{v}_r = \frac{\partial}{\partial x_2} \underset{\sim}{v}_r = 0 \quad \text{and} \quad \underset{\sim}{\psi}_r = 0 \text{ on } \gamma.$$

If we introduce the notation $z^{\frac{1}{2}+\sigma}$, $z^{3/2+\sigma}$ for the set of all
distributions having the form (2.21), (2.22) with the assigned
regularity in Theorem 2.9, and with $\alpha \in H^{3/2+\sigma}(\gamma)$ then there
holds the following conformity for the augmented finite element
spaces $z_h^{1/2}(S)$, $z_h^{3/2}(S)$ in (3.6):

$$z_h^{1/2}(S) \subset z^{\frac{1}{2}+\sigma}(S) \subset \tilde{H}^{-1/2}(S)$$

(3.8)

and

$$z_h^{3/2}(S) \subset z^{3/2+\sigma}(S) \subset \tilde{H}^{1/2}(S)$$

Our <u>improved Galerkin schemes</u> now read as:
<u>For the Dirichlet screen problem find</u> $\psi_h = \beta_h \rho^{-1/2}\chi + \psi_h^r \in z_h^{1/2}(S)$
such that

$$\langle V_s \psi_h, \underset{\sim}{\phi} \rangle_{L^2(S)} = \langle 2g, \underset{\sim}{\phi} \rangle_{L^2(S)}$$

(3.9)

<u>for all</u> $\underset{\sim}{\phi} = \beta \rho^{-1/2}\chi + \underset{\sim}{\phi}^r \in z_h^{1/2}(S)$.
<u>For the Neumann screen problem find</u> $v_h = \alpha_h \rho^{1/2}\chi + v_h^r \in z_h^{3/2}(S)$
such that

$$\langle D_s v_h, \underset{\sim}{v} \rangle_{L^2(S)} = -\langle 2h, \underset{\sim}{w} \rangle_{L^2(S)}$$

(3.10)

<u>for all</u> $\underset{\sim}{w} = \underset{\sim}{\alpha} \rho^{1/2}\chi + \underset{\sim}{v}^r \in z_h^{3/2}(S)$.

Now the above Galerkin equations with test functions
$\underset{\sim}{w} \in z_h^{3/2}(S)$, $\underset{\sim}{\phi} \in z_h^{1/2}(S)$ define quadratic systems of linear
equations for the unknown coefficients of $\alpha_h, \beta_h \in S_h^{t',\ell}(\gamma)$,
$v_h^r \in \overset{\circ}{S}_h^{t,k}(S)$ and $\psi_h^r \in \overset{\circ}{S}_h^{t-1,k-1}(S)$. There holds the following
result:

THEOREM 3.2.*(i) There exists a meshwidth* $h_0 > 0$ *such
that the Galerkin equations* (3.9), (3.10) *in the augmented
finite element spaces* $z_h^{1/2}(S)$, $z_h^{3/2}(S)$, *are uniquely solvable
for any h,* $0 < h \le h_0$.

[+] $\underset{\sim}{\psi}_r$ is a linear polynomial in two variables on each triangle of
a quasi-uniform triangulation of S.

(ii) For decreasing meshsize h → 0 we have for the exact solutions ψ of (2.13) and v of (2.16) and for the Galerkin solutions

$$\psi_h = \beta_h\, \rho^{-1/2}\chi + \psi_h^r \in Z_h^{1/2}(S), \qquad v_h = \alpha_h \rho^{1/2}\chi + v_h^r \in Z_h^{3/2}(S)$$

of (3.9), (3.10)

$$\|\psi - \psi_h\|_{-1/2,S} \;\le\; c \inf_{\eta \in Z_h^{1/2}(S)} \|\psi - \eta\|_{-1/2,S} \;\le\; ch^{1+\sigma}\, \|\psi\|_{Z^{\frac{1}{2}+\sigma}(S)} ,$$

$$\tag{3.11}$$

$$\|v - v_h\|_{1/2,S} \;\le\; c \inf_{\underline{u} \in Z_h^{3/2}(S)} \|v - \underline{u}\|_{1/2,S} \;\le\; ch^{1+\sigma}\, \|v\|_{Z^{3/2+\sigma}(S)} .$$

The arising norms are defined via the notation in Theorem 3.1 as follows ($\|\cdot\|_{q,\gamma}$ denotes the Sobolev norm in $H^q(\gamma)$):

$$\|v\|_{Z^q(S)} := \begin{cases} \|\alpha\sqrt{\rho}\chi(\rho) + v_r\|_{q,S}\,, & \frac{1}{2} \le q < 1-\varepsilon,\ \varepsilon>0 \text{ arbitrary} \\[2mm] \|\alpha\|_{q,\gamma} + \|v_r\|_{q,S}\,, & 1 \le q \le \frac{3}{2} + \sigma. \end{cases} \tag{3.12}$$

$$\|\psi\|_{Z^p(S)} := \begin{cases} \|-\frac{\alpha}{2}\rho^{-1/2} + \psi_r\|_{p,S}\,, & -\frac{1}{2} \le p < -\varepsilon, \\[2mm] \|\alpha\|_{p,\gamma} + \|\psi_r\|_{p,S}\,, & 0 \le p \le \frac{1}{2} + \sigma. \end{cases}$$

PROOF. (i) By the inclusions (3.8) the augmented Galerkin schemes (3.9), (3.10) are uniquely solvable for small enough h since the Gårding inequalities for V_s and D_s hold in the respective energy space $\tilde{H}^{-1/2}(S)$ and $\tilde{H}^{1/2}(S)$. Furthermore, the integral equations (2.13), (2.16) are uniquely solvable in thos spaces.

The estimate in (ii) is a consequence of Theorem 3.1 due to the conformity (3.8) and of the following convergence property (3.13), (3.14) of the augmented finite element spaces $Z_h^{3/2}(S)$, $Z_h^{1/2}(S)$.   □

LEMMA 3.3. *The finite element spaces $Z_h^{3/2}(S)$, $Z_h^{1/2}(S)$ have the approximation properties: For any $U = \alpha\sqrt{\rho}\chi(\rho) + v_r \in Z^s(S)$ there exists a $\tilde{U} = \underline{\alpha}\sqrt{\rho}\chi(\rho) + \underline{v}^r \in Z_h^{3/2}(S)$ with $t \ge s$ and a*

constant c > 0 independent of h and U such that for q ≤ min{k,s}

$$\|U-\tilde{U}\|_{Z^q(S)} \leq ch^{s-q}\|U\|_{Z^s(S)} \quad , \quad \frac{1}{2} \leq q \leq s \leq 2-\varepsilon \quad (3.13)$$

For any $\Psi = \beta\rho^{-1/2}\chi(\rho) + \psi_r \in Z^{s'}(S)$ *there exists a*
$\tilde{\Psi} = \beta\rho^{-1/2}\chi(\rho) + \underset{\sim}{\psi}^r \in Z_h^{1/2}(S)$ *with t-1 ≥ s' and a constant*
c > 0 independent of h and Ψ such that for $q' \leq min\{k-1,s'\}$

$$\|\Psi-\tilde{\Psi}\|_{Z^{q'}(S)} \leq ch^{s-q'}\|\Psi\|_{Z^{s'}(S)} \quad . \quad (3.14)$$

PROOF. For $|\sigma| < 1/2$ and arbitrary $\varepsilon > 0$ there holds with S substituted by $I \times \gamma$, $I = [0,1]$,

$$\|\sqrt{\rho}(\alpha-\underset{\sim}{\alpha})\|_{L^2(I,H^{\frac{1}{2}+\sigma}(\gamma))} \leq c\|\alpha-\underset{\sim}{\alpha}\|_{H^{\frac{1}{2}+\sigma}(\gamma)}$$

$$\|\sqrt{\rho}(\alpha-\underset{\sim}{\alpha})\|_{H^{\frac{1}{2}+\sigma}(I,L^2(\gamma))} \leq c\|\alpha-\underset{\sim}{\alpha}\|_{H^\varepsilon(\gamma)} \quad . \quad (3.15)$$

The estimates (3.15) imply for $\frac{1}{2} \leq q < 1-\varepsilon$ and $1-\varepsilon \leq s < 2-\varepsilon$ with (3.12) and the triangle inequality

$$\|U-\tilde{U}\|_{Z^q(S)} \leq \|\sqrt{\rho}(\alpha-\underset{\sim}{\alpha})\|_{q,S} + \|v_r-\underset{\sim}{v}^r\|_{q,S}$$

$$\leq ch^{s-q}\{(1+h^{q-\varepsilon})\|\alpha\|_{s,\gamma} + \|v_r\|_{s,S}\}$$

$$\leq ch^{s-q}\|U\|_{Z^s(S)} \quad (3.16)$$

Here we have used $\|\alpha-\underset{\sim}{\alpha}\|_{H^q(\gamma)} \leq ch^{s-q}\|\alpha\|_{H^s(\gamma)}$ and
$\|v_r-\underset{\sim}{v}^r\|_{q,S} \leq ch^{s-q}\|v_r\|_{s,S}$ which hold by the standard approximation property of $S_h^{t',\ell}(\gamma)$ and $\overset{\circ}{S}_h^{t,k}(S)$, respectively [21].

For $1 \leq q \leq s \leq 3/2 + \sigma$ we have with (3.12)

$$\|U-\tilde{U}\|_{Z^q(S)} = \|\alpha-\underset{\sim}{\alpha}\|_{q,\gamma} + \|v_r-\underset{\sim}{v}^r\|_{q,S}$$

$$\leq ch^{s-q}(\|\alpha\|_{s,\gamma} + \|v_r\|_{s,S})$$

$$= ch^{s-q}\|U\|_{Z^s(S)} \quad .$$

For $s < 1-\varepsilon$ we take $\tilde{U} = \underset{\sim}{v}^r \in \overset{\circ}{S}_h^{t,k}(S)$ in (3.16) completing (3.13). The estimate (3.14) holds analogously. Finally we

note that the estimates (3.15) are verified as in the proof
of Theorem 1.4.4 in [16]. And for completeness we recall from
[11] for any real number $s \geq 0$

$$H^s(I \times \gamma) = \{u: u \in L^2(I;H^s(\gamma)), u \in H^s(I;L^2(\gamma))\},$$

with the equivalence of norms

$$\|u\|^2_{H^s(I \times \gamma)} \approx \|u\|^2_{L^2(I;H^s(\gamma))} + \|u\|^2_{H^s(I;L^2(\gamma))} . \qquad \square$$

Finally, we remark that for problems in acoustics we
have in (1.2)

$$g = -u_i , \quad h = - \frac{\partial u_i}{\partial n}$$

where u_i denotes the incident field with a source away from
the scattering screen S. Therefore the given data for the scat-
tered field u in (1.2) are infinitely smooth. In this case the
proof of Theorem 2.9 (compare (2.25)-(2.29)) shows that the
unknowns in our integral equations (2.13), (2.16) have still
the form (2.21), (2.22) but with smooth line layers $\alpha, \beta \in C^\infty(\gamma)$
for a smooth edge curve γ. Following our above analysis we
expect further improvement for the Galerkin method if further
singular elements like $\rho^{3/2}, \rho^{5/2}, \ldots$ are used. This is indi-
cated for the two-dimensional case in [19]. For an analysis of
errors arising from a finite element approximation of the screen
surface S we refer to [13].

REFERENCES

[1]   Copson, E. T.: On the problem of the electrified disc,
        Proc. Edin. Math. Soc. 8, 14 (1947).

[2]   Costabel, M., Stephan, E.: A direct boundary integral
        equation method for transmission problems, J. Math.
        Anal. Appl. 106 (1985), 367-413.

[3]   Durand, M.: Layer potentials and boundary value prob-
        lems for the Helmholtz equation in the complement of
        a thin obstacle, Math. Meth. Appl. Sci. 5 (1983),
        389-421.

[4] Eskin, G. I.: Boundary problems for elliptic pseudo-
     differential operators, Transl. of Math. Mon., American
     Mathematical Society 52, Providence Rhode Island
     (1981).

[5] Hayashi, J.: The expansion theory of the Dirichlet prob-
     lem for the Helmholtz equation for an open boundary,
     J. Math. Anal. Appl. 61 (1977), 331-340.

[6] Hayashi, J.: Three dimensional Dirichlet problem for
     the Helmholtz equation for an open boundary, Proc.
     Japan Acad. 53, Ser. A (1977), 159-162.

[7] Hildebrandt, St., Wienholtz, E.: Constructive proofs of
     representation theorems in separable Hilbert space, Comm.
     Pure Appl. Math. 17 (1964), 369-373.

[8] Hönl, H., Maue, A. W., Westpfahl, K.: Theorie der
     Beugung, "Handbuch der Physik" 25/1, S. Flügge, ed.,
     Springer-Verlag, Berlin (1961).

[9] Hörmander, L.: Linear Partial Differential Operators,
     Springer-Verlag, Berlin (1969).

[10] Hsiao, G. C., Wendland, W. L.: A finite element method
     for some integral equations of the first kind, J. Math.
     Anal. Appl. 58 (1977), 449-481.

[11] Lions, J. L., Magenes, E.: Non-Homogeneous Boundary
     Value Problems and Applications I, Berlin -Heidelberg -
     New York, Springer (1972).

[12] MacCamy, R. C., Stephan, E.: Solution procedures for
     three-dimensional eddy current problems, J. Math. Anal.
     Appl. 101 (1984), 348-379.

[13] Nedelec, J. C.: Curved finite methods for the solution of
     singular integral equations on surfaces in $\mathbb{R}^3$, Comp.
     Meth. Appl. Mech. Engin. 8 (1976), 61-80.

[14] Seeley, R.: Topics in pseudo-differential operators,
     Pseudo-Differential Operators, L. Nirenberg, ed., Roma,
     C.I.M.E., Cremonese (1969), 168-305.

[15] Stephan, E.: Solution procedures for interface problems
     in acoustics and electromagnetics, in CISM, Courses and
     Lectures No. 277, Springer-Verlag, Wien - New York
     (1983), 291-348.

[16] Stephan, E. P.: Boundary integral equations for mixed
     boundary value problems, screen and transmission prob-
     lems in $\mathbb{R}^3$, Habilitationsschrift (THD-Preprint 848,
     Darmstadt) (1984).

[17]   Stephan, E. P.: Boundary integral equations for magnetic
       screens in $\mathbb{R}^3$, Proceedings A of the Royal Society
       Edinburgh (1985), in print.

[18]   Stephan, E., Wendland, W. L.: Remarks to Galerkin and
       least squares methods with finite elements for general
       elliptic problems, Lecture Notes Math., Springer,
       Berlin, 564 (1976), 461-471, Manuscripta Geodaetica 1
       (1976), 93-123.

[19]   Stephan, E., Wendland, W. L.: An augmented Galerkin
       procedure for the boundary integral method applied to
       two-dimensional screen and crack problems, Applicable
       Analysis 18 (1984), 183-219.

[20]   Taylor, M.: Pseudodifferential Operators, Princeton,
       University Press, 1981.

[21]   Wendland, W. L.: On the asymptotic convergence of some
       boundary element methods, Mafelap IV, J. Whiteman,
       ed., London - New York - San Francisco, Academic Press
       (1982), 281-312.

[22]   Wendland, W. L.: Boundary element methods and their
       asymptotic convergence in CISM, Courses and Lectures No.
       277, Springer Verlag, Wien - New York (1983), 135-216.

[23]   Wilcox, C. H.: Scattering Theory for the d'Alembert
       Equation in Exterior Domains, Lecture Notes Math. 442,
       Springer Verlag, Berlin - Heidelberg - New York (1969).

School of Mathematics,
Georgia Institute of Technology,
Atlanta, GA   30332,
U.S.A.