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Large deviations for Langevin spin glass dynamics

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Summary. We study the asymptotic behaviour of asymmetrical spin glass dynamics in a Sherrington–Kirkpatrick model as proposed by Sompolinsky–Zippelius. We prove that the annealed law of the empirical measure on path space of these dynamics satisfy a large deviation principle in the high temperature regime. We study the rate function of this large deviation principle and prove that it achieves its minimum value at a unique probability measure Q which is not markovian. We deduce that the quenched law of the empirical measure converges to δ_Q. Extending then the preceeding results to replicated dynamics, we investigate the quenched behavior of a single spin. We get quenched convergence to Q in the case of a symmetric initial law and even potential for the free spin.

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1 Introduction

The Sherrington–Kirkpatrick model is a mean field simplification of the spin glass model of Edwards–Anderson. The behaviour of its static characteristics such as its partition function has been intensively studied by physicists (see [9] for a broad survey). There are few mathematical results available (except for [1, 3, 17]).

In [9], it is argued that studying dynamics might be simpler since it avoidsusing the "replica trick" and the Parisi ansatz for symmetry-breaking which are yet to be put on firm ground. It seems that, in the physics literature, the first attempt to study the dynamics of Sherrington–Kirkpatrick is due to Sompolinsky and Zippelius (see [16]), who chose a Langevin dynamics scheme.

We follow here this strategy with some technical restrictions explained below.

Our aim was to understand Chap. V of [9] from a mathematical point of view.

Roughly speaking, the first conclusion to be drawn from [9] is that the limiting dynamics are not markovian and seem rather mysterious. One of our goals is to derive the law of those dynamics by means of a large deviation principle.

Our approach builds upon the strategy developed for a much simpler mean-field dynamics problem; i.e. the large deviation approach to study propagation of chaos for mean field interacting diffusions, and subsequently convergence to McKean–Vlasov dynamics (see [2, 4, 15, 18]). To be more specific, let us recall that the Sherrington–Kirkpatrick hamiltonian is given by $H_J(x) = \frac{1}{\sqrt{N}}\sum_{i,j} J_{ij}x_i x_j$, for $x = (x_1, \dots, x_N) \in \{-1, 1\}^N$, where the randomness in the spin glass is here modelled by the $(J_{ij})_{i\leqq j}$ which are i.i.d. standard centered gaussian random variables, and where $J_{ij} = J_{ji}$. The Gibbs measure one would like to study (for N large) is given by

$$\frac{e^{-\beta H_J(x)}}{Z_N(J)}\alpha^{\otimes N}(dx),$$

where $\alpha = \frac{1}{2}(\delta_{-1} + \delta_1)$ and β is the inverse of temperature.
$Z_N(J)$ is the partition function:

$$Z_N(J) = \frac{1}{2^N}\sum_{x\in\{-1,1\}^N} e^{-\beta H_J(x)}.$$

If one replaces the hard spins $\{-1, +1\}$ by continuous spins, i.e. by spins taking values in $\mathbb{R}$, or as we shall see in a bounded interval of $\mathbb{R}$, and if one replaces the measure $\alpha = \frac{1}{2}(\delta_{-1} + \delta_1)$ by $\alpha = e^{-2U(x)}/\int e^{-2U(x)dx}\,dx$, where U is, for instance, a double well potential on $\mathbb{R}$, then, the Langevin dynamics for this problem are given by

$$dx_t^j = dB_t^j - \nabla U(x_t^j)\,dt - \frac{\beta}{\sqrt{N}}\sum_{1\leqq i\leqq N} J_{ji}x_t^i\,dt\,, \tag{1}$$

where B is a N-dimensional brownian motion.

We want to understand the limiting behavior (for large N) of this system of randomly interacting diffusions.

We will need two simplifying features:

First, we will study only bounded spins, i.e. we will assume that $U(x)$ is defined on a bounded interval $[-A, A]$ and tends to infinity when $|x| \to A$ sufficiently fast to insure our spins x^j stay in the interval $[-A, A]$.

The second simplifying feature is that we will assume that the whole matrix $(J_{ij})_{i,j}$ is made of i.i.d $N(0, 1)$ random variables and we will not impose the symmetry $J_{ij} = J_{ji}$.

Our first goal is to study the empirical measure $\hat{\mu}^N = \frac{1}{N}\sum_{i=1}^N \delta_{x^i}$ on path space. There is no reason for this to be a simple problem, since, for fixed interaction J, the variables $(x^1, \dots, x^N)$ are not exchangeable. So, we first study the law of the empirical measure $\hat{\mu}^N$ averaged on the interaction.

The main result of this paper is a large deviation principle for this averaged law in a large temperature (or short time) regime $\beta^2 A^2 T < 1$ which entails the convergence of the empirical measure to the unique minimum of the good rate function which governs this large deviation principle. This minimum is a probability measure, say Q, on path space that we describe explicitly as the law of a non-markovian, highly non-linear, solution of a stochastic differential equation. The existence and uniqueness problems for this limit law are not obvious and are the analogue here of the existence and uniqueness problem for McKean–Vlasov diffusions in the mean field interacting diffusion context as obtained in [14].

As a consequence, we show that the quenched law (i.e. the law with given interaction) of the *empirical measure* converges exponentially fast to δ_Q in the high temperature (or short time) regime. In particular, if $\beta^2 A^2 T < 1$, Q describes the asymptotic mean behavior of spin glass dynamics since, for any bounded continuous function f on W_T^A, for almost all J, for almost all $(x^i)_{1 \leqq i \leqq N}$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} f(x^i) = \int f \, dQ \, .$$

Since the variables $(x^1, \ldots, x^N)$ are not exchangeable, this result is not enough to get convergence for the quenched law of a *single* spin. Thus, we investigate the quenched behavior of a single spin using *replicated* systems and get only very preliminary results.
It might well be that the model we have chosen is unnecessarily complicated. The difficult features are here due to the fact that we are working in continuous time, and on a continuous spin state space with boundary problems. The same study on discrete space, or compact manifolds, is of course possible and might be easier and more transparent.

The organization of the paper is as follows:

We give the notations and the results in Sect. 2.

In Sect. 3, we establish a large deviation principle for a time discretization of the system, which represents only a necessary technical step.

In Sect. 4, we get from Sect. 3 the full large deviation principle in the high temperature (or short time) regime.

In Sect. 5, we study the minima of the rate function which governs those large deviations results, show their uniqueness, and give a pathwise non-markovian description of this probability measure on path space.

Finally, in Sect. 6, we introduce replica to get a first understanding (in tune with [9]) of the nature of this limit law and also to get some preliminary quenched results. We thank gratefully the referees for their very competent reading and their suggestions.

2 Statement of the results

We begin by describing the system of randomly interacting diffusions we want to study.

Let A be a strictly positive real ($A > 1$) and U be a C^2 function on the interval $]-A, A[$ such that U tends to infinity, when $|x| \to A$, sufficiently fast to insure that

$$\lim_{\substack{|x|\to A \\ <}} k_U(x) = +\infty ,$$

where $k_U(x) = 2\int_0^x \exp 2U(y)\left(\int_0^y \exp - 2U(z)\,dz\right)dy$.

Remark.. We can take $U(x) = -\log(A^2 - x^2)$.
For any number N of particles, any temperature $(= \frac{1}{\beta})$ and $J = (J_{ij})_{1 \leqq i,j \leqq N} \in \mathbb{R}^{N\times N}$, we consider the following system $\mathscr{S}_\beta^N(J)$ of interacting diffusions:

$$\mathscr{S}_\beta^N(J) = \begin{cases} dx_t^j = -\nabla U(x_t^j)\,dt + dB_t^j + \dfrac{\beta}{\sqrt{N}}\displaystyle\sum_{i=1}^N J_{ji}x_t^i\,dt & \forall 1 \leqq i \leqq N , \\ \text{Law of } x_0 = \mu_0^{\otimes N} , \end{cases}$$

where $(B^j)_{1\leqq j \leqq N}$ is a N-dimensional brownian motion and μ_0 a probability measure on $[-A,A]$ which does not put mass on the boundary $\{-A,+A\}$.

Proposition 2.1 *For each $J \in \mathbb{R}^{N\otimes N}$, $\mathscr{S}_\beta^N(J)$ has a unique weak solution.*

In the following pages, we shall focus on the evolution of this system until a time T. We will call $P_\beta^N(J)$ the weak solution of $\mathscr{S}_\beta^N(J)$ restricted to the σ-algebra $\sigma(x_s^i,\ 1 \leqq i \leqq N,\ s \leqq T)$.

Proof of Proposition 2.1. Proposition 2.1 is a direct consequence of Girsanov theorem and of the following very classical lemma (see [13, p. 357] (Criterion for explosions)).

Lemma 2.2 *Let $\left(\Gamma,(\mathscr{G}_t)_{t\geqq 0},\mathscr{G},p\right)$ be a probability space on which a brownian motion $(B_t)_{t\geqq 0}$ lives. Then there exists a unique strong solution to the stochastic differential equation $(\mathscr{S})$:*

$$(\mathscr{S}) = \begin{cases} dx_t = -\nabla U(x_t)\,dt + dB_t , \\ \textit{Law of } x_0 = \mu_0 . \end{cases}$$

Moreover, if $T_\varepsilon = \inf\{s/|x_s| \geqq A-\varepsilon\}$. Then $\forall T$, $\quad P(T_\varepsilon \leqq T) \leqq \frac{\exp T}{1+k_U(A-\varepsilon)}$, so that $P\left(\lim_{\varepsilon\downarrow 0} T_\varepsilon = +\infty\right) = 1$.

Notation. We shall note P the law of this solution restricted to the σ-algebra $\mathscr{F}_T = \sigma(x_s, s \leqq T)$. Lemma 2.2 implies that P is a probability measure on the space W_T^A of continuous functions on $[0,T]$ with values in $[-A,A]$.

As a consequence, Girsanov theorem shows that, for any $J \in \mathbb{R}^{N\otimes N}$, $P_\beta^N(J)$ is a probability measure on $(W_T^A)^N$.

We want to study the behaviour of the empirical law $\widehat{\mu}_N = \frac{1}{N}\sum_{i=1}^N \delta_{x^i}$, under $P_\beta^N(J)$, when J is a random matrix with independent standard centered gaussian entries ($J_{ij} \sim N(0,\,1)$).

We will first study the law of $\widehat{\mu}_N$ averaged on J, and then deduce some quenched results, i.e. the results we can find for a given interaction (or disorder) shape (i.e. the J almost sure properties).

More precisely, let $(\Omega,\mathscr{A},\gamma)$ be a probability space and J_{ij} be i.i.d random variables on Ω such that the J_{ij} are, under γ, standard centered gaussian variables.

We first remark that $P_\beta^N(J)$ is a measurable function of J, indeed Girsanov theorem shows that $P_\beta^N(J)$ is absolutely continuous with respect to $P^{\otimes N}$ and that:

$$\frac{dP_\beta^N(J)}{dP^{\otimes N}} = \exp\sum_{j=1}^{N}\left\{\beta\int_0^T\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}J_{ji}x_t^i\right)dB_t^j - \frac{\beta^2}{2}\int_0^T\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}J_{ji}x_t^i\right)^2 dt\right\}$$

and it is obvious that this density is a measurable function of J.
Hence, we can define a probability measure Q_β^N on $(W_T^A)^N$ by

$$Q_\beta^N = \int P_\beta^N(J(\omega))\,d\gamma(\omega)\,.$$

Let $\Pi_{\beta,T}^N$ be the law of the empirical measure under Q_β^N. $\Pi_{\beta,T}^N$ is a probability measure on the set $\mathcal{M}_1^+(W_T^A)$ of the probability measures on W_T^A which is defined by

$$\begin{aligned}\Pi_{\beta,T}^N(B) &= Q_\beta^N(\hat{\mu}^N \in B)\\ &= \int P_\beta^N(J(\omega))(\hat{\mu}^N \in B)\,d\gamma(\omega)\end{aligned}$$

for any measurable subset B of $\mathcal{M}_1^+(W_T^A)$.
The main result of this paper is:

Theorem 2.3 *There exists a good rate function H on $\mathcal{M}_1^+(W_T^A)$ such that if $\beta^2A^2T < 1$, $\Pi_{\beta,T}^N$ satisfies a full large deviation principle with rate function H, i.e*:

For any open subset O of $\mathcal{M}_1^+(W_T^A)$ $\quad \underline{\lim}_{N\to\infty}\frac{1}{N}\log\Pi_{\beta,T}^N(O) \geqq -\inf_O H$

For any closed subset F of $\mathcal{M}_1^+(W_T^A)$ $\quad \overline{\lim}_{N\to\infty}\frac{1}{N}\log\Pi_{\beta,T}^N(F) \leqq -\inf_F H$

A complete description of H is given in Sect. 4, Theorem 4.1.

The proof is given in Sect. 4, Theorem 4.1, after a preliminary study in Sect. 3 of a time discretized version of the dynamics.

To get a convergence result for $\Pi_{\beta,T}^N$, we need to investigate the minima of H.

Theorem 2.4 *H achieves its minimal value at a unique probability measure Q on W_T^A which is implicitly given by the following procedure*:

Let $P(h)$ be the law of the diffusion on W_T^A solution of

$$\begin{cases} dx_t = -\nabla U(x_t)\,dt + dB_t + \beta h_t\,dt\,,\\ \textit{Law of } x_0 = \mu_0\end{cases}$$

for a determinist process h in $L^2([0,T],dt)$.

Then, Q satisfy the non-linear equation

$$Q = \int P(G(\omega))\,d\gamma_Q(\omega)\,,$$

where γ_Q is the law of a centered gaussian process G with covariance

$$\int G_s G_t \, d\gamma_Q = \int x_s x_t \, dQ .$$

We can elucidate the non-markovian character of Q by

Theorem 2.5 *Q is the unique solution of the stochastic differential system:*

$$\begin{cases} x_t = x_0 - \int_0^t \nabla U(x_s)\, ds + B_t , \\ B_t = W_t + \beta^2 \int_0^t \int_0^s \widetilde{K}_Q^s(s,u)\, dB_u \, ds , \\ K_Q(t,s) = \int x_t x_s \, dQ(x) , \\ \text{Law of } x = Q, \ Q|_{\mathscr{F}_0} = \mu_0 , \end{cases}$$

where $(W_t)_{t \geqq 0}$ is a brownian motion under Q, and, for any continuous covariance K on $[O,T]^2$, for any $t \leqq T$, $\widetilde{K^t}$ is the covariance given by

$$\widetilde{K^t}(s,u) = \frac{\int \exp\left\{-\frac{\beta^2}{2}\int_0^t G_u^2(\omega)\, du\right\} G_s(\omega) G_u(\omega)\, d\gamma_K(\omega)}{\int \exp\left\{-\frac{\beta^2}{2}\int_0^t G_u^2(\omega)\, du\right\} d\gamma_K(\omega)} ,$$

where G is, under γ_K, a centered gaussian process with covariance K.

Theorems 2.4 and 2.5 are proved in Sect. 5.

As a consequence, one finally gets the propagation of chaos result:

Theorem 2.6 (1) *If $\beta^2 A^2 T < 1$, then $\Pi_{\beta,T}^N$ converges weakly to δ_Q i.e.*

$$\forall F \in C_b(\mathscr{M}_1^+(W_T^A)) \lim_{N\to\infty} \int \left(\int F\left(\frac{1}{N}\sum_1^N \delta_{x^i}\right) P_\beta^n(J(\omega))\,(dx)\right) d\gamma(\omega) = F(Q) . \tag{2}$$

In particular, if $f \in C_b(W_T^A)$,

$$\lim_{N\to\infty} \int \left(\int \frac{1}{N}\sum_1^N f(x^i) P_\beta^N(J(\omega))(dx)\right) d\gamma(\omega) = \int f(x)\, dQ(x) . \tag{3}$$

(2) *As a consequence, if $\beta^2 A^2 T < 1$, $\forall k \in \mathbb{N}$, $\forall (f_1,\ldots,f_k) \in C_b(W_T^A)^k$,*

$$\lim_{N\to\infty} \int \left(\int f_1(x^1)\ldots f_k(x^k) P_\beta^N(J(\omega))(dx)\right) d\gamma(\omega) = \prod_{i=1}^k \int f_i(x)\, dQ(x) . \tag{4}$$

The proof of Theorem 2.6 is very classical, its main arguments are recalled in Appendix C.

Of course, one can deduce from Theorem 2.3 that the quenched law of the empirical measure satisfies a large deviation upper bound, i.e.:

Theorem 2.7 *There exists a good rate function H such that if $\beta^2 A^2 T < 1$, for any closed subset F of $\mathscr{M}_1^+(W_T^A)$, for almost all J,*

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log P_\beta^N(J)(\widehat{\mu}^N \in F) \leqq -\inf_F H. \tag{5}$$

The proof of Theorem 2.7 relies on Borel Cantelli lemma and is given in Appendix C.

A consequence of Theorems 2.7 and 2.4 is that the quenched law of the empirical measure converges exponentially fast to δ_Q so that:

Theorem 2.8 *If $\beta^2 A^2 T < 1$*

(1) *For any bounded continuous function F on $\mathscr{M}_1^+(W_T^A)$, for almost all J,*

$$\lim_{N\to\infty} \int F(\widehat{\mu}^N)\, dP_\beta^N(J) = \int f\, dQ.$$

(2) *For any bounded continuous function f on W_T^A, for almost all J, for almost all $((x^i)_{1\leqq i\leqq N})_{N\in\mathbb{N}}$,*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N f(x^i) = \int f\, dQ.$$

Theorem 2.8 is proved in Appendix C.

Since $P_\beta^N(J)$ is not exchangeable, we cannot deduce from Theorem 2.8(1) that $P_\beta^N(J)$ is Q chaotic as in Theorem 2.6. Thus, we introduce replica to get a better understanding of the quenched asymptotic behavior of a single spin.

We will identify in Sect. 6 a gaussian external magnetic field H and a probability measure P_H on W_T^A which depends on H such that:

Theorem 2.9 *For any integer r such that $r\beta^2 A^2 T < 1$, for any functions $(f_1,\dots,f_m) \in \mathscr{C}_b^0(W_T^A)$*

$$\int \left(\int \prod_{i=1}^m f_i(x^i) P_\beta^N(J(\omega))(dx)\right)^r d\gamma(\omega) \underset{N\to\infty}{\longrightarrow} \prod_{i=1}^m \mathscr{E}^H\left[\left(\int f_i\, dP_H\right)^r\right],$$

where $\mathscr{E}^H$ denotes the expectation on the gaussian process H.

The law of H and P_H are described in Sect. 6 as the unique solution of the following non-linear procedure: Let (H, G) be two independent centered gaussian processes and denote $\mathscr{E}^H$ (resp. $\mathscr{E}^G$) the expectation over H (resp. G). For f in $L^2([0,T])$, let $P(f)$ be the restriction on $[0,T]$ of the law of the diffusion

$$\begin{cases} dx_t = -\nabla U(x_t)\, dt + dB_t + \beta f(t)\, dt \\ \text{Law of } x_0 = \mu_0. \end{cases}$$

Then, the covariances of (H, G) are defined non-linearly by

$$\mathscr{E}^G[G_t G_s] = \mathscr{E}^H \mathscr{E}^G\left[\int x_s x_t\, dP(G+H)\right]$$
$$- \mathscr{E}^H\left[\mathscr{E}^G\left[\int x_s\, dP(G+H)\right] \mathscr{E}^G\left[\int x_t\, dP(G+H)\right]\right],$$

$$\mathscr{E}^H[H_t H_s] = \mathscr{E}^H\left[\mathscr{E}^G\left[\int x_s\, dP(G+H)\right] \mathscr{E}^G\left[\int x_t\, dP(G+H)\right]\right].$$

Finally P_H is given by

$$P_H = \mathscr{E}^G[P(G+H)]\,.$$

Theorem 2.9 enables us to prove that, in general, $P^N_\beta(J)$ is not Q chaotic but that, if U is even and μ_0 is a symmetric law, $P^N_\beta(J)$ is Q chaotic in the following weak sense:

Theorem 2.10 *Suppose that $2\beta^2A^2T < 1$, that U is even and μ_0 is a symmetric law. Then, for any times $(t_1,\dots,t_k)\in[0,T]^k$, there exists a subsequence $(N_p)_{p\geqq 0}$ such that, for almost all J, for any integer m, the law of $(x^1_{t_1},\dots,x^1_{t_k},\dots,x^m_{t_1},\dots,x^m_{t_k})$ under $P^{N_p}_\beta(J)$ converges to the law of $(x^1_{t_1},\dots,x^1_{t_k},\dots,x^m_{t_1},\dots,x^m_{t_k})$ under $Q^{\otimes m}$ when p tends to infinity.*

A more detailed result is given in Sect. 6.

3 A technical step: large deviation principles for discretized systems

As mentioned at the end of Sect. 1, we prove here a large deviation principle for the measures $\Pi^N_{\beta,T}$ after a time discretization. In this section, an integer n will be fixed. We introduce a version $\mathscr{S}^{N,n}_\beta(J)$ of the stochastic system $\mathscr{S}^N_\beta(J)$ where the interaction has been discretized in time. Let $\Delta^n = \{0 = t_0 < t_1 \cdots < t_{n+1} = T\}$ be a partition of $[0,T]$ and define

$$\mathscr{S}^{N,n}_\beta(J)\begin{cases} dx^j_t = -\nabla U(x^j_t)\,dt + dB^j_t + \dfrac{\beta}{\sqrt{N}}\sum\limits_{i=1}^N J_{ji}\,x^i_{t^{(n)}}\,dt, \quad 1 \leqq j \leqq N\,,\\ t^{(n)} = \sup\{t_k \in \Delta^n / t_k \leqq t\}\,,\\ \text{Law of } x_0 = \mu_0^{\otimes N}\,. \end{cases}$$

As in Proposition 2.1, it is clear that $\mathscr{S}^{N,n}_\beta(J)$ has a unique weak solution for any $J \in \mathbb{R}^{N\times N}$. We will denote $P^{N,n}_\beta(J)$ its restriction to $(W^A_T)^N$.

We will call $Q^{N,n}_\beta$ the probability measure on $(W^A_T)^N$ defined by

$$Q^{N,n}_\beta = \int P^{N,n}_\beta(J(\omega))\,d\gamma(\omega)\,.$$

Let finally $\Pi^{N,n}_{\beta,T}$ be the law of the empirical measure under $Q^{N,n}_\beta$:

$$\forall A \in \mathscr{B}(\mathscr{M}^+_1(W^A_T)),\quad \Pi^{N,n}_{\beta,T}(A) = Q^{N,n}_\beta\left(\frac{1}{N}\sum_{p=1}^N \delta_{x^p} \in A\right)\,.$$

To state the main result of this section, i.e. a large deviation principle for $\Pi^{N,n}_{\beta,T}$, we first introduce the rate function.

Recall here that P is the law of the diffusion process solution of

$$\begin{cases} dx_t = -\nabla U(x_t)dt + dB_t, \quad 0 \leqq t \leqq T\\ \text{Law of } x_0 = \mu_0\,. \end{cases}$$

If $\mu \in \mathscr{M}_1^+(W_T^A)$, let $I(\mu|P)$ be the relative entropy with respect to P, i.e.:

$$I(\mu|P) = \begin{cases} \int \log \frac{d\mu}{dP} d\mu & \text{if } \mu \ll P\,, \\ +\infty & \text{otherwise}\,. \end{cases}$$

And define

$$\Gamma^n(\mu) = \int \left(\log \int \exp \left\{ \beta \sum_{k=0}^{n} G_{t_k}(\omega)(B_{t_{k+1}} - B_{t_k})(x) \right.\right.$$
$$\left.\left. - \frac{\beta^2}{2} \sum_{k=0}^{n} G_{t_k}^2(\omega)(t_{k+1} - t_k) \right\} d\gamma_{K_\mu}(\omega) \right) d\mu(x)\,,$$

where G is, under γ_{K_μ}, a centered gaussian process with covariance $K_\mu(s,t) = \int x_s x_t \, d\mu(x)$ and $B_t(x) = x_t - x_0 + \int_0^t \nabla U(x_s)\, ds$.

We then define, for $\mu \in \mathscr{M}_1^+(W_T^A)$,

$$H^n(\mu) = \begin{cases} I(\mu|P) - \Gamma^n(\mu) & \text{if } \quad I(\mu|P) < \infty\,, \\ +\infty & \text{otherwise}\,. \end{cases}$$

The aim of this section is to prove the following large deviation theorem for the discretized systems.

Theorem 3.1
(1) *H^n is a good rate function, i.e. $\forall L > 0$, $\{H^n \leqq L\}$ is a compact set.*
(2) *$\Pi_{\beta,T}^{N,n}$ satisfies a weak large deviation principle with rate function H^n.*
(3) *If $\beta^2 A^2 T < 1$, $\Pi_{\beta,T}^{N,n}$ satisfies a full large deviation principle with rate function H^n.*

We first give another description of Γ^n for which we need some preliminary notations.

We recall that, for any probability measure μ on W_T^A, we define the covariance K_μ by

$$K_\mu(s,t) = \int x_s x_t \, d\mu(x)\,.$$

Moreover, for any $t \leqq T$, we define a map $\widetilde{\bullet}^{t,n}$ in the set of covariances (i.e. of symmetric positive kernels) on $[0,T] \times [0,T]$ such that, for any covariance K, $\widetilde{K}^{t,n}$ is given by

$$\widetilde{K}^{t,n}(s,u) = \int G_s G_u \frac{\exp\left\{-\frac{\beta^2}{2}\int_0^t G_{s(n)}^2 \, ds\right\}}{\int \exp\left\{-\frac{\beta^2}{2}\int_0^t G_{s(n)}^2 \, ds\right\} d\gamma_K} \, d\gamma_K$$
$$= \int G_s G_u \frac{\exp\left\{-\frac{\beta^2}{2}\sum_0^n G_{t_k}^2 (t_{k+1} \wedge t - t_k \wedge t)\right\}}{\int d\gamma_K \exp\left\{-\frac{\beta^2}{2}\sum_0^n G_{t_k}^2 (t_{k+1} \wedge t - t_k \wedge t)\right\}} \, d\gamma_K\,.$$

In particular, for any probability measure μ on W_T^A, for any $t \leqq T$, we denote $\widetilde{K_\mu}^{t,n} = \widetilde{\bullet}^{t,n}(K_\mu)$.

We remark that, for any probability measure μ on W_T^A, for any $s,t \leqq T$, $|\widetilde{K_\mu}^{t,n}(s,s)| \leqq |K_\mu(s,s)|$ (see the proof in Appendix A, Lemma A.5), so that the covariance $\widetilde{K_\mu}^{t,n}$ is bounded by A^2.
Let

$$\Gamma_1^n(\mu) = \log \int \exp\left\{-\frac{\beta^2}{2}\sum_{k=0}^{n} G_{t_k}^2(\omega)(t_{k+1}-t_k)\right\} d\gamma_{K_\mu}(\omega)$$

$$\Gamma_2^n(\mu) = \frac{\beta^2}{2}\int\int \left(\sum_{k=0}^{n} G_{t_k}(\omega)(B_{t_{k+1}}(x)-B_{t_k}(x))\right)^2 d\gamma_{\widetilde{K_\mu}^{T,n}}(\omega)\, d\mu(x)\,.$$

Standard gaussian calculus gives (see Neveu [10, Proposition 8.4]):

Property 3.2

$$\Gamma^n(\mu) = \Gamma_1^n(\mu) + \Gamma_2^n(\mu)\,.$$

Notation: In short, we shall write γ_μ for γ_{K_μ}.

In order to prove Theorem 3.1, we first study the continuity properties of the applications Γ_1^n and Γ_2^n:

We will denote d_T the Vaserstein distance on $\mathcal{M}_1^+(W_T^A)$, i.e.

$$d_T(\mu,\nu) = \inf\left\{\int \sup_{t\leqq T}|x_t - y_t|^2\, d\xi(x,y)\right\}^{1/2}$$

The infimum being taken on the laws ξ with marginals ν and μ.
d_T is a distance on $\mathcal{M}_1^+(W_T^A)$ which is compatible with the weak topology (see [6, Theorem 2]).

Lemma 3.3
(1) *Γ_1^n is a bounded Lipschitz function from $(\mathcal{M}_1^+(W_T^A), d_T)$ to $(\mathbb{R}, |\ |)$. It is therefore continuous. More precisely:*
(a) $-\frac{1}{2}\beta^2 A^2 T \leqq \Gamma_1^n \leqq 0$.
(b) *There exists a positive constant C_T, depending on T but not on n, such that: $|\Gamma_1^n(\mu) - \Gamma_1^n(\nu)| \leqq C_T d_T(\mu,\nu)$.*
(2) *Γ_2^n is lower semi-continuous.*
(3) *$\Gamma^n \leqq I(|P)$, i.e. H^n is positive. Hence Γ^n is finite whenever $I(|P)$ is.*
(4) *There exist real constants $\alpha < 1$ and $\eta > 0$ such that $\Gamma^n \leqq \alpha I(|P) + \eta$.*

Proof. We recall here that $t^{(n)} = \sup\{t_k / t_k \leqq t\}$ so that $\int_0^T G_{t^{(n)}}^2\, dt = \sum_{k=0}^n G_{t_k}^2(t_{k+1}-t_k)$ for instance.

Proof of Lemma 3.3(1). By Jensen inequality

$$\begin{aligned}\Gamma_1^n(\mu) &= \log\int\exp\left\{-\frac{1}{2}\beta^2\int_0^T G_{t^{(n)}}^2\, dt\right\} d\gamma_\mu \\ &\geqq -\frac{1}{2}\beta^2\int\left\{\int_0^T G_{t^{(n)}}^2\, dt\right\} d\gamma_\mu = -\frac{1}{2}\beta^2\int\left\{\int_0^T x_{t^{(n)}}^2\, dt\right\} d\mu \\ &\geqq -\frac{1}{2}\beta^2 A^2 T\,.\end{aligned}$$

So that

$$-\frac{1}{2}\beta^2 A^2 T \leqq \Gamma_1^n(\mu) \leqq 0 \quad \forall \mu \in \mathscr{M}_1^+(W_T^A).$$

Moreover

$$|\Gamma_1^n(\mu) - \Gamma_1^n(\nu)| = \left|\log\left(1 + \frac{\int \exp\left\{-\frac{\beta^2}{2}\int_0^T G_{t(n)}^2\,dt\right\}\,d(\gamma_\mu - \gamma_\nu)}{\int \exp\left\{-\frac{\beta^2}{2}\int_0^T G_{t(n)}^2\,dt\right\}\,d\gamma_\nu}\right)\right|$$

$$\leqq e^{1/2\beta^2 A^2 T}\left|\int \exp\left\{-\frac{\beta^2}{2}\int_0^T G_{t(n)}^2\,dt\right\}\,d(\gamma_\mu - \gamma_\nu)\right|.$$

Let ξ be a probability measure on $W_T^A \times W_T^A$ with marginals ν and μ. Then

$$K_\xi(s,t) = \begin{pmatrix} \int x_s x_t\,d\xi(x,y) & \int x_s y_t\,d\xi(x,y) \\ \int x_t y_s\,d\xi(x,y) & \int y_s y_t\,d\xi(x,y) \end{pmatrix}$$

defines the covariance of a bidimensional centered gaussian process (G, G'). Remark that the law of G (resp. G') is γ_μ (resp. γ_ν) and denote γ_ξ the law of (G, G'). Then

$$\left|\int \exp -\frac{\beta^2}{2}\int_0^T G_{t(n)}^2\,dt\,d(\gamma_\mu - \gamma_\nu)\right|$$

$$= \left|\int\left\{\exp\left\{-\frac{\beta^2}{2}\int_0^T G_{t(n)}^2\,dt\right\} - \exp\left\{-\frac{\beta^2}{2}\int_0^T G_{t(n)}'^2\,dt\right\}\right\}\,d\gamma_\xi\right|$$

$$\leqq \frac{\beta^2}{2}\int\int_0^T \left|G_{t(n)}^2 - G_{t(n)}'^2\right|\,dt\,d\gamma_\xi$$

$$\leqq \frac{\beta^2}{2}\prod_{\varepsilon=\pm 1}\left(\int\int_0^T (G_{t(n)} + \varepsilon G'_{t(n)})^2\,dt\,d\gamma_\xi\right)^{1/2}$$

by Cauchy–Schwarz inequality.
But

$$\int\int_0^T \left(G_{t(n)} + G'_{t(n)}\right)^2\,dt\,d\gamma_\xi = \int\int_0^T \left(x_{t(n)} + y_{t(n)}\right)^2\,dt\,d\xi(x,y) \leqq 4A^2 T,$$

$$\int\int_0^T \left(G_{t(n)} - G'_{t(n)}\right)^2\,dt\,d\gamma_\xi = \int\int_0^T (x_{t(n)} - y_{t(n)})^2\,dt\,d\xi(x,y).$$

Hence,

$$|\Gamma_1^n(\mu) - \Gamma_1^n(\nu)| \leqq \beta^2 A\sqrt{T}\exp\frac{1}{2}\beta^2 A^2 T\left(\int\int_0^T (x_{t(n)} - y_{t(n)})^2\,dt\,d\xi(x,y)\right)^{1/2}. \tag{6}$$

So that, taking the infimum on the measures ξ, we find

$$|\Gamma_1^n(\mu) - \Gamma_1^n(\nu)| \leqq \left\{\beta^2 AT \exp \frac{1}{2}\beta^2 A^2 T\right\} d_T(\mu, \nu).$$

Proof of Lemma 3.3(2). We define

$$\Gamma_2^{n,M}(\mu) = \int\int \left(\int_0^T G_{t(n)}(\omega)\, dB_t(x)\right)^2 \wedge M \, d\gamma_{\widetilde{K}_\mu^{T,n}}(\omega)\, d\mu(x).$$

We state in Appendix A, Lemma A.5, that $\mu \to \widetilde{K_\mu}^{T,n}$ is Lipschitz for the weak distance so that it is continuous for the weak topology. Hence $\mu \to \gamma_{\widetilde{K}_\mu^{T,n}}$ is continuous for the weak topology.

But $(\omega, x) \to \left(\int_0^T G_{t(n)}(\omega)\, dB_t(x)\right)^2 \wedge M$ is a bounded continuous function. Therefore, $\Gamma_2^{n,M}$ is continuous from $\mathscr{M}_1^+(W_T^A)$ into $\mathbb{R}$.

Finally, by monotone convergence theorem, when M grows to infinity, $\Gamma_2^{n,M}$ grows to Γ_2^n, so that Γ_2^n is lower semi-continuous.

Proof of Lemma 3.3(3). Let

$$F_\mu(x) = \log \int \exp\left\{\beta \int_0^T G_{t(n)}\, dB_t(x) - \frac{\beta^2}{2}\int_0^T G_{t(n)}^2\, dt\right\} d\gamma_\mu.$$

It is well known (see [5, Lemma 3.2.13] for instance) that

$$I(\mu|P) = \sup\left(\int_{W_T^A} \phi\, d\mu - \log \int_{W_T^A} \exp \phi\, dP;\ \phi \in \mathscr{C}_b(W_T^A)\right)$$

so that, by bounded convergence, for any bounded measurable function ϕ on W_T^A, we have

$$\int_{W_T^A} \phi\, d\mu - \log \int_{W_T^A} \exp \phi\, dP \le I(\mu|P). \tag{7}$$

For instance, if we define, for a positive real number M, $F_\mu^M(x) = \log \int \left(M \wedge \exp\left\{\beta \int_0^T G_{t(n)}\, dB_t(x) - \frac{\beta^2}{2}\int_0^T G_{t(n)}^2\, dt\right\}\right) d\gamma_\mu$, we find that, for any positive real number a,

$$a \int F_\mu^M(x) \mu(dx) \leqq I(\mu|P) + \log \int \exp aF_\mu^M(x)\, dP(x).$$

So that monotone convergence gives

$$a \int F_\mu(x) \mu(dx) \leqq I(\mu|P) + \log \int \exp aF_\mu(x)\, dP(x). \tag{8}$$

By Jensen inequality, $\forall a \geqq 1$,

$$\int \exp aF_\mu(x) dP(x) \leqq \int\int \exp\left\{a\beta \int_0^T G_{t(n)}\, dB_t(x) - a\frac{\beta^2}{2}\int_0^T G_{t(n)}^2\, dt\right\} dP(x)\, d\gamma_\mu.$$

But, under P, B_t is a brownian motion and

$$\int \exp\left\{a\beta\int_0^T G_{t(n)}\,dB_t(x)\right\}dP(x) = \exp\left\{\frac{a^2\beta^2}{2}\int_0^T G_{t(n)}^2\,dt\right\}$$

so that, $\forall a \geqq 1$, (8) implies

$$a\int F_\mu(x)\,d\mu(dx) \leqq I(\mu|P) + \log\int \exp\left\{(a^2-a)\frac{\beta^2}{2}\int_0^T G_{t(n)}^2\,dt\right\}d\gamma_\mu\,. \quad (9)$$

We have chosen F_μ so that $\Gamma^n(\mu) = \int F_\mu(x)\mu(dx)$; letting $a = 1$ in (9) proves that $\Gamma^n \leqq I(|P)$.

Proof of Lemma 3.3(4). According to Lemma A.3(2) of the Appendix, if $b = (a^2-a)\beta^2 A^2 T < 1$, we can find a finite constant c such that

$$\int \exp\left\{(a^2-a)\frac{\beta^2}{2}\int_0^T G_{t(n)}^2\,dt\right\}\,d\gamma_\mu \leqq \exp cb\,,$$

so that, (9) becomes

$$\Gamma^n(\mu) \leqq \frac{1}{a}I(\mu|P) + \frac{cb}{a}\,. \qquad \square$$

To circumvent the fact that Γ_2^n is not continuous, we approach this function by linear applications

Lemma 3.4 *Let ν be a probability measure on W_T^A.*
(1) Define $\Gamma_{2,\nu}^n : \mathcal{M}_1^+(W_T^A) \to \mathbb{R}^+$ by

$$\Gamma_{2,\nu}^n(\mu) = \frac{\beta^2}{2}\int\int\left(\sum_{k=0}^n G_{t_k}(\omega)(B_{t_{k+1}}(x) - B_{t_k}(x))\right)^2 d\gamma_{\widetilde{K}_\nu^{T,n}}(\omega)\,d\mu(x)\,.$$

Then, there exists a constant C_T, such that, for any integer n,

$$|\Gamma_2^n(\mu) - \Gamma_{2,\nu}^n(\mu)| \leqq C_T(1 + I(\mu|P))\,d_T(\mu,\nu)\,.$$

(2) Let $\Gamma_\nu^n(\mu) = \Gamma_1^n(\nu) + \Gamma_{2,\nu}^n(\mu)$. Then we can define a probability measure Q_ν^n on W_T^A by

$$dQ_\nu^n(x) = \exp\Gamma_\nu^n(\delta_x)\,dP(x) \quad (10)$$

$$= \int \exp\left\{\beta\int_0^T G_{t(n)}\,dB_t(x) - \frac{\beta^2}{2}\int_0^T G_{t(n)}^2\,dt\right\}\,d\gamma_\nu\,dP(x)\,. \quad (11)$$

Then, the relative entropy, $I(\mu|Q_\nu^n)$, of μ with respect to Q_ν^n, is equal to H_ν^n:

$$H_\nu^n : \mathcal{M}_1^+(W_T^A) \to [0,\infty]$$

$$\mu \to \begin{cases} I(\mu|P) - \Gamma_\nu^n(\mu) & \text{if } I(\mu|P) < \infty, \\ +\infty & \text{otherwise.}\end{cases}$$

As a consequence, H_ν^n is lower semi-continuous.

Proof of Lemma 3.4(1).

$$\Gamma_2^n(\mu) - \Gamma_{2,\nu}^n(\mu) = \frac{\beta^2}{2}\int\int \left(\int_0^T G_{t(n)}\, dB_t\right)^2 d(\gamma_{\widetilde{K}_\mu^{T,n}} - \gamma_{\widetilde{K}_\nu^{T,n}})\, d\mu\,.$$

Pick, as in the proof of Lemma 3.3(1), a probability measure ξ on $W_T^A \times W_T^A$ with marginals μ and ν and let γ_ξ be the law of a bidimensional centered gaussian process (G, G') with covariance K_ξ.

Let

$$\Lambda_T^n(G) = \frac{\exp\left\{-\frac{\beta^2}{2}\int_0^T G_{t(n)}^2\, dt\right\}}{\int \exp\left\{-\frac{\beta^2}{2}\int_0^T G_{t(n)}^2\, dt\right\} d\gamma_\xi}\,.$$

Then

$$|\Gamma_2^n(\mu) - \Gamma_{2,\nu}^n(\mu)|$$

$$= \frac{\beta^2}{2}\left|\int\int\left\{\Lambda_T^n(G)\left(\int_0^T G_{t(n)} dB_t\right)^2 - \Lambda_T^n(G')\left(\int_0^T G'_{t(n)}\, dB_t\right)^2\right\} d\gamma_\xi\, d\mu\right|$$

$$\leqq \frac{\beta^2}{2}\int\int |\Lambda_T^n(G) - \Lambda_T^n(G')| \left(\int_0^T G_{t(n)}\, dB_t\right)^2 d\gamma_\xi\, d\mu$$

$$+ \frac{\beta^2}{2}\int\int \Lambda_T^n(G')\left|\int_0^T \left(G_{t(n)} + G'_{t(n)}\right) dB_t\right| \times \left|\int_0^T \left(G_{t(n)} - G'_{t(n)}\right) dB_t\right| d\gamma_\xi\, d\mu\,. \tag{12}$$

Let

$$B_1 = \frac{\beta^2}{2}\int\int |\Lambda_T^n(G) - \Lambda_T^n(G')| \left(\int_0^T G_{t(n)}\, dB_t\right)^2 d\gamma_\xi\, d\mu\,,$$

$$B_2 = \frac{\beta^2}{2}\prod_{\varepsilon=\pm 1}\left(\int \Lambda_T^n(G')^2 \left(\int_0^T (G_{t(n)} + \varepsilon G'_{t(n)})\, dB_t\right)^2 d\gamma_\xi\, d\mu\right)^{1/2}\,.$$

If we apply Cauchy–Schwarz inequality in the second term of the right hand of (12), we find that

$$\left|\Gamma_2^n(\mu) - \Gamma_{2,\nu}^n(\mu)\right| \leqq B_1 + B_2\,. \tag{13}$$

We first bound B_1. Remark that

$$|\Lambda_T^n(G) - \Lambda_T^n(G')|$$

$$= \left|\exp\left\{-\Gamma_1^n(\mu) - \frac{\beta^2}{2}\int_0^T G_{t(n)}^2\, dt\right\} - \exp\left\{-\Gamma_1^n(\nu) - \frac{\beta^2}{2}\int_0^T G'^2_{t(n)}\, dt\right\}\right|$$

$$\leqq \exp\left\{\frac{1}{2}\beta^2 A^2 T\right\}\left(\frac{\beta^2}{2}\int_0^T \left|G_{t(n)}^2 - G'^2_{t(n)}\right| dt + |\Gamma_1^n(\mu) - \Gamma_1^n(\nu)|\right) \tag{14}$$

so that, if $c = \frac{1}{2}\beta^2(1 + \frac{1}{2}\beta^2)\exp\frac{1}{2}\beta^2 A^2 T$,

$$B_1 \leqq c|\Gamma_1^n(\mu) - \Gamma_1^n(\nu)| \int\int \left(\int_0^T G_{t(n)}\, dB_t\right)^2 d\gamma_\xi\, d\mu$$

$$+ c\int\int\int_0^T \left|G_{t(n)}^2 - G_{t(n)}'^2\right| dt \left(\int_0^T G_{t(n)}\, dB_t\right)^2 d\gamma_\xi\, d\mu\,. \tag{15}$$

Moreover, one can deduce from (7) that, for any probability measure μ on W_T^A, for any function h in $L^2([0,T], dt)$, for any $C > 2$, we have

$$\int \left(\int_0^T h_t\, dB_t\right)^2 d\mu \leqq C(1 + I(\mu|P))\int_0^T h_t^2\, dt\,. \tag{16}$$

We come back to Eq. (15). We first integrate with respect to μ and use the independence of G and B to apply Eq. (16). We then find a constant c_T such that

$$B_1 \leqq c_T(1 + I(\mu|P))\left(|\Gamma_1^n(\mu) - \Gamma_1^n(\nu)|\right.$$

$$\left. + \int \left(\int_0^T G_{t(n)}^2\, dt\right) \int_0^T \left|G_{t(n)}^2 - G_{t(n)}'^2\right| dt\, d\gamma_\xi\right)\,.$$

So that, using Lemma 3.3(1), we deduce that we can find a constant c_T' such that

$$B_1 \leqq c_T' d_T(\mu, \nu)(1 + I(\mu|P))\,.$$

We can bound similarly B_2 so that inequality (13) gives the result, i.e. that we can find a finite constant C_T such that, for any integer number n and $(\mu, \nu) \in \mathcal{M}_1^+(W_T^A)$:

$$\left|\Gamma_2^n(\mu) - \Gamma_{2,\nu}^n(\mu)\right| \leqq C_T d_T(\mu, \nu)(1 + I(\mu|P))\,.$$

Remark. 3.5. We could also have remarked that

$$\left|\Gamma_2^n(\mu) - \Gamma_{2,\nu}^n(\mu)\right| \leqq \frac{\beta^2}{2} \sum_{k,k'=1}^n \left|\widetilde{K_\mu}^{T,n} - \widetilde{K_\nu}^{T,n}\right| (k,k')$$

$$\times \int \left|(B_{t_{k+1}} - B_{t_k})\left(B_{t_{k'+1}} - B_{t_{k'}}\right)\right| d\mu\,.$$

And, since we state in Appendix A, Lemma A.5, that $\widetilde{K_\mu}^{T,n}$ is Lipschitz for the Vaserstein distance, we should have found a finite constant C_T such that

$$\left|\Gamma_2^n(\mu) - \Gamma_{2,\nu}^n(\mu)\right| \leqq \frac{\beta^2}{2} n^2 C_T\, d_T(\mu, \nu) \sup_{1 \leqq k \leqq n} \int (B_{t_{k+1}} - B_{t_k})^2\, d\mu\,. \tag{17}$$

Proof of Lemma 3.4(2). The equality between the two definitions, (10) and (11), of the density dQ_ν^n/dP is due to standard gaussian computations (see, for more details, the proof of Lemma 5.15). We deduce from the martingale properties of this density that Q_ν^n is in fact a probability measure. The equality between $I(|Q_\nu^n)$ and H_ν^n is proved in Appendix B.

We can now prove Theorem 3.1.

Proof of Theorem 3.1(1). We first prove that H^n is lower semi-continuous. Take a sequence (μ_p) of probability measures on W_T^A converging to μ and choose a subsequence $(\mu_{p_m})_m$ such that $\lim_m H^n(\mu_{p_m}) = \underline{\lim}_p H^n(\mu_p)$.

Then, we distinguish the case where the sequence $I(\mu_{p_m}|P)$ stays bounded for large m from the case where we can find a subsequence $(p_{m(M)})_{M\in\mathbb{N}}$ such that $I(\mu_{p_{m(M)}}|P)$ tends to infinity when M does.
– In the first case, with the notations of Lemma 3.4,

$$\underline{\lim}_p H^n(\mu_p) = \lim_m (I - \Gamma^n)(\mu_{p_m}) \geqq \underline{\lim}_m \left(I - \Gamma_\mu^n\right)(\mu_{p_m}) + \underline{\lim}_m \left(\Gamma_\mu^n - \Gamma^n\right)(\mu_{p_m}),$$

As $I(\mu_{p_m}|P)$ is finite for large m, Lemma 3.4(2) implies

$$\underline{\lim}_m \left(I(\mu_{p_m}|P) - \Gamma_\mu^n(\mu_{p_m})\right) = \underline{\lim}_m H_\mu^n(\mu_{p_m}) \geqq H_\mu^n(\mu) = H^n(\mu).$$

Moreover, Lemmas 3.3(1) and 3.4(1) imply that we can find a finite constant C such that

$$\left|\Gamma^n(\mu_{p_m}) - \Gamma_\mu^n(\mu_{p_m})\right| \leqq C(1 + I(\mu_{p_m}|P))\, d_T(\mu, \mu_{p_m}).$$

Therefore, $I(\mu_{p_m}|P)$ being bounded for large m, $\underline{\lim}_m \left(\Gamma_\mu^n - \Gamma^n\right)(\mu_{p_m}) = 0$. Hence $\underline{\lim}_p H^n(\mu_p) \geqq H^n(\mu)$.
– Suppose now that we can find a subsequence $(p_{m(M)})_{M\in\mathbb{N}}$ such that $\lim_{M\to\infty} I\left(\mu_{p_{m(M)}}|P\right) = \infty$. Then, according to Lemma 3.3(4), $\lim_M H^n\left(\mu_{p_{m(M)}}\right) = +\infty$, so that

$$\underline{\lim}_p H^n(\mu_p) = \lim_m H^n(\mu_{p_m}) = \lim_M H^n\left(\mu_{p_{m(M)}}\right) = +\infty \geqq H^n(\mu).$$

Hence, we proved that, for any sequence $(\mu_p)_{p\in\mathbb{N}}$ converging to μ, $\underline{\lim}\, H^n(\mu_p) \geqq H^n(\mu)$. This means that H^n is lower semi-continuous, and, equivalently that, for any positive real number M, $\{H^n \leqq M\}$ is closed. Moreover, by Lemma 3.3(4), we also know that the entropy relative to P is bounded on $\{H^n \leqq M\}$ so that this set is in fact compact.

Proof of Theorem 3.1(2). The demonstration of the large deviation principle will follow the following classical steps:

First of all, we shall compare our system with the system without interaction $\mathscr{S}_0^N$. We shall state

$$d\Pi_{\beta,T}^{N,n}(\mu) = \exp\{N\Gamma^n(\mu)\}\, d\Pi_{0,T}^N(\mu). \tag{18}$$

Then, we shall prove, without any restriction as $\beta^2 A^2 T < 1$, that a weak large deviation principle holds; we give a lower bound inequality:

$$\text{For any open set } O \text{ of } \mathscr{M}_1^+(W_T^A), \quad -\inf_O H^n \leqq \underline{\lim}_{N\to\infty} \frac{1}{N} \log \Pi_{\beta,T}^{N,n}(O). \tag{19}$$

And an upper bound for any compact set K:

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log \Pi^{N,n}_{\beta,T}(K) \leqq -\inf_K H^n . \tag{20}$$

In a second step, we shall prove an exponential tightness lemma for which we need the condition $\beta^2 A^2 T < 1$:

$$\exists \alpha > 1 \ \exists C < \infty \quad \sup_{N,n} \left(\int \exp\{\alpha N \Gamma^n(\widehat{\mu}^N)\} \, dP^{\otimes N} \right)^{1/N} < \exp C . \tag{21}$$

So that, if $\delta = 1 - \alpha^{-1}, \forall B \in \mathscr{B}(\mathscr{M}_1^+(W_T^A))$,

$$\Pi^{N,n}_{\beta,T}(B) \leqq \exp \frac{1}{\alpha} CN \times \Pi^N_{0,T}(B)^{\delta} . \tag{22}$$

But, by Sanov exponential tightness property, we know that, for any positive integer L, we can find a compact set K_L such that

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log \Pi^N_{0,T}(K_L^c) \leqq -L .$$

Hence, $\forall L$, there exists a compact set $K_{\frac{1}{\delta}(L+C/\alpha)}$ such that

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log \Pi^{N,n}_{\beta,T}\left(K^c_{\frac{1}{\delta}(L+C/\alpha)} \right) \leqq -L . \tag{23}$$

Then, Eqs. (20) and (23) give the upper bound for any closed set F;

$$\forall F = \overline{F} \quad \overline{\lim_{N\to\infty}} \frac{1}{N} \log \Pi^{N,n}_{\beta,T}(F) \leqq -\inf_F H^n . \tag{24}$$

The following lemma allows us to compare our system with the system without interaction and is the key of the whole approach of this paper:

Lemma 3.6 *Note* $Q^{N,n}_\beta = \int P^{N,n}_\beta(J(\omega))\, d\gamma(\omega)$. *Then* $Q^{N,n}_\beta \ll P^{\otimes N}$ *and*

$$\frac{dQ^{N,n}_\beta}{dP^{\otimes N}} = \exp\{N\Gamma^n(\widehat{\mu}^N)\} .$$

Proof. By Girsanov theorem, for all $J \in \mathbb{R}^{N\times N}$, $P^{N,n}_\beta(J) \ll P^{\otimes N}$ and

$$\frac{dP^{N,n}_\beta(J)}{dP^{\otimes N}} = \exp \sum_{j=1}^N \left\{ \beta \int_0^T \left(\frac{1}{\sqrt{N}} \sum_{i=1}^N J_{ji} x^i_{t(n)} \right) dB^j_t - \frac{\beta^2}{2} \int_0^T \left(\frac{1}{\sqrt{N}} \sum_{i=1}^N J_{ji} x^i_{t(n)} \right)^2 dt \right\} .$$

Applying Fubini theorem to the positive function $\frac{dP^{N,n}_\beta(J)}{dP^{\otimes N}}$, we find that $Q^{N,n}_\beta \ll P^{\otimes N}$ and

$$\frac{dQ^{N,n}_\beta}{dP^{\otimes N}} = \int \exp \sum_{j=1}^N \left\{ \beta \int_0^T \left(\frac{1}{\sqrt{N}} \sum_{i=1}^N J_{ji}(\omega) x^i_{t(n)} \right) dB^j_t - \frac{\beta^2}{2} \int_0^T \left(\frac{1}{\sqrt{N}} \sum_{i=1}^N J_{ji}(\omega) x^i_{t(n)} \right)^2 dt \right\} d\gamma(\omega) .$$

But, under γ, the J_{ij} are independent, so that

$$\frac{dQ_\beta^{N,n}}{dP^{\otimes N}} = \prod_{j=1}^N \int \exp\left\{\beta \int_0^T \left(\frac{1}{\sqrt{N}}\sum_{i=1}^N J_{ji}(\omega)x^i_{t(n)}\right) dB^j_t - \frac{\beta^2}{2}\int_0^T \left(\frac{1}{\sqrt{N}}\sum_{i=1}^N J_{ji}(\omega)x^i_{t(n)}\right)^2 dt\right\} d\gamma(\omega).$$

Moreover, under γ, the law of $\left(\frac{1}{\sqrt{N}}\sum_{i=1}^N J_{ji}x^i_{t(n)}\right)$ is $\gamma_{\widehat{\mu}^N}$, i.e. $\left\{\frac{1}{\sqrt{N}}\sum_{i=1}^N J_{ji}x^i_{t(n)}, t \leqq T\right\}$ is a centered gaussian process with covariance $\int x_{t(n)}x_{s(n)}\, d\widehat{\mu}^N(x) = \frac{1}{N}\sum_{i=1}^N x^i_{t(n)}x^i_{s(n)}$.
Hence

$$\frac{dQ_\beta^{N,n}}{dP^{\otimes N}} = \exp\left(\sum_{j=1}^N \log\int \exp\left\{\beta\int_0^T G_{t(n)}\, dB^j_t - \frac{\beta^2}{2}\int_0^T G^2_{t(n)}\, dt\right\} d\gamma_{\widehat{\mu}^N}\right) = \exp\{N\Gamma^n(\widehat{\mu}^N)\}. \qquad \square$$

We prove here the lower bound (19):

Lemma 3.7 *If O is an open set of $\mathcal{M}_1^+(W_T^A)$, then*

$$-\inf_O H^n \leqq \varliminf_{N\to\infty} \frac{1}{N}\log \Pi^{N,n}_{\beta,T}(O).$$

Proof. According to Lemma 3.6

$$\Pi^{N,n}_{\beta,T}(O) = \int_O \exp\{N\Gamma^n(\widehat{\mu}^N)\}\, dP^{\otimes N}(x).$$

But, by Lemma 3.2, $\Gamma^n = \Gamma_1^n + \Gamma_2^n$ and $\Gamma_1^n + \Gamma_2^n$ is lower semi-continuous according to Lemma 3.3(1) and (2). We can therefore apply [5, Theorem. 2.1.7], to obtain the result. $\square$

The next step is to prove the weak upper bound (20):

Lemma 3.8 *For any compact subset K of $\mathcal{M}_1^+(W_T^A)$,*

$$\varlimsup_{N\to\infty} \frac{1}{N}\log \Pi^{N,n}_{\beta,T}(K) \leqq -\inf_K H^n.$$

Proof. Take $\delta > 0$. We can find an integer p and a family $(v_i)_{1\leqq i\leqq p}$ of probability measures such that

$$K \subset \bigcup_{i=1}^p B(v_i,\delta),$$

where $B(v_i,\delta) = \{\mu / d_T(\mu, v_i) < \delta\}$ is an open ball in $\mathcal{M}_1^+(W_T^A)$ for the Vaserstein's metric, so that

$$\varlimsup \frac{1}{N}\log \Pi^{N,n}_{\beta,T}(K) \leqq \max_{1\leqq i\leqq p} \varlimsup \frac{1}{N}\log \Pi^{N,n}_{\beta,T}(K\cap B(v_i,\delta)). \tag{25}$$

Let ν be a probability measure on W_T^A.

$$\Pi_{\beta,T}^{N,n}(K \cap B(\nu,\delta)) = \int_{K\cap B(\nu,\delta)} \exp\{N\Gamma^n(\mu)\}\, d\Pi_0^N(\mu)\,.$$

We noticed in Remark 3.5 that there exists a finite constant C such that

$$|\Gamma_\nu^n(\mu) - \Gamma^n(\mu)| \leqq C \max_{1\leqq k\leqq n} \int (B_{t_{k+1}} - B_{t_k})^2\, d\mu\, d_T(\mu,\nu)\,.$$

Thus

$$\Pi_{\beta,T}^{N,n}(K \cap B(\nu,\delta))$$
$$\leqq \int_{K\cap B(\nu,\delta)} \exp\left\{NC\delta \max_{1\leqq k\leqq n} \int (B_{t_{k+1}} - B_{t_k})^2\, d\mu + N\Gamma_\nu^n(\mu)\right\} d\Pi_{0,T}^N(\mu)\,. \qquad (26)$$

But, for any probability measure ν, $Q_\nu^{N,n} = \exp\left\{N\Gamma_\nu^n(\widehat{\mu}^N)\right\}.P^{\otimes N} = (Q_\nu^n)^{\otimes N}$ is a probability measure on $(W_T^A)^N$ (see Lemma 3.4.2).

Hence, (26) implies that for any conjugate exponents (p,q),

$$\begin{aligned}\Pi_{\beta,T}^{N,n}(K \cap B(\nu,\delta)) &\leqq \int_{\widehat{\mu}^N\in K\cap B(\nu,\delta)} \exp\left\{NC\delta \max_{1\leqq k\leqq n} \int (B_{t_{k+1}} - B_{t_k})^2\, d\widehat{\mu}^N\right\} dQ_\nu^{N,n}\\ &\leqq Q_\nu^{N,n}(\widehat{\mu}^N \in K\cap B(\nu,\delta))^{1/p}\\ &\quad\times\left(\int \exp\left\{NqC\delta \max_{1\leqq k\leqq n} \int (B_{t_{k+1}} - B_{t_k})^2\, d\widehat{\mu}^N\right\} dQ_\nu^{N,n}\right)^{1/q}.\end{aligned} \qquad (27)$$

To get an upper bound for the right hand side of (27), we need:

Lemma 3.9 *Whenever* $\xi = 2\max_k |t_{k+1} - t_k|\, qC\delta(1+\beta^2A^2) < 1$,

$$\int \exp\left\{NqC\delta \max_{0\leqq k\leqq n} \int (B_{t_{k+1}} - B_{t_k})^2 d\widehat{\mu}^N\right\} dQ_\nu^{N,n} \leqq \left(\frac{1}{\sqrt{1-\xi}}\right)^N.$$

Proof. As we remark in Appendix B, (79), the processes B^i are, under $(Q_\nu^n)^{\otimes N}$, independent gaussian processes.

Moreover, its covariance is given by

$$\int (B_{t_{k+1}} - B_{t_k})^2\, dQ_\nu^n$$
$$= \int \left(\int (B_{t_{k+1}} - B_{t_k})^2 \exp\left\{\beta\int_0^T G_{t(n)}\, dB_t - \frac{\beta^2}{2}\int_0^T G_{t(n)}^2\, dt\right\} dP\right) d\gamma_\nu\,.$$

But Girsanov theorem implies that, for given G, if W denotes the Wiener measure:

$$\int (B_{t_{k+1}} - B_{t_k})^2 \exp\left\{\beta\int_0^T G_{t(n)}\, dB_t - \frac{\beta^2}{2}\int_0^T G_{t(n)}^2\, dt\right\} dP$$
$$= \int \left(w_{t_{k+1}} - w_{t_k} - \beta\int_{t_k}^{t_{k+1}} G_{s(n)}\, ds\right)^2 dW(w)$$

so that

$$\int (B_{t_{k+1}} - B_{t_k})^2 \, dQ_\nu^n = \int \left(\int \left(w_{t_{k+1}} - w_{t_k} - \beta \int_{t_k}^{t_{k+1}} G_{s(n)} \, ds \right)^2 dW \right) d\gamma_\nu$$

$$\leqq (1 + \beta^2 A^2)(t_{k+1} - t_k) \leqq (1 + \beta^2 A^2) \max_k |t_{k+1} - t_k| .$$

Classical integrability properties of gaussian variables (see [8, Lemma 3.1], for instance) end the proof. □

To bound the first term in the right hand side of (27), we finally remark that, according to Sanov theorem, $\{Q_\nu^{N,n} \circ (\widehat{\mu}^N)^{-1}\}_N$ satisfies a large deviation principle with good rate function $I(|Q_\nu^n)$. Moreover, we saw in Lemma 3.4(2) that

$$I(\mu \mid Q_\nu^n) = H_\nu^n(\mu) = \begin{cases} I(\mu \mid P) - \Gamma_\nu^n(\mu) & \textit{whenever } I(\mu \mid P) < \infty , \\ +\infty & \textit{otherwise} . \end{cases}$$

As a consequence:

$$\overline{\lim} \frac{1}{N} \log Q_\nu^{N,n}(\widehat{\mu}^N \in K \cap B(\nu, \delta)) \leqq - \inf_{K \cap \overline{B(\nu,\delta)}} (I - \Gamma_\nu^n) . \tag{28}$$

Hence, if we fix (p, q) and choose δ small enough, Eqs. (27), (28) and Lemma 3.9 give

$$\overline{\lim} \frac{1}{N} \log \Pi_{\beta,T}^{N,n}(K \cap B(\nu, \delta)) \leqq -\frac{1}{p} \inf_{K \cap \overline{B(\nu,\delta)}} (I - \Gamma_\nu^n) - \frac{1}{2q} \log(1 - \xi) .$$

Thus, Eq. (25) implies

$$\overline{\lim} \frac{1}{N} \log \Pi_{\beta,T}^{N,n}(K) \leqq \max_{1 \leqq i \leqq p} \left(-\frac{1}{p} \inf_{K \cap \overline{B(\nu_i,\delta)}} \left(I(|P) - \Gamma_{\nu_i}^n \right) \right) - \frac{1}{2q} \log(1 - \xi) .$$

But, as a consequence of Lemmas 3.3 and 3.4:

$$|\Gamma_{\nu_i}^n(\mu) - \Gamma^n(\mu)| \leqq C d_T(\mu, \nu_i)(1 + I(\mu \mid P)) .$$

Therefore

$$\overline{\lim} \frac{1}{N} \log \Pi_{\beta,T}^{N,n}(K) \leqq -\frac{1}{p} \inf_K ((1 - C\delta) I(|P) - \Gamma^n) + C\delta - \frac{1}{2q} \log(1 - \xi) .$$

Let $\delta \searrow 0$: as $\Gamma^n \leqq \alpha I(|P) + \eta$ for an $\alpha < 1$, we can prove that

$$\lim_{\delta \to 0} \inf_K ((1 - C\delta) I(|P) - \Gamma^n) = \inf_K (I(|P) - \Gamma^n) .$$

Letting $p \searrow 1$, we proved Lemma 3.8. □

Proof of Theorem 3.1.3. To prove Theorem 3.1.3, we have seen (see (24)) that it is enough to prove the following exponential tightness lemma:

Lemma 3.10 *If* $\beta^2 A^2 T < 1, \exists \alpha > 1,$

$$\sup_{N,n} \left(\int \exp\left\{ \alpha N \Gamma^n(\widehat{\mu}^N) \right\} dP^{\otimes N} \right)^{1/N} < \infty .$$

Proof.

Let $B_N = \int \exp\left\{ \alpha N \Gamma^n(\widehat{\mu}^N) \right\} dP^{\otimes N}$

$$= \int \Pi_{j=1}^N \left(\int \exp\left\{ \beta \int_0^T G^j_{t(n)}\, dB_t - \frac{\beta^2}{2} \int_0^T \left(G^j_{t(n)} \right)^2 dt \right\} d\gamma \right)^{\alpha} dP^{\otimes N} ,$$

where $G^j_t = \frac{1}{\sqrt{N}} \sum_{i=1}^N J_{ji} x^i_t$. If $\alpha \geqq 1$, we use Jensen inequality and Hölder inequality with conjugate exponents (p, q) to get

$$B_N \leqq \int \exp\left\{ \alpha\beta \sum_{j=1}^N \int_0^T G^j_{t(n)}\, dB^j_t - \frac{\beta^2}{2} \alpha \sum_{j=1}^N \int_0^T \left(G^j_{t(n)} \right)^2 dt \right\} d\gamma\, dP^{\otimes N}$$

$$\leqq \left(\int \exp\left\{ \alpha\beta q \sum_{j=1}^N \int_0^T G^j_{t(n)}\, dB^j_t - \frac{(\alpha\beta q)^2}{2} \sum_{j=1}^N \int_0^T \left(G^j_{t(n)} \right)^2 dt \right\} d\gamma\, dP^{\otimes N} \right)^{1/q}$$

$$\times \left(\int \exp\left\{ \frac{1}{2} \alpha\beta^2 p(q\alpha - 1) \sum_{j=1}^N \int_0^T \left(G^j_{t(n)} \right)^2 dt \right\} d\gamma\, dP^{\otimes N} \right)^{1/p} .$$

By supermartingale properties, the first term is bounded by one.

Moreover, since the G^j are i.i.d,

$$\int \exp\left\{ \frac{\alpha\beta^2}{2} p(q\alpha - 1) \sum_{j=1}^N \int_0^T \left(G^j_{t(n)} \right)^2 dt \right\} d\gamma\, dP^{\otimes N}$$

$$= \int \left(\int \exp\left\{ \frac{\alpha\beta^2}{2} p(q\alpha - 1) \int_0^T \left(G^1_{t(n)} \right)^2 dt \right\} d\gamma \right)^N dP^{\otimes N} .$$

Furthermore,

$$\int \int_0^T \left(G^1_{t(n)} \right)^2 dt\, d\gamma = \int_0^T \frac{1}{N} \sum_{p=1}^N \left(x^p_{t(n)} \right)^2 dt \leqq A^2 T ,$$

$P^{\otimes N}$-almost surely, so that, whenever $\alpha\beta^2 p(q\alpha - 1) T A^2 < 1$, we can find a finite constant $C(\alpha, p)$ (see Appendix A, Lemma A.3.2)), which does not depend on n, such that

$$\int \exp\left\{ \frac{\alpha\beta^2}{2} p(q\alpha - 1) \int_0^T \left(G^1_{t(n)} \right)^2 dt \right\} d\gamma \leqq e^{C(\alpha,p)} , \quad P^{\otimes N} \text{ a.s.}$$

But $\alpha \beta^2\, p(q\,\alpha - 1)\, TA^2 \underset{\alpha,\, q \to 1}{\longrightarrow} \beta^2 A^2 T$ so that, if $\beta^2 A^2 T < 1$, there exists a real number α, $\alpha > 1$, and conjugate exponents (p, q) such that $C(\alpha, p)$ is finite.

Then, for any integer n,

$$\int \exp\left\{\alpha N\Gamma^n(\hat{\mu}^N)\right\} dP^{\otimes N} \leqq \exp C(\alpha, p)N . \qquad \square$$

4 Large deviation principles in the high temperature regime

We remove here the cut-off in the time variable, i.e. the discretization in the interaction and prove a large deviation principle for $\Pi^N_{\beta,T}$.

Theorem 4.1
(a) *Let*

$$\Gamma : \{I(|P) < \infty\} \to \mathbb{R}$$
$$\mu \to \int \log\left(\int \exp\left\{\beta\int_0^T G_t\, dB_t - \frac{\beta^2}{2}\int_0^T G_t^2\, dt\right\} d\gamma_\mu\right) d\mu ,$$

and define

$$H(\mu) = \begin{cases} I(\mu|P) - \Gamma(\mu) & \text{if } \mu \in \{I(|P) < \infty\} , \\ +\infty & \text{otherwise} . \end{cases}$$

Then H is a good rate function.
(b) *If $\beta^2A^2T < 1$, $\Pi^N_{\beta,T}$ satisfies a full large deviation principle with rate function H.*

Remark. 4.2: Let μ be in $\{I(|P) < \infty\}$.
(1) $\mu \ll P$ so that Girsanov theorem imply that $\{B_t\}_{t\leqq T}$ is a semimartingale under μ. In particular, the stochastic integral $\int_0^T G_t\, dB_t$ is well defined for any G in $L^2([0,T])$, i.e. for γ_μ-almost all $G(\int_0^T G_t\, dB_t$ is, given G, a centered gaussian variable with covariance $\int_0^T G_t^2\, dt)$. In particular, it implies that Γ is well defined when the entropy relative to P is finite. Moreover, we can see, as in Lemma 3.3, that Γ is finite whenever $I(|P)$ is finite.
(2) Moreover, we notice in the proof of Lemma A.1, Appendix A, that there exists a sequence of centered gaussian process G^M which converges in probability (and even almost surely) to G such that:

$$G_s^M = \sum_{0\leqq n\leqq M} g_n^\mu(s)\xi_n^\mu ,$$

where $(\xi_n^\mu)_{n\geqq 0}$ are i.i.d $N(0,1)$ and $(g_n^\mu(s))_{n\geqq 0}$ are determinist functions in $L^2([0,T])$.

It is obvious that $\int_0^T G_s^M\, dB_s$ is, conditionally to B, a centered gaussian variable and that

$$\int \exp\left\{\alpha\int_0^T G_s^M\, dB_s - \frac{\beta^2}{2}\int_0^T \left(G_s^M\right)^2\, ds\right\} d\gamma_\mu$$
$$= \exp\left\{\frac{\alpha^2}{2}\sum_{0\leqq n\leqq M}\frac{\left(\int_0^T g_n^\mu(s)\, dB_s\right)^2}{1+\beta^2\lambda_n^\mu}\right\}\int \exp\left\{-\frac{\beta^2}{2}\int_0^T \left(G_s^M\right)^2\, ds\right\} d\gamma_\mu .$$

The same formula shows that, for almost all B, $\left(\exp\left\{\alpha\int_0^T G_s^M\,dB_s - \frac{\beta^2}{2}\int_0^T \left(G_s^M\right)^2 ds\right\}\right)_{M\geqq 0}$ is bounded in $L^{1+\delta}(\Omega,\mathscr{A},\gamma)$, for any positive real number δ, so that this sequence is uniformly integrable. Thus

$$\int \exp\left\{\alpha\int_0^T G_s\,dB_s - \frac{\beta^2}{2}\int_0^T (G_s)^2\,ds\right\} d\gamma_\mu$$
$$= \exp\left\{\frac{\alpha^2}{2}\sum_{n\geqq 0}\frac{\left(\int_0^T g_n^\mu(s)\,dB_s\right)^2}{1+\beta^2\lambda_n^\mu}\right\}\int \exp\left\{-\frac{\beta^2}{2}\int_0^T (G_s)^2\,ds\right\} d\gamma_\mu\,.$$

Hence, under the new law

$$\tilde{\gamma}_\mu^T = \frac{\exp\left\{-\frac{\beta^2}{2}\int_0^T G_s^2\,ds\right\}}{\mathscr{E}\left[\exp\left\{-\frac{\beta^2}{2}\int_0^T G_s^2\,ds\right\}\right]}\gamma_\mu\,.$$

$\int_0^T G_s dB_s$ is, conditionally to B, a centered gaussian variable with covariance $\int\left(\int_0^T G_s\,dB_s\right)^2 d\tilde{\gamma}_\mu^T$.

We will here follow the pattern of the proof used in Sect. 3. We first dwelve on the properties of the rate function H.

In the following pages, we will choose the subdivisions Δ_n such that $|\Delta_n| = \max_{0\leqq k\leqq n}|t_{k+1}-t_k|$ tends to zero when n tends to infinity.

Proposition 4.3 (1) *On the compact set $K_L = \{I(|P) \leqq L\}$, Γ^n converges uniformly to Γ.*
As a consequence, $\lim_{n\to\infty}\inf_F H^n = \inf_F H$, *for any closed set F, and* $\overline{\lim}_{n\to\infty}\inf_O H^n \leqq \inf_O H$ *for any open set O.*
(2) $\forall\mu\in\{I(|P)<\infty\}$, $\Gamma(\mu) = \Gamma_1(\mu)+\Gamma_2(\mu)$ *where*

$$\Gamma_1(\mu) = \log\int\exp\left\{-\frac{\beta^2}{2}\int_0^T G_s^2\,ds\right\} d\gamma_\mu\,,$$

$$\Gamma_2(\mu) = \frac{\beta^2}{2}\int\int\left(\int_0^T G_t\,dB_t\right)^2 d\gamma_{\tilde{K}_\mu^T}\,d\mu\,.$$

(3) $\Gamma \leqq I(|P)$ *and* $\exists\alpha<1,\ \eta>0/\Gamma\leqq\alpha I+\eta$.
(4) *H is a good rate function.*

Proof.
(1) We will show that Γ_1^n and Γ_2^n introduced in Property 3.2 are uniformly Cauchy on K_L:
- By an argument similar to the one used in the proof of Lemma 3.3(1), we see that there exists a finite constant C such that

$$\left|\left(\Gamma_1^n-\Gamma_1^{n+p}\right)(\mu)\right| \leqq C\left\{\int\sup_{t\leqq T}\left|x_{t(n+p)}-x_{t(n)}\right|^2 d\mu\right\}^{1/2}$$
$$\leqq C\left\{\int\sup_{|t-s|\leqq|\Delta_n|}|x_t-x_s|^2\,d\mu\right\}^{1/2}. \qquad (29)$$

– Similarly, as in Lemma 3.4, we find a finite constant C such that

$$\left|\Gamma_2^n(\mu) - \Gamma_2^{n+p}(\mu)\right| \leqq C(1 + I(\mu\,|\,P))\left\{\int \sup_{|t-s| \leqq |\Delta_n|} |x_t - x_s|^2 \, d\mu\right\}^{1/2} . \tag{30}$$

But, according to (7), for any α, we know that

$$\alpha \int \sup_{|t-s| \leqq |\Delta_n|} |x_t - x_s|^2 \, d\mu \leqq I(\mu\,|\,P) + \log \int \exp\left\{\alpha \sup_{|t-s| \leqq |\Delta_n|} |x_t - x_s|^2\right\} dP . \tag{31}$$

And, by bounded convergence theorem, for any α,

$$\lim_{n\to\infty} \log \int \exp\left\{\alpha \sup_{|t-s| \leqq |\Delta_n|} |x_t - x_s|^2\right\} dP = 0 . \tag{32}$$

Let $\varepsilon > 0$, choosing $\alpha = \frac{1}{\varepsilon^2}$ in (31), one sees that there exists an integer $n(\varepsilon)$ depending on ε but not on μ such that, for $n \geqq n(\varepsilon)$:

$$\int \sup_{|t-s| \leqq |\Delta_n|} |x_t - x_s|^2 \, d\mu \leqq (I(\mu\,|\,P) + 1)\varepsilon^2 .$$

Using this last estimate in (29) and (30), one gets that, for any $n \geqq n(\varepsilon)$, any integer p, and any $\mu \in K_L$:

$$\left|\Gamma_1^n(\mu) - \Gamma_1^{n+p}(\mu)\right| \leqq C(1 + L)^{1/2}\varepsilon ,$$

$$\left|\Gamma_2^n(\mu) - \Gamma_2^{n+p}(\mu)\right| \leqq C(1 + L)^{3/2}\varepsilon .$$

As a consequence, Γ^n converges uniformly in K_L to a limit which is obviously Γ.

We now study the behaviour of $\inf_B H^n$, when n grows to infinity, for a measurable set B. We distinguish the case where $\inf_B H$ is finite from the case where it is not.

Suppose first that $\inf_B H = \infty$. Then, for all μ in B, $I(\mu\,|\,P) = \infty$ so that $H^n(\mu) = \infty$, and, of course, $\inf_B H^n = \infty$. Hence $\inf_B H = \inf_B H^n$.

If $\inf_B H < \infty$, we can find a positive number M such that $\inf_B H = \inf_{B\cap\{I\leqq M\}} H$. But, recalling Eqs. (29) and (30), one sees that

$$\lim_{n\to\infty} \inf_{B\cap\{I\leqq M\}} H^n = \inf_{B\cap\{I\leqq M\}} H . \tag{33}$$

But $\inf_{B\cap\{I\leqq M\}} H^n \geqq \inf_B H^n$ so that Eq. (33) implies

$$\overline{\lim_{n\to\infty}} \inf_B H^n \leqq \inf_{B\cap\{I\leqq M\}} H = \inf_B H . \tag{34}$$

Moreover, we stated in Theorem 3.1 that H^n is a good rate function so that it achieves its minimal value on B, if B is closed. Let μ^n be a probability measure such that $\inf_B H^n = H^n(\mu^n)$. Then Lemma 3.3(4) shows that there exists a real α, $\alpha < 1$, and a finite real η such that $H^n(\mu^n) \geqq (1-\alpha)I(\mu^n\,|\,P) - \eta$. So that, using Eq. (34), we see that we can find a finite constant M' such that, for n large enough,

$$I(\mu^n|P) \leqq M' . \tag{35}$$

Hence, for n large enough, $\inf_B H^n = \inf_{B\cap\{I\leqq M'\}} H^n$, and Eq. (33) gives, for any closed set B,

$$\lim_{n\to\infty} \inf_B H^n = \inf_B H . \tag{36}$$

(2) We can obviously identify the limits

$$\lim_{n\to\infty} \Gamma_1^n(\mu) = \log \int \exp\left\{-\frac{\beta^2}{2}\int_0^T G_s^2\, ds\right\} d\gamma_\mu = \Gamma_1(\mu),$$

$$\lim_{n\to\infty} \Gamma_2^n(\mu) = \frac{\beta^2}{2}\int \left(\int_0^T G_t\, dB_t\right)^2 d\mu\, d\gamma_{\widetilde{K}_\mu^T} = \Gamma_2(\mu).$$

Similarly,

$$\lim_{n\to\infty} \Gamma^n(\mu) = \int \left(\log \int \exp\{\beta\int_0^T G_s\, dB_s - \frac{\beta^2}{2}\int_0^T G_s^2\, ds\}\, d\gamma_\mu\right) d\mu = \Gamma(\mu)$$

which proves that $\Gamma(\mu) = \Gamma_1(\mu) + \Gamma_2(\mu)$, using Lemma 3.2. Another proof would be to use directly Remark 4.2(2).

(3) The proof is identical to Lemma 3.3(3) and (4).

(4) The proof of this last point is very similar to the proof of Theorem 3.1(1); it relies on the convergence of the lower semi-continuous rate functions H^n to H on the compact sets K_L (Theorem 3.1(1) and 4.3(1)) and then on Proposition 4.3(3). We leave it to the reader. □

We now turn to the proof of Theorem 4.1(b). We recall that, if $\beta^2 A^2 T < 1$, we have the following exponential tightness lemma (see Lemma 3.10):

Lemma 4.4 $\exists \alpha > 1$

$$\sup_N \left(\int \exp\{\alpha N \Gamma(\widehat{\mu}^N)\}\, dP^{\otimes N}\right)^{1/N} < \infty .$$

Therefore $\exists C$, $\exists \eta > 0$, $\forall B \in \mathscr{B}(\mathscr{M}_1^+(W_T^A))$,

$$\Pi_{\beta,T}^N(B) \leqq e^{CN}\Pi_0^N(B)^\eta .$$

Moreover, we have the following crucial result:

Lemma 4.5 *Let* $\delta > 0$ *be given*:

$$\lim_{n\to\infty} \overline{\lim_{N\to\infty}} \frac{1}{N} \log P^{\otimes N}(|\Gamma - \Gamma^n|(\widehat{\mu}^N) > \delta) = -\infty .$$

Proof of Lemma 4.5. Let $Y_N^n = \frac{1}{N}\sum_{j=1}^N \sup_{|t-s|\leqq|\Delta_n|}\left(x_t^j - x_s^j\right)^2$. Then, for any real number η:

$$\begin{aligned} &P^{\otimes N}(|\Gamma - \Gamma^n|(\widehat{\mu}^N) > \delta) \\ &\leqq P^{\otimes N}(|\Gamma - \Gamma^n|(\widehat{\mu}^N) > \delta;\ Y_N^n < \eta) + P^{\otimes N}(Y_N^n > \eta) . \end{aligned} \tag{37}$$

We first estimate the tail of Y_N^n:

Lemma 4.6 $\forall \eta > 0\ \forall R \in \mathbb{R}^+\ \exists n(R,\eta)\ \forall n \geqq n(R,\eta)$

$$P^{\otimes N}(Y_N^n > \eta) \leqq \exp -RN\,. \tag{38}$$

Proof. $\forall \xi \in \mathbb{R}^+$

$$P^{\otimes N}(Y_N^n > \eta) \leqq \exp -\eta\xi N \int \exp\{N\xi Y_N^n\}\, dP^{\otimes N}$$

$$= \exp -\eta\xi N \left(\int \exp\{\xi \sup_{|t-s| \leqq |\Delta_n|} (x_t - x_s)^2\}\, dP \right)^N$$

so that (32) gives (38).

We now estimate the first term of the right hand side of (37). □

Lemma 4.7 *For any $R > 0$, there exists $\eta(R) > 0$ and $n(R) \in \mathbb{N}$ depending on R but not on N, such that, for any $\eta \leqq \eta(R)$, any $n \geqq n(R)$,*

$$P^{\otimes N}(|\Gamma - \Gamma^n|(\widehat{\mu}^N) > \delta;\ Y_N^n < \eta) \leqq \exp\{-RN\}\,. \tag{39}$$

Proof. By Tchebyshev inequality, it is enough to show that we can find a constant C such that, for any α, when η is small and n large,

$$E_{P^{\otimes N}}\left[\mathbb{1}_{Y_N^n < \eta} \exp\{\alpha N |\Gamma - \Gamma^n|(\widehat{\mu}^N)\}\right] < C^N\,.$$

It is of course, since $|\Gamma - \Gamma^n| \leqq |\Gamma_1 - \Gamma_1^n| + |\Gamma_2 - \Gamma_2^n|$, enough to prove, for $i = 1,2$:

$$E_{P^{\otimes N}}\left[\mathbb{1}_{Y_N^n < \eta} \exp\{\alpha N |\Gamma_i - \Gamma_i^n|(\widehat{\mu}^N)\}\right] < C^N\,. \tag{40}$$

Inequality (40) with $i = 1$ is obvious since we can show as in the proof of Lemma 3.3(1) that

$$\exists C_1 / |\Gamma_1 - \Gamma_1^n|(\widehat{\mu}^N) \leqq C_1 \left(\frac{1}{N} \sum_{j=1}^N \sup_{|t-s| \leqq |\Delta_n|} \left| x_t^j - x_s^j \right|^2 \right)^{1/2} = C_1 (Y_N^n)^{1/2}\,. \tag{41}$$

Moreover, to prove inequality (40) with $i = 2$, we follow the lines of the proof of Lemma 3.4(1) so that we find a finite constant $c\left(c = (\beta^2/2 + 1)\exp \frac{1}{2}\beta^2 A^2 T\right)$ such that

$$\begin{aligned}
&\frac{2}{\beta^2} |\Gamma_2 - \Gamma_2^n|(\widehat{\mu}^N) \\
&\leqq c \int\int |\Gamma_1 - \Gamma_1^n| \left(\int_0^T G_{t(n)}\, dB_t \right)^2 d\gamma_{\widehat{\mu}^N}\, d\widehat{\mu}^N \\
&\quad + c\int\int \left| \int_0^T G_{t(n)}^2\, dt - \int_0^T G_t^2\, dt \right| \left(\int_0^T G_{t(n)}\, dB_t \right)^2 d\gamma_{\widehat{\mu}^N}\, d\widehat{\mu}^N \\
&\quad + \int\int \Lambda_T(G) \left| \int_0^T (G_{t(n)} - G_t)\, dB_t \right| \left| \int_0^T (G_{t\,(n)} + G_t)\, dB_t \right| d\gamma_{\widehat{\mu}^N}\, d\widehat{\mu}^N\,.
\end{aligned} \tag{42}$$

According to (41), the first term in the right hand side of (42) is bounded by

$$|\Gamma_1 - \Gamma_1^n| \int\int \left(\int_0^T G_{t(n)}\, dB_t\right)^2 d\gamma_{\widehat{\mu}^N}\, d\widehat{\mu}^N \leqq C_1 (Y_N^n)^{1/2} \left(\frac{1}{N^2}\sum_{i,j=1}^N \left(\int_0^T x^i_{t(n)}\, dB^j_t\right)^2\right). \tag{43}$$

We now focus on the second term in the right hand side of (42):

$$\begin{aligned}
&\int\int \left|\int_0^T G^2_{t(n)}\, dt - \int_0^T G_t^2\, dt\right| \left(\int_0^T G_{t(n)}\, dB_t\right)^2 d\gamma_{\widehat{\mu}^N}\, d\widehat{\mu}^N \\
&= \frac{1}{N}\sum_{j=1}^N \int \left|\int_0^T G^2_{t(n)} dt - \int_0^T G_t^2 dt\right| \left(\int_0^T G_{t(n)} dB^j_t\right)^2 d\gamma_{\widehat{\mu}^N} \\
&\leqq \frac{1}{N}\sum_{j=1}^N \left(\int\int_0^T (G_{t(n)} - G_t)^2\, dt\, d\gamma_{\widehat{\mu}^N}\right)^{1/2} \\
&\quad \times \left(\int\int_0^T (G_{t(n)} + G_t)^2\, dt \left(\int_0^T G_{t(n)}\, dB^j_t\right)^4 d\gamma_{\widehat{\mu}^N}\right)^{1/2}.
\end{aligned}$$

But

$$\int\int_0^T (G_{t(n)} - G_t)^2\, dt = \frac{1}{N}\sum_{i=1}^N \int_0^T (x^i_t - x^i_{t(n)})^2\, dt\, d\gamma_{\widehat{\mu}^N} \leqq T\, Y_N^n$$

and, for any centered gaussian variables X and Z, we know that

$$\mathscr{E}[Z^2X^4] \leqq 15\mathscr{E}[Z^2]\mathscr{E}[X^2]^2$$

so that, for any $t \leqq T$:

$$\begin{aligned}
&\int (G_{t(n)} + G_t)^2 \left(\int_0^T G_{t(n)}\, dB^j_t\right)^4 d\gamma_{\widehat{\mu}^N} \\
&\leqq 15\left(\int (G_{t(n)} + G_t)^2\, d\gamma_{\widehat{\mu}^N}\right)\left(\int \left(\int_0^T G_{t(n)}\, dB^j_t\right)^2 d\gamma_{\widehat{\mu}^N}\right)^2 \\
&\leqq 60A^2 \left(\frac{1}{N}\sum_{i=1}^N \left(\int_0^T x^i_{t(n)}\, dB^j_t\right)^2\right)^2.
\end{aligned}$$

We conclude that we can find a finite constant C_2 so that

$$\begin{aligned}
&\int\int \left|\int_0^T G^2_{t(n)}\, dt - \int_0^T G_t^2\, dt\right| \left(\int_0^T G_{t(n)}\, dB_t\right)^2 d\gamma_{\widehat{\mu}^N}\, d\widehat{\mu}^N \\
&\leqq C_2 (Y_N^n)^{1/2} \left(\frac{1}{N^2}\sum_{i,j=1}^N \left(\int_0^T x^i_{t(n)}\, dB^j_t\right)^2\right).
\end{aligned} \tag{44}$$

Similarly, we find

$$\int d\widehat{\mu}^N \int d\gamma_{\widehat{\mu}^N} \Lambda_T(G) \left| \int_0^T (G_{t(n)} - G_t)\, dB_t \right| \left| \int_0^T (G_{t(n)} + G_t)\, dB_t \right|$$
$$\leqq C_3 \left(\frac{1}{N^2} \sum_{i,j=1}^N \left(\int_0^T (x^i_{t(n)} + x^i_t)\, dB^j_t \right)^2 \right)^{1/2}$$
$$\times \left(\frac{1}{N^2} \sum_{i,j=1}^N \left(\int_0^T (x^i_{t(n)} - x^i_t)\, dB^j_t \right)^2 \right)^{1/2}$$
$$\leqq \frac{1}{2} C_3 \eta^{1/2} \left\{ \frac{1}{N^2} \sum_{i,j=1}^N \left(\int_0^T (x^i_{t(n)} + x^i_t)\, dB^j_t \right)^2 + \frac{1}{\eta T N^2} \sum_{i,j=1}^N \left(\int_0^T (x^i_{t(n)} - x^i_t)\, dB^j_t \right)^2 \right\}. \qquad (45)$$

Thus, if we recall the main steps (42), (43), (44) and (45), we can find a finite constant C_4 such that, on the subset $\{Y^n_N < \eta\} = \left\{ \frac{1}{N} \sum_{j=1}^N \sup_{|t-s| \leqq \Delta_n} \left| x^j_t - x^j_s \right|^2 < \eta \right\}$:

$$\frac{2}{\beta^2} |\Gamma_2 - \Gamma^n_2| (\widehat{\mu}^N)$$
$$\leqq C_4 \eta^{1/2} \frac{1}{N^2} \sum_{i,j=1}^N \left(\int_0^T x^i_{t(n)}\, dB^j_t \right)^2 + C_4 \eta^{1/2}$$
$$\times \left\{ \frac{1}{N^2} \sum_{i,j=1}^N \left(\int_0^T (x^i_{t(n)} + x^i_t)\, dB^j_t \right)^2 + \frac{1}{\eta T N^2} \sum_{i,j=1}^N \left(\int_0^T (x^i_{t(n)} - x^i_t)\, dB^j_t \right)^2 \right\}. \qquad (46)$$

It is now quite easy to deduce (40) with $i = 2$ from (46) since, for any previsible processes $(h^i)_{1 \leqq i \leqq N}$ such that $\frac{1}{N} \sum_{i=1}^N (h^i)^2$ is uniformly bounded by one, for any $\varepsilon < \frac{1}{2\sqrt{T}}$,

$$\int \exp \left\{ \frac{\varepsilon^2}{2N} \sum_{i,j=1}^N \left(\int_0^T h^i_t\, dB^j_t \right)^2 \right\} dP^{\otimes N} \leqq (1 - 4\varepsilon^2 T)^{-N/4}. \qquad \square$$

We can now prove Theorem 4.1(b). Let us first verify that the upper bound of the large deviation principle holds.

Let B be a closed set of $\mathcal{M}_1^+(W^A_T)$. For any integer number n, for any positive real number δ,

$$\Pi^N_{\beta,T}(B) = \int_B \exp\{N\Gamma(\widehat{\mu}^N)\}\, dP^{\otimes N}$$
$$\leqq \int_{\{|\Gamma - \Gamma^n| \leqq \delta\} \cap B} \exp\{N\Gamma(\widehat{\mu}^N)\}\, dP^{\otimes N} + \int_{|\Gamma^n - \Gamma| > \delta} \exp\{N\Gamma(\widehat{\mu}^N)\}\, dP^{\otimes N}$$

so that

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log \int_B \exp\{N\Gamma(\widehat{\mu}^N)\}\, dP^{\otimes N}$$

$$\leqq \max\left\{\delta + \overline{\lim_{N\to\infty}} \frac{1}{N} \log \int_B \exp\{N\Gamma^n(\widehat{\mu}^N)\}\, dP^{\otimes N};\right.$$

$$\left.\overline{\lim_{N\to\infty}} \frac{1}{N} \log \int_{|\Gamma-\Gamma^n|>\delta} \exp\{N\Gamma(\widehat{\mu}^N)\}\, dP^{\otimes N}\right\}.$$

But, by Theorem 3.1(3),

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log \int_B \exp\{N\Gamma^n(\widehat{\mu}^N)\}\, dP^{\otimes N} = \overline{\lim_{N\to\infty}} \frac{1}{N} \log \Pi^{N,n}_{\beta,T}(B) \leqq -\inf_B H^n$$

so that

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log \Pi^N_{\beta,T}(B)$$

$$\leqq \max\left\{\delta - \inf_B H^n;\ \overline{\lim_{N\to\infty}} \frac{1}{N} \log \int_{|\Gamma-\Gamma^n|>\delta} \exp\{N\Gamma(\widehat{\mu}^N)\} dP^{\otimes N}\right\}. \tag{47}$$

Moreover, by exponential tightness Lemma 4.4, $\exists\eta > 0\ \exists C < \infty$

$$\int_{|\Gamma^n-\Gamma|>\delta} \exp N\Gamma(\widehat{\mu}^N)\, dP^{\otimes N} = \Pi^N_{\beta,T}(|\Gamma^n - \Gamma| > \delta)$$

$$\leqq \exp CN \times \Pi^N_0(|\Gamma^n - \Gamma| > \delta)^\eta.$$

which, according to Lemma 4.7, implies

$$\lim_{n\to\infty} \overline{\lim_{N\to\infty}} \frac{1}{N} \log \Pi^N_{\beta,T}(|\Gamma^n - \Gamma| > \delta) = -\infty. \tag{48}$$

Finally recalling that we proved in Proposition 4.3(1) that $\lim_{n\to\infty} \inf_B H^n = \inf_B H$ and letting $n \to +\infty$, and then letting $\delta \to 0$, (47) becomes

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log \Pi^N_{\beta,T}(B) \leqq -\inf_B H.$$

We shall now prove the lower bound; i.e. if O is an open set of $\mathscr{M}_1^+(W_T^A)$,

$$\underline{\lim_{N\to\infty}} \frac{1}{N} \log \Pi^N_{\beta,T}(O) = \underline{\lim_{N\to\infty}} \frac{1}{N} \log \int_O \exp\{N\Gamma(\widehat{\mu}^N)\}\, dP^{\otimes N} \geqq -\inf_O H.$$

But, by Theorem 3.1, for any integer number n, for any positive real number δ,

$$-\inf_O H^n \leqq \underline{\lim}_{N\to\infty} \frac{1}{N} \log \int_O \exp\{N\Gamma^n(\widehat{\mu}^N)\}\, dP^{\otimes N}$$
$$\leqq \max\left\{\delta + \underline{\lim}_{N\to\infty} \frac{1}{N} \log \int_O \exp\{N\Gamma(\widehat{\mu}^N)\}\, dP^{\otimes N};\right.$$
$$\left.\overline{\lim}_{N\to\infty} \frac{1}{N} \log \Pi^{N,n}_{\beta,T}(|\Gamma_n - \Gamma| > \delta)\right\}.$$

So that, using Proposition 4.3(1) and again Lemmas 4.7 and 3.10 as in (48), and letting $n \to \infty$, $\delta \to 0$, we get,

$$-\inf_O H \leqq \underline{\lim}_{N\to\infty} \frac{1}{N} \log \Pi^N_{\beta,T}(O). \qquad \square$$

5 Existence, uniqueness and description of the limit system

In this section, we study the minima of H. First, we characterize these minima through a variational study of H. We show that any minimum of H is solution of the non-linear equation:

$$Q \ll P \quad \frac{dQ}{dP} = \int \exp\left\{\beta \int_0^T G_s\, dB_s - \frac{\beta^2}{2} \int_0^T G_s^2\, ds\right\} d\gamma_Q. \tag{49}$$

Secondly, we prove that there exists a unique probability measure Q on W_T^A which satisfies (49). We proved in Sect. 2, Theorem 2.6, that this implies that Q_β^N is Q-chaotic. We finally give a pathwise description of Q.

5.1 Variational characterization of the minima of H

We shall prove:

Theorem 5.1 *H achieves its minimum value $(= 0)$ on the subset M of probability measures on W_T^A which is given by*

$$M = \left\{Q \in \mathscr{M}_1^+(W_T^A)/Q \ll P \right.$$
$$\left. \frac{dQ}{dP} = \int \exp\left\{\beta \int_0^T G_s\, dB_s - \frac{\beta^2}{2} \int_0^T G_s^2\, ds\right\} d\gamma_Q \right\}.$$

Proof of Theorem 5.1. We first establish that any minimum of H is equivalent to P. To do so, we give the following technical lemma:

Lemma 5.2 *Let Q be a probability measure on W_T^A which minimizes H. Then: (1) $Q \ll P$. (2) Denotes $B = \{\widetilde{\omega}/\frac{dQ}{dP}(\widetilde{\omega}) = 0\}$ and $\delta = P(B)$. Then*

(a) $I\left(\frac{Q + s\,\mathbb{1}_B P}{1+s\delta} \mid P\right) = I(Q \mid P) + s\delta \log s + O(s)$.

(b) $\Gamma\left(\frac{Q + s\,\mathbb{1}_B P}{1+s\delta}\right) = \Gamma(Q) + O(s)$.

Remark. Since Q minimizes $H, I(Q \mid P)$ is finite so that $I\left(\frac{Q+s\mathbb{1}_B P}{1+s\delta} \mid P\right)$ is also finite for any s. According to Proposition 4.3, we then know that $\Gamma\left(\frac{Q+s\,\mathbb{1}_B P}{1+s\delta}\right)$ is well defined and finite. We recall that this was in part made possible by the semi-martingale properties of B under the measures of entropy relative to P finite, which allowed us to define stochastic integrals against this process.

Proof of Lemma 5.2. We denote $Q^s = \frac{Q+s\,\mathbb{1}_B P}{1+s\delta}$.

Proof of (1). Since $I(Q \mid P)$ is finite, $Q \ll P$.

Proof of (2)(a). One can compute:

$$I(Q^s \mid P) = \frac{1}{1+s\delta} I(Q \mid P) - \frac{\log(1+s\delta)}{1+s\delta} + \frac{s\delta}{1+s\delta} \log \frac{s}{1+s\delta}$$

which gives (2)(a).

Proof of (2)(b). To get the Taylor expansion for Γ at Q, we remark first that, if G and V are independent centered gaussian processes with covariances $E_Q[x_u x_t]$ and $E_{\mathbb{1}_B.P}[x_u x_t]$, then $G^s = \dfrac{G+\sqrt{s}V}{\sqrt{1+s\delta}}$ is a centered gaussian process with covariance $\int x_u x_t \, dQ_s(x)$.
Hence, we can write

$$\Gamma(Q^s) = \int \left(\log \int \exp\left\{ \beta \int_0^T G_t^s \, dB_t - \frac{\beta^2}{2} \int_0^T (G_t^s)^2 \, dt \right\} d\gamma_Q \otimes \gamma_{\mathbb{1}_B.P} \right) dQ^s .$$

We compute the Taylor expansion of the last term in the right hand side of the last quality so that we find, for any real number s, a random variable $R(s)$ such that

$$\begin{aligned}
&\exp\left\{ \beta \int_0^T G_t^s \, dB_t - \frac{\beta^2}{2} \int_0^T (G_t^s)^2 \, dt \right\} \\
&= \exp\left\{ \beta \int_0^T \frac{G_t + \sqrt{s}V_t}{\sqrt{1+s\delta}} \, dB_t - \frac{\beta^2}{2} \int_0^T \left(\frac{G_t + \sqrt{s}V_t}{\sqrt{1+s\delta}} \right)^2 dt \right\} \\
&= (1+sR(s)) \left(1 + \sqrt{s} \left\{ \beta \int_0^T V_t \, dB_t - \beta^2 \int_0^T V_t \, G_t \, dt \right\} \right) \\
&\quad \times \exp\left\{ \beta \int_0^T G_t \, dB_t - \frac{\beta^2}{2} \int_0^T (G_t)^2 \, dt \right\} .
\end{aligned}$$

Then, a detailed analysis of $R(s)$ using that G and V are gaussian processes with bounded covariances and that the mean quadratic variation of B under Q^s is also bounded[1] shows that

$$\int \int |\log(1+sR(s))| \, d\gamma_Q \otimes \gamma_{\mathbb{1}_B.P} dQ^s = O(s) .$$

[1] Since $Q^s \ll P$, Girsanov theorem (see Chap. 12 in [7]) implies that we can find a previsible process b and a brownian motion w such that $B_t = w_t + \int_0^t b_s \, ds$. Then, $E_{Q^s}\left[\frac{1}{2}\int_0^T b_s^2 \, ds\right] \leqq I(Q^s \mid P)$

We then get

$$\Gamma(Q^s) = \int \log \int \exp\left\{\beta \int_0^T G_t\, dB_t - \frac{\beta^2}{2}\int_0^T (G_t)^2\, dt\right\} \times \left(1 + \sqrt{s}\left\{\beta \int_0^T V_t\, dB_t - \beta^2 \int_0^T V_t\, G_t\, dt\right\}\right) \times d\gamma_Q \otimes \gamma_{\mathbb{1}_B.P} dQ^s + O(s)\,.$$

Integrating with respect to $\gamma_{\mathbb{1}_B.P}$, and taking into account the fact that G and V are independent and centered, we prove the result, i.e.:

$$\Gamma(Q^s) = \int \left(\log \int \exp\left\{\beta \int_0^T G_t\, dB_t - \frac{\beta^2}{2}\int_0^T (G_t)^2\, dt\right\} d\gamma_Q\right) dQ + O(s) = \Gamma(Q) + O(s)\,. \quad \square$$

Lemma 5.3 *If Q minimizes H, $Q \simeq P$.*

Proof. If Q minimizes H,

$$\lim_{s\to 0} \frac{1}{s}\left(H\left(\frac{Q + s\mathbb{1}_B.P}{1+s\delta}\right) - H(Q)\right) = 0\,.$$

But, Lemma 5.2(2) implies that

$$\frac{1}{s}\left(H\left(\frac{Q + s\mathbb{1}_B.P}{1+s\delta}\right) - H(Q)\right) = \delta \log s + O(1)$$

so that $\delta = P(B) = 0$, which is just what we need to prove the claim. $\square$

To characterize Q, we study the Taylor expansion of $H\left(\frac{1+s\phi}{1+s}Q\right)$ for positive bounded measurable functions ϕ such that $\int \phi\, dQ = 1$. We denote Q^s_ϕ the probability measure $Q^s_\phi = \frac{1+s\phi}{1+s}Q$.

Lemma 5.4 *Let ϕ be a positive and bounded measurable function on W^A_T such that $\int \phi\, dQ = 1$. Denote $\psi = \phi - 1$.*

(1) $I\left(Q^s_\phi \,|\, P\right) = I(Q\,|\,P) + s\int \psi \log \dfrac{dQ}{dP}\, dQ + O\left(s^2\right)$.

(2) $\Gamma\left(Q^s_\phi\right) = \Gamma(Q)$

$$+s\int\left(\log\int \exp\{\beta \textstyle\int_0^T G_t\, dB_t - \frac{\beta^2}{2}\int_0^T G_t^2\, dt\} d\gamma_Q + Y_T\right)\psi\, dQ + O\left(s^2\right).$$

where $(Y_t)_{t\leqq T}$ is an adapted process with finite variation.

Proof. The first point is left to the reader. To prove the second point, we consider, as in Lemma 5.2(2), the independent centered gaussian processes G and V with covariances $E_Q[x_s x_t]$ and $E_{\phi\cdot Q}[x_s x_t]$, and write

$$\Gamma(Q^s_\phi) = \int \left(\log \int \exp\left\{ \beta \int_0^T \frac{G_t + \sqrt{s}V_t}{\sqrt{1+s\delta}}\, dB_t - \frac{\beta^2}{2} \int_0^T \left(\frac{G_t + \sqrt{s}V_t}{\sqrt{1+s\delta}} \right)^2 dt \right\} d\gamma_Q \otimes \gamma_{\phi \cdot Q} \right) dQ^s_\phi .$$

We repeat the proof of Lemma 5.2(2):

$$\int \exp\left\{ \beta \int_0^T \frac{G_t + \sqrt{s}V_t}{\sqrt{1+s\delta}}\, dB_t - \frac{\beta^2}{2} \int_0^T \left(\frac{G_t + \sqrt{s}V_t}{\sqrt{1+s\delta}} \right)^2 dt \right\} d\gamma_Q \otimes \gamma_{\phi \cdot Q}$$

$$= \int d\gamma_Q \otimes \gamma_{\phi \cdot Q} \exp\left\{ \beta \int_0^T G_t\, dB_t - \frac{\beta^2}{2} \int_0^T G_t^2\, dt \right\}$$

$$\times \left(1 + \sqrt{s} \left\{ \beta \int_0^T V_t\, dB_t - \beta^2 \int_0^T V_t G_t\, dt \right\} + s \left\{ -\frac{\beta}{2} \int_0^T G_t\, dB_t - \frac{\beta^2}{2} \int_0^T V_t^2\, dt \right\} \right) + O\left(s^2\right) .$$

So that, integrating with respect to $\gamma_Q \otimes \gamma_{\phi \cdot Q}$ and taking into account the fact that G and V are independent and centered, we find that

$$\Gamma\left(Q^s_\phi\right) = \Gamma(Q) + s \left\{ \int \left(\log \int \exp\left\{ \beta \int_0^T G_t\, dB_t - \frac{\beta^2}{2} \int_0^T G_t^2\, dt \right\} d\gamma_Q \right) \psi\, dQ + \int X_T(x, \phi)\, dQ(x) \right\} + O\left(s^2\right)$$

where

$$X_T(x, \phi) = \int \left\{ -\frac{\beta^2}{2} \int_0^T G_s(\omega_2)\, dB_s(x) + \frac{\beta^2}{2} \int_0^T G_s(\omega_2)^2\, ds + \frac{1}{2} \left(\beta \int_0^T V_s(\omega_1)\, dB_s(x) - \beta^2 \int_0^T G_s(\omega_2) V_s(\omega_1)\, ds \right)^2 \right\}$$
$$\times\, d\gamma_{\phi \cdot Q}(\omega_1)\, d\gamma^x(\omega_2)$$

and

$$\frac{d\gamma^x}{d\gamma_Q}(\omega_2) = \frac{\exp\left\{ \beta \int_0^T G_s(\omega_2)\, dB_s(x) - \frac{\beta^2}{2} \int_0^T G_s(\omega_2)^2\, ds \right\}}{\int \exp\left\{ \beta \int_0^T G_s\, dB(x) - \frac{\beta^2}{2} \int_0^T G_s(\omega_2)^2\, ds \right\} d\gamma_Q(\omega_2)} .$$

We now observe that $X_T(x, \phi)$ must be linear in $\psi = \phi - \int \phi\, dQ$ so that

$$\int X_T(x, \phi)\, dQ(x) = \int \psi(y) Y_T(y)\, dQ(y) ,$$

where, according to the definition of $X_T(x, \phi)$, Y_T is given by

$$Y_T(y) = \frac{1}{2}\int\int \left(\beta\int_0^T y_s\, dB_s(x) - \beta^2 \int_0^T G_s(\omega_2) y_s\, ds\right)^2 d\gamma^x(\omega_2)\, dQ(x)\,.$$

Since B is, under Q, a semi-martingale with bounded quadratic variation, it is clear that $(Y_t)_{t \leqq T}$ has finite variations. □

We can now prove that Q satisfies (49), i.e. Theorem 5.1.
Since Q minimizes H,

$$\lim_{s\to 0} \frac{1}{s}\left(H\left(Q_\phi^s\right) - H(Q)\right) = 0\,.$$

But, Lemma 5.4 implies that

$$\begin{aligned} &H\left(Q_\phi^s\right) - H(Q) \\ &= \int \left\{\log\frac{dQ}{dP} - \log\int \exp\left\{\beta\int_0^T G_t\, dB_t - \frac{\beta^2}{2}\int_0^T G_t^2\, dt\right\} d\gamma_Q - Y_T\right\} \\ &\times \psi\, dQ + O\left(s^2\right) \end{aligned}$$

so that we can find a constant C_Q such that, Q-almost surely, and so P-almost surely by Lemma 5.2,

$$\log\frac{dQ}{dP}(x) = \log\int \exp\left\{\beta\int_0^T G_s\, dB_s(x) - \frac{\beta^2}{2}\int_0^T G_s^2\, ds\right\} d\gamma_Q + Y_T(x) + C_Q\,.$$

But $\left(\frac{dQ}{dP}\Big|_{\mathscr{F}_t}\right)_{t\leqq T}$ must be a $\left(W_T^A, (\mathscr{F}_t)_{t\leqq T}, \mathscr{F}_T, P\right)$ local martingale (see [13, Chap. VIII]).
Since $\left(\int \exp\left\{\beta\int_0^t G_s\, dB_s - \frac{\beta^2}{2}\int_0^t G_s^2\, ds\right\} d\gamma_Q\right)_{t\leqq T}$ is a local martingale and $(Y_t)_{t\leqq T}$ a process with finite variation, by uniqueness of semimartingale decomposition, we find

$$\frac{dQ}{dP} = \int \exp\left\{\beta\int_0^T G_t\, dB_t - \frac{\beta^2}{2}\int_0^T G_t^2\, dt\right\} d\gamma_Q\,. \qquad \square$$

5.2 M is reduced to one probability measure Q

We shall use a fixed point argument to prove that H admits a unique minimum, i.e.:

Theorem 5.5 *The set M is reduced to a probability measure Q, i.e., there exists a unique probability measure Q on W_T^A which is implicitly defined by*

$$Q \ll P \quad \frac{dQ}{dP} = \int \exp\left\{\beta\int_0^T G_s(\omega)\, dB_s - \frac{\beta^2}{2}\int_0^T G_s^2(\omega)\, ds\right\} d\gamma_Q(\omega)\,.$$

For $\mu \in \mathcal{M}_1^+\left(W_T^A\right)$, let $L(\mu)$ be the measure defined by

$$dL(\mu) = \int \exp\left\{\beta \int_0^T G_t\, dB_t - \frac{\beta^2}{2}\int_0^T G_t^2\, dt\right\} d\gamma_\mu\, dP\,,$$

where γ_μ is the law of a centered gaussian process G with covariance

$$\int G_s G_t\, d\gamma_\mu = \int x_s x_t\, d\mu(x)\,.$$

We want to characterize M as the set of the fixed points of the map L, which needs that L maps $\mathcal{M}_1^+\left(W_T^A\right)$ into $\mathcal{M}_1^+\left(W_T^A\right)$, i.e.:

Lemma 5.6 *For any probability measure μ on W_T^A, $L(\mu)$ is a probability measure on W_T^A.*

Proof. For any $\mu \in \mathcal{M}_1^+\left(W_T^A\right)$, it is clear that $L(\mu)$ is a positive measure on W_T^A so that we only need to prove that $L(\mu)\left(W_T^A\right) = 1$. But Fubini Theorem implies that

$$\begin{aligned} L(\mu)\left(W_T^A\right) &= \int\left\{\int \exp\left\{\beta\int_0^T G_s\, dB_s - \frac{\beta^2}{2}\int_0^T G_s^2\, ds\right\} d\gamma_\mu\right\} dP \\ &= \int\left\{\int \exp\left\{\beta\int_0^T G_s\, dB_s - \frac{\beta^2}{2}\int_0^T G_s^2\, ds\right\} dP\right\} d\gamma_\mu\,. \end{aligned}$$

And $\left(\exp\left\{\beta\int_0^t G_s\, dB_s - \frac{\beta^2}{2}\int_0^t G_s^2\, ds\right\}\right)_{t\leqq T}$ is a uniformly integrable P-martingale as soon as $\int_0^T G_s^2\, ds$ is finite (see, for instance, Novikov criterion, Proposition 1.15 in [13]) so that, γ_Q almost surely, $\int \exp\left\{\beta\int_0^T G_s\, dB_s - \frac{\beta^2}{2}\int_0^T G_s^2\, ds\right\}$ $dP \equiv 1$. Hence $L(\mu)\left(W_T^A\right) \equiv 1$. □

Thus, it is clear that Theorem 5.5 is equivalent to:

Theorem 5.7 *L admits a unique fixed point Q.*

We shall prove Theorem 5.7 through a contraction argument.
Let $(\mathcal{F}_t)_{t\leqq T}$ be the natural filtration on $\mathcal{M}_1^+\left(W_T^A\right)$ defined by

$$\mathcal{F}_t = \sigma\left(x_s, \quad s \leqq t\right)$$

We will denote in this section P_t the restriction of P to the σ-algebra $\mathcal{F}_t\,(P = P_T)$.
Let $\mathcal{A}_T$ be the subset of $\mathcal{M}_1^+\left(W_T^A\right)$ made of the probability measures which are absolutely continuous with respect to P_T.
Let D_T be the variational distance on $\mathcal{M}_1^+\left(W_T^A\right)$ defined by

$$\forall(\mu,\nu) \in \mathcal{M}_1^+\left(W_T^A\right) \quad D_T(\mu,\nu) = \sup\left|\int f\, d\mu - \int f\, d\nu\right|\,,$$

where the supremum is taken on the measurable functions on W_T^A which are uniformly bounded by one.
It is well known (see [12, Corollary 6.1.1]) that the variational distance is stronger than the Vaserstein distance, i.e. that:

$$\forall(\mu,\nu) \in \mathcal{M}_1^+\left(W_T^A\right) \quad d_T(\mu,\nu) \leqq D_T(\mu,\nu)\,. \tag{50}$$

On $\mathscr{A}_T$, one can see that

$$\forall(\mu, \nu) \in \mathscr{A}_T \quad D_T(\mu, \nu) = \int \left| \frac{d\mu}{dP_T} - \frac{d\nu}{dP_T} \right| dP_T \,.$$

In the following pages, for probability measures μ and ν in $\mathscr{A}_T$ and for $t \leqq T$, we will denote $D_t(\mu, \nu)$ instead of $D_t(\mu|_{\mathscr{F}_t}, \nu|_{\mathscr{F}_t})$ for simplification.

We then prove that:

Proposition 5.8 *We can find a strictly positive real number q and a finite constant λ such that for any probability measures μ and ν absolutely continuous with respect to P_T, for any $t \leqq T$,*

$$D_t(L(\mu), L(\nu))^q \leqq \lambda \int_0^t D_s(\mu, \nu)^q \, ds \,.$$

Proof of Proposition 5.8. We first give another formula for $\left.\frac{dL^1(\mu)}{dP_T}\right|_{\mathscr{F}_t}$:

For $v \in [0, T]$ and $\mu \in \mathscr{M}_1^+\left(W_T^A\right)$, we recall that we define a covariance $\widetilde{K}_\mu^v$ by

$$\widetilde{K}_\mu^v(s, u) = \int G_s G_u \left\{ \frac{\exp\left\{-\frac{\beta^2}{2} \int_0^v G_s^2 \, ds\right\}}{\int \exp\left\{-\frac{\beta^2}{2} \int_0^v G_s^2 \, ds\right\} d\gamma_\mu} \right\} d\gamma_\mu \,.$$

Then:

Lemma 5.9 *For any $\mu \in \mathscr{M}_1^+(W_T^A)$ and any $t \leqq T$,*

$$\left.\frac{dL^1(\mu)}{dP_T}\right|_{\mathscr{F}_t} = \exp\left\{ \beta^2 \int_0^t \int_0^s \widetilde{K}_\mu^s(s,u) \, dB_u \, dB_s - \frac{\beta^4}{2} \int_0^t \left(\int_0^s \widetilde{K}_\mu^s(s,u) \, dB_u \right)^2 ds \right\} .$$

Proof. We first notice that, for any time $t \leqq T$,

$$\left.\frac{dL^1(\mu)}{dP}\right|_{\mathscr{F}_t} = \int \exp\left\{ \beta \int_0^t G_s \, dB_s - \frac{\beta^2}{2} \int_0^t G_s^2 \, ds \right\} d\gamma_\mu \,,$$

i.e. that $\left(\mathscr{E}_t = \int \exp\left\{\beta \int_0^t G_s \, dB_s - \frac{\beta^2}{2} \int_0^t G_s^2 \, ds\right\} d\gamma_\mu\right)_{t \leqq T}$ is a true $\mathscr{F}_t$-martingale.

In fact, $(\mathscr{E}_t)_{t \leqq T}$ is a supermartingale such that $\mathscr{E}_0 = 1$, so that it is enough to prove that $E[\mathscr{E}_t] \equiv 1$, which can be proved as in Lemma 5.6.

Finally, we prove in Lemma 5.15 (in the case $\mu = Q$) that

$$\int \exp\left\{ \beta \int_0^t G_s \, dB_s - \frac{\beta^2}{2} \int_0^t G_s^2 \, ds \right\} d\gamma_\mu$$

$$= \exp\left\{ \beta^2 \int_0^t \int_0^s \widetilde{K}_\mu^s(s,u) \, dB_u \, dB_s - \frac{\beta^4}{2} \int_0^t \left(\int_0^s \widetilde{K}_\mu^s(s,u) \, dB_u \right)^2 ds \right\} . \qquad \square$$

We denote $\widetilde{\gamma}^{v}_{\mu}$ the law of a centered gaussian process with covariance $\widetilde{K}^{v}_{\mu}$. Let

$$X^{\mu}_t(x) = \beta^2 \int_0^t \int_0^s \widetilde{K}^s_{\mu}(s,u)\, dB_u\, dB_s - \frac{\beta^4}{2} \int_0^t \left(\int_0^s \widetilde{K}^s_{\mu}(s,u)\, dB_u \right)^2 ds$$

Then:

Lemma 5.10 *For any conjugate exponents (p,q), for any probability measures μ and v on W^A_T:*

$$D_t(L(\mu),L(v)) \leqq \left(\int |X^{\mu}_t - X^{v}_t|^q \, dP_t\right)^{1/q}$$
$$\times \left(\int_0^1 \left(\int \exp pX^{\mu}_t \, dP_t\right)^{1-\alpha} \left(\int \exp pX^{v}_t \, dP_t\right)^{\alpha} d\alpha \right)^{1/p} .$$

Proof. Let $\mu, v \in \mathcal{M}^+_1(W^A_T)$.

$$D_t(L^1(\mu),L^1(v)) = \int \left| \left.\frac{dL^1(\mu)}{dP_T}\right|_{\mathcal{F}_t} - \left.\frac{dL(v)}{dP_T}\right|_{\mathcal{F}_t} \right| dP_t \, .$$

But, according to Lemma 5.9,

$$\left.\frac{dL^1(\mu)}{dP_T}\right|_{\mathcal{F}_t} = \exp X^{\mu}_t \, .$$

Hence:

$$D_t(L^1(\mu),L^1(v)) = \int |\exp X^{\mu}_t(x) - \exp X^{v}_t(x)| \, dP_t(x)$$
$$= \int |X^{\mu}_t(x) - X^{v}_t(x)| \int_0^1 \exp\{X^{\mu}_t(x) + \alpha(X^{v}_t(x) - X^{\mu}_t(x))\} \, d\alpha \, dP_t(x) \, .$$

Let (p,q) be conjugate exponents (i.e. $p^{-1} + q^{-1} = 1$). Hölder inequality gives

$$D_t(L^1(\mu),L^1(v)) \leqq \left(\int |X^{\mu}_t - X^{v}_t)|^q \, dP_t\right)^{1/q}$$
$$\times \left(\int \int_0^1 \exp p\,\{X^{\mu}_t + \alpha(X^{v}_t - X^{\mu}_t)\} \, d\alpha \, dP_t \right)^{1/p}$$

We obtain Lemma 5.10 using Hölder inequality with conjugate exponents $\left(\frac{1}{1-\alpha}, \frac{1}{\alpha}\right)$:

$$\int \exp p\{X^{\mu}_t + \alpha(X^{v}_t - X^{\mu}_t)\} \, dP_t \leqq \left(\int \exp pX^{\mu}_t \, dP_t\right)^{1-\alpha} \left(\int \exp pX^{v}_t \, dP_t\right)^{\alpha} . \qquad \square$$

We first bound the second term in the right hand side of the inequality of Lemma 5.10:

Lemma 5.11 *If $p-1$ is small enough ($\beta^2 p(p-1)A^2 T < 1$), we can find a finite constant C_1 such that, for any $\mu \in \mathcal{M}^+_1(W^A_T)$, for any $t \leqq T$,*

$$\int \exp\{pX^{\mu}_t\} \, dP_t \leqq C_1 \, .$$

Proof.

$$\int \exp\{pX_t^\mu\}\,dP_t = \int\left(\int \exp\left\{\beta\int_0^t G_s\,dB_s(x) - \frac{\beta^2}{2}\int_0^t G_s^2\,ds\right\}d\gamma_\mu\right)^p dP_t(x)$$
$$\leqq \int\int \exp\left\{p\beta\int_0^t G_s\,dB_s(x) - p\frac{\beta^2}{2}\int_0^t G_s^2\,ds\right\}d\gamma_\mu\,dP_t(x)$$
$$= \int \exp\left\{\frac{\beta^2}{2}p(p-1)\int_0^t G_s^2\,ds\right\}d\gamma_\mu\,. \tag{51}$$

Moreover, if $p-1$ is small enough so that

$$\beta^2 p(p-1)\int\int_0^t G_s^2\,ds\,d\gamma_\mu = \beta^2 p(p-1)\int\int_0^t x_s^2\,ds\,d\mu \leqq \beta^2 p(p-1)A^2T < 1\,.$$

Then, we prove in Appendix A, Lemma A.3, that we can bound (51) by a finite constant C_1 which only depends on p. □

We will suppose in the following pages that p has been chosen so that Lemma 5.11 holds and, for later convenience, so that $q = \frac{p}{p-1} \geqq 2$.
We now focus on the first term in the right hand side of the inequality of Lemma 5.10:

Lemma 5.12 *We can find a finite constant C_2 such that for any $(\mu,\nu) \in \mathcal{M}_1^+(W_T^A)$*

$$\int |X_t^\mu - X_t^\nu|^q\,dP_t \leqq C_2\int_0^t \sup_{s\leqq u}\left|\widetilde{K}_\mu^u(u,s) - \widetilde{K}_\nu^u(u,s)\right|^q du\,.$$

Proof.

$$\int |X_t^\mu - X_t^\nu|^q dP_t$$
$$= \beta^{2q}\int\Bigg|\int_0^t\int_0^u \left(\widetilde{K}_\mu^u(u,s) - \widetilde{K}_\nu^u(u,s)\right)dB_s\,dB_u$$
$$-\frac{\beta^2}{2}\int_0^t\int_0^u \left(\widetilde{K}_\mu^u(u,s) - \widetilde{K}_\nu^u(u,s)\right)dB_s\int_0^u \left(\widetilde{K}_\mu^u(u,s) + \widetilde{K}_\nu^u(u,s)\right)dB_s\,du\Bigg|^q dP_t$$
$$\leqq \beta^{2q}2^{q-1}\int\left|\int_0^t\int_0^u \left(\widetilde{K}_\mu^u(u,s) - \widetilde{K}_\nu^u(u,s)\right)dB_s\,dB_u\right|^q dP_t$$
$$+\frac{\beta^{4q}}{2}\int\Bigg|\int_0^t\int_0^u \left(\widetilde{K}_\mu^u(u,s) - \widetilde{K}_\nu^u(u,s)\right)dB_s$$
$$\times\int_0^u \left(\widetilde{K}_\mu^u(u,s) + \widetilde{K}_\nu^u(u,s)\right)dB_s\,du\Bigg|^q dP_t\,. \tag{52}$$

We first focus on the first term in the right hand side of (52). We apply Burkholder–Davis–Gundy inequality to the martingale $M_t = \int_0^t \int_0^u (\widetilde{K}_\mu^u(u,s) - \widetilde{K}_\nu^u(u,s))\, dB_s\, dB_u$ so that we find a finite constant c_q such that

$$\int \left| \int_0^t \int_0^u \left(\widetilde{K}_\mu^u(u,s) - \widetilde{K}_\nu^u(u,s) \right) dB_s\, dB_u \right|^q dP_t$$
$$\leqq c_q \int \left(\int_0^t \left(\int_0^u \left(\widetilde{K}_\mu^u(u,s) - \widetilde{K}_\nu^u(u,s) \right) dB_s \right)^2 du \right)^{q/2} dP_t$$

Since we supposed $q \geqq 2$, we can use Jensen inequality:

$$\int \left| \int_0^t \int_0^u \left(\widetilde{K}_\mu^u(u,s) - \widetilde{K}_\nu^u(u,s) \right) dB_s\, dB_u \right|^q dP_t$$
$$\leqq c_q T^{q/2-1} \int_0^t \int \left| \int_0^u \left(\widetilde{K}_\mu^u(u,s) - \widetilde{K}_\nu^u(u,s) \right) dB_s \right|^q dP_t\, du\,. \tag{53}$$

Moreover, since B is a brownian motion,

$$G_u = \int_0^u \left(\widetilde{K}_\mu^u(u,s) - \widetilde{K}_\nu^u(u,s) \right) dB_s$$

is a centered gaussian process with covariance

$$\int \left(\int_0^u \left(\widetilde{K}_\mu^u(u,s) - \widetilde{K}_\nu^u(u,s) \right) dB_s \right)^2 dP_t = \int_0^u \left(\widetilde{K}_\mu^u(u,s) - \widetilde{K}_\nu^u(u,s) \right)^2 ds$$
$$\leqq T \sup_{s \leqq u} \left| \widetilde{K}_\mu^u(u,s) - \widetilde{K}_\nu^u(u,s) \right|^2$$

so that we can find a finite constant c_2 such that

$$\int \left| \left(\int_0^u \left(\widetilde{K}_\mu^u(u,s) - \widetilde{K}_\nu^u(u,s) \right) dB_s \right) \right|^q dP_t \leqq c_2 \sup_{s \leqq u} \left| \widetilde{K}_\mu^u(u,s) - \widetilde{K}_\nu^u(u,s) \right|^q \tag{54}$$

(53) and (54) allow us to conclude that we can find a finite constant c_3 such that

$$\int \left| \int_0^t \int_0^u \left(\widetilde{K}_\mu^u(u,s) - \widetilde{K}_\nu^u(u,s) \right) dB_s\, dB_u \right|^q dP_t \leqq c_3 \int_0^t \sup_{s \leqq u} \left| \widetilde{K}_\mu^u(u,s) - \widetilde{K}_\nu^u(u,s) \right|^q du\,. \tag{55}$$

Similarly, we can bound the second term in the right hand side of (52) and find a finite constant c_4 such that, for every $t \leqq T$,

$$\int \left| \int_0^t \int_0^u \left(\widetilde{K}_\mu^u(u,s) - \widetilde{K}_\nu^u(u,s) \right) dB_s \int_0^u \left(\widetilde{K}_\mu^u(u,s) + \widetilde{K}_\nu^u(u,s) \right) dB_s\, du \right|^q dP_t$$
$$\leqq c_4 \int_0^t \sup_{s \leqq u} \left| \widetilde{K}_\mu^u(u,s) - \widetilde{K}_\nu^u(u,s) \right|^q du\,. \tag{56}$$

Thus, (52), (55), and (56) achieves the proof of Lemma 5.12. □

Thanks to Lemmas 5.10, 5.11 and 5.12, we conclude that we can find a positive real number q ($q \geqq 2$) and a finite constant K such that, for any $t \leqq T$, for any $\mu, \nu \in \mathcal{M}_1^+(W_T^A)$,

$$D_t(L^1(\mu), L^1(\nu))^q \leqq K \int_0^t \sup_{s \leqq u} \left| \widetilde{K}_\mu^u(u,s) - \widetilde{K}_\nu^u(u,s) \right|^q du \,. \tag{57}$$

Finally, we prove in Appendix A, Lemma A.4, that $\mu \to \widetilde{K}_\mu^u$ is Lipschitz for the distance D_u:

Lemma 5.13 *We can find a finite constant k such that, for any $u \leqq T$, for any probability measures $\mu, \nu \in \mathcal{A}_T$,*

$$\sup_{s \leqq u} \left| \widetilde{K}_\mu^u(u,s) - \widetilde{K}_\nu^u(u,s) \right| \leqq k D_u(\mu, \nu) \,.$$

Hence, inequality (57) implies that we can find a positive real number q and a finite constant λ such that, for any $t \leqq T$:

$$D_t(L(\mu), L(\nu))^q \leqq \lambda \int_0^t D_u(\mu, \nu)^q du \,. \quad \square \tag{58}$$

Proof of Theorem 5.7. It is now classical to deduce Theorem 5.7 from Proposition 5.8:
We construct a sequence $(\mu_n)_{n \in \mathbb{N}}$ of probability measures on W_T^A as follows:

$$\mu_0 = P_T, \qquad \mu_{n+1} = L(\mu_n) \,.$$

We notice that, for any integer n, μ_n belongs to $\mathcal{A}_T$.
We deduce from Proposition 5.8 that, for any integer n,

$$D_T(\mu_{n+1}, \mu_n)^q \leqq \lambda^n \int_{0 \leqq s_1 \leqq \cdots \leqq s_n \leqq T} D_{s_1}(L(P), P)^q \prod_{i=1}^n ds_i \leqq 2^q \frac{(\lambda T)^n}{(n-1)!} \,. \tag{59}$$

Thus, inequality (50) implies

$$d_T(\mu_{n+1}, \mu_n) \leqq 2 \left(\frac{(\lambda T)^n}{(n-1)!} \right)^{1/q} .$$

Thus, $(\mu_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in the complete metric space $(\mathcal{M}_1^+(W_T^A), d_T)$, so that this sequence converges to a probability measure Q. Q is a fixed point of L.
Using Gronwall Lemma and Proposition 5.8, we see that L admits a unique fixed point. $\square$

5.3 Characterization of M as the set of weak solutions of a (non-markovian) stochastic differential equation

We interpret here the elements of M as the weak solutions of a non-linear, non-markovian, stochastic system S defined, on the time interval $[0, T]$, by

$$S:\begin{cases} X_t = X_0 - \int_0^t \nabla U(X_s)ds + B_t\,, \\ B_t = W_t + \beta^2 \int_0^t \int_0^s \widetilde{K}_Q^s(s,u)\,dB_u\,dt\,, \\ \text{Law of } (X) = Q,\ Q|_{\mathscr{F}_0} = \mu_0\,, \end{cases} \tag{60}$$

where W is a brownian motion.

Theorem 5.14

- (a) *If Q belongs to M, then Q is solution of S.*
- (b) *Reciprocally, If S admits a weak solution Q, then $Q \in M$.*

To prove Theorem 5.14, we first give another formula for $\frac{dQ}{dP}$.

Lemma 5.15

$$\frac{dQ}{dP} = \exp\left\{\beta^2 \int_0^T \int_0^t \widetilde{K}_Q^t(t,s)\,dB_s\,dB_t - \frac{\beta^4}{2}\int_0^T \left(\int_0^t \widetilde{K}_Q^t(t,s)\,dB_s\right)^2 dt\right\}.$$

Proof of Lemma 5.15. Gaussian properties (see [10, Proposition 8.4]) show that, under the probability measure

$$\tilde{\gamma}_Q^t = \left\{\frac{\exp - \frac{\beta^2}{2}\int_0^t G_s^2\,ds}{\int \exp - \frac{\beta^2}{2}\int_0^t G_s^2\,ds\,d\gamma_Q}\right\}\gamma_Q\,.$$

G is a centered gaussian process with covariance $\widetilde{K}_Q^t$. Hence, according to Remark 4.2:

$$\frac{dQ}{dP} = \int \exp\left\{\beta \int_0^T G_t\,dB_t - \frac{\beta^2}{2}\int_0^T G_t^2\,dt\right\} d\gamma_Q$$

$$= \left(\int \exp\left\{-\frac{\beta^2}{2}\int_0^T G_t^2\,dt\right\} d\gamma_Q\right) \exp\frac{\beta^2}{2}\int_0^T\int_0^T \widetilde{K}_Q^T(t,s)\,dB_s\,dB_t\,. \tag{61}$$

Let

$$\Lambda_T = \frac{\exp\left\{-\frac{\beta^2}{2}\int_0^T G_t^2\,dt\right\}}{\int \exp\left\{-\frac{\beta^2}{2}\int_0^T G_t^2\,dt\right\} d\gamma_Q}\,.$$

Then

$$\widetilde{K}_Q^T(t,s) = \int G_t G_s \Lambda_T\,d\gamma_Q$$

so that

$$\int_0^T\int_0^T \widetilde{K}_Q^T(t,s)\,dB_s\,dB_t = \int \left(\int_0^T G_s\,dB_s\right)^2 \Lambda_T\,d\gamma_Q\,. \tag{62}$$

Its formula implies that

$$\left(\int_0^T G_s\,dB_s\right)^2 = 2\int_0^T G_t \int_0^t G_s\,dB_s\,dB_t + \int_0^T G_t^2\,dt\,, \tag{63}$$

$$\Lambda_T = 1 + \frac{\beta^2}{2}\int_0^T \Lambda_t \left(\widetilde{K}_Q^t(t,t) - G_t^2\right) dt \tag{64}$$

So that (62) becomes

$$\begin{aligned}\int_0^T\int_0^T \widetilde{K}_Q^T(t,s)\,dB_s\,dB_t &= \int \left(2\int_0^T \Lambda_t G_t \int_0^t G_s\,dB_s\,dB_t + \int_0^T \Lambda_t G_t^2\,dt \right.\\ &\quad \left. + \frac{\beta^2}{2}\int_0^T \left(\int_0^t G_s dB_s\right)^2 \Lambda_t \left(\widetilde{K}_Q^t(t,t) - G_t^2\right) dt\right) d\gamma_Q\\ &= 2\int_0^T\int_0^t \widetilde{K}_Q^t(t,s)\,dB_s\,dB_t + \int_0^T \widetilde{K}_Q^t(t,t)\,dt\\ &\quad + \frac{\beta^2}{2}\int d\gamma_Q \int_0^T \left(\int_0^t G_s\,dB_s\right)^2 \Lambda_t \left(\widetilde{K}_Q^t(t,t) - G_t^2\right) dt\,. \end{aligned} \tag{65}$$

We now observe that for any $(s,r,t,u) \in [0,T]^4$,

$$\int \left[G_s^2 G_u G_r\right] d\tilde{\gamma}_Q^t = \widetilde{K}_Q^t(s,s)\widetilde{K}_Q^t(r,u) + 2\widetilde{K}_Q^t(r,s)\widetilde{K}_Q^t(s,u)$$

So that we can compute the last term in the right hand side of (65):

$$\int\int_0^T \left(\int_0^t G_s\,dB_s\right)^2 \Lambda_t \left(\widetilde{K}_Q^t(t,t) - G_t^2\right) dt\,d\gamma_Q = -2\int_0^T \left(\int_0^t \widetilde{K}_Q^t(t,s)\,dB_s\right)^2 dt\,. \tag{66}$$

Finally, we can prove by the integration by parts formula that

$$\int \exp\left\{-\frac{\beta^2}{2}\int_0^T G_t^2\,dt\right\} d\gamma_Q = \exp\left\{-\frac{\beta^2}{2}\int_0^T \widetilde{K}_Q^t(t,t)\,dt\right\}\,. \tag{67}$$

Hence, according to (61), (65), (66) and (67), we have proved that

$$\begin{aligned}\frac{dQ}{dP} &= \int \exp\left\{\beta\int_0^T G_t\,dB_t - \frac{\beta^2}{2}\int_0^T G_t^2\,dt\right\} d\gamma_Q\\ &= \left(\int \exp\left\{-\frac{\beta^2}{2}\int_0^T G_t^2\,dt\right\} d\gamma_Q\right) \exp\frac{\beta^2}{2}\int_0^T\int_0^T \widetilde{K}_Q^T(t,s)\,dB_s\,dB_t\\ &= \exp\left\{\beta^2\int_0^T\int_0^t \widetilde{K}_Q^t(t,s)\,dB_s\,dB_t - \frac{\beta^4}{2}\int_0^T \left(\int_0^t \widetilde{K}_Q^t(t,s)\,dB_s\right)^2 dt\right\}\\ &\quad \times \left(\int \exp\left\{-\frac{\beta^2}{2}\int_0^T G_t^2\,dt\right\} d\gamma_Q\right) \exp\left\{-\frac{\beta^2}{2}\int_0^T \widetilde{K}_Q^t(t,t)\,dt\right\}\\ &= \exp\left\{\beta^2\int_0^T\int_0^t \widetilde{K}_Q^t(t,s)\,dB_s\,dB_t - \frac{\beta^4}{2}\int_0^T \left(\int_0^t \widetilde{K}_Q^t(t,s)\,dB_s\right)^2 dt\right\}\,. \qquad \square\end{aligned}$$

Proof of Theorem 5.14.

- *Proof* of (a): Since $Q \ll P$, this point is a direct consequence of Girsanov theorem according to Lemma 5.15 (see [13], Theorem 1.12, p. 306).
- *Proof* of (b): Reciprocally, if Q is a weak solution of S, Q belongs to M as soon as

$$A_T = \int_0^T \left(\int_0^s \widetilde{K}_Q^s(s,u)\, dB_u \right)^2 ds \tag{68}$$

is Q almost surely finite (see Jacod [7, Theorem 12–57b]).

Since Q is solution of S,

$$W_t = B_t - \beta^2 \int_0^t \int_0^s \widetilde{K}_Q^s(s,u)\, dB_u\, ds$$

is a Q brownian motion. Then

$$\int \left(\int_0^s \widetilde{K}_Q^s(s,u)\, dB_u \right)^2 dQ \leqq 2 \int \left(\int_0^s \widetilde{K}_Q^s(s,u)\, dW_u \right)^2 dQ$$
$$+ 2\beta^4 \int \left(\int_0^s \widetilde{K}_Q^s(s,u) \int_0^u \widetilde{K}_Q^u(u,v)\, dB_v\, du \right)^2 dQ\,.$$

We recall that Lemma A.2 in Appendix A, implies that $\widetilde{K}_Q^s(s,u)$ is bounded by A^2, so that we conclude

$$\int \left(\int_0^s \widetilde{K}_Q^s(s,u)\, dB_u \right)^2 dQ \leqq 2A^2 s + 2\beta^4 A^2 \int_0^s \int \left(\int_0^u \widetilde{K}_Q^u(u,v)\, dB_v \right)^2 dQ\, du\,.$$

Thus, Gronwall lemma shows that $\int \left(\int_0^s \widetilde{K}_Q^s(s,u)\, dB_u \right)^2 dQ$ is uniformly bounded when $s \leqq T$ so that $\int A_T\, dQ$ is finite, and so A_T is Q almost surely finite.
□

6 Higher moments and quenched results using Replica

To improve our study of the quenched laws $P_\beta^N(J)$, we shall study the asymptotic behaviour of annealed replica laws $Q_\beta^{r,N}$ on $\left(W_T^A\right)^{\otimes r}$

$$Q_\beta^{r,N} = \int P_\beta^N(J(\omega))^{\otimes r}\, d\gamma(\omega)\,.$$

$Q_\beta^{r,N}$ is the annealed law of r independent replica of spin glass dynamics.

This analysis enables us to get convergence results for higher moments of random variables like $\int \prod_{i=1}^m f_i(x^i) dP_\beta^N(J)$, where $(f_1, \ldots, f_m)$ are bounded continuous functions on W_T^A. Namely:

Theorem 6.1 *For any integer number r, there exists a probability measure Q_r on $\left(W_T^A\right)^r$ such that, whenever $r\beta^2 A^2 T < 1$, for any bounded continuous functions $(f_1, \ldots, f_m)$ on W_T^A,*

$$\int \left(\int \prod_{i=1}^m f_i(x^i)\, dP_\beta^N(J(\omega)) \right)^r d\gamma(\omega) \underset{N\to\infty}{\longrightarrow} \prod_{i=1}^m \int \prod_{j=1}^r f_i(x^j)\, dQ_r\,.$$

Moreover, Q_r is characterized as the unique solution of the non-linear equation

$$Q_r \ll P^{\otimes r} \quad \frac{dQ_r}{dP^{\otimes r}} = \int \exp\left\{\beta \int_0^T \langle G_t, dB_t\rangle - \frac{\beta^2}{2}\int_0^T \|G_t\|^2\, dt\right\} d\gamma_{Q_r} \tag{69}$$

where B is a r-dimensional brownian motion under $P^{\otimes r}$ and G is, under γ_{Q_r} a centered r-dimensional centered gaussian process with covariance

$$\int G_s^i G_t^j d\gamma_{Q_r} = \int x_s^i x_t^j dQ_r(x), \quad 1 \leqq i,\ j \leqq r.$$

The proof is derived from a large deviation principle for the law of the empirical measure under $Q_\beta^{r,N}$ and is similar to the case $r = 1$ (see Theorem 2.6); we omit it.

If Q_2 is the unique solution of (69) (with $r = 2$), let (G,H) be independent gaussian centered processes with covariance

$$\mathscr{E}^G[G(t)G(s)] = \frac{1}{2}E_{Q_2}[(x_t^1 - x_t^2)(x_s^1 - x_s^2)], \quad \mathscr{E}^H[H(t)H(s)] = E_{Q_2}[x_t^1 x_s^2].$$

For f in $L^2([0,T])$, let $P(f)$ be the restriction on $[0,T]$ of the law of the diffusion

$$\begin{cases} dx_t = -\nabla U(x_t)dt + dB_t + \beta f(t)dt\,, \\ \text{Law of } x_0 = \mu_0\,, \end{cases}$$

and let P_H be the partially averaged law

$$P_H = \mathscr{E}^G[P(G+H)]\,.$$

Then, one can improve Theorem 6.1:

Theorem 6.2 *Let r be an integer, and suppose that $r\beta^2 A^2 T < 1$. Then, for any functions $(f_1,\ldots,f_m) \in \mathscr{C}_b^0(W_T^A)$*

$$\lim_{N\to\infty} \int \left(\int \prod_{i=1}^m f_i(x^i) dP_\beta^N(J(\omega))\right)^r d\gamma(\omega) = \prod_{i=1}^m \mathscr{E}^H\left[\left(\int f_i\, dP_H\right)^r\right].$$

Proof. Theorem 6.2 is based on the observation that, for any number r of replica, $Q_r = \mathscr{E}^H[P_H^{\otimes r}]$. □

Theorem 6.2 is a good lead to the following:

Conjecture 6.3: The law of a single spin x^1 converges to P_H, almost surely with respect to the random interaction.
There is one case where Theorem 6.1 can be very simply used:

Theorem 6.4 *If $Q_2 = (Q_1)^{\otimes 2}$ and $2\beta^2 A^2 T < 1$, for any bounded measurable functions $(f_1,\ldots,f_m)$ on W_T^A,*

$$\lim_{N\to\infty} \gamma\left(\omega \,/\, \left|\int \prod_{i=1}^m f_i(x^i)\, dP_\beta^N(J(\omega)) - \prod_{i=1}^m \int f_i\, dQ_1\right| > \varepsilon\right) = 0\,.$$

Proof. If $Q_2 = Q_1^{\otimes 2}$, then H is null and $P_H = Q_1$. Then, by Theorem 6.1, $\int f_1(x^1)\cdots f_m(x^m)dP_\beta^N$ converges in $L^2(\gamma)$ to the constant $\prod_{i=1}^m \int f_i\, dQ$, and so converges in probability as stated in Theorem 6.4. □

It remains of course to understand the condition $Q_2 = (Q_1)^{\otimes 2}$. For this, we state (without proof).

Proposition 6.5
(1) $Q_r = Q_1^{\otimes r}$ *iff* $\int x_t dQ_1 = 0\ \forall t \leqq T$.
(2) *In particular, if the potential* U *is even and the initial distribution* μ_0 *symmetric then* $Q_r = Q_1^{\otimes r}$ *and Theorem* 6.4 *applies.*

Remark 6.6. If one is interested by the convergence of the law of a single spin, one can prove that, if $Q_2 = (Q_1)^{\otimes 2}$ and $2\beta^2 A^2 T < 1$, for any $t \leqq T$,

$$\lim_{N\to\infty} \gamma\left(\omega \;/\; \|P_\beta^N(J(\omega))\circ (x_t^1)^{-1} - Q_1\circ (x_t)^{-1}\|_{\mathscr{L}} > \varepsilon\right) = 0\,,$$

where $\|\cdot\|_{\mathscr{L}}$ is the norm on $\mathscr{M}_1^+([-A,+A])$ defined by

$$\|\mu\|_{\mathscr{L}} = \sup\left|\int f\, d\mu\right|$$

where the supremum is taken on the Lipschitz functions f such that

$$\sup_{x\in[-A,A]} |f(x)| + \sup_{x,y\in[-A,A]} \frac{|f(x)-f(y)|}{|x-y|} \leqq 1\,.$$

Of course this statement is also valid for n times and m spins, as stated in Theorem 2.10.

Appendix A

Let μ be a probability measure on W_T^A. Let K_μ be a symmetric definite positive kernel on $[0,T]^2$ defined by:

$$K_\mu(s,t) = \int x_s x_t\, d\mu(x)\,.$$

Obviously, K_μ is continuous and bounded on $[0,T]^2$.

We introduce a covariance operator $\overline{K}_\mu$ on $L^2([0,T])$.

$$\overline{K}_\mu f(s) = \int_0^T K_\mu(s,t) f(t)\, dt$$

$\overline{K}_\mu$ is a positive self adjoint trace class operator on $L^2([0,T])$. Let $(h_n^\mu)_{n\geqq 0}$ be an orthonormal basis of eigenfunctions of $\overline{K}_\mu$ associated with the eigenvalues $(\lambda_n^\mu)_{n\geqq 0}$. Let $g_n^\mu = \sqrt{\lambda_n^\mu} h_n^\mu$.

$$\text{Then:}\quad K_\mu(s,t) = \sum_{n\geqq 0} g_n^\mu(s) g_n^\mu(t)$$

and the convergence is uniform (by Mercer theorem, see [10, Proposition 3.7, p. 42]).

$$\text{As a consequence:} \quad K_\mu(s,s) = \sum_{n \geqq 0} (g_n^\mu(s))^2$$

and the trace of the operator $\overline{K}_\mu$ is equal to $tr(\overline{K}_\mu) = \sum_{n \geqq 0} \lambda_n^\mu$.

Of course, one knows that, given a covariance K_μ, there exists a gaussian centered process $(\Omega, \mathscr{A}, \gamma, G_t)$ with covariance K_μ, and that, for any such process, if H_μ denotes the gaussian space associated, i.e. the $L^2(\Omega, \mathscr{A}, \gamma)$ closed linear span of $(G_t)_{0 \leqq t \leqq T}$, then H_μ is isomorphic to the autoreproducing Hilbert space $\mathscr{H}_\mu$ associated to K_μ by

$$\phi : H_\mu \to \mathscr{H}_\mu ,$$

$$Z \to \mathscr{E}[ZG] .$$

Here, the space $\mathscr{H}_\mu \subset L^2([0,T])$ (more precisely, $\mathscr{H}_\mu \subset C([0,T])$) admits $(g_n^\mu)_{n \geqq 0}$ as an orthonormal basis. If $\xi_n^\mu = \phi^{-1}(g_n^\mu)$, then $(\xi_n^\mu)_{n \geqq 0}$ is a sequence of i.i.d $N(0,1)$ random variables in H_μ and one has

$$G_s = \sum_{n \geqq 0} g_n^\mu(s) \xi_n^\mu ,$$

where the convergence is in H^μ (or in $L^2(\Omega, \mathscr{A}, \gamma)$, see [10, Prop. 3.7]).
Moreover, let us consider the new trace class symmetric operator $\overline{\widetilde{K_\mu}}$ on $L^2([0,T])$ given by

$$\overline{\widetilde{K_\mu}} = \left(Id + \beta^2 \overline{K}_\mu\right)^{-1} \overline{K}_\mu$$

and let $\widetilde{K_\mu}$ be its kernel:

$$\widetilde{K_\mu}(s,t) = \sum_{n \geqq 0} \frac{g_n^\mu(s) g_n^\mu(t)}{1 + \beta^2 \lambda_n^\mu} . \tag{70}$$

The autoreproducing Hilbert space associated to $\widetilde{K_\mu}$ is $\left(Id + \beta^2 \overline{K}_\mu\right)^{-1} \mathscr{H}_\mu$; so that one sees that if G is a centered gaussian process with covariance K_μ, then

Lemma A.1

$$\frac{\mathscr{E}\left[G_s G_t \exp\left\{-\frac{\beta^2}{2} \int_0^T G_s^2 ds\right\}\right]}{\mathscr{E}\left[\exp\left\{-\frac{\beta^2}{2} \int_0^T G_s^2 ds\right\}\right]} = \widetilde{K_\mu}(s,t) .$$

Let us give a very short proof.

Let α be a real number and $G_s^N = \sum_{0 \leqq n \leqq N} g_n^\mu(s) \xi_n^\mu$.
Then a simple computation shows that

$$\mathscr{E}\left[\exp\left\{\alpha G_s^N - \frac{\beta^2}{2}\int_0^T (G_s^N)^2\, ds\right\}\right]$$

$$= \mathscr{E}\left[\exp\left\{\alpha \sum_{0\leqq n\leqq N} g_n^\mu(s)\xi_n^\mu - \frac{\beta^2}{2}\sum_{0\leqq n\leqq N}\lambda_n^\mu(\xi_n^\mu)^2\right\}\right]$$

$$= \prod_{n=0}^N \mathscr{E}\left[\exp\left\{\alpha g_n^\mu(s)\xi_n^\mu - \frac{\beta^2}{2}\lambda_n^\mu(\xi_n^\mu)^2\right\}\right]$$

$$= \left(\prod_{n=0}^N (1+\beta^2\lambda_n^\mu)\right)^{-1/2} \exp\frac{\alpha^2}{2}\left\{\sum_{0\leqq n\leqq N}\frac{g_n^\mu(s)^2}{1+\beta^2\lambda_n^\mu}\right\}.$$

The same formula shows that the sequence $\exp\left\{\alpha G_s^N - \frac{\beta^2}{2}\int_0^T (G_s^N)^2\, ds\right\}$ is bounded in $L^{1+\delta}(\Omega,\mathscr{A},\gamma)$, for any positive real number δ, so that this sequence is uniformly integrable. It converges in probability to $\exp\left\{\alpha G_s - \frac{\beta^2}{2}\int_0^T (G_s)^2\, ds\right\}$. Hence, we deduce

$$\mathscr{E}\left[\exp\left\{\alpha G_s - \frac{\beta^2}{2}\int_0^T (G_s)^2\, ds\right\}\right] = \left(\prod_n (1+\beta^2\lambda_n^\mu)\right)^{-1/2}\exp\frac{\alpha^2}{2}\left\{\sum_{n\geqq 0}\frac{g_n^\mu(s)^2}{1+\beta^2\lambda_n^\mu}\right\}.$$

In particular,

$$\mathscr{E}\left[\exp\left\{-\frac{\beta^2}{2}\int_0^T (G_s)^2\, ds\right\}\right] = \left(\prod_n (1+\beta^2\lambda_n^\mu)\right)^{-1/2} \tag{71}$$

$$\frac{\mathscr{E}\left[\exp\left\{\alpha G_s - \frac{\beta^2}{2}\int_0^T G_s^2\, ds\right\}\right]}{\mathscr{E}\left[\exp\left\{-\frac{\beta^2}{2}\int_0^T G_s^2\, ds\right\}\right]} = \exp\frac{\alpha^2}{2}\left\{\sum_{n\geqq 0}\frac{g_n^\mu(s)^2}{1+\beta^2\lambda_n^\mu}\right\} \tag{72}$$

Hence, the process G, under the new law $\tilde{\gamma}_\mu^T$:

$$\tilde{\gamma}_\mu^T = \frac{\exp\left\{-\frac{\beta^2}{2}\int_0^T G_s^2\, ds\right\}}{\mathscr{E}\left[\exp\left\{-\frac{\beta^2}{2}\int_0^T G_s^2\, ds\right\}\right]}\gamma_\mu$$

is a centered gaussian process. Its covariance is easy to compute since it is enough to derive (72) two times in $\alpha = 0$ to obtain

$$\int G_s^2\, d\tilde{\gamma}_\mu^T = \frac{\mathscr{E}\left[G_s^2\exp\left\{-\frac{\beta^2}{2}\int_0^T G_s^2\, ds\right\}\right]}{\mathscr{E}\left[\exp\left\{-\frac{\beta^2}{2}\int_0^T G_s^2\, ds\right\}\right]} = \exp\frac{\alpha^2}{2}\left\{\sum_{n\geqq 0}\frac{g_n^\mu(s)^2}{1+\beta^2\lambda_n^\mu}\right\}$$

which gives Lemma A.1 by polarization. □

It is clear that the last results do not depend on the choice of the time T. More precisely, let $t \leqq T$. Let $\overline{K}_\mu^t$ be the integral operator on $L^2([0,t])$ with kernel K_μ. If we define a new trace class symmetric operator $\overline{\widetilde{K}_\mu}^t$ on $L^2([0,t])$ by

$$\overline{\widetilde{K}_\mu}^t = \left(Id + \beta^2 \overline{K}_\mu^t\right)^{-1} \overline{K}_\mu^t$$

and if we denote $\widetilde{K}_\mu^t$ its kernel, then:

Lemma A.1 bis

$$\frac{\mathscr{E}\left[G_s G_u \exp\left\{-\frac{\beta^2}{2}\int_0^t G_s^2\, ds\right\}\right]}{\mathscr{E}\left[\exp\left\{-\frac{\beta^2}{2}\int_0^t G_s^2\, ds\right\}\right]} = \widetilde{K}_\mu^t(s,u)\,.$$

We can also deduce from the proof of Lemma A.1 (see, for instance, (70)) the following:

Lemma A.2

$$\forall s \leqq t \quad \widetilde{K}_\mu^t(s,s) \leqq K_\mu(s,s)$$

Moreover:

Lemma A.3
(1) *For any real number* α, $\alpha A^2 T < 1$

$$\mathscr{E}\left[\exp\left\{\frac{\alpha}{2}\int_0^T (G_s)^2\, ds\right\}\right] = \left(\prod_n (1 - \alpha\lambda_n^\mu)\right)^{-1/2}$$

(2) *For any real number* α, $\alpha A^2 T < 1$, *there exists a finite constant* C_α *such that*

$$\mathscr{E}\left[\exp\left\{\frac{\alpha}{2}\int_0^T G_s^2\, ds\right\}\right] \leqq \exp\left\{C_\alpha \alpha \mathscr{E}\left[\int_0^T (G_s)^2\, ds\right]\right\} \leqq \exp\{\alpha C_\alpha A^2 T\}\,.$$

Proof. The first point is a direct consequence of the proof of Lemma A.1, (71).

For the second point, we notice that, since the λ_n^μ are positive, $\lambda_n^\mu \leqq \sum_{n \geqq 0} \lambda_n^\mu = tr\,\overline{K}_\mu = \mathscr{E}\left[\int_0^T (G_s)^2\, ds\right] \leqq A^2 T$. But, for any strictly positive real number $\delta < 1, \sqrt{1-x} \geqq e^{-\gamma_\delta x}$ if $x < 1 - \delta$ with $\gamma_\delta = \frac{-\log\delta}{2(1-\delta)}$, so that Lemma A.3(1) implies Lemma A.3(2) with $C = C_{1-\alpha A^2 T}$. □

Let d_t be the Vaserstein distance on W_t^A, i.e.:

$$d_t(\mu,\nu) = \sup\left\{\int \sup_{s \leqq t} |x_s - y_s|\, d\xi(x,y)\right\}$$

where the supremum is taken on the probability measure ξ on $W_t^A \times W_t^A$ with marginals μ and ν. We prove that $\widetilde{K}_\mu^t$ is Lipschitz for this metric:

Lemma A.4 *There exists a finite constant k such that, for any $t \in [0,T]$, for any probability measures (μ,ν) on W_T^A,*

$$\sup_{s,u \leqq t} |\widetilde{K_\mu}^t(s,u) - \widetilde{K_\nu}^t(s,u)| \leqq k d_t(\mu,\nu)\,.$$

Proof. We can give another formula for $\widetilde{K_\mu}^t$ which does not depend on the choice of the basis $(h_n^\mu)_{n\geqq 0}$ of $L^2([0,T])$. Since any $x \in W_T^A$ is in $L^2([0,T])$, we can write

$$\widetilde{K_\mu}^t(s,u) = \int x(s)\left(Id + \beta^2 \overline{K}_\mu^t\right)^{-1} x(u)\,d\mu(x)\,.$$

Hence, for any probability measures ν and μ on W_T^A:

$$\begin{aligned}
&\left|\widetilde{K}_\mu^t(s,u) - \widetilde{K_\nu}^t(s,u)\right| \\
&\leqq \left|\int x(s)\left(Id + \beta^2\overline{K}_\mu^t\right)^{-1} x(u)\,d(\mu-\nu)(x)\right| \\
&\quad + \int \left| x(s)\left\{\left(Id + \beta^2\overline{K}_\mu^t\right)^{-1} - \left(Id + \beta^2\overline{K}_\nu^t\right)^{-1}\right\} x(u)\right| d\nu(x)\,. \qquad (73)
\end{aligned}$$

To bound the first term in the right hand side of (73), we first prove that, for any $u \leqq t$ $f_\mu^u(x) = \left(Id + \beta^2\overline{K}_\mu^t\right)^{-1} x(u)$ is a lipschitz function on W_t^A, endowed with the uniform topology, and bound its lipschitz constant independently of the probability measure μ on W_T^A.

If we denote $\|\|_2$ the norm in $L^2([0,t])$; $\|f\|_2 = \sqrt{\int_0^t f^2(s)ds}$, for any $x \in L^\infty([0,t])$, we have

$$\begin{aligned}
\left|f_\mu^u(x)\right| &= \left|x(u) - \beta^2\overline{K}_\mu^t\left(Id + \beta^2\overline{K}_\mu^t\right)^{-1} x(u)\right| \\
&= \left|x(u) - \beta^2\int_0^t \left(\int y(u)y(s)\,d\mu(y)\right)\left(Id + \beta^2\overline{K}_\mu^t\right)^{-1} x(s)\,ds\right| \\
&\leqq |x(u)| + \beta^2 A^2\sqrt{t}\left\|\left(Id + \beta^2\overline{K}_\mu^t\right)^{-1} x\right\|_2\,.
\end{aligned}$$

But $\overline{K}_\mu^t$ is a positive operator so that

$$\left|f_\mu^u(x)\right| \leqq |x(u)| + \beta^2A^2\sqrt{t}\|x\|_2 \leqq |x(u)| + \beta^2A^2t\sup_{s\leqq t}|x(s)|\,. \qquad (74)$$

Thus, f_μ^u is Lipschitz, with lipschitz constant $(1+\beta^2A^2t)$. Hence, for any probability measure ξ with marginals μ and ν, we have:

$$\begin{aligned}
&\left|\int x(s)\left(Id + \beta^2\overline{K}_\mu^t\right)^{-1} x(u)d(\mu-\nu)(x)\right| \\
&= \left|\int (x(s)-y(s))f_\mu^u(x) + y(s)\left(f_\mu^u(x) - f_\mu^u(y)\right)d\xi(x,y)\right| \\
&\leqq (A + \beta^2A^3t)\int |x(s)-y(s)|\,d\xi(x,y) \\
&\quad + A\int\left(|x(u)-y(u)| + \beta^2A^2t\sup_{s\leqq t}|x(s)-y(s)|\right)d\xi(x,y) \\
&\leqq 2(A+\beta^2A^3t)\int \sup_{s\leqq t}|x(s)-y(s)|\,d\xi(x,y)
\end{aligned}$$

Since this inequality holds for any ξ with marginals ν and μ, we get:

$$\left|\int x(s)\left(Id+\beta^2\overline{K}^t_\mu\right)^{-1}x(u)d(\mu-\nu)(x)\right| \leqq 2A^2(1+\beta^2A^2t)d_t(\mu,\nu)\,. \quad (75)$$

We now bound the second term in the right hand side of (73).

$$\left(Id+\beta^2\overline{K}^t_\mu\right)^{-1}-\left(Id+\beta^2\overline{K}^t_\nu\right)^{-1}=\beta^2\left(Id+\beta^2\overline{K}^t_\mu\right)^{-1}\left(\overline{K}^t_\mu-\overline{K}^t_\nu\right)\times\left(Id+\beta^2\overline{K}^t_\nu\right)^{-1}$$

So that, for any $x\in W^A_T$,

$$\left|x(s)\left\{\left(Id+\beta^2\overline{K}^t_\mu\right)^{-1}-\left(Id+\beta^2\overline{K}^t_\mu\right)^{-1}\right\}x(u)\right| \leqq \beta^2A\left|\left(Id+\beta^2\overline{K}^t_\mu\right)^{-1}\left(\overline{K}^t_\mu-\overline{K}^t_\nu\right)\left(Id+\beta^2\overline{K}^t_\nu\right)^{-1}x(u)\right|\,.$$

But

$$\left|\left(\overline{K}^t_\mu-\overline{K}^t_\nu\right)\left(Id+\beta^2\overline{K}^t_\nu\right)^{-1}x(u)\right| = \left|\int_0^t\int y_sy_ud(\mu-\nu)(y)\left(Id+\beta^2\overline{K}^t_\nu\right)^{-1}x(s)\,ds\right| \leqq 2A\sqrt{t}\|x\|_2d_t(\mu,\nu)\leqq 2A^2td_t(\mu,\nu)\,. \quad (76)$$

So that, for any $x\in W^A_T$,

$$\left|x(s)\left\{\left(Id+\beta^2\overline{K}^t_\mu\right)^{-1}-\left(Id+\beta^2\overline{K}^t_\mu\right)^{-1}\right\}x(u)\right| \leqq 2\beta^2A^3t(1+\beta^2A^2t)d_t(\mu,\nu)\,.$$

According to (74), we conclude that, for any probability measures ν and μ on W^A_T:

$$\int\left|x(s)\left\{\left(Id+\beta^2\overline{K}^t_\mu\right)^{-1}-\left(Id+\beta^2\overline{K}^t_\mu\right)^{-1}\right\}x(u)\right|d\nu(x) \leqq 2\beta^2A^3t(1+\beta^2A^2t)d_t(\mu,\nu)\,. \quad (77)$$

Equations (73), (75), (77) give Lemma A.4. □

We state Lemmas A.2, A.3 and A.4 in the time discretized setting. Let $\Delta_n=\{0=t_0<t_1<\cdots<t_{n+1}=T\}$ be a subdivision of $[0,T]$. We recall that we denote $t^{(n)}=\max\{t_k\in\Delta_n, t_k\leqq t\}$. We define a covariance $\widetilde{K_\mu}^{t,n}$ by:

$$\widetilde{K_\mu}^{t,n}(s,u)=\frac{\mathscr{E}\left[G_{s^{(n)}}G_{u^{(n)}}\exp\left\{-\frac{\beta^2}{2}\int_0^{t^{(n)}}G^2_{s^{(n)}}ds\right\}\right]}{\mathscr{E}\left[\exp\left\{-\frac{\beta^2}{2}\int_0^{t^{(n)}}G^2_{s^{(n)}}ds\right\}\right]}$$

It is obvious that the lemmas of this appendix are also valid for $\widetilde{K_\mu}^{t,n}$:

Lemma A.5
(1) $\forall s \leqq t, \quad \widetilde{K_\mu}^{t,n}(s,s) \leqq K_\mu(s,s)$.
(2) *There exists a finite constant* k *such that, for any* $t \in [0,T]$, *for any probability measures* μ, ν *on* W_T^A,

$$\sup_{s,u \leqq t} |\widetilde{K_\mu}^{t,n}(s,u) - \widetilde{K_\nu}^{t,n}(s,u)| \leqq k d_t(\mu,\nu) .$$

Appendix B

Lemma B: *Define*

$$H_\nu^n : \mathscr{M}_1^+(W_T^A) \to [0,\infty]$$

$$\mu \to \begin{cases} I(\mu|P) - \Gamma_\nu^n(\mu) & \text{when } I(\mu|P) < \infty , \\ +\infty & \text{otherwise,} \end{cases}$$

where $\Gamma_\nu^n(\mu) = \int \log \left(\int \exp\{\beta \int_0^T G_{t(n)} dB_t(x) - \frac{\beta^2}{2} \int_0^T G_{t(n)}^2 \, dt\} \, d\gamma_\nu \right) d\mu(x)$.
Then H_ν^n *is the entropy relative to the probability measure* Q_ν^n *on* W_T^A:

$$Q_\nu^n = \int \exp \left\{ \beta \int_0^T G_{t(n)} \, dB_t - \frac{\beta^2}{2} \int_0^T G_{t(n)}^2 \, dt \right\} d\gamma_\nu P .$$

Proof.
- It is clear that $Q_\nu^n \simeq P$. So that whenever $\mu \not\ll P$, then $\mu \not\ll Q_\nu^n$ and $I(\mu|P)$, $I(\mu|Q_\nu^n)$ are both infinite so that $H_\nu^n(\mu) = I(\mu|Q_\nu^n)$.
- If, on the contrary, $\mu \ll P$, we first remark that the proof would be clear if $\frac{dQ_\nu^n}{dP}$ was bounded. In fact, we would then write

$$I(\mu|P) = \int \log \frac{d\mu}{dP} \, d\mu = \int \left\{ \log \frac{d\mu}{dQ_\nu^n} + \log \frac{dQ_\nu^n}{dP} \right\} d\mu = I(\mu|Q_\nu^n) + \Gamma_\nu^n(\mu) .$$

Though $\Gamma_\nu^n(\mu)$ would then also be bounded, we would get $H_\nu^n = I(|Q_\nu^n)$.
Our proof will need a deeper study of the probability measures Q_ν^n.

Lemma B.1 *Let* $\nu \in \mathscr{M}_1^+(W_T^A)$. *Then*
(a) $\dfrac{dQ_\nu^n}{dP} \geqq \exp - \dfrac{\beta^2}{2} A^2 T$,
(b) $I(Q_\nu^n|P) < \infty$.

Proof.
(a) Gaussian computations give

$$\frac{dQ_\nu^n}{dP} = \left(\int \exp \left\{ -\frac{\beta^2}{2} \int_0^T G_{t(n)}^2 dt \right\} d\gamma_\nu \right) \times \exp \left\{ \frac{\beta^2}{2} \int \left(\int_0^T G_{t(n)} \, dB_t \right)^2 d\gamma_{\widetilde{K_\nu}^{T,n}} \right\} \tag{78}$$

so that, Jensen inequality implies

$$\frac{dQ_v^n}{dP} \geqq \exp - \left(\int \frac{\beta^2}{2} \int_0^T G_{t(n)}^2 \, dt \, d\gamma_v \right) \geqq \exp - \frac{\beta^2}{2} A^2 T \, .$$

(b) Equation (78) becomes, after some gaussian computations (see Lemma 5.15):

$$\frac{dQ_v^n}{dP} = \exp \left\{ \beta^2 \int_0^T H_{t(n)}(Q_v^n) \, dB_t - \frac{\beta^4}{2} \int_0^T H_{t(n)}(Q_v^n)^2 \, dt \right\} \, ,$$

where $H_{t(n)}(Q_v^n) = \int G_{t(n)} \int_0^{t^{(n)}} G_{s(n)} \, dB_s \, d\gamma_{\widetilde{K}_v^{t(n)},n} = \int_0^{t^{(n)}} \widetilde{K}_v^{t^{(n)},n}(t^{(n)}, s^{(n)}) \, dB_s$.

Thus, Girsanov theorem implies that, under Q_v^n, there exists a brownian motion W such that

$$B_t = W_t + \beta^2 \int_0^t H_{s(n)}(Q_v^n) \, ds \tag{79}$$

which shows that B is a linear function of W, so that, it is, under Q_v^n, a centered gaussian variable.

Hence B has finite moments and, in particular,

$$\begin{aligned} I(Q_v^n|P) &= \int \log \frac{dQ_v^n}{dP} \, dQ_v^n \\ &= \log \int \exp \left\{ -\frac{\beta^2}{2} \int_0^T G_{t(n)}^2 dt \right\} d\gamma_v + \frac{\beta^2}{2} \int\int \left(\int_0^T G_{t(n)} \, dB_t \right)^2 d\gamma_{\widetilde{K}_v^{T,n}} \, dQ_v^n \end{aligned}$$

is finite. □

We now prove that, when $\mu \ll P$,

$$I(\mu|P) = I(\mu|Q_v^n) + \Gamma_v^n(\mu) \, . \tag{80}$$

Let $\mu_\theta = \theta Q_v^n + (1-\theta)\mu$ for $\theta \in [0,1]$. We notice that $\frac{d\mu_\theta}{dQ_v^n} \geqq \theta$, which allows us to write

$$I(\mu_\theta|P) = \int \log \frac{d\mu_\theta}{dP} \, d\mu_\theta = \int \left\{ \log \frac{d\mu_\theta}{dQ_v^n} + \log \frac{dQ_v^n}{dP} \right\} d\mu_\theta \, .$$

But $\frac{d\mu_\theta}{dQ_v^n} \geqq \theta$ and $\frac{dQ_v^n}{dP} \geqq \exp - \frac{\beta^2}{2} A^2 T$ by Lemma B.1(a), so that both integrals in the last formula are lower bounded and

$$I(\mu_\theta|P) = \int \log \frac{d\mu_\theta}{dQ_v^n} \, d\mu_\theta + \int \log \frac{dQ_v^n}{dP} \, d\mu_\theta = I(\mu_\theta|Q_v^n) + \Gamma_v^n(\mu_\theta) \, . \tag{81}$$

The next step is to check the two following points:

$$\lim_{\theta \to 0} I(\mu_\theta|Q_v^n) = I(\mu|Q_v^n) \, . \tag{82}$$

$$\lim_{\theta \to 0} I(\mu_\theta|P) = I(\mu|P) \, . \tag{83}$$

– Equation (82) was proved in [5, Lemma 3.2.13].
– Using the convexity of $x \log x$ and Jensen's inequality, one finds

$$I(\mu_\theta|P) \leqq (1-\theta)I(\mu|P) + \theta I(Q_v^n|P) .$$

Hence, according to Lemma B.1(b), $\overline{\lim}_{\theta\to 0} I(\mu_\theta|P) \leqq I(\mu|P)$.
Moreover

$$\frac{d\mu_\theta}{dP} = \theta\frac{dQ_v^n}{dP} + (1-\theta)\frac{d\mu}{dP} \geqq \theta \exp -\frac{1}{2}\beta^2A^2T + (1-\theta)\frac{d\mu}{dP}$$

so that, using the concavity of $\log x$, we get

$$\begin{aligned} I(\mu_\theta|P) &= \theta \int \log\frac{d\mu_\theta}{dP}\, dQ_v^n + (1-\theta)\int \log\frac{d\mu_\theta}{dP}\, d\mu \\ &\geqq \theta\left\{\log\theta\exp -\frac{1}{2}\beta^2A^2T\right\} \\ &\quad + (1-\theta)\left\{(1-\theta)I(\mu|P) + \theta\left(-\frac{1}{2}\beta^2A^2T\right)\right\} \end{aligned}$$

and $\underline{\lim}_{\theta\to 0} I(\mu_\theta|P) \geqq I(\mu|P)$, which achieves the proof of (83).
For the same reasons, as Γ_v^n is linear,

$$\Gamma_v^n(\mu_\theta) = (1-\theta)\Gamma_v^n(\mu) + \theta I(Q_v^n|P) \underset{\theta\to 0}{\longrightarrow} \Gamma_v^n(\mu) . \tag{84}$$

Hence, letting $\theta \to 0$ in (81), (82), (83), (84) imply (80), for all $\mu \ll P$:

$$I(\mu|P) = I(\mu|Q_v^n) + \Gamma_v^n(\mu) .$$

But, we remark as in Lemma 3.3(4) that $\Gamma_v^n(\mu) \leqq \alpha I(\mu|P) + \eta$ for an $\alpha < 1$ so that if $I(\mu|P) < \infty$, then $H_v^n(\mu) = I(\mu|P) - \Gamma_v^n(\mu)$ and $\Gamma_v^n(\mu) < \infty$, and finally, using (80)

$$I(\mu|Q_v^n) = H_v^n(\mu) . \tag{85}$$

We finally handle the case where $\mu \ll P$ but $I(\mu|P) = \infty$. Then $H_v^n(\mu) = \infty$.
Let $A_M = \left\{\frac{dQ_v^n}{dP} \leqq M\right\}$ and $\mu^M = \frac{\mathbb{1}_{A_M}}{\mu(A_M)}\mu$. Then, as $P(A_M^c) \underset{M\to\infty}{\longrightarrow} 0$, $\mu(A_M^c) \underset{M\to\infty}{\longrightarrow} 0$, so that μ^M converges to μ when $M \to \infty$.
Using standard monotone convergence arguments, one finds that

$$I(\mu^M|P) \underset{M\to\infty}{\longrightarrow} I(\mu|P) = \infty\, ; \quad I(\mu^M|Q_v^n) \underset{M\to\infty}{\longrightarrow} I(\mu|Q_v^n)$$

But

$$\Gamma_v^n(\mu_M) = \frac{1}{\mu(A_M)}\int \mathbb{1}_{A_M}\frac{dQ_v^n}{dP}\, d\mu \leqq M .$$

Hence, using (85) and again the fact that $\Gamma_v^n(\mu) \leqq \alpha I(\mu|P) + \eta$ for an $\alpha < 1$, we get

$$I(\mu^M|Q_v^n) = I(\mu^M|P) - \Gamma_v^n(\mu_M) \geqq (1-\alpha)I(\mu^M|P) - \eta$$

which implies that

$$\lim_{M\to\infty} I(\mu^M|Q_v^n) = I(\mu|Q_v^n) = \infty = H_v^n(\mu) .$$

Thus, we proved that

$$\forall \mu \in \mathcal{M}_1^+(W_T^A) \quad H_\nu^n(\mu) = I(\mu|Q_\nu^n)\,. \qquad \square$$

Appendix C

Proof of Theorem 2.6
(1) Let $\delta > 0$, and $B(Q,\delta)$ be an open ball of radius δ in a metric which is compatible with the weak topology, for instance the Vaserstein's metric (see its definition in Sect. 3). $B(Q,\delta)$ is an open subset of $\mathcal{M}_1^+(W_T^A)$ for the weak topology, which contains Q. Hence, $\inf_{B(Q,\delta)^c} H$ is strictly positive since H is a good rate function. But, according to the large deviation principle established in Theorem 2.3(2),

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log Q_\beta^N(\widehat{\mu}^N \in B(Q,\delta)^c) \leqq - \inf_{B(Q,\delta)^c} H < 0$$

which proves the convergence result.
(2) Since Q_β^N are symmetric measures, the propagation of chaos may be deduced from (1) as in Sznitman [15, Lemma 3.1]. $\square$

Proof of Theorem 2.7. Theorem 2.7 can be deduced from Theorem 2.3 by Borel Cantelli lemma. Let F be a closed subset of $\mathcal{M}_1^+(W_T^A)$. If $\inf_F H = 0$, it is clear that (5) is satisfied. Otherwise, $\inf_F H > 0$. Then, Tchebyshev inequality implies that, for any integer p,

$$\begin{aligned}\gamma\left(J/P_\beta^N(J)(\widehat{\mu}^N \in F) > \exp\left\{-N\left(1-\frac{2}{p}\right)\inf_F H\right\}\right) \\ \leqq \exp\left\{N\left(1-\frac{2}{p}\right)\inf_F H\right\} \int P_\beta^N(J)(\widehat{\mu}^N \in F)\, d\gamma \\ = \exp\left\{N\left(1-\frac{2}{p}\right)\inf_F H\right\} \Pi_{\beta,T}^N(F)\,.\end{aligned}$$

But Theorem 2.3(1) implies that, for any integer p, for N large enough,

$$\Pi_{\beta,T}^N(F) \leqq \exp\left\{-N\left(1-\frac{1}{p}\right)\inf_F H\right\}\,.$$

Thus, for N large enough,

$$\gamma\left(J/P_\beta^N(J)(\widehat{\mu}^N \in F) > \exp\left\{-N\left(1-\frac{2}{p}\right)\inf_F H\right\}\right) \leqq \exp\left\{-N\frac{1}{p}\inf_F H\right\}\,.$$

As a consequence $\sum_N \gamma\left(J/P_\beta^N(J)(\widehat{\mu}^N \in F) > \exp\left\{-N\left(1-\frac{2}{p}\right)\inf_F H\right\}\right)$ is finite so that Borel Cantelli lemma implies that, for almost all J,

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log P_\beta^B(J)(\widehat{\mu}^N \in F) \leqq -\left(1-\frac{2}{p}\right)\inf_F H\,.$$

Letting $p \to \infty$, we get Theorem 2.7. $\square$

Proof of Theorem 2.8
(1) The proof is very similar to that of Theorem 2.6(1), we omit it.
(2) Let f be a bounded continuous function on W_T^A such that $\int f\, dQ = 0$ and let Ω_p be the set of all the $J's$ such that

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log P_\beta^N(J) \left(\left| \int f d\widehat{\mu}^N \right| \geqq \frac{1}{p} \right) \leqq - \inf_{\{\mu / |\int f d\mu| \geqq \frac{1}{p}\}} H .$$

But $\inf_{\{\mu/|\int f d\mu| \geqq \frac{1}{p}\}} H$ is strictly positive for any finite integer p by Theorem 2.4. Thus for any $J \in \bigcap_p \Omega_p$, Borel Cantelli lemma implies that $\frac{1}{N}\sum_{i=1}^N f(x^i)$ converges to zero when N tends to infinity almost surely. Finally, if $\beta^2 A^2 T < 1$, Theorem 2.7(2) implies that $\gamma(\Omega_p) = 1$ so that $\Omega = \bigcap_p \Omega_p$ has probability one. □

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