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Periodic solutions of second order differential equations with superlinear asymmetric nonlinearities

By

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1. Introduction. Consider the scalar boundary value problem

(1) $$x'' + f(t, x) = 0,$$

(2) $$x(0) = x(2\pi), \quad x'(0) = x'(2\pi).$$

We are interested in cases where f is asymmetric. Such systems were considered first by N. Dancer [1] and S. Fučík [7] who called them *jumping nonlinearities.* A simple situation of that type occurs, for instance, when

(3) $$f(t, x) = m_+ x^+(t) - m_- x^-(t) + \psi(x) + e(t),$$

where $x^+(t) = \max(x(t), 0)$, $x^-(t) = \max(-x(t), 0)$, ψ being a bounded function. S. Fučík [7] has shown that, with such a function f, problem (1), (2) has a solution if there exists an integer n such that

(4) $$m_+ > 0, m_- > 0 \quad \text{and} \quad \frac{2}{n+1} < \frac{1}{\sqrt{m_+}} + \frac{1}{\sqrt{m_-}} < \frac{2}{n}.$$

The following generalization appears in P. Drábek and S. Invernizzi [4]. Instead of supposing f to be of the form (3), assume that positive numbers A_-, A_+, B_-, B_+ exist such that

$$A_+ \leqq \liminf_{x \to +\infty} \frac{f(t, x)}{x} \leqq \limsup_{x \to +\infty} \frac{f(t, x)}{x} \leqq B_+,$$

$$A_- \leqq \liminf_{x \to -\infty} \frac{f(t, x)}{x} \leqq \limsup_{x \to -\infty} \frac{f(t, x)}{x} \leqq B_-,$$

the limits being uniform in t. If for some integer n, the relations

(5) $$\frac{1}{\sqrt{A_+}} + \frac{1}{\sqrt{A_-}} < \frac{2}{n},$$

(6) $$\frac{1}{\sqrt{B_+}} + \frac{1}{\sqrt{B_-}} > \frac{2}{n+1},$$

hold, problem (1), (2) has a solution. The result of S. Fučík [7] corresponds to the particular case

$$A_+ = B_+ = m_+, \quad A_- = B_- = m_-.$$

A complementary result was obtained by J. Mawhin and J. Ward [12] who considered a Liénard equation together with an assumption of the type

$$A_+ > 0, \; A_- > 0 \quad \text{and} \; B_- < 1/4. \tag{7}$$

A similar case refers to $A_+ > 0$, $A_- > 0$ and $B_+ < 1/4$. In these situations the function f is possibly superlinear on one side. Concerning this problem, one must also notice the early work of K. Schmitt [15] and R. Reissig [14].

Conditions (5), (6) or (7) are easily understandable in the m_+, m_- plane. In assumption (5), (6) one imposes that the box $[A_+, B_+] \times [A_-, B_-]$ does not intersect the Fučík spectrum which consists of the lines

$$C_0 = \{(m_+, m_-) \mid m_+ = 0 \text{ or } m_- = 0\}$$

and

$$C_n = \left\{(m_+, m_-) \left| \frac{1}{\sqrt{m_+}} + \frac{1}{\sqrt{m_-}} = \frac{2}{n} \right.\right\}, \quad n = 1, 2, 3 \dots .$$

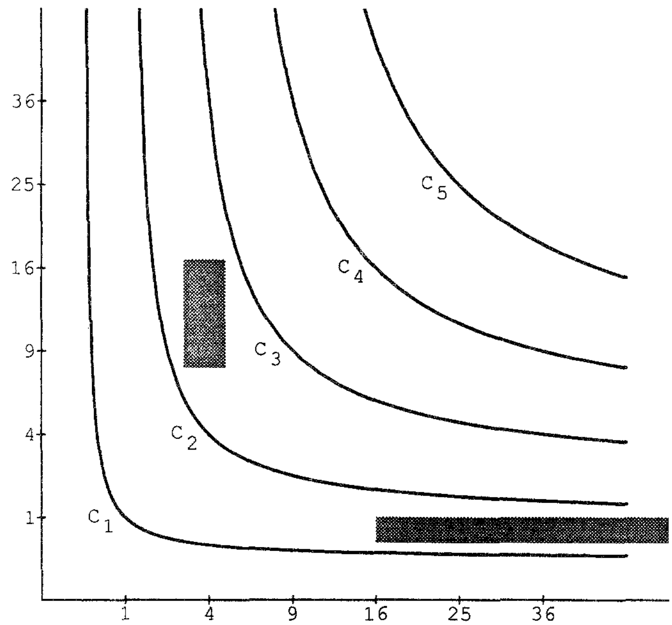

Figure 1: Fučík curves C_n

Condition (7) means that the box $[A_+, +\infty[\times[A_-, B_-]$ remains between C_0 and C_1. Looking at the Fučík spectrum, one can expect that the same idea applies between two successive curves C_n and C_{n+1}. The aim of this paper is to study such a situation.

Let us also mention that another type of generalization was considered by L. Fernandez and F. Zanolin [6]. These authors considered a boundary value problem

$$x'' + f(x) = e(t),$$
$$x(0) = x(2\pi), \quad x'(0) = x'(2\pi),$$

together with an assumption such as

$$\liminf_{x\to+\infty} \frac{2F(x)}{x^2} < \frac{1}{4},$$

where $F(x) = \int_0^x f(x)\,dx$. We do not consider here such possible assumptions.

Our main result requires f to satisfy the following *L^∞-Carathéodory conditions*:
(a) $f(.,x)$ is measurable on $[0, 2\pi]$, for all $x \in \mathbb{R}$;
(b) $f(t,.)$ is continuous on $\mathbb{R}$, for a.e. $t \in [0, 2\pi]$;
(c) for all $R > 0$, there exists a positive constant H such that $|f(t,x)| \leqq H$, for all x with $|x| \leqq R$ and for a.e. $t \in [0, 2\pi]$.

Theorem 1. *Assume that the function $f: [0, 2\pi]\times\mathbb{R} \to \mathbb{R}$ satisfies L^∞-Carathéodory conditions. Let a_+, b_+, a_- be L^∞-functions such that*

$$a_+(t) \leqq \liminf_{x\to+\infty} \frac{f(t,x)}{x} \leqq \limsup_{x\to+\infty} \frac{f(t,x)}{x} \leqq b_+(t), \tag{8}$$

$$a_-(t) \leqq \liminf_{x\to-\infty} \frac{f(t,x)}{x}, \tag{9}$$

the limits being uniform in t. Suppose also that

$$\liminf_{|x|\to+\infty} \operatorname{sgn}(x) f(t,x) > 0, \tag{10}$$

uniformly in t. Moreover, assume that positive numbers A_+, B_+, A_- exist, with $A_+ \leqq B_+$ such that, for some integer n,

$$\frac{1}{2\pi}\int_0^{2\pi} \min\left\{\frac{a_+(t)}{A_+}, \frac{a_-(t)}{A_-}, 1\right\} dt > \frac{n}{2}\left(\frac{1}{\sqrt{A_+}} + \frac{1}{\sqrt{A_-}}\right), \tag{11}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \max\left\{\frac{b_+(t)}{B^+}, 1\right\} dt < \frac{n+1}{2}\,\frac{1}{\sqrt{B_+}}. \tag{12}$$

Then, problem (1), (2) *has a solution.*

Remarks. Assumption (10) implies that

$$\liminf_{x \to +\infty} \frac{f(t,x)}{x} \geqq 0, \quad \liminf_{x \to -\infty} \frac{f(t,x)}{x} \geqq 0$$

and we can assume $a_+(t) \geqq 0$, $a_-(t) \geqq 0$. However, we must notice that (10) does not follow from these last conditions. This is clear from the example $f(t,x) = -\tan^{-1} x$.

The following Corollary, which is an immediate consequence of the above Theorem, relates it to the result of P. Drábek and S. Invernizzi [4].

Corollary 1. *Assume that f satisfies L^∞-Carathéodory conditions, as well as condition* (10). *Let a_+, b_+, a_- be L^∞-functions such that* (8), (9) *hold. Assume that there exist positive numbers A_+, B_+, A_- such that, for a.e. $t \in [0, 2\pi]$,*

$$A_+ \leqq a_+(t) \leqq b_+(t) \leqq B_+, \quad A_- \leqq a_-(t).$$

Then, problem (1), (2) *has a solution if for some positive integer n,*

$$\frac{1}{\sqrt{A_+}} + \frac{1}{\sqrt{A_-}} < \frac{2}{n}, \quad B_+ < \left(\frac{n+1}{2}\right)^2. \tag{13}$$

The above result clearly appears as a limiting case of the result of P. Drábek and S. Invernizzi [4], when B_- is allowed to go to infinity. If we interpret (13) in the plane of Figure 1, one sees that the box $[A_+, B_+] \times [A_-, +\infty[$ has to stay between the two sucessive Fučík curves C_n and C_{n-1}. Actually, if f satisfies (8), (9) and if there exists an integrable function b_- such that

$$\limsup_{x \to -\infty} \frac{f(t,x)}{x} \leqq b_-(t),$$

S. Invernizzi [9] has proven the existence of a solution for problem (1), (2), assuming that, for a.e. $t \in [0, 2\pi]$, the (variable) rectangle $[a_+(t), b_+(t)] \times [a_-(t), b_-(t)]$ is included in a (fixed) rectangle $[A_+, B_+] \times [A_-, +\infty[$, which does not intersect Fučík curves. Although that result also involves an unbounded rectangle between Fučík lines, it differs from Corollary 1 by the fact that it requires $f(t,x)$ to grow at most linearly in x.

With respect to the existence conditions of Corollary 1, the conditions (11), (12) of Theorem 1 even allow the rectangle $[a_+(t), b_+(t)] \times [a_-(t), +\infty[$ to cross Fučík curves for some values of t. Such integral conditions have already been considered by C. Fabry [5] for problems where f is growing at most linearly.

2. A priori bounds for solutions having at most $2n$ zeros. The proof of Theorem 1 is based on an auxiliary result, which is of independent interest. Roughly speaking, it states that, if $x f(t,x)$ is positive and bounded away from 0 for large $|x|$, a solution of (1) cannot escape to infinity without having an infinite number of zeros. The idea of this result can be traced back to P. Hartman [8]. Notice that a similar argument can be found in T. Ding and F. Zanolin [3].

A priori bounds will be needed not only for equation (1), but for a family of equations with a parameter $\lambda \in [0, 1]$. So, we consider the differential equation

$$x'' + F(t, x, \lambda) = 0. \tag{14}$$

In the sequel, F will be defined on $[0, 2\pi] \times \mathbb{R} \times [0, 1]$. However, it is convenient to extend F to $\mathbb{R} \times \mathbb{R} \times [0, 1]$, by periodicity in t. In the next lemma, we will assume such a function F to be 2π- periodic in its first variable, and to satisfy a *uniform L^∞-Carathéodory condition*, by which we mean here that

(a) $F(., x, \lambda)$ is measurable on $[0, 2\pi]$ for all $(x, \lambda) \in \mathbb{R} \times [0, 1]$;

(b) $F(t, ., \lambda)$ is continuous on $\mathbb{R}$ for a.e. $t \in [0, 2\pi]$ and for all $\lambda \in [0, 1]$;

(c) for all $R > 0$, there exists H such that, for a.e. $t \in [0, 2\pi]$, for all $\lambda \in [0, 1]$, for all x with $|x| \leqq R$,

$$|F(t, x, \lambda)| \leqq H.$$

Lemma 1. *Assume that $F: \mathbb{R} \times \mathbb{R} \times [0, 1] \to \mathbb{R}$ is 2π-periodic in its first variable and satisfies uniform L^∞-Carathéodory conditions. Assume that there exists a number $\eta > 0$ such that*

$$\liminf_{|x| \to \infty} \operatorname{sgn}(x)\, F(t, x, \lambda) \geqq \eta, \tag{15}$$

uniformly in t, λ. Then, for any $\varrho > 0$, there exists $R > 0$, such that, for any solution $x: [t_0, \omega] \to \mathbb{R}$ of (14) with $\omega > t_0$, $|x(t_0)| \geqq R$, $x'(t_0) = 0$ and $x^2(\omega) + x'^2(\omega) < \varrho^2$, there exists $t_1 \in (t_0, \omega)$ such that:

(a) *x has at least two zeros in $[t_0, t_1]$,*

(b) *for all $t \in [t_0, t_1]$, $x^2(t) + x'^2(t) \geqq \varrho^2$,*

(c) *$|x(t_1)| \geqq \varrho$, $x'(t_1) = 0$.*

Proof. Take $\varepsilon \in (0, \eta/2)$. Define a function $g_0: \mathbb{R} \to \mathbb{R}$ by

$$g_0(x) = \min\{\eta/2, \operatorname{essinf}\{F(t, \xi, \lambda) - \varepsilon \mid t \in \mathbb{R}, \xi \geqq x, \lambda \in [0, 1]\}\}. \tag{16}$$

Notice that $F(t, \xi, \lambda) - \varepsilon$ becomes larger than $\eta/2$ for large positive ξ, so that $g_0(x) = \eta/2$ for large positive values of x. By construction, g_0 is nondecreasing and such that, for all $\lambda \in [0, 1]$, a.e. $t \in [0, 2\pi]$ and all $x \in \mathbb{R}$,

$$g_0(x) \leqq F(t, x, \lambda) - \varepsilon. \tag{17}$$

It is easy to deduce from g_0 a continuous nondecreasing function g such that $g(x) \leqq g_0(x)$ for all $x \in \mathbb{R}$ and $g(x) = \eta/2$ for large positive values of x. For example, one can take g piecewise linear such that, for all $n \in \mathbb{N}$, $g(n+1) = g_0(n)$. Similarly, a continuous nondecreasing function h can be built such that, for all $\lambda \in [0, 1]$, a.e. $t \in [0, 2\pi]$ and all $x \in \mathbb{R}$,

$$F(t, x, \lambda) + \varepsilon \leqq h(x)$$

and $h(x) = -\eta/2$ for large negative values of x. Introduce the convex functions G, H defined by

$$G(x) = \int_0^x g(u)\, du, \quad H(x) = \int_0^x h(u)\, du.$$

It is clear that

$$G(x) < H(x) \quad \text{for } x > 0 \quad \text{and } G(x) > H(x) \quad \text{for } x < 0. \tag{18}$$

Moreover, since $g(x) = \eta/2$ for large positive values of x, we have

$$\lim_{x \to +\infty} G(x) = +\infty;$$

similarly, we can write

$$\lim_{x \to -\infty} H(x) = +\infty.$$

Let $B_\varrho = \{(x, y) \mid x^2 + y^2 \leqq \varrho^2\}$ and choose $k > 0$ such that, for all $(x, y) \in B_\varrho$,

$$\frac{y^2}{2} + H(x) < k, \quad \frac{y^2}{2} + G(x) < k.$$

Let $\alpha < 0$ be such that $H(\alpha) = k$; such a number exists since $H(0) = 0$ and $\lim_{x \to -\infty} H(x) = +\infty$. Define next curves Γ_1, Γ_2 in the (x, y) plane by

$$\Gamma_1 = \left\{(x, y) \,\middle|\, \frac{y^2}{2} + H(x) = H(\alpha),\ y \geqq 0\right\}$$

$$\Gamma_2 = \left\{(x, y) \,\middle|\, \frac{y^2}{2} + G(x) = G(\alpha),\ y \leqq 0\right\}.$$

Possible curves are shown in Fig. 2. Since the function H is convex, for any $y \in \mathbb{R}$, there are at most two points x_1, x_2 such that $(x_1, y) \in \Gamma_1, (x_2, y) \in \Gamma_1$. The same holds true for Γ_2. Clearly, there exists $\beta > \varrho$ such that $H(\beta) = H(\alpha)$. There also exists $\gamma > \varrho$ such that $G(\gamma) = G(\alpha) > H(\alpha)$. Since $G(x) < H(x)$ for $x < 0$, we will have $\gamma > \beta$. Now, let x be a solution of (14). If the curve $t \mapsto (x(t), x'(t))$ crosses Γ_1, the crossing must be from the "inside" towards the "outside". Indeed, along solutions of (14), we have, for $x' > 0$,

$$\frac{d}{dt}\left(\frac{x'^2}{2} + H(x)\right) = x'(h(x) - F(t, x, \lambda)) > 0,$$

showing that, at points of Γ_1, the vector field associated to the differential equations (14) points outwards. A similar result holds for Γ_2. Also, the vector field points downwards along the half-line $\{(x, 0) \mid x \geqq \gamma\}$. Indeed, if a solution curve crosses that half-line at a point $x \geqq \gamma$, we have

$$x'' = -F(t, x, \lambda) \leqq -g(x) \leqq -g(\gamma)$$

and $g(\gamma) > 0$, since, otherwise, we would have $G(\gamma) \leqq 0$. Consequently, if $x: [t_0, \omega] \to \mathbb{R}$ is a solution of (14) with $\omega > t_0$, $x(t_0) \geqq \gamma$, $x'(t_0) = 0$, one sees that the curve $t \mapsto (x(t), x'(t))$ must circle at least once around B_ϱ before crossing the segment

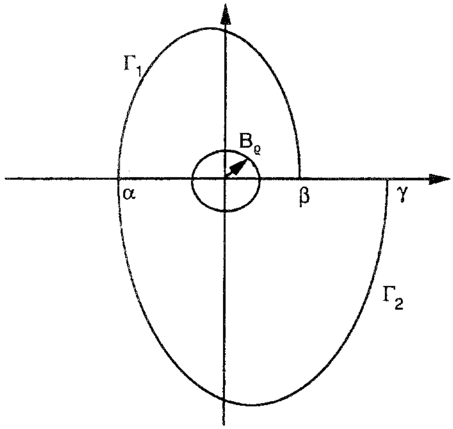

Figure 2: The curves Γ_1 and Γ_2

$\{(x, 0) \mid \beta \leqq x \leqq \gamma\}$ and entering the ball B_ϱ. A similar construction takes place for solutions x with $x(t_0) < 0$, $x'(t_0) = 0$. Hence, by choosing R large enough, the conclusion follows. □

By iteration of Lemma 1, we can prove the next lemma.

Lemma 2. *Assume that* $F: \mathbb{R} \times \mathbb{R} \times [0, 1] \to \mathbb{R}$ *is* 2π*-periodic in its first variable and satisfies uniform* L^∞*-Carathéodory conditions. Assume further there exists a number* $\eta > 0$ *such that* (15) *holds uniformly in* t, λ.

Then, for any $n \in \mathbb{N}$ *and any* $\varrho_0 > 0$, *there exists a number* $R_0 > 0$ *such that, for any solution* $x: [t_0, \omega] \to \mathbb{R}$ *of* (14), *with* $t_0 < \omega$, $|x(t_0)| \geqq R_0, x'(t_0) = 0$, *either* $x^2(t) + x'^2(t) \geqq \varrho_0^2$ *for all* $t \in [t_0, \omega]$, *or* x *has at least* $2n$ *zeros on an interval* $[t_0, t_n] \subset [t_0, \omega]$ *and, for all* $t \in [t_0, t_n]$, $x^2(t) + x'^2(t) \geqq \varrho_0^2$.

Proof. Let $R_n = \varrho_0$. Given $R_i (i = n, n-1, \ldots, 1)$, we choose $\varrho = R_i$ in Lemma 1, from which we get R and let $R_{i-1} = R$. Then, if $x: [t_0, \omega] \to \mathbb{R}$ is a solution of (14) with

$$t_0 < \omega, |x(t_0)| \geqq R_0, \quad x'(t_0) = 0,$$

either $x^2(t) + x'^2(t) \geqq \varrho_0^2$ for all $t \in [t_0, \omega]$, or there exists $t_1 < t_2 < \ldots < t_n$ such that
(a) x has at least two zeros in $[t_{i-1}, t_i]$ for $i = 1, \ldots, n$;
(b) for all $t \in [t_{i-1}, t_i]$, $x^2(t) + x'^2(t) \geqq R_i^2 \geqq \varrho_0^2$;
(c) $|x(t_i)| \geqq R_i$, $x'(t_i) = 0$. □

Since the hypotheses about F are unaffected by a time reversal, the above Lemma can be rephrased as follows, after some easy modifications.

Lemma 3. *Let* $F: \mathbb{R} \times \mathbb{R} \times [0, 1] \to \mathbb{R}$ *be* 2π*-periodic in its first variable. Assume it satisfies uniform* L^∞*-Carathéodory conditions and there exists a number* $\eta > 0$ *such that* (15) *holds uniformly in* t, λ.

Then, for any $n \in \mathbb{N}$ *and any* $\varrho_0 > 0$, *there exists a number* $R_0 > 0$ *such that, if* $x: [t_0, \omega] \to \mathbb{R}$ *is a solution of* (14) *with* $x^2(t_0) + x'^2(t_0) \leqq \varrho_0^2$, *having at most* $2n$ *zeros, we have, for all* $t \in [t_0, \omega]$,

$$x^2(t) + x'^2(t) \leqq R_0^2.$$

Lemma 3 shows that, under condition (15) an a priori bound can be found for the solutions of equation (14), which enter the ball B_ϱ at some time t_0 and have less than some given number of zeros. Lemma 2 and 3 have been written using the euclidean norm in the phase plane. That norm could obviously be replaced by any other norm.

3. Proof of Theorem 1. Take $m_+ \in [A_+, B_+]$, $m_- \geqq A_-$. It results immediately from hypotheses (11), (12) that

$$\frac{2}{n+1} < \frac{1}{\sqrt{m_+}} + \frac{1}{\sqrt{m_-}} < \frac{2}{n}. \tag{19}$$

For $\lambda \in [0, 1]$, define the function F by

$$F(t, x, \lambda) = \lambda f(t, x) + (1 - \lambda)[m_+ x^+ - m_- x^-];$$

by 2π-periodicity in t, that function will be extended to $\mathbb{R} \times \mathbb{R} \times [0, 1]$. By degree theoretic arguments, the theorem will be established if we can find a priori bounds, in the sup-norm, for the solutions of

$$x'' + F(t, x, \lambda) = 0 \tag{20}$$

$$x(0) = x(2\pi), \quad x'(0) = x'(2\pi), \tag{21}$$

independently of $\lambda \in [0, 1]$ (see [9] or [10]). It results indeed from (19) that, for $\lambda = 0$, the degree is equal to 1. By adding or substracting a small positive constant to the functions a_+, b_+, A_-, we can assume that (11), (12) still hold and replace (8), (9) by the stronger assumption that, for some constant $K > 0$,

$$a_+(t) x^2 - K \leqq x f(t, x) \leqq b_+(t) x^2 + K \quad \text{for all } x \geqq 0, \quad \text{for a.e. } t \in [0, 2\pi], \tag{22}$$

$$a_-(t) x^2 - K \leqq x f(t, x) \text{ for all } x \leqq 0, \quad \text{for a.e. } t \in [0, 2\pi]. \tag{23}$$

Let x be a solution of (20), (21) such that, for some $t_0 \in [0, 2\pi]$, $x^2(t_0) + x'^2(t_0) \geqq R_0^2$. We will show that such a solution cannot exist if R_0 is large enough. The method used is based on a count of the number of revolutions of the orbit in the phase plane or, equivalently, in a plane $(\mu x, x')$, where μ is an arbitrary positive number. Using Prüfer's change of variables [13]

$$\mu x = \varrho \sin\theta, \quad x' = \varrho \cos\theta$$

it is easy to see that

$$\theta' = \frac{\mu x' \cos\theta - x'' \sin\theta}{\varrho} = \mu \frac{-x x'' + x'^2}{\mu^2 x^2 + x'^2}.$$

Hence if an orbit $(x(t), x'(t))$ makes k revolutions in the phase plane on the interval $[0, 2\pi]$, and if $x^2 + x'^2$ does not vanish, we will have

$$k = \frac{\mu}{2\pi}\int_0^{2\pi} \frac{-xx'' + x'^2}{\mu^2 x^2 + x'^2}\,dt = \frac{\mu}{2\pi}\int_0^{2\pi} \frac{xF(t,x,\lambda) + x'^2}{\mu^2 x^2 + x'^2}\,dt$$

for any $\mu \geqq 0$. Looking at the half planes $x \geqq 0$ and $x \leqq 0$, and letting

$$I_+ = \{t \in [0, 2\pi] \mid x(t) \geqq 0\}, \quad I_- = \{t \in [0, 2\pi] \mid x(t) \leqq 0\},$$

we also have

$$\frac{k}{2} = \frac{\mu}{2\pi}\int_{I_+} \frac{xF(t,x,\lambda) + x'^2}{\mu^2 x^2 + x'^2}\,dt, \tag{24}$$

$$\frac{k}{2} = \frac{\nu}{2\pi}\int_{I_-} \frac{xF(t,x,\lambda) + x'^2}{\nu^2 x^2 + x'^2}\,dt, \tag{25}$$

where μ and ν are arbitrary positive numbers. We will find a priori bounds of the solutions of (20), (21), distinguishing two cases, depending on the number of zeros of the possible solution in $[0, 2\pi]$. In the sequel, the number n is the integer appearing in hypotheses (11), (12).

1st Case. The solution x has at most $2n$ zeros in $[0, 2\pi[$.

Take ϱ_0 large enough so that

$$\frac{1}{2\pi}\int_0^{2\pi} \min\left\{\frac{a_+(t)}{A_+}, \frac{a_-(t)}{A_-}, 1\right\} dt - \frac{K}{\varrho_0^2} > \frac{n}{2}\left(\frac{1}{\sqrt{A_+}} + \frac{1}{\sqrt{A_-}}\right). \tag{26}$$

Because of hypothesis (10), we can apply Lemma 2. Since the solution x is assumed to have at most $2n$ zeros in $[0, 2\pi]$, a number R can be found, using Lemma 2, such that, if $|x(t_0)| \geqq R$, $x'(t_0) = 0$ for some $t_0 \in [0, 2\pi]$, then $A_+ x^2(t) + x'^2(t) \geqq \varrho_0^2$ and $A_- x^2(t) + x'^2(t) \geqq \varrho_0^2$, for all $t \in [0, 2\pi]$. If k is the number of revolutions of that solution in the phase plane, we see, using (22) and (24) with $\mu = \sqrt{A_+}$, that

$$\begin{aligned}\frac{k}{2} &\geqq \frac{\sqrt{A_+}}{2\pi}\int_{I_+} \frac{\lambda[a_+(t)x^2 - K] + (1-\lambda)m_+ x^2 + x'^2}{A_+ x^2 + x'^2}\,dt \\ &\geqq \frac{\sqrt{A_+}}{2\pi}\int_{I_+} \min\left\{\frac{a_+(t)}{A_+}, 1\right\} dt - \frac{\sqrt{A_+}}{2\pi}\int_{I_+} \frac{K}{A_+ x^2 + x'^2}\,dt \\ &\geqq \frac{\sqrt{A_+}}{2\pi}\int_{I_+} \min\left\{\frac{a_+(t)}{A_+}, 1\right\} dt - \frac{\sqrt{A_+}}{2\pi}\frac{K}{\varrho_0^2}\,\mathrm{mes}(I_+).\end{aligned} \tag{27}$$

Similarly, by (23) and (25) with $\nu = \sqrt{A_-}$, we get

$$\frac{k}{2} \geqq \frac{\sqrt{A_-}}{2\pi}\int_{I_-} \min\left\{\frac{a_-(t)}{A_-}, 1\right\} dt - \frac{\sqrt{A_-}}{2\pi}\frac{K}{\varrho_0^2}\,\mathrm{mes}(I_-). \tag{28}$$

Combining (27) and (28), we obtain

$$\left(\frac{1}{\sqrt{A_-}}+\frac{1}{\sqrt{A_-}}\right)\frac{k}{2}\geqq\frac{1}{2\pi}\int_0^{2\pi}\min\left\{\frac{a_+(t)}{A_+},\frac{a_-(t)}{A_-},1\right\}dt-\frac{K}{\varrho_0^2}. \tag{29}$$

Confrontation with (26) shows that $k > n$, leading to a contradiction. Hence, we conclude that $|x(t)| \leqq R$ for all $t \in [0, 2\pi]$, if x has at most $2n$ zeros in $[0, 2\pi]$.

2nd case. The solution x has at least $2n + 2$ zeros in $[0, 2\pi[$.

Take ϱ_0 large enough so that

$$\frac{1}{2\pi}\int_0^{2\pi}\max\left\{\frac{b_+(t)}{B_+},1\right\}dt+\frac{K}{\varrho_0^2}<\frac{n+1}{2}\frac{1}{\sqrt{B_+}}. \tag{30}$$

The solution x is now assumed to have at least $2n + 2$ zeros in $[0, 2\pi]$. Using Lemma 2 again, a number R can be found such that, if for some $t_0 \in [0, 2\pi]$ we have $|x(t_0)| \geqq R$, $x'(t_0) = 0$ then $B_+ x^2(t) + x'^2(t) \geqq \varrho_0^2$ on some interval J, on which x has at least $2n + 2$ zeros. Since x is assumed to have at least $2n + 2$ zeros in $[0, 2\pi]$, using the periodicity, we can take $J \subset [t_0, t_0 + 2\pi]$. As above, if $J_+ = \{t \in J \mid x(t) \geqq 0\}$, we have

$$\frac{k}{2}=\frac{v}{2\pi}\int_{J_+}\frac{xF(t,x,\lambda)+x'^2}{v^2x^2+x'^2}dt,$$

k being the number of revolutions in the phase plane when t travels through J. We will use that formula with $v = \sqrt{B_+}$. Taking (23) into account, this leads to

$$\frac{k}{2}\leqq\frac{\sqrt{B_+}}{2\pi}\int_{J_+}\frac{\lambda[b_+(t)x^2+K]+(1-\lambda)m_+x^2+x'^2}{B_+x^2+x'^2}dt$$

$$\leqq\frac{\sqrt{B_+}}{2\pi}\int_{J_+}\max\left\{\frac{b_+(t)}{B_+},1\right\}dt+\frac{\sqrt{B_+}}{2\pi}\frac{K}{\varrho_0^2}\operatorname{mes}(J_+).$$

But, because of (30), this would imply $k < n + 1$, leading to a contradiction. Hence, we must have, in this case also, $|x(t)| \leqq R$ for all $t \in [0, 2\pi]$. □

Remark 1. The hypotheses of Theorem 1 can be slightly weakened. Indeed, a careful analysis of the proof shows that, the inequality (27) can be replaced by a strict one if $a_+(t) \neq A_+$ on a subset of I_+ of positive measure. A similar Remark holds for (28). On the other hand, either $\operatorname{mes}(I_+) \geqq \pi$ or $\operatorname{mes}(I_-) \geqq \pi$. Consequently, if $\operatorname{mes}\{t \in [0, 2\pi] \mid a_+(t) \neq A_+\} > \pi$ and $\operatorname{mes}\{t \in [0, 2\pi] \mid a_-(t) \neq A_-\} > \pi$, relation (29) can be replaced by a strict inequality, which allows hypothesis (11) to be replaced by a nonstrict inequality.

Remark 2. A result similar to Theorem 1 can also be obtained for other boundary value problems, when the boundary conditions can be put in relation with a count of the number of revolutions in the phase plane. This is the case, for instance, of the Neumann problem

$$x'' + f(t, x) = 0,$$

$$x'(0) = x'(2\pi) = 0.$$

Remark 3. It is also possible to extend the result of Theorem 1 to differential equations with a damping linear term, i.e.

$$x'' + cx' + f(t, x) = 0.$$

Indeed, the change of variable $x(t) = \exp(-ct/2)\,u(t)$ transforms the differential equation into

$$u'' + \exp(ct/2)\,f(t, u\exp(-ct/2)) - c^2 u/4 = 0,$$

whereas the periodic boundary conditions become

$$u(0) = \exp(-cT/2)\,u(T), \quad u'(0) = \exp(-cT/2)\,u'(T).$$

Although the problem is no longer a periodic boundary value problem, the number of revolutions in the phase plane must still be an integer. Hence, the method of Theorem 1 applies and existence conditions can be written, replacing $f(t, x)$ by $f(t, x) - c^2 x/4$.

Remark 4. At last, let us point out that results in the line of Corollary 1 have been recently obtained by D. de Figueiredo and B. Ruf [2].

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