Arch. Math., Vol. 61, 277–284 (1993)

0003-889X/93/6103-0277 $ 3.10/0

New characterizations of the Clifford tori and the Veronese surface

By

CHUANXI WU

1. Introduction. Let M be an n-dimensional compact minimally immersed submanifold in the unit sphere S^{n+p}, and let h denote the second fundamental form of M. We denote the square of the length of h by S. It is well-known that if $S \leqq \frac{n}{2-\frac{1}{p}}$, then $S = 0$ or $S = \frac{n}{2-\frac{1}{p}}$ (J. Simons [6]). The minimal submanifold M satisfying $S = 0$ is totally geodesic. S. S. Chern, Do Carmo and S. Kobayashi [3] proved that the Veronese surface in S^4 and the Clifford tori in S^{n+1} are the only compact minimal submanifolds of dimension n in S^{n+p} with $S = \frac{n}{2-\frac{1}{p}}$. For the case of the codimension of the submanifold $p \geqq 2$, A. Li and J. Li [4] proved that if $S \leqq \frac{2n}{3}$, then M is either a totally geodesic submanifold or the Veronese surface in S^4.

Let M be an n-dimensional compact minimally immersed hypersurface in S^{n+1}. We denote the Schrödinger operator $-\Delta - S$ by L_1, where Δ is the Laplacian on M. The operator L_1 is said to have the eigenvalue λ, if the equation

$$L_1 f = \lambda f$$

has a non-trivial solution, where $f: M \to R$ is a smooth function. We denote the first eigenvalue of L_1 by λ_1. If M is totally geodesic, then $\lambda_1 = 0$. J. Simons [6] proved that $\lambda_1 \leqq -n$ if M is not totally geodesic. Recently the author [7] got the following result: If $\lambda_1 \geqq -n$, then M is either totally geodesic or a Clifford torus. This result give a new characterization of Clifford tori.

The purpose of this paper is to continue the work of [6] and [7] by extending the above results to compact minimally immersed submanifolds in the unit sphere S^{n+p}. The main result is the following

Theorem A. *Let M be an n-dimensional compact minimally immersed submanifold in the unit sphere S^{n+p}, and let μ_1 be the first eigenvalue of the Schrödinger operator*

$L_2 = -\Delta - \left(2 - \frac{1}{p}\right) S$. *Then*

$$\mu_1 = 0, \quad \textit{if } M \textit{ is totally geodesic},$$

$$\mu_1 \leqq -n, \quad \textit{otherwise}.$$

Moreover, if $\mu_1 \geqq -n$, *then either* $\mu_1 = 0$ *and* M *is totally geodesic or* $\mu_1 = -n$ *and* M *is either the Veronese surface in* S^4 *or the Clifford torus in* S^{n+1}.

For the case of the codimension of submanifolds $p \geqq 2$, we have the following sharp result.

Theorem B. *Let* M *be an* n*-dimensional compact minimally immersed submanifold in* S^{n+p} *and* $p \geqq 2$, *and let* σ_1 *be the first eigenvalue of the Schrödinger operator* $L_3 = -\Delta - \frac{3S}{2}$. *Then*

$$\sigma_1 = 0, \quad \textit{if } M \textit{ is totally geodesic},$$

$$\sigma_1 \leqq -n, \quad \textit{otherwise}.$$

Moreover, if $\sigma_1 \geqq -n$, *then either* $\sigma_1 = 0$ *and* M *is totally geodesic or* $\sigma_1 = -n$ *and* M is the *Veronese surface in* S^4.

2. Preliminaries. Let M be an n-dimensional Riemannian manifold immersed in the unit sphere S^{n+p}. We choose a local orthonormal frame field $e_1, \ldots, e_{n+p}$ in S^{n+p} such that, restricted to M, the vectors $e_1, \ldots, e_n$ are tangent to M. We shall make use of the following convention on the ranges of indices:

$$1 \leqq A, B, C, \ldots, \leqq n + p;$$

$$1 \leqq i, j, k, \ldots, \leqq n;$$

$$n + 1 \leqq \alpha, \beta, \gamma, \ldots, \leqq n + p,$$

and we shall agree that repeated indices are summed over the respective ranges. With respect to the frame field chosen above, let $\omega^1, \ldots, \omega^{n+p}$ be the field of dual frames. Then the structure equations of S^{n+p} are given by

$$d\omega^A = -\sum \omega^A_B \wedge \omega^B, \quad \omega^A_B + \omega^B_A = 0, \tag{2.1}$$

$$d\omega^A_B = -\sum \omega^A_C \wedge \omega^C_B + \Phi^A_B, \quad \Phi^A_B = \tfrac{1}{2} \sum K^A_{BCD} \omega^C \wedge \omega^D, \tag{2.2}$$

$$K^A_{BCD} = \delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC}. \tag{2.3}$$

Restriction of these forms to M gives

$$\omega^\alpha = 0. \tag{2.4}$$

Since $0 = d\omega^\alpha = -\sum \omega^\alpha_i \wedge \omega^i$, by Cartan's lemma we may write

$$\omega^\alpha_i = \sum h^\alpha_{ij}, \quad h^\alpha_{ij} = h^\alpha_{ji}. \tag{2.5}$$

From these formulas, we obtain

$$d\omega^i = -\sum \omega^i_j \wedge \omega^j, \quad \omega^i_j + \omega^j_i = 0, \tag{2.6}$$

$$d\omega^i_j = -\sum \omega^i_k \wedge \omega^k_j + \Omega^i_j, \quad \Omega^i_j = \tfrac{1}{2}\sum R^i_{jkl}\,\omega^k \wedge \omega^l, \tag{2.7}$$

$$R^i_{jkl} = K^i_{jkl} + \sum_\alpha (h^\alpha_{ik} h^\alpha_{lj} - h^\alpha_{il} h^\alpha_{jk}), \tag{2.8}$$

$$d\omega^\alpha_\beta = -\sum \omega^\alpha_\gamma \wedge \omega^\gamma_\beta + \Omega^\alpha_\beta, \quad \Omega^\alpha_\beta = \tfrac{1}{2}\sum R^\alpha_{\beta kl}\,\omega^k \wedge \omega^l, \tag{2.9}$$

$$R^\alpha_{\beta kl} = K^\alpha_{\beta kl} + \sum_i (h^\alpha_{ik} h^\beta_{il} - h^\alpha_{il} h^\beta_{ik}). \tag{2.10}$$

We call $h = \sum h^\alpha_{ij}\,\omega^i \omega^j e_\alpha$ the second fundamental form of the immersed submanifold M, and $\frac{1}{n}\sum_\alpha \left(\sum_i h^\alpha_{ii}\right) e^\alpha$ the mean curvature vector. An immersion is said to be minimal if its mean curvature vector vanishes identically, i.e., if $\sum_i h^\alpha_{ii} = 0$ for all α.

We define h^α_{ijk} and h^α_{ijkl} by

$$\sum h^\alpha_{ijk}\,\omega^k = dh^\alpha_{ij} - \sum h^\alpha_{il}\,\omega^l_j - \sum h^\alpha_{lj}\,\omega^l_i + \sum h^\beta_{ij}\,\omega^\alpha_\beta, \tag{2.11}$$

$$\sum h^\alpha_{ijkl}\,\omega^l = dh^\alpha_{ijk} - \sum h^\alpha_{ljk}\,\omega^l_i - \sum h^\alpha_{ilk}\,\omega^l_j - \sum h^\alpha_{ijl}\,\omega^l_k + \Sigma\, h^\beta_{ijk}\,\omega^\alpha_\beta. \tag{2.12}$$

The Laplacian Δh^α_{ij} of the second fundamental form h^α_{ij} is defined by

$$\Delta h^\alpha_{ij} = \sum_k h^\alpha_{ijkk}. \tag{2.13}$$

3. Proofs of theorems. In order to prove our results, we need the following

Lemma 1 [5]. *Let M be an n-dimensional minimally immersed submanifold in S^{n+p}. Then*

$$|\nabla S|^2 \leqq \frac{4n}{n+2} S\,|\nabla h|^2, \tag{3.1}$$

where

$$|\nabla h|^2 = \sum_{\alpha, i, j, k} (h^\alpha_{ijk})^2.$$

Now we prove

Theorem 1. *Let M be an n-dimensional compact minimally immersed submanifold in S^{n+p} such that M is not totally geodesic, and let μ_1 be the first eigenvalue of the operator $L_2 = -\Delta - \left(2 - \frac{1}{p}\right) S$. Then*

$$\mu_1 \leqq -n - \frac{2}{n+2} \cdot \frac{\int_M |\nabla h|^2 * 1}{\int_M S * 1}, \tag{3.2}$$

where $*1$ denotes the volume element of M.

Proof. It is known that

$$\mu_1 = \inf_{\substack{f \in C^\infty(M) \\ f \not\equiv 0}} \left\{ \int_M L_2(f) f * 1 \Big/ \int_M f^2 * 1 \right\}. \tag{3.3}$$

For any constant $\varepsilon > 0$. We set

$$f_\varepsilon = (S + \varepsilon)^{\frac{1}{2}}.$$

Clearly $f_\varepsilon \in C^\infty(M)$, and if M is not totally geodesic, then

$$\lim_{\varepsilon \to 0} \int_M f_M^2 * 1 = \int_M S * 1 > 0. \tag{3.4}$$

By a simple calculation we obtain

$$\begin{aligned} \Delta f_\varepsilon &= -(S+\varepsilon)^{-\frac{3}{2}} \sum_k \Big(\sum_{\alpha,i,j} h_{ij}^\alpha h_{ijk}^\alpha \Big)^2 + (S+\varepsilon)^{-\frac{1}{2}} \sum_{\alpha,i,j,k} (h_{ijk}^\alpha)^2 \\ &\quad + (S+\varepsilon)^{-\frac{1}{2}} \sum_{\alpha,i,j} h_{ij}^\alpha \Delta h_{ij}^\alpha \\ &\geqq (S+\varepsilon)^{-\frac{3}{2}} \left[S \sum_{\alpha,i,j,k} (h_{ijk}^\alpha)^2 - \sum_k \Big(\sum_{\alpha,i,j} h_{ij}^\alpha h_{ijk}^\alpha \Big)^2 \right] \\ &\quad + (S+\varepsilon)^{-\frac{1}{2}} \sum h_{ij}^\alpha \Delta h_{ij}^\alpha. \end{aligned} \tag{3.5}$$

By using Lemma 1, we have

$$\Delta f_\varepsilon \geqq (S+\varepsilon)^{-\frac{3}{2}} \frac{2}{n+2} S |\nabla h|^2 + (S+\varepsilon)^{-\frac{1}{2}} \sum_{\alpha,i,j} h_{ij}^\alpha \Delta h_{ij}^\alpha. \tag{3.6}$$

From [3] we know that

$$\sum_{\alpha,i,j} h_{ij}^\alpha \Delta h_{ij}^\alpha \geqq nS - \left(2 - \frac{1}{p}\right) S^2. \tag{3.7}$$

It follows that

$$\begin{aligned} \Delta f_\varepsilon &\geqq \frac{2}{n+2} (S+\varepsilon)^{-\frac{3}{2}} S |\nabla h|^2 \\ &\quad + (S+\varepsilon)^{-\frac{1}{2}} \left[nS - \left(2 - \frac{1}{p}\right) S^2 \right], \end{aligned} \tag{3.8}$$

and hence

$$\begin{aligned} L_2(f_\varepsilon) f_\varepsilon &= -f_\varepsilon \Delta f_\varepsilon - \left(2 - \frac{1}{p}\right) S f_\varepsilon^2 \\ &\leqq -nS - \frac{2}{n+2} (S+\varepsilon)^{-1} S |\nabla h|^2. \end{aligned} \tag{3.9}$$

From (3.3) and (3.9), we obtain

$$\mu_1 \leqq \int_M L_2(f_\varepsilon) f_\varepsilon * 1 \Big/ \int_M f_\varepsilon^2 * 1 \tag{3.10}$$
$$\leqq -n \frac{\int_M S * 1}{\int_M (S+\varepsilon) * 1} - \frac{2}{n+2} \cdot \frac{\int_M (S+\varepsilon)^{-1} S |\nabla h|^2 * 1}{\int_M (S+\varepsilon) * 1}.$$

Letting $\varepsilon \to 0$ and taking the limit in (3.10), we have

$$\mu_1 \leqq -n - \frac{2}{n+2} \cdot \frac{\int_M |\nabla h|^2 * 1}{\int_M S * 1},$$

which is just (3.2). $\square$

Theorem 2. *Let M be a n-dimensional compact minimally immersed submanifold in S^{n+p}, and let μ_1 denote the first eigenvalue of the operator $L_2 = -\Delta - \left(2 - \frac{1}{p}\right) S$. Then*

i) $\mu_1 \geqq -n$ *if and only if* $S \leqq \dfrac{n}{2 - \frac{1}{p}}$;

ii) $\mu_1 = 0$ *if and only if* $S = 0$;

iii) $\mu_1 = -n$ *if and only if* $S = \dfrac{n}{2 - \frac{1}{p}}$.

Proof. i) If $S \leqq \dfrac{n}{2 - \frac{1}{p}}$, then we obtain $\mu_1 \geqq -n$ from (3.3). Vice versa, if $\mu_1 \geqq -n$, then we obtain $|\nabla h|^2 = 0$ from Theorem 1. Thus S is constant, and hence the first eigenvalue μ_1 of the operator $L_2 = -\Delta - \left(2 - \frac{1}{p}\right) S$ is $-\left(2 - \frac{1}{p}\right) S$. Therefore $\mu_1 \geqq -n$ implies $S \leqq \dfrac{n}{2 - \frac{1}{p}}$.

ii) If $S = 0$, clearly we have

$$L_2 = -\Delta, \quad \mu_1 = 0.$$

If $\mu_1 = 0$, then M must be totally geodesic from Theorem 1.

iii) If $S = \dfrac{n}{2 - \frac{1}{p}}$, then

$$L_2 = -\Delta - n, \ \mu_1 = -n.$$

If $\mu_1 = -n$, then we know that $S = \dfrac{n}{2 - \frac{1}{p}}$ from the proof of Theorem 1. $\square$

Theorem A follows from Theorem 1 and 2.

For a matrix $A = (a_{ij})$ we denote the square of the norm of A by $N(A)$, i.e.,

$$N(A) = \text{trace } A^t A = \sum_{i,j} (a_{ij})^2.$$

Recently Li [4] proved the following inequality.

Lemma 2 [4]. *Let $A_1, A_2, \ldots, A_p$ be symmetric $(n \times n)$-matrices and $p \geqq 2$. Denote* $S_{\alpha\beta} = \text{trace } A_\alpha^t A_\beta$, $S_\alpha = S_{\alpha\alpha} = N(A_\alpha)$, $S = S_1 + \cdots + S_p$. *Then*

$$\sum_{\alpha,\beta} N(A_\alpha A_\beta - A_\beta A_\alpha) + \sum_{\alpha,\beta} S_{\alpha\beta}^2 \leqq \tfrac{3}{2} S^2, \tag{3.11}$$

and the equality holds if and only if one of the following conditions holds:

i) $A_1 = A_2 = \cdots = A_p = 0$,

ii) *Only two of the matrices $A_1, A_2, \ldots, A_p$ are different from zero. Moreover, assuming $A_1 \neq 0$, $A_2 \neq 0$, $A_3 = \cdots = A_p = 0$, then $S_1 = S_2$, and there exists an orthogonal $(n \times n)$-matrix T such that*

$$T A_1^t T = \sqrt{\frac{S_1}{2}} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & \\ \hline & 0 & 0 \end{array}\right), \quad T A_2^t T = \sqrt{\frac{S_1}{2}} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & \\ \hline & 0 & 0 \end{array}\right).$$

As an application of the above inequality, they proved

Theorem 3 [4]. *Let M be an n-dimensional compact minimally immersed submanifold in S^{n+p}, $p \geqq 2$. If $S \leqq \frac{2n}{3}$, then M is either a totally geodesic submanifold or the Veronese surface in S^4.*

Using Lemma 2, we can prove the following sharp result if the codimension p of the submanifold satisfies $p \geqq 2$.

Theorem 4. *Let M be an n-dimensional compact minimally immersed submanifold in S^{n+p}, and let σ_1 be the first eigenvalue of the operator $L_3 = -\Delta - \frac{3S}{2}$. If M is not totally geodesic and $p \geqq 2$, then*

$$\sigma_1 \leqq -n - \frac{2}{n+2} \frac{\int_M |\nabla h|^2 * 1}{\int_M S * 1}. \tag{3.12}$$

Proof. As before we have

$$\sigma_1 = \inf_{\substack{f \in C^\infty(M) \\ f \not\equiv 0}} \left\{ \int_M L_3(f) f * 1 \Big/ \int_M f^2 * 1 \right\}, \tag{3.13}$$

where $L_3 = -\Delta - \frac{3S}{2}$. For any $\varepsilon > 0$, we set

$$f_\varepsilon = (S + \varepsilon)^{\frac{1}{2}}.$$

From [3], we have

$$\sum_{\alpha, i, j} h_{ij}^\alpha \Delta h_{ij}^\alpha = -\sum_{\alpha, \beta} N(H_\alpha H_\beta - H_\beta H_\alpha) - \sum_{\alpha, \beta} S_{\alpha\beta}^2 + nS, \tag{3.14}$$

where $H_\alpha = (h_{ij}^\alpha)$, $S_{\alpha\beta} = \operatorname{trace} H_\alpha^t H_\beta$. By Lemma 2, we obtain

$$\sum_{\alpha, i, j} h_{ij}^\alpha \Delta h_{ij}^\alpha \geqq -\frac{3}{2} S^2 + nS. \tag{3.15}$$

Using (3.1) and (3.15), it follows from (3.5) that

$$\Delta f_\varepsilon \geqq \frac{2}{n+2} (S + \varepsilon)^{-\frac{3}{2}} S \, |\nabla h|^2 + (S + \varepsilon)^{-\frac{1}{2}} \left(nS - \frac{3}{2} S^2 \right), \tag{3.16}$$

and hence

$$\begin{aligned} L_3(f_\varepsilon) f_\varepsilon &= -f_\varepsilon \Delta f_\varepsilon - \frac{3}{2} S f_\varepsilon^2 \\ &\leqq -nS - \frac{2}{n+2} (S + \varepsilon)^{-\frac{1}{2}} S \, |\nabla h|^2. \end{aligned} \tag{3.17}$$

Thus

$$\sigma_1 \leqq -n \frac{\int_M S * 1}{\int_M (S + \varepsilon) * 1} - \frac{2}{n+2} \cdot \frac{\int_M (S + \varepsilon)^{-1} S \, |\nabla h|^2 * 1}{\int_M (S + \varepsilon) * 1}. \tag{3.18}$$

Letting $\varepsilon \to 0$ and taking the limit, we obtain

$$\sigma_1 \leqq -n - \frac{2}{n+2} \cdot \frac{\int_M |\nabla h|^2 * 1}{\int_M S * 1}. \quad \square.$$

Theorem 5. *Let M be an n-dimensional compact minimally immersed submanifold in S^{n+p} and $p \geqq 2$, and let σ_1 denote the first eigenvalue of the operator $L_3 = -\Delta - \frac{3}{2} S$.*

Then

(i) $\sigma_1 \geqq -n$ *if and only if* $S \leqq \frac{2}{3}n$;

(ii) $\sigma_1 = 0$ *if and only if* $S = 0$;

(iii) $\sigma_1 = -n$ *if and only if* $S = n$.

The proof of Theorem 5 is completely similar to the proof of Theorem 2, and therefore is omitted here.

Theorem B follows from Theorem 3, 4 and 5.

Acknowledgement. This work was done while the author was a guest at Lehigh University and MSRI (Berkeley). The author would like to express his sincere gratitute to professors C. C. Hsiung, R. Osserman and S. S. Chern.

References

[1] K. Benko, M. Kothe, K. D. Semmler and U. Simon, Eigenvalue of the Laplacian and curvature. Colloq. Math. **42**, 19–31 (1979).
[2] S. S. Chern, Minimal submanifolds in a Riemannian manifold. Kansas 1968.
[3] S. S. Chern, M. Do Carmo and S. Kobayashi, Minimal submanifolds of a sphere with second fundamental form of constant length. In: Selected Papers, S. S. Chern ed., 393–409, Berlin-Heidelberg-New York 1978.
[4] A.-M. Li and J. M. Li, An intrinsic rigidity theorem for minimal submanifolds in a sphere. Arch. Math. **58**, 582–594 (1992).
[5] Y. B. Shen, Curvature and stability for minimal submanifolds. Sci. Sinica Ser. A **31**, 787–797 (1988).
[6] J. Simons, Minimal varieties in Riemannian manifolds. Ann. of Math. (2) **88**, 62–105 (1968).
[7] C. X. Wu, A characterization of Clifford minimal hypersurfaces (Chinese). Adv. in Math. (Beijing) **18**, 352–355 (1989).

Eingegangen am 26. 6. 1992*)

Anschrift des Autors:

Chuanxi Wu
Department of Mathematics
Hubei University
Wuhan 430062
People's Republic of China

*) Eine Neufassung ging am 18. 10. 1992 ein.