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Probability Theory and Related Fields

The exact hausdorff dimension of a branching set

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Summary. We obtain a critical function for which the Hausdorff measure of a branching set generated by a simple Galton–Watson process is positive and finite.

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1 Introduction

1.1 Sequences, trees and branching sets

Let $\mathbb{N}$ (respectively $\mathbb{N}_+$) be the set of non-negative integers (respectively positive) with the discrete topology, let $T = \bigcup_{k=0}^{\infty}\mathbb{N}^k$ be the set of all finite sequences and let $I = \mathbb{N}^{\mathbb{N}_+}$ be the set of all infinite sequences $\mathbf{i} = (i_1, i_2, \ldots)$ with the product topology; we make the convention that $\mathbb{N}^0$ contains the null sequence $\emptyset$. If $\mathbf{i} = (i_1, i_2, \ldots, i_n)$ $(n \leqq \infty)$, we write $|\mathbf{i}| = n$ for the length of $\mathbf{i}$, $\mathbf{i}|k = (i_1, i_2, \ldots, i_k)$ $(k \leqq n)$ for the curtailement of $\mathbf{i}$ after k terms and, if $n < \infty$, we put $\mathbf{i}^* = (i_1, \ldots, i_{n-1}, i_n + 1)$; for convenience, we define $|\emptyset| = 0$ and $\mathbf{i}|0 = \emptyset$. If $\sigma \in T$ and $\tau \in T \cup I$, we write $\sigma^*\tau$ for the sequence obtained by juxtaposition of the terms of σ and τ. We partially order T by writing $\sigma < \tau$ (or $\tau > \sigma$) to mean that the sequence τ is an extension of σ, that is, $\tau = \sigma^*\tau'$ for some sequence $\tau' \in T$; we use a similar notation if $\sigma \in T$ and $\tau \in I$. We remark that the null sequence $\emptyset < \mathbf{i}$ for any sequence $\mathbf{i}$. Finally, if $\mathbf{i}$ and $\mathbf{j}$ are two sequences of T or I, we write $\mathbf{i} \wedge \mathbf{j}$ for the common sequence of $\mathbf{i}$ and $\mathbf{j}$, that is, the maximal sequence σ such that $\sigma < \mathbf{i}$ and $\sigma < \mathbf{j}$. It is easily seen that I is metrisable, and a possible choice of metric is given by

$$d(\mathbf{i}, \mathbf{j}) = 2^{-|\mathbf{i}\wedge\mathbf{j}|} .$$

A *tree* $\mathscr{T}$ is a subset of T such that

(a) $\emptyset \in \mathscr{T}$;

(b) if $\sigma \in \mathscr{T}$ then $\sigma^* i \in \mathscr{T}$ if and only if $0 \leqq i < Z^\sigma$ for some $Z^\sigma \in \mathbb{N}$;
(c) $\sigma \in \mathscr{T}$ implies $\sigma' \in \mathscr{T}$ for any $\sigma' < \sigma$

(See Neveu 1986). The sequences σ of $\mathscr{T}$ may be identified with the vertices of a directed graph with σ joined to $\sigma^* i$ in the obvious way; the null sequence $\emptyset$ corresponds to the root of the tree; for all $\sigma \in \mathscr{T}, Z^\sigma$ represents the number of edges going out from σ. Let

$$\tilde{\mathscr{T}} = \{\mathbf{i} \in I: \ \forall n \in \mathbb{N}, \ \mathbf{i}|n \in \mathscr{T}\}$$

be the *boundary* of $\mathscr{T}$. If $Z^\sigma < \infty$ for all $\sigma \in \mathscr{T}$, then as a subspace of $I, \tilde{\mathscr{T}}$ *is a separable compact topological space* (cf. Liu 1993, chap.1, Lemma 2.2).

Let $(\Omega, \mathscr{A}, P)$ be a probability space. Suppose that $p_k \geqq 0$, that $\sum_{k=0}^{\infty} p_k = 1$ and let (Z^σ) $(\sigma \in T)$ be a countable family of independent random variables defined on Ω, each distributed according to the law $P(Z = k) = p_k$. Let $\mathscr{T} = \mathscr{T}(\omega)$ be the associated random tree; the corresponding boundary $\tilde{\mathscr{T}} = \tilde{\mathscr{T}}(\omega)$ is then called a *branching set*. Write

$$\mathscr{T}_n = \{\sigma \in \mathscr{T}: \ |\sigma| = n\} .$$

$(n \geqq 0)$, then

$$Z_n = \text{the cardinality of } \mathscr{T}_n$$

is a Galton–Watson process with $Z_0 = 1$ and offspring having the same distribution as Z. We only consider the (supercritical) case where

$$1 < m := \sum k p_k < \infty .$$

We know that the limit

$$W = \lim_{n \to \infty} Z_n / m^n$$

almost surely (a.s.) exists, and, if

$$\sum p_k k \log k < \infty , \qquad (Z \log Z)$$

then $\mathbb{E}(W) = 1$ and the extinction probability satisfies

$$P(Z_n \to 0) = P(W = 0) = P(\tilde{\mathscr{T}} = \emptyset) \qquad (1.1)$$

(see e.g. Athrey–Ney 1972). Our interest centers on the Hausdorff measures of the branching set $\tilde{\mathscr{T}}$.

1.2 Hausdorff measures

Let (E, d) be a metric space and $f = f(t)$ be a *dimension function*, that is, a positive function defined for all $t > 0$ sufficiently small, non-decreasing and continuous on the right such that $f(0+) = 0$. For all $A \subseteq E$, the *Hausdorff* (*outer*) *measure* of A with respect to the dimension function f is defined as

$$\mathscr{H}^f(A) = \lim_{\delta \to 0+} \mathscr{H}_\delta^f(A) ,$$

where

$$\mathscr{H}_\delta^f(A) = \inf\left\{\sum_{i=1}^{\infty} f(|U_i|):\ A \subset \bigcup_{i=1}^{\infty} U_i, \quad |U_i| \leqq \delta\right\},$$

with $|U_i|$ representing the diameter of U_i. It is known that the quantity $\mathscr{H}^f(A)$ does not change if in the definition we use covers of just open sets or just closed sets, or again just subsets of A, see for example Rogers (1970). If we use covers of just balls, we obtain the *spherical Hausdorff measure* of A which will be denoted by $\bar{\mathscr{H}}^f(A)$; we shall also write $\bar{\mathscr{H}}_\delta^f(A)$ for the corresponding number of $\mathscr{H}_\delta^f(A)$. The two measures $\mathscr{H}^f(\cdot)$ and $\bar{\mathscr{H}}^f(\cdot)$ are not identical in general (see Besicovitch 1928, chap. 3); however, we have

Lemma 0 (a) *Let (E,d) be a metric space and $f(t) \geqq 0$ be a dimension function such that for some $c > 0$ and all $t > 0$ sufficiently small, $f(2t) \leqq cf(t)$. Then for all $A \subseteq E$,*

$$\mathscr{H}^f(A) = \bar{\mathscr{H}}^f(A) \leqq c\mathscr{H}^f(A).$$

(b) *For all dimension functions f and all $A \subseteq I$,*

$$\bar{\mathscr{H}}^f(A) = \mathscr{H}^f(A).$$

A proof of this result can be found in Liu (1993, chap. 1, Lemmas 2.1 and 2.3). If $0 < \mathscr{H}^f(A) < \infty$, we say that f is an *exact dimension function* of A; if $f(t) = t^a$ $(a > 0)$, we write $\mathscr{H}^a(A)$ instead of $\mathscr{H}^f(A)$, and we call it the a-dimensional Hausdorff measure of A. The *Hausdorff dimension* of A is defined as

$$\dim A = \sup\{a > 0 \mid \mathscr{H}^a(A) = +\infty\} = \inf\{a > 0 \mid \mathscr{H}^a(A) = 0\}.$$

Then $\mathscr{H}^a(A) = +\infty$ if $a < \dim A$ and $\mathscr{H}^a(A) = 0$ if $a > \dim A$. All the statements above hold if $\mathscr{H}^f(\cdot)$ is replaced by $\bar{\mathscr{H}}^f(\cdot)$, provided that $A \subseteq I$ or f satisfies the regularity condition in Lemma 0.

1.3 Main results and examples

Throughout the entire paper, we use the following conventions, notations or definitions:

$$\frac{a}{\infty} = 0 \quad \text{and} \quad \frac{a}{0} = a^\infty = \infty \quad \text{if } 0 < a < \infty; \tag{1.2a}$$

$$\alpha = \frac{\log m}{\log 2} \quad \text{and} \quad \beta = 1 - \frac{\log m}{\log \|Z\|_\infty}, \quad \text{where } \|Z\|_\infty = \operatorname{ess\,sup} Z \leqq \infty; \tag{1.2b}$$

$$\phi_\theta(t) = t^\alpha\left(\log\log\frac{1}{t}\right)^\theta \quad \text{and} \quad \psi_\theta(t) = t^\alpha\left(\log\frac{1}{t}\right)^\theta \quad \text{for all } \theta \in \mathbb{R}; \tag{1.2c}$$

$$r_\theta = \sup\{t \geqq 0: \mathbb{E}(e^{tW^\theta}) < \infty\} \quad \text{if } 0 < \theta < \infty; \tag{1.2d}$$

$$\gamma = \sup\{p > 0: \mathbb{E}Z^p < \infty\}. \tag{1.2e}$$

Thus,

$$\phi_\beta(t) = t^\alpha \left(\log\log\frac{1}{t}\right)^\beta, \phi_1(t) = t^\alpha \log\log\frac{1}{t},$$

etc.; r_θ is *the radius of convergence of the moment generating function of* W^θ; $\gamma \in [1, \infty]$ is the critical value for existence of moments of Z or W.

We first gather some known results as follows:

Theorem 0 (i) $\mathscr{H}^\alpha(\tilde{\mathscr{T}}(\omega)) \leqq W$ *a.s.* (ii) $\dim \tilde{\mathscr{T}}(\omega) = \alpha$ *a.s. on* $\tilde{\mathscr{T}}(\omega) \neq \emptyset$. (iii) $0 < \mathscr{H}^\alpha(\tilde{\mathscr{T}}(\omega)) < \infty$ *a.s. on* $\tilde{\mathscr{T}}(\omega) \neq \emptyset$ *if and only if* Z *is a.s. a constant.*

Part (i) is immediate by considering the natural covers of Z_n balls $B(\sigma) = \{i \in I\colon\ \mathbf{i} > \sigma\}$ $(\sigma \in \mathscr{T}_n)$ of diameter 2^{-n} and the fact that $Z_n(2^{-n})^\alpha = Z_n/\mu^n \to W$. Part (ii) was first found by Hawkes (1981) under the condition that $\sum p_k k \log^2 k < \infty$; it was also proved by Falconer (1986, Corollary 5.7) and Lyons (1990, Proposition 6.4) in different languages under the only condition that $1 < m < \infty$ (see also Lyons and Pemantle 1992). The fact that the condition $1 < m < \infty$ suffices for the dimension result can also be seen by an easy truncation argument from Hawkes' result. Part (iii) is a special case of a result of Falconer (1987, Lemma 4.4).

Theorem 0 shows that in the non-degenerate case the α-dimensional measure of the branching set vanishes and so the function t^α is too small to measure the set. In the following, we calculate the exact value of the Hausdorff measure of $\tilde{\mathscr{T}}$ with respect to the function ϕ_θ $(0 < \theta < \infty)$; this leads us to a general criterion for a function of the form $\phi_\theta(t)$ to be an exact dimension function of $\tilde{\mathscr{T}}$. From now on, we always suppose that the moment condition $(Z \log Z)$ is satisfied. Thus by (1.1), the events "$\tilde{\mathscr{T}} \neq \emptyset$" and "$W > 0$" coincide a.s.

Theorem 1 *For all* $0 < \theta < \infty$, $\mathscr{H}^{\phi_\theta}(\tilde{\mathscr{T}}) = (r_{1/\theta})^\theta W$ *a.s. on* $\tilde{\mathscr{T}} \neq \emptyset$.

We notice that by the convention (1.2a), the number $(r_{1/\theta})^\theta W \in [0,\infty]$ is a.s. well defined on $\mathscr{T} \neq \emptyset$; the result shows in particular that *a.s. on* $\tilde{\mathscr{T}} \neq \emptyset, \mathscr{H}^{\phi_\theta}(\tilde{\mathscr{T}})$ *is zero, positive and finite, or infinite if and only if the same is true for* $r_{1/\theta}$; if $\mathbb{E}(Z^p) = \infty$ for some $p > 1$, then $r_{1/\theta} = 0$ for all $\theta \in (0,\infty)$, and so $\mathscr{H}^{\phi_\theta}(\tilde{\mathscr{T}}) = 0$ a.s. for all $\theta > 0$; in this case, Theorem 4 below will give more precise conclusions.

Theorem 2 *If* $\|Z\|_\infty < \infty$, *then it is almost sure that on* $\mathscr{T} \neq \emptyset$,

$$\mathscr{H}^{\phi_\beta}(\tilde{\mathscr{T}}) = 1 \ \textit{if}\ \beta = 0 \quad \textit{and} \quad \mathscr{H}^{\phi_\beta}(\tilde{\mathscr{T}}) = (r_{1/\beta})^\beta W$$
$$\textit{with}\ 0 < r_{1/\beta} < \infty \quad \textit{if}\ \beta > 0.$$

Therefore ϕ_β is the exact dimension function of $\tilde{\mathscr{T}}$ if $\tilde{\mathscr{T}} \neq \emptyset$. Noting that $\beta = 0$ if and only if Z is a.s. a constant, we see that the iterated logarithmic term disappears in the deterministic case.

Theorem 3 *If* $\|Z\|_\infty = \infty$, *then it is almost sure that on* $\tilde{\mathscr{T}} \neq \emptyset$,

(i) $\mathscr{H}^{\phi_\theta}(\tilde{\mathscr{T}}) = 0$ *if* $\theta < 1$;

(ii) $\mathscr{H}^{\phi_1}(\tilde{\mathscr{T}}) > 0$ *if for some* $t > 0$, $\mathbb{E}(e^{tZ}) < \infty$;

(iii) $\mathscr{H}^{\phi_1}(\tilde{\mathscr{T}}) < \infty$ *if for some* $t > 0$, $\mathbb{E}(e^{tZ}) = \infty$;

(iv) $\mathscr{H}^{\phi_1}(\tilde{\mathscr{T}}) = r_1 W$ *with* r_1 *positive and finite, provided that for some but not all* $t > 0$, $\mathbb{E}(e^{tZ}) < \infty$.

In the case where Z is of geometric distribution (thus $r_1 = 1$), part (iv) of the theorem was proved by Hawkes (1981).

Theorem 4 *It is almost sure that on* $\tilde{\mathscr{T}} \neq \emptyset$,

(i) $\mathscr{H}^{\psi_\theta}(\tilde{\mathscr{T}}) = 0$ *if* $\theta < 1/\gamma$ *and* $\mathscr{H}^{\psi_\theta}(\tilde{\mathscr{T}}) = \infty$ *if* $\theta > 1/(\gamma - 1)$;

(ii) $\mathscr{H}^{\psi_{1/(\gamma-1)}}(\tilde{\mathscr{T}}) = \infty$ *if* $1 < \gamma < \infty$ *and* $\mathbb{E}(Z^\gamma) < \infty$;

(iii) $\mathscr{H}^{\psi_{1/\gamma}}(\tilde{\mathscr{T}}) < \infty$ *if* $\gamma < \infty$ *and* $\limsup_{k\to\infty}\{\sum_{v=[\log k]}^{k} P(W \geqq v^{1/\gamma}) - \frac{1}{\gamma}\log k\} > -\infty$.

If $\gamma = \infty$, that is, if for all $p > 1, \mathbb{E}(Z^p) < \infty$, then part (i) of the theorem is interpreted as "$\mathscr{H}^{\psi_\theta}(\tilde{\mathscr{T}}) = 0$ if $\theta < 0$ and $\mathscr{H}^{\psi_\theta}(K) = \infty$ a.s. on $\tilde{\mathscr{T}} \neq \emptyset$ if $\theta > 0$"; in this case, Theorems 1 and 2 give more precise conclusions under stronger conditions. The theorem shows that if $1 < \gamma \leqq \infty$, that is, if for some $p > 1$, $\mathbb{E}(Z^p) < \infty$, then there exists a critical value $\chi \in [1/\gamma, 1/(\gamma - 1)]$ such that $\mathscr{H}^{\psi_\theta}(\tilde{\mathscr{T}}) = 0$ if $\theta < \chi$ and $\mathscr{H}^{\psi_\theta}(\tilde{\mathscr{T}}) = \infty$ a.s. on $\tilde{\mathscr{T}} \neq \emptyset$ if $\theta > \chi$. The author believes that this would also be the case if $\gamma = 1$ (thus $\chi = \infty$) and that $\chi = 1/(\gamma - 1)$ in all cases:

Conjecture 5 $\mathscr{H}^{\psi_\theta}(\tilde{\mathscr{T}}) = 0$ *if* $\theta < 1/(\gamma - 1)$ *and* $\mathscr{H}^{\psi_\theta}(\tilde{\mathscr{T}}) = \infty$ *a.s. on* $\tilde{\mathscr{T}} \neq \emptyset$ *if* $\theta > 1/(\gamma - 1)$.

Remark 6. If the distance $d(\mathbf{i}, \mathbf{j}) = 2^{-|\mathbf{i} \wedge \mathbf{j}|}$ on I is replaced by

$$d_M(\mathbf{i}, \mathbf{j}) := M^{-|\mathbf{i} \wedge \mathbf{j}|}$$

for some $M > 1$, then all the results above hold with α replaced by

$$\alpha(M) := \log m / \log M .$$

As applications of the theorems, we give some examples below.

Example 1 (*Embedding in Euclidean space*). Suppose that the distribution of $Z = Z_1$ has compact support, that is, $\|Z\|_\infty < \infty$ or $p_k = 0$ for k sufficiently large. Let M be an integer such that $M \geqq \|Z\|_\infty$ (namely $p_k = 0$ for $k > M$). If $Z_1 = k$ we choose at random k distinct integers $j_1, j_2, \ldots, j_k$ with $0 \leqq j_i \leqq M - 1$, and let

$$I_1 = \bigcup_{i=1}^{Z_1} [j_i/M, (j_i + 1)/M) .$$

We now treat each interval in I_1 as the vertex of a tree and proceed inductively in the same fashion; at the n-th stage we have I_n as a union of Z_n intervals of length M^n. The limit set $K = \bigcap_{n=0}^{\infty} I_n \subseteq [0, 1]$ can be described by the associated branching set $\tilde{\mathscr{T}}$ of the process under the mapping

$$f : \tilde{\mathscr{T}} \to K, \quad \mathbf{i} \to \sum i_k M^{-k} .$$

If we consider covers of K by M-adic sets and if $\tilde{\mathscr{T}}$ carries the matric $d_M(\mathbf{i},\mathbf{j}) = M^{-|\mathbf{i}\wedge\mathbf{j}|}$, it is then easily seen that the Cantor set K has the same exact dimension function as $\tilde{\mathscr{T}}$, given by $\phi_\beta^M(t) = t^{\alpha(M)}(\log\log\frac{1}{t})^\beta$, where $\alpha(M) = \log m/\log M$ (cf. Theorem 2 and Remark 6).

We now give a more explicit construction to explain this. Divide the unit interval into three equal parts and retain each independently with probability p. Repeat this with the parts that remain, and so on. In this case $M = 3, m = \mathbb{E}(Z) = 3p$ and $\|Z\|_\infty = 3$. Then $\alpha = \log(3p)/\log 3 = 1 + \log p/\log 3$ and $\beta = 1 - \log(3p)/\log 3 = 1 - \alpha$. The exact Hausdorff dimension function of the resulting fractal set is then $\phi_{1-\alpha}(t) = t^\alpha(\log\log\frac{1}{t})^{1-\alpha}$ with $\alpha = 1 + \log p/\log 3$. Graf et al. (1988, p. 89) also calculate this function. So we see that our results here are closely related to those of Graf et al. (1988). In fact, the author has recently developed the ideas of the present paper to Euclidian space, and thus improved the classical results of Graft et al. (see Liu 1993).

Example 2 (*On the conjecture of Hawkes*). Hawkes (1981) conjectured that an exact dimension function of $\tilde{\mathscr{T}}$ would be of the form $h(t) = t^\alpha R^{-1}(\log\log\frac{1}{t})$ if $R(x) = -\log P(W \geqq x)$ satisfies some regularity conditions at $+\infty$. We say that this is in fact the case if, for example, for some $\lambda > 0$, $a > 0$ and all sufficiently large $x > 0$,

$$P(W > x) = e^{-\lambda x^a}\ .$$

To see this, let us first calculate r_a. If $0 < t < \lambda$, then by integration by parts,

$$\mathbb{E}e^{tW^a} = \int_{[0,\infty)} e^{tx}\, dP(W^a \leqq x) = 1 + t\int_0^\infty e^{tx} P(W^a > x)\, dx < \infty\ ;$$

if $t > \lambda$, then for all $r > 0$ sufficiently large,

$$\mathbb{E}e^{tW^a} \geqq \int_{[r,\infty)} e^{tx}\, dP(W^a \leqq x) \geqq e^{tr} P(W^a > r) = e^{(t-\lambda)r}\ ,$$

and so $\mathbb{E}e^{tW^a} = \infty$ by letting $r \to \infty$. Therefore, $r_a = \lambda \in (0,\infty)$ and, by Theorem 1,

$$\phi_{1/a}(t) = t^\alpha \left(\log\log\frac{1}{t}\right)^{1/a} = \lambda\, h(t)$$

is an exact dimension function of $\tilde{\mathscr{T}}$. Moreover, we have the following more general result: *if for some positive constants* λ_1, λ_2, c_1, c_2, a, Δ *and all* $x \geqq \Delta$,

$$c_1 e^{-\lambda_1 x^a} \leqq P(W \geqq x) \leqq c_2 e^{-\lambda_2 x^a}\ ,$$

then $\phi_{1/a}(t) = t^\alpha(\log\log\frac{1}{t})^{1/a}$ *is an exact dimension function of* $\tilde{\mathscr{T}}$. This also follows from Theorem 1 since a similar calculation as above shows that

$$0 < \lambda_2 \leqq r_a \leqq \lambda_1 < \infty\ .$$

Example 3 (*Case where the distribution of reproduction decreases geometrically*). If for some positive constants $\lambda_1 \geqq \lambda_2$, c_1, c_2 and Δ, either

$$c_1 e^{-\lambda_1 k} \leqq P(Z = k) \leqq c_2 e^{-\lambda_2 k}, \quad \forall k \geqq \Delta\ ,$$

or

$$c_1 e^{-\lambda_1 k} \leqq P(Z \geqq k) \leqq c_2 e^{-\lambda_2 k}, \quad \forall k \geqq \Delta ,$$

then $\phi_1(t) = t^\alpha(\log\log \frac{1}{t})$ is an exact dimension function of $\tilde{\mathscr{T}}$. This follows from Theorem 3(iv) since $\mathbb{E}(e^{tZ}) < \infty$ if $t < \lambda_2$ and $\mathbb{E}(e^{tZ}) = \infty$ if $t > \lambda_1$ (cf. the calculation in Example 2). This result covers of course the case of geometric distribution.

Example 4 (*Case where the distribution of reproduction decreases polynomially*). If for some constants $c_1 > 0, c_2 > 0$, $\theta > 1$ and $\Delta > 0$, either

$$c_1 k^{-(\theta+1)} \leqq P(Z = k) \leqq c_2 k^{-(\theta+1)}, \quad \forall k > \Delta ,$$

or

$$c_1 k^{-\theta} \leqq P(Z \geqq k) \leqq c_2 k^{-\theta}, \quad \forall k > \Delta ,$$

then

$$\mathscr{H}^{\psi_b}(\tilde{\mathscr{T}}) = 0 \quad \text{if } b < 1/\theta \text{ and } \mathscr{H}^{\psi_b}(\tilde{\mathscr{T}}) = \infty \text{ a.s. on } \tilde{\mathscr{T}} \neq \emptyset \quad \text{if } b > 1/(\theta - 1) .$$

The result follows from Theorem 5 since $\mathbb{E}(Z^p) = \int_0^\infty P(Z \geqq x^{1/p})\,dx$ is finite if $p < \theta$ and infinite if $p \geqq \theta$. In this case, the existence of an exact dimension function remains open and the author thinks that the answer would be negative.

2 Growth of moments of the limit of a supercritical branching process

Let (Z_n) $(n \geqq 0)$ be a Galton–Watson process with $Z_0 = 1$ and $1 < m = \mathbb{E}Z_1 < \infty$. We shall need some results concerning the growth of the moments of the limit

$$W = \lim_{n \to \infty} Z_n / m^n .$$

These results are of some interest by their own.

Theorem 2.1 (*Comparison theorem for the radii of convergence of W and Z_1*). *Denote by $r(Z_1) = \sup\{t \geqq 0\colon \mathbb{E}[e^{tZ_1}] < \infty\}$ the radius of convergence of the moment generating function $\mathbb{E}[e^{tZ_1}]$ of Z_1 and $r(W)$ that of W, then*

$r(W)$ is zero, positive and finite, or infinite

if and only if the same is true for $r(Z_1)$.

Proof. Since

$$\mathbb{E}[e^{tZ_1}] = 1 + t\mathbb{E}[Z_1] + \frac{t^2}{2!}\mathbb{E}[Z_1^2] + \dots$$

$$\mathbb{E}[e^{tW}] = 1 + t\mathbb{E}[W] + \frac{t^2}{2!}\mathbb{E}[W^2] + \dots$$

and

$$\mathbb{E}[W^n] = \mathbb{E}[\mathbb{E}(W^n|\mathbb{F}_1)] \geqq \mathbb{E}[\mathbb{E}(W|\mathbb{F}_1)^n] = \mathbb{E}[(Z_1/m)^n] = m^{-n}\mathbb{E}[Z_1^n] ,$$

where $\mathbb{F}_1$ is the σ-algebra generated by Z_1, we see by the well known formula on the radius of convergence of Taylor series that

$$r(W) \leqq m\, r(Z_1)\,.$$

So $r(Z_1) < +\infty$ implies $r(W) < +\infty$, and $r(Z_1) = 0$ implies $r(W) = 0$.

We now prove that $r(Z_1) > 0$ implies $r(W) > 0$. Put

$$p(t) = \mathbb{E}[t^{Z_1}]\,,$$

then

$$\mathbb{E}[t^{Z_n}] = (p^0)^n(t)\,,$$

where $(p^0)^1(t) = p(t)$ and $(p^0)^{k+1}(t) = p((p^0)^k(t))$ $(k \geqq 1)$. Since $r(Z_1) > 0$, there is some $r_1 > 1$ such that $p(r_1) < \infty$. We now choose by induction $r_2, r_3, \ldots$ so that

$$p(r_2) = r_1, p(r_3) = r_2, \ldots, p(r_n) = r_{n-1}, \ldots$$

Since $p(s) > s$ for all $s > 1$, we see that $1 < r_{n+1} < r_n$ for all $n \geqq 1$. As

$$r_{n-1} - 1 = p(r_n) - 1 = \int_1^{r_n} p'(t)\,dt \leqq (r_n - 1)p'(r_n)$$

[where $p'(t)$ is the derivative of $p(t)$], by induction on n we obtain that

$$r_n \geqq [p'(r_n)\ldots p'(r_2)]^{-1}(r_1 - 1) + 1\,. \tag{2.1a}$$

Notice that for any $t > 0$ and all $n \geqq 2$, if $e^{t/m^n} \leqq r_n$ then

$$\mathbb{E}[e^{tZ_n/m^n}] = p^{0n}(e^{t/m^n}) \leqq p^{0n}(r_n) = r_1\,, \tag{2.1b}$$

where the last step holds by the definition of $\{r_n\}$. So by (2.1b) and (2.1a), to see that $r(W) > 0$, i.e., $\mathbb{E}(e^{tW}) < \infty$ for some $t > 0$, it suffices to prove that for some $t > 0$ and all $n \geqq 2$,

$$t/m^n \leqq \log\{1 + [p'(r_n)\ldots p'(r_2)]^{-1}(r_1 - 1)\}\,.$$

Since $\log(1+x) \geqq \frac{1}{2}x$ for $0 \leqq x \leqq 1$, this will be the case if for some $t > 0$ and all $n \geqq 2$,

$$t/m^n \leqq \tfrac{1}{2}[p'(r_n)\ldots p'(r_2)]^{-1}(r_1 - 1)\,.$$

So it suffices to prove that

$$\prod_{n=2}^{\infty} \frac{p'(r_n)}{m} = \prod_{n=2}^{\infty}\left(1 + \frac{1}{m}\int_1^{r_n} p''(t)\,dt\right) < +\infty\,.$$

We see that this is true because

$$\int_1^{r_n} p''(t)\,dt \leqq (r_n - 1)p''(r_n) \leqq (r_n - 1)p''(r_1)$$

and

$$r_n - 1 \leqq \frac{1}{p'(1)}(p(r_n) - 1) = \frac{1}{m}(r_{n-1} - 1) \leqq \ldots \leqq \frac{1}{m^{n-1}}(r_1 - 1)\,.$$

It remains to prove that $r(Z_1) = \infty$ implies $r(W) = \infty$. To see this, we use the well-known functional equation

$$f(mt) = g(f(t)), \tag{2.2a}$$

where $f(t) = \mathbb{E}[e^{tW}]$ and $g(t) = \mathbb{E}[e^{tZ_1}]$ [see for example Athreya and Ney (1972)]. Since we have shown that $r(Z_1) > 0$ implies $r(W) > 0$, we know that $f(t) < \infty$ for some $t > 0$. From the functional equation and the fact that $g(t) < \infty$ for all $t > 0$ ($r(Z_1) = \infty$), we know immediately that $f(t) < \infty$ for all $t > 0$. □

The following result can be compared to that of Kahane–Peyrière (1976) obtained for a model of turbulence of Mandelbrot. We recall that

$$\beta = 1 - \log m / \log \|Z_1\|_\infty$$

and that for $0 < \theta < \infty$, r_θ denotes the radius of convergence of the moment generating function $\mathbb{E}(e^{tW^\theta})$ of W^θ [so $r_1 = r(W)$, and $0 < \beta < 1$ if and only if Z_1 is not a.s. a constant].

Theorem 2.2 *If Z_1 is not a.s. a constant with $\|Z_1\|_\infty < \infty$, then*

$$\lim_{k\to+\infty} \frac{\log \mathbb{E}(W^k)}{k \log k} = \beta \quad \textit{and} \quad 0 < r_{1/\beta} < +\infty .$$

Proof. (i) We first prove that in both cases $\|Z_1\|_\infty < \infty$ and $\|Z_1\|_\infty = \infty$,

$$\liminf_{k\to\infty} \frac{\log \mathbb{E}(W^k)}{k \log k} \geqq \beta .$$

Let $n \geqq 2$ be an integer such that $p_n = P(Z_1 = n) > 0$. Since

$$W = \frac{1}{m} \sum_{i=1}^{Z_1} W_i \tag{2.2b}$$

with W_i ($i \geqq 1$) being independent copies of W which are also independent of Z_1 [cf. (2.2a)], we have

$$\begin{aligned} \mathbb{E}[W^k | Z_1 = n] &= \frac{n}{m^k} \mathbb{E}[W^k] + \frac{1}{m^k} \sum_{\substack{k_1+\ldots+k_n=k \\ 0\le k_i \le k-1}} \frac{k!}{k_1! \ldots k_n!} \prod_{i=1}^n \mathbb{E}[W^{k_i}] \\ &\geqq \frac{n}{m^k} \mathbb{E}[W^k] + \frac{1}{m^k} (n^k - n) \inf \prod_{i=1}^n \mathbb{E}[W^{k_i}] , \end{aligned} \tag{2.3}$$

where $\mathbb{E}[W^k | Z_1 = n]$ denotes the conditional expectation of W^k given $Z_1 = n$ and the infimum is taken over all $(k_1, \ldots, k_n)$ such that $k_1 + \ldots + k_n = k$ and $0 \leqq k_i \leqq k-1$; if $k = n\tilde{k}$, this infimum is $(\mathbb{E}[W^{\tilde{k}}])^n$. Hence

$$\begin{aligned} \mathbb{E}[W^{n\tilde{k}} | Z_1 = n] &\geqq \frac{n}{m^k} \mathbb{E}[W^{n\tilde{k}}] + m^{-n\tilde{k}} (n^{n\tilde{k}} - n)(\mathbb{E}[W^{\tilde{k}}])^n \\ &\geqq \left(\frac{n}{m}\right)^{n\tilde{k}} (\mathbb{E}[W^{\tilde{k}}])^n . \end{aligned}$$

Therefore

$$\mathbb{E}[W^{n\tilde{k}}] \geqq p_n \left(\frac{n}{m}\right)^{n\tilde{k}} (\mathbb{E}[W^{\tilde{k}}])^n ,$$

and so

$$\frac{1}{n\tilde{k}} \log \mathbb{E}[W^{n\tilde{k}}] \geqq \log \frac{n}{m} + \frac{1}{\tilde{k}} \log \mathbb{E}[W^{\tilde{k}}] + \frac{\log p_n}{n\tilde{k}} .$$

Choosing $\tilde{k} = n^r$ ($r \in \mathbb{N}$) and using repeatedly this inequality, we see that

$$n^{-(r+1)} \log \mathbb{E}[W^{n^{r+1}}] \geqq (r+1) \log \frac{n}{m} + \log \mathbb{E}[W] + \frac{\log p_n}{n} \sum_{i=0}^{r} \frac{1}{n^i} .$$

Thus for all $r \geqq 0$,

$$n^{-r} \log \mathbb{E}[W^{n^r}] \geqq r \log \frac{n}{m} + C(n) , \tag{2.4}$$

where $C(n) > -\infty$ is a constant independent of r. Hence

$$\liminf_{r\to\infty} \frac{\log \mathbb{E}(W^{n^r}]}{n^r \log n^r} \geqq 1 - \frac{\log m}{\log n} . \tag{2.5}$$

Now for each $k \in \mathbb{N}$ sufficiently large, choose $r \in \mathbb{N}$ such that $n^r \leqq k < n^{r+1}$. Thus

$$\frac{\log \mathbb{E}(W^k)}{k \log k} = \frac{\log [\mathbb{E}(W^k)]^{1/k}}{\log k} \geqq \frac{\log [\mathbb{E}(W^{n^r})]^{1/n^r}}{\log n^{r+1}} = \frac{\log \mathbb{E}[W^{n^r}]}{n^r \log n^{r+1}} . \tag{2.6}$$

It follows from (2.5) and (2.6) that

$$\liminf_{k\to\infty} \frac{\log \mathbb{E}(W^k)}{k \log k} \geqq 1 - \frac{\log m}{\log n} .$$

As this holds for all $n \geqq 2$ with $P(Z_1 = n) > 0$, we obtain that

$$\liminf_{k\to\infty} \frac{\log \mathbb{E}(W^k)}{k \log k} \geqq 1 - \frac{\log m}{\log \|Z_1\|_\infty} = \beta .$$

(ii) We now prove that if $\|Z_1\|_\infty < \infty$, then

$$\limsup_{k\to\infty} \frac{\log \mathbb{E}(W^k)}{k \log k} \leqq \beta .$$

For convenience, we write $n = \|Z_1\|_\infty$. By (2.2b), for all $k > 1$

$$\mathbb{E}[W^k] \leqq \frac{n}{m^k} \mathbb{E}[W^k] + \frac{1}{m^k} \sum_{\substack{k_1+\ldots+k_n=k \\ 0\leqq k_i\leqq k-1}} \frac{k!}{k_1!\ldots k_n!} \prod_{i=1}^{n} \mathbb{E}[W^{k_i}] .$$

Thus for all $k \in \mathbb{N}$ sufficiently large (so that $m^k - n > 0$),

$$\mathbb{E}[W^k] \leqq \frac{1}{m^k - n} \sum_{\substack{k_1+\ldots+k_n=k \\ 0\leqq k_i\leqq k-1}} \frac{k!}{k_1!\ldots k_n!} \prod_{i=1}^{n} \mathbb{E}[W^{k_i}] .$$

Since the number of the terms in the sum $\sum$ above is less than k^n, writing $B_k = \sup_{\ell<k}(\mathbb{E}[W^\ell]/\ell!)^{1/\ell}$, we have, for all $k \in \mathbb{N}$ sufficiently large,

$$B_{k+1}^k \leqq \sup\left(\frac{k^n}{m^k - n} B_k^k, B_k^k\right).$$

Therefore B_k is bounded. This shows that $\mathbb{E}(e^{tW}) < +\infty$ for sufficiently small $t > 0$. Again from the recursive relation (2.2b), we obtain

$$\mathbb{E}(e^{Wmt}) \leqq (\mathbb{E}(e^{Wt}))^n.$$

So $\mathbb{E}(e^{tW}) < +\infty$ for all $t > 0$ and

$$\mathbb{E}[e^{Wm^kt}] \leqq (\mathbb{E}(e^{Wt}))^{n^k} = (\mathbb{E}(e^{Wt}))^{m^{kL}}$$

for all $k \in \mathbb{N}$, where

$$L = \log n/\log m.$$

Put

$$\eta(t) = \log \mathbb{E}(e^{tW}),$$

then $\eta(mt) \leqq n\eta(t)$ and consequently $\eta(m^k) \leqq n^k\eta(1)$. For each $k \in \mathbb{N}$, choose an integer $i \geqq 0$ such that

$$m^i \leqq k^{1/L} < m^{i+1}.$$

Thus

$$\mathbb{E}\left[e^{k^{1/K}W}\right] \leqq \mathbb{E}\left[e^{m^{i+1}W}\right] \leqq \left(\mathbb{E}\left(e^W\right)\right)^{n^{i+1}} = \left(\mathbb{E}\left(e^W\right)\right)^{m^{(i+1)L}} \leqq \left[\mathbb{E}\left(e^W\right)\right]^{m^Lk}.$$

Therefore, using Markov's inequality, we obtain

$$\mathbb{E}(W^k) = \int_0^\infty P(W^k > t)\,dt = \int_0^\infty P\left(e^{k^{1/L}W} > e^{k^{1/L}t^{1/k}}\right) dt$$

$$\leqq \mathbb{E}\left(e^{k^{1/L}W}\right] \int_0^\infty e^{-k^{1/L}t^{1/k}}\,dt = \mathbb{E}\left(e^{k^{1/L}W}\right] k!/k^{k/L}$$

$$\leqq [\mathbb{E}(e^W)]^{m^Lk} k!/k^{k/L}.$$

That is, for $B := (\mathbb{E}(e^W))^{m^L} \in (0,\infty)$ and all $k \geqq 1$,

$$\mathbb{E}(W^k) \leqq B^k(k!)/k^{k/L}. \tag{2.7}$$

Since $1 - 1/L = \beta$, it follows that

$$\limsup_{k\to\infty} \frac{\log \mathbb{E}(W^k)}{k \log k} \leqq \beta.$$

So we have proved the limit result of the lemma.

(iii) It remains to prove that if $\|Z_1\|_\infty < \infty$, then $0 < r_{1/\beta} < +\infty$. By (2.4), we have, for all $r \geqq 0$,

$$\mathbb{E}\left[W^{n^r}\right] \geqq \left[\left(\frac{n}{m}\right)^r e^{C(n)}\right]^{n^r}.$$

For all $k \in \mathbb{N}$ sufficiently large, choose $r \in \mathbb{N}$ such that $n^r \leqq k/\beta < n^{r+1}$. So by Stirling's formula we obtain

$$\limsup_{k\to\infty}\left(\frac{\mathbb{E}(W^{k/\beta})}{k!}\right)^{1/k} \geqq \limsup_{k\to\infty}\frac{\mathbb{E}[W^{n^r}])^{1/(n^r\beta)}}{k/e}$$
$$\geqq \limsup_{r\to\infty}\frac{\left(\frac{n}{m}\right)^{r/\beta}e^{C(n)/\beta}}{\beta n^{r+1}/e} = \frac{e^{1+C(n)/\beta}}{\beta n},$$

where the last step holds since $(n/m)^{r/\beta} = n^r$. Thus $r_{1/\beta} < \infty$. On the other hand, by (2.7), we have, for $\theta = 1/\beta$,

$$\limsup_{k\to\infty}\left(\frac{\mathbb{E}(W^{k\theta})}{k!}\right)^{1/k} \leqq \limsup_{k\to\infty}\frac{(\mathbb{E}(W^{[k\theta]+1}))^{\theta/([k\theta]+1)}}{k!^{1/k}}$$
$$\leqq \limsup_{k\to\infty}\frac{B^\theta([k\theta]+1)!^{\theta/([k\theta]+1)}}{k!^{1/k}([k\theta]+1)^{\theta/L}} = B^\theta\theta e^{1-\theta} < +\infty .$$

This shows that $r_{1/\beta} > 0$. □

Theorem 2.3 *If* $\|Z_1\|_\infty = +\infty$, *then*

(i) $\liminf_{k\to\infty}\frac{\log\mathbb{E}(W^k)}{k\log k} \geqq 1$ *and* $r_\theta = 0$ for all $\theta > 1$;

(ii) *If* $\mathbb{E}(e^{tZ_1}) < \infty$ *for some* $t > 0$, then $\lim_{k\to+\infty}\frac{\log\mathbb{E}(W^k)}{k\log k} = 1$ *and* $r_1 = r(W) > 0$.

Proof. (i) The result concerning the limit inferior was already shown in part (i) of the proof of Theorem 2.2. To see that $r_\theta = 0$ for all $\theta > 1$, let us fix $\theta > 1$ and choose $n \in \mathbb{N}$ sufficiently large such that $P(Z_1 = n) > 0$ and $\theta\beta_n > 1$, where $\beta_n = 1 - \log m/\log n$. By (2.5), we have, for all $\varepsilon > 0$ and sufficiently large $r \in \mathbb{N}$,

$$\log\mathbb{E}[W^{n^r}] \geqq (\beta_n - \varepsilon)n^r\log n^r .$$

Choose $\varepsilon > 0$ such that $\theta(\beta_n - \varepsilon) > 1$. For $k \in \mathbb{N}$ sufficiently large, choose $r \in \mathbb{N}$ such that $n^r \leqq k\theta < n^{r+1}$. Thus by Stirling's formula,

$$\limsup_{k\to\infty}\left(\frac{\mathbb{E}(W^{k\theta})}{k!}\right)^{1/k} \geqq \limsup_{k\to\infty}\frac{\mathbb{E}[W^{n^r}]^{\theta/n^r}}{k/e} \geqq \limsup_{r\to\infty}\frac{(n^r)^{\theta(\beta_n-\varepsilon)}}{n^{r+1}/(e\theta)} = +\infty .$$

So $r_\theta = 0$, as desired.

(ii) If $\mathbb{E}(e^{tZ_1}) < \infty$ for some $t > 0$, then $r(Z_1) > 0$ and, by Lemma 3.1, $r(W) > 0$. Thus

$$\infty > \limsup_{k\to\infty}\left(\frac{\mathbb{E}(W^k)}{k!}\right)^{1/k} = \limsup_{k\to\infty}\frac{\mathbb{E}[(W^k)]^{1/k}}{k/e} .$$

Therefore, $\limsup_{k\to\infty}\frac{\log\mathbb{E}(W^k)}{k\log k} \leqq 1$ and so, by (i), $\lim_{k\to\infty}\frac{\log\mathbb{E}(W^k)}{k\log k} = 1$. □

Remark. The limit result in part (ii) of Lemma 3.3 does not hold in general. For example, if $\mathbb{E}(W^k) = \infty$ for some $k > 0$, then it is evident that

$$\lim_{k\to\infty} \frac{\log \mathbb{E}(W^k)}{k \log k} = \infty .$$

3 Proof of Theorems

3.1 A random measure μ_ω on I and the Q-measure on $\Omega \times I$

If $\sigma \in \mathscr{T}_n$, let

$$Z_{\sigma,p} = \sum_{\tau\in\mathscr{T}_p, \tau>\sigma} Z^\tau$$

denote the number of descendents of σ in the generation p and define

$$W_\sigma = \lim_{p\to\infty} \frac{Z_{\sigma,p}}{m^{p-n}} ;$$

if $\sigma \in \mathbb{N}^n - \mathscr{T}_n$, we choose W_σ as an independent copy of W such that $\{W_\sigma\colon \sigma \in \mathbb{N}^n - \mathscr{T}_n\}$ is a family of independent random variables which are also independent of the family $\{W_\tau :\ \tau \in \mathscr{T}_n\}$. Then $\{W_\sigma\colon\ \sigma \in T\}$ is a family of random variables, each distributed as $W_\emptyset = W = \lim_{n\to\infty} Z_n/m^n$, and W_σ and W_τ are independent if neither $\sigma < \tau$ nor $\tau < \sigma$. It is easily verified that almost surely

$$m^{-|\sigma|} W_\sigma = \sum_{0\leqq i<Z^\sigma} m^{-|\sigma * i|} W_{\sigma * i} \tag{3.1}$$

if $\sigma \in \mathscr{T}$, where the sum is interpreted as 0 if $Z_\sigma = 0$. For $\sigma \in T$, let

$$B(\sigma) = \{\tau \in I\colon\ \sigma < \tau\}$$

be a ball in I of radius $|B(\sigma)| = 2^{-|\sigma|}$ and define

$$\mu^*_\omega(B(\sigma)) = 1_{\{\sigma\in\mathscr{T}\}}(\omega) m^{-|\sigma|} W_\sigma , \tag{3.2a}$$

where $1_{\{\sigma\in\mathscr{T}\}}$ is the indicator function of the set $\{\sigma \in \mathscr{T}\} = \{\omega\colon\ \sigma \in \mathscr{T}(\omega)\}$ (for any set Δ, 1_Δ will represent the indicator function of Δ); if $A \subseteq I$, we let

$$\mu^*_\omega(A) = \inf\left\{\mu^*_\omega(B(\tau))\colon\ A \subseteq \bigcup_\tau B(\tau)\right\} . \tag{3.2b}$$

By (2.1) it is easily verified that μ^*_ω is a.s. a metric outer measure on I and so the Borel sets are measurable. Let μ_ω be the corresponding measure. This measure is concentrated on the branching set $\tilde{\mathscr{T}}(\omega)$, and $\mu_\omega(I) = \mu_\omega(\tilde{\mathscr{T}}(\omega)) = W(\omega)$.

It will prove very useful to consider the product space $\Omega \times I$ with the product σ-field and with probability law Q defined by

$$Q(A) = \mathbb{E} \int 1_A(\omega, \mathbf{i})\, d\mu_\omega(\mathbf{i}) . \tag{3.3}$$

To obtain some density theorems about the measures μ_ω, we will need the distributions of the random variables $\widehat{W}_n(\omega, \mathbf{i}) := W_{\mathbf{i}|n}$ $(n \geqq 0)$ defined on $\Omega \times I$.

Lemma 3.1 *For all $n \geqq 0$ and all Borel measurable functions $f : \mathbb{R} \to [0,\infty)$,*

$$\mathbb{E}_Q f(\hat{W}_n) = \mathbb{E}\, W f(W)\,,$$

where $\mathbb{E}_Q$ represents the integration with respect to Q.

Proof. From the definition of Q, we have, for all $n \geqq 0$,

$$\mathbb{E}_Q f(\hat{W}_n) = \mathbb{E}_Q f(W_{\mathbf{i}|n}) = \mathbb{E} \sum_{\sigma \in \mathcal{T}_n} f(W_\sigma) m^{-n} W_\sigma\,.$$

If $n = 0$, the result follows immediately; if $n > 0$, it follows that

$$\mathbb{E}_Q f(W_{\mathbf{i}|n}) = \mathbb{E} \sum_{\sigma \in \mathcal{T}_{n-1}} m^{-n} \sum_{0 \leqq i < Z^\sigma} f(W_\sigma) W_\sigma$$

$$= \mathbb{E}\left\{ \sum_{\sigma \in \mathcal{T}_{n-1}} m^{-n} m\, \mathbb{E}[f(W)W] \right\} = \mathbb{E} W f(W)\,. \qquad \square$$

The following result concerns with the density of the measure μ_ω.

Proposition 3.1 *Let $\theta \in (0,\infty)$. (i) If $\mathbb{E}\left(e^{rW^\theta}\right) < \infty$ for some $r \in (0,\infty)$, then*

$$\limsup_{n\to\infty} \frac{m^n \mu_\omega[B(\mathbf{i}|n)]}{(\log n)^{1/\theta}} \leqq r^{-1/\theta} \quad \text{for } P\text{-a.e. } \omega \text{ and } \mu_\omega\text{-a.e. } \mathbf{i}\,, \tag{3.4a}$$

or equivalently,

$$\limsup_{n\to\infty} \frac{\mu_\omega[B(\mathbf{i}|n)]}{\phi_{1/\theta}(|B(\mathbf{i}|n)|)} \leqq r^{-1/\theta} \quad \text{for } P\text{-a.e. } \omega \text{ and } \mu_\omega\text{-a.e. } \mathbf{i}\,. \tag{3.4b}$$

(ii) *If $\mathbb{E}(W^{\theta+1}) < \infty$, then*

$$\lim_{n\to\infty} \frac{m^n \mu_\omega[B(\mathbf{i}|n)]}{n^{1/\theta}} = 0 \quad \text{for } P\text{-a.e. } \omega \text{ and } \mu_\omega\text{-a.e. } \mathbf{i}\,, \tag{3.5a}$$

or equivalently,

$$\lim_{n\to\infty} \frac{\mu_\omega[B(\mathbf{i}|n)]}{\psi_{1/\theta}(|B(\mathbf{i}|n)|)} = 0 \quad \text{for } P\text{-a.e. } \omega \text{ and } \mu_\omega\text{-a.e. } \mathbf{i}\,. \tag{3.5b}$$

Proof. (i) It is easily seen that both (3.4a) and (3.4b) are equivalent to

$$\limsup_{n\to\infty} W_{\mathbf{i}|n}^{\theta}/\log n \leqq r^{-1} \quad Q\text{-a.e.} \tag{3.6}$$

By Lemma 3.1, for all $\varepsilon > 0$,

$$Q\left(e^{rW_{\mathbf{i}|n}^\theta} \geqq n^{1+\varepsilon}\right) = \mathbb{E}[W 1_{\{e^{rW^\theta} \geqq n^{1+\varepsilon}\}}]\,,$$

and consequently

$$\sum_{n=1}^{\infty} Q\left(e^{rW_{\mathbf{i}|n}^{\theta}} \geqq n^{1+\varepsilon}\right) = O\left\{\mathbb{E}\left(We^{rW^{\theta}/(1+\varepsilon)}\right)\right\} < \infty ,$$

where the last step holds since $\mathbb{E}(e^{rW^{\theta}}) < \infty$. So by the Borel–Cantelli lemma,

$$\limsup_{n\to\infty} \frac{rW_{\mathbf{i}|n}^{\theta}}{\log n} \leqq (1+\varepsilon) \quad Q\text{-a.e.}$$

This gives (3.6).

(ii) In the same way as shown above, both (3.5a) and (3.5b) are equivalent to

$$\lim_{n\to\infty} \frac{W_{\mathbf{i}|n}^{\theta}}{n} = 0 \quad Q\text{-a.e.} \tag{3.7}$$

and (3.7) also follows from the Borel–Cantelli lemma since for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} Q(W_{\mathbf{i}|n}^{\theta} \geqq n\varepsilon) = \sum_{n=1}^{\infty} \mathbb{E}(W1_{\{W^{\theta} \geqq n\varepsilon\}}) = O\left\{\frac{1}{\varepsilon}\,\mathbb{E}[W^{1+\theta}]\right\} < \infty . \qquad \square$$

3.2 *The lower bound*

Proposition 3.2 *Let* $\theta \in (0,\infty)$. (i) *If* $\mathbb{E}(e^{rW^{\theta}}) < \infty$ *for some* $r \in (0,\infty)$, *then*

$$\bar{\mathscr{H}}^{\phi_{1/\theta}}(\tilde{\mathscr{T}}) \geqq r^{1/\theta} W \quad a.s.$$

(ii) *If* $\mathbb{E}(W^{\theta+1}) < \infty$ *then*

$$\bar{\mathscr{H}}^{\psi_{1/\theta}}(\tilde{\mathscr{T}}) = \infty \quad a.s. \text{ on } W > 0 .$$

Proof. (i) We recall that $\mu_\omega(\tilde{\mathscr{T}}(\omega)) = W$ a.s. By Proposition 3.1(i), for each $\varepsilon > 0$ and almost all $\omega \in \Omega$, we can choose a compact subset $K = K(\omega)$ of $\tilde{\mathscr{T}}$ and an integer $N_0 = N_0(\omega)$ such that $\mu_\omega(K) \geqq W - \varepsilon$ and that for all $\mathbf{i} \in K$ and all $n \geqq N_0$,

$$\mu_\omega[B(\mathbf{i}|n)] \leqq (1+\varepsilon) r^{-1/\theta} \phi_{1/\theta}(|B(\mathbf{i}|n)|) .$$

This means that for almost all $\omega \in \Omega$ and all $n \geqq N_0$,

$$\mu_\omega(B(\mathbf{i}|n) \cap K) \leqq (1+\varepsilon) r^{-1/\theta} \phi_{1/\theta}(|B(\mathbf{i}|n)|) .$$

Let (S_j) be any cover of K by balls with diameter $|S_j| \leqq 2^{-N_0}$. Then

$$\mu_\omega(S_j \cap K) \leqq (1+\varepsilon) r^{-1/\theta} \phi_{1/\theta}(|S_j|) .$$

Hence

$$W - \varepsilon \leqq \mu_\omega(K) \leqq \mu_\omega\left[\bigcup_j (S_j \cap K)\right] \leqq \sum_j \mu_\omega(S_j \cap K) \leqq (1+\varepsilon) r^{-1/\theta} \sum_j \phi_{1/\theta}(|S_j|) .$$

This implies that, for all $\varepsilon > 0$ and almost all ω,

$$\bar{\mathscr{H}}^{\phi_{1/\theta}}(\tilde{\mathscr{T}}) \geqq \bar{\mathscr{H}}^{\phi_{1/\theta}}(K) \geqq \frac{1}{1+\varepsilon} r^{1/\theta}(W - \varepsilon) .$$

Letting $\varepsilon \to 0$ gives the result desired.

(ii) The same idea as above: by Proposition 3.1(ii), for each $\eta > 0$ and almost all $\omega \in \Omega$, we can choose a compact subset $K' = K'(\omega)$ of $\tilde{\mathscr{T}}$ and an integer $N_0 = N_0(\omega)$ such that $\mu_\omega(K') \geqq W - \eta$ and, for all $\mathbf{i} \in K'$ and $n \geqq N_0$,

$$\mu_\omega[B(\mathbf{i}|n) \cap K'] \leqq \eta \psi_{1/\theta}(|B(\mathbf{i}|n)|) .$$

Thus for any cover $\{S_j\}$ of balls of K' with $|S_j| \leqq 2^{-N_0}$,

$$W - \eta \leqq \mu_\omega(K') \leqq \mu_\omega \left[\bigcup_j (S_j \cap K')\right] \leqq \sum_j \mu_\omega(S_j \cap K') \leqq \eta \sum_j \psi_{1/\theta}(|S_j|) .$$

So $\bar{\mathscr{H}}^{\psi_{1/\theta}}(\tilde{\mathscr{T}}) \geqq \bar{\mathscr{H}}^{\psi_{1/\theta}}(K) \geqq \frac{1}{\eta}(W - \eta)$. Letting $\eta \to 0$, we see that $\bar{\mathscr{H}}^{\psi_{1/\theta}}(\tilde{\mathscr{T}}) = \infty$ a.s. on $W > 0$. □

3.3 The upper bound

Lemma 3.2 *Suppose that* g: $\mathbb{R} \to [0, 1]$ *is a non-increasing function with* $\int_1^\infty g(t)\,dt = +\infty$ *and that* $j : \mathbb{N} \to \mathbb{R}$ *is a function satisfying* $\limsup_{k\to\infty} \frac{j(k)}{k} < 1$, *then for all* $\varepsilon > 0$ *and all* $\bar{\varepsilon} \in (0, \varepsilon)$,

$$\limsup_{k\to\infty} \left\{ \int_{j(k)^{1/(1+\varepsilon)}}^{k^{1/(1+\varepsilon)}} g(t) t^\varepsilon\,dt - k^{\bar{\varepsilon}/(1+\varepsilon)} \right\} = +\infty . \tag{3.8}$$

Proof. Since

$$\infty = \int_1^\infty g(t)\,dt = \sum_{k=1}^\infty \int_k^{k+1} g(t)\,dt \leqq \sum_{k=1}^\infty g(k) ,$$

for all $\varepsilon' > 0$, we can choose an increasing sequence $(k_v)_{v\in\mathbb{N}}$ of integers such that, for all $v = 1, 2, \ldots$

$$g(k_v) \geqq k_v^{-(1+\varepsilon')} .$$

For each k_v, choose $K_v \in \mathbb{N}$ such that $(k_v - 1)^{1+\varepsilon} < K_v \leqq k_v^{1+\varepsilon}$. This is possible since $k_v^{1+\varepsilon} - (k_v - 1)^{1+\varepsilon} \geqq 1$. Therefore, choosing $\varepsilon' \in (0, \varepsilon - \bar{\varepsilon})$, we obtain that

$$\begin{aligned}
&\limsup_{k\to\infty} \left\{ \int_{[j(k)]^{1/(1+\varepsilon)}}^{k^{1/(1+\varepsilon)}} g(t) t^\varepsilon\,dt - k^{\bar{\varepsilon}/(1+\varepsilon)} \right\} \\
&\geqq \limsup_{v\to\infty} \left\{ \int_{[j(K_v)]^{1/(1+\varepsilon)}}^{K_v^{1/(1+\varepsilon)}} g(t) t^\varepsilon\,dt - K_v^{\bar{\varepsilon}/(1+\varepsilon)} \right\} \\
&\geqq \limsup_{v\to\infty} \left\{ \frac{1}{1+\varepsilon} g(k_v)[K_v - j(K_v)] - K_v^{\bar{\varepsilon}/(1+\varepsilon)} \right\} \\
&\geqq \limsup_{v\to\infty} \left\{ \frac{1}{1+\varepsilon} k_v^{-(1+\varepsilon')} K_v[1 - j(K_v)/K_v] - K_v^{\bar{\varepsilon}/(1+\varepsilon)} \right\}
\end{aligned}$$

$$\geqq \limsup_{v\to\infty} \left\{ \frac{1}{1+\varepsilon} k_v^{-(1+\varepsilon')}(k_v - 1)^{1+\varepsilon}[1 - j(K_v)/K_v] - k_v^{\bar{\varepsilon}} \right\}$$

$$= \limsup_{v\to\infty} k_v^{\bar{\varepsilon}} \left\{ \frac{1}{1+\varepsilon} k_v^{\varepsilon-\varepsilon'-\bar{\varepsilon}}(1 - 1/k_v)^{1+\varepsilon}[1 - j(K_v)/K_v] - 1 \right\} = +\infty . \qquad \square$$

Lemma 3.3 *For $x \in \mathbb{R}$, let $[x]$ be the integral part of x; for $\sigma \in T$ and $\omega \in \Omega$, let*

$$\ell_\sigma = \ell_\sigma(\omega) = 2^{-|\sigma|\alpha} 1_{\{\sigma \in \mathcal{T}\}}(\omega) .$$

(i) Fix $\theta \in (0, \infty)$. *For $t > 0$, $k \in \mathbb{N}_+$ and $\omega \in \Omega$, write*

$$B_k^* = B_k^*(\theta, t) = \left\{ \sigma \in \mathbb{N}^k : \ W_{(\sigma|v)^*}(\omega) < \left(\frac{1}{t} \log\log \frac{1}{2^{-v}} \right)^\theta \right.$$
$$\left. \text{for all } v = [\log k], \ [\log k] + 1, \dots, k \right\}$$

[recall that $\tau^* = (\tau_1, \dots, \tau_{n-1}, \tau_n + 1)$ if $\tau = (\tau_1, \dots, \tau_n)$] and

$$I_k^* = I_k^*(\theta, t) = \int_\Omega \sum_{\sigma \in B_k^*} \ell_\sigma(\omega) \left(\log\log \frac{1}{2^{-k}} \right)^\theta dP .$$

If for some $r \in (0, \infty)$, $\mathbb{E}(e^{rW^{1/\theta}}) = \infty$, then for all $t > r$,

$$\liminf_{k\to\infty} I_k^* = 0 .$$

(ii) *For all $\theta \in (0, \infty)$, put*

$$\bar{B}_k^* = \bar{B}_k^*(\theta) = \left\{ \sigma \in \mathbb{N}^k : \ W_{(\sigma|v)^*}(\omega) < \left(\log \frac{1}{2^{-v}} \right)^\theta \right.$$
$$\left. \text{for all } v = [\log k], \ [\log k] + 1, \dots, k \right\}$$

and

$$\bar{I}_k^*(\theta) = \int_\Omega \sum_{\sigma \in \bar{B}_k^*} \ell_\sigma(\omega) \left(\log \frac{1}{2^{-k}} \right)^\theta dP .$$

If

$$\limsup_{k\to\infty} \left\{ \sum_{v=[\log k]}^{k} P[W^{1/\theta} \geqq v] - \theta \log k \right\} > -\infty \tag{3.9}$$

then

$$\liminf_{k\to\infty} \bar{I}_k^*(\theta) < +\infty .$$

In particular, if $\mathbb{E}(W^{1/\theta}) = \infty$, then for all $\varepsilon \in (0, \theta)$,

$$\liminf_{k\to\infty} \bar{I}_k^*(\theta - \varepsilon) = 0 .$$

Proof. (i) We first remark that

$$I_k^* = \sum_{\sigma\in\mathbb{N}^k} \mathbb{E}\left[\ell_\sigma\left(\log\log\frac{1}{2^{-|\sigma|}}\right)^\theta 1_{\cap_{v=[\log k]}^k\{W_{(\sigma|v)^*}<(\frac{1}{t}\log\log\frac{1}{2^{-v}})^\theta\}}\right].$$

Recall that $W_{(\sigma|v)^*}([\log k] \leqq v \leqq k)$ are independent and identically distributed as W, which are also independent of $\{Z^{(\sigma|i)}: 0 \leqq i \leqq |\sigma|\}$, so writing the indicator function of the intersection set as product of indicator functions of sets and first taking conditional expectations given $\{Z^{(\sigma|i)}: 0 \leqq i \leqq |\sigma|\}$, we see that

$$\begin{aligned}
I_k^* &= \sum_{\sigma\in\mathbb{N}^k} \mathbb{E}\left[\ell_\sigma\left(\log\log\frac{1}{2^{-|\sigma|}}\right)^\theta \prod_{v=[\log k]}^k P\left\{W<\left(\frac{1}{t}\log\log\frac{1}{2^{-v}}\right)^\theta\right\}\right] \\
&= \left(\log\log\frac{1}{2^{-k}}\right)^\theta \mathbb{E}\left[\sum_{\sigma\in\mathbb{N}^k}\ell_\sigma\right] \prod_{v=[\log k]}^k P\left\{W<\left(\frac{1}{t}\log\log\frac{1}{2^{-v}}\right)^\theta\right\} \\
&= \left(\log\log\frac{1}{2^{-k}}\right)^\theta \prod_{v=[\log k]}^k P\left\{W<\left(\frac{1}{t}\log\log\frac{1}{2^{-v}}\right)^\theta\right\} \\
&\leqq \left(\log\log\frac{1}{2^{-k}}\right)^\theta \exp\left\{-\sum_{v=[\log k]}^k P\left\{W\geqq\left(\frac{1}{t}\log\log\frac{1}{2^{-v}}\right)^\theta\right\}\right\} \\
&\leqq (\log k)^\theta \exp\left\{-\sum_{v=[\log k]}^k P\left[W\geqq\left(\frac{1}{t}\log v\right)^\theta\right]\right\}.
\end{aligned}$$

Therefore,

$$I_k^* \leqq \exp\left\{-\sum_{v=[\log k]}^k P\left[e^{tW^{1/\theta}}\geqq v\right]+\theta\log\log k\right\}. \tag{3.10}$$

Let $\varepsilon\in\mathbb{R}$ be determined by $t/(1+\varepsilon)=r$. Then $\varepsilon>0$ (since $t>r$) and

$$\begin{aligned}
\sum_{v=[\log k]}^k P\left[e^{tW^{1/\theta}}\geqq v\right] &\geqq \sum_{v=[\log k]}^{k-1}\int_v^{v+1} P\left[e^{tW^{1/\theta}}\geqq x\right]dx \\
&= \int_{[\log k]}^k P[e^{tW^{1/\theta}}\geqq x]\,dx \\
&= (1+\varepsilon)\int_{[\log k]^{1/(1+\varepsilon)}}^{k^{1/(1+\varepsilon)}} f(y)y^\varepsilon\,dy\,,
\end{aligned} \tag{3.11}$$

where

$$f(y)=P\left[e^{rW^{1/\theta}}\geqq y\right]$$

and the last step holds by changing variables $x=y^{1+\varepsilon}$. Since

$$\int_1^\infty f(y)\,dy=\int_0^\infty f(y)\,dy+O(1)=\mathbb{E}\left[e^{rW^{1/\theta}}\right]+O(1)=+\infty\,,$$

we obtain by Lemma 3.2 that, for $\bar{\varepsilon} = \varepsilon/2$,

$$\limsup_{k\to\infty} (1+\varepsilon)\left\{\int_{[\log k]^{1/(1+\varepsilon)}}^{k^{1/(1+\varepsilon)}} f(y)y^{\varepsilon}\,dy - k^{\bar{\varepsilon}/(1+\varepsilon)}\right\} = +\infty\,.$$

As $\theta \log\log k \leqq (1+\varepsilon)k^{\bar{\varepsilon}/(1+\varepsilon)}$ for sufficiently large k, it follows from (3.11) that

$$\limsup_{k\to\infty}\left\{\sum_{v=[\log k]}^{k} P\left[e^{tW^{1/\theta}} \geqq v\right] - \theta\log\log k\right\} = +\infty\,.$$

Therefore, by (3.10), $\liminf_{k\to\infty} I_k^* = 0$. This ends the proof of part (i).

(ii) For part (ii), the same argument as above shows that

$$\bar{I}_k^*(\theta) \leqq \exp\left\{-\sum_{v=[\log k]}^{k} P\left[W^{1/\theta} \geqq v\right] + \theta\log k\right\}\,.$$

So the first conclusion follows immediately; to show the second, using the above inequality for $\theta - \varepsilon$, we need only to prove that for all $\varepsilon > 0$

$$\limsup_{k\to\infty}\left\{\sum_{v=[\log k]}^{k} P\left[W^{1/(\theta-\varepsilon)} \geqq v\right] - (\theta-\varepsilon)\log k\right\} = +\infty\,. \tag{3.12}$$

As

$$\begin{aligned}
&\sum_{v=[\log k]}^{k} P[W^{1/(\theta-\varepsilon)} \geqq v] \\
&\geqq \sum_{v=[\log k]}^{k-1} \int_{v}^{v+1} P[W^{1/(\theta-\varepsilon)} \geqq x]\,dx \\
&= \int_{[\log k]}^{k} P[W^{1/(\theta-\varepsilon)} \geqq x]\,dx = \int_{[\log k]}^{k} P[W^{1/\theta} \geqq x^{1-\varepsilon/\theta}]\,dx\,,
\end{aligned}$$

by changing variables $t = x^{1-\varepsilon/\theta}$ we see that

$$\sum_{v=[\log k]}^{k} P[W^{1/(\theta-\varepsilon)} \geqq v] \geqq (1+\varepsilon')\int_{[\log k]^{1/(1+\varepsilon')}}^{k^{1/(1+\varepsilon')}} g(t)t^{\varepsilon'}\,dt\,, \tag{3.13}$$

where $\varepsilon' = \varepsilon/(\theta-\varepsilon)$ and $g(t) = P(W^{1/\theta} \geqq t)$. Since

$$\int_1^{\infty} g(y)\,dy = \int_0^{\infty} g(y)\,dy + O(1) = \mathbb{E}[W^{1/\theta}] + O(1) = +\infty\,,$$

we obtain by Lemma 3.2 (with $j(x) = [\log x]$) that for $\bar{\varepsilon} = \varepsilon'/2$,

$$\limsup_{k\to\infty}\left\{\int_{[\log k]^{1/(1+\varepsilon')}}^{k^{1/(1+\varepsilon')}} g(t)t^{\varepsilon'}\,dt - k^{\bar{\varepsilon}/(1+\varepsilon')}\right\} = +\infty\,. \tag{3.14}$$

As $(\theta-\varepsilon)\log k \leqq k^{\bar{\varepsilon}/(1+\varepsilon')}$ for all sufficiently large k, (3.12) follows from (3.13) and (3.14). This ends the proof of the lemma. □

Proposition 3.3 *Fix* $\theta \in (0,\infty)$. (i) *If for some* $r \in (0,\infty)$, $\mathbb{E}[e^{rW^{1/\theta}}] = +\infty$, *then*

$$\mathbb{E}[\bar{\mathscr{H}}^{\phi_\theta}(\tilde{\mathscr{T}})] \leqq r^\theta . \tag{3.15}$$

(ii) *If* $\mathbb{E}(W^{1/\theta}) = +\infty$, *then for all* $\varepsilon > 0$,

$$\bar{\mathscr{H}}^{\psi_{\theta-\varepsilon}}(\tilde{\mathscr{T}}) = 0 \quad a.s. \tag{3.16}$$

Moreover, if (3.9) *holds, then* $\mathbb{E}[\bar{\mathscr{H}}^{\psi_\theta}(\tilde{\mathscr{T}})] < \infty$ [*and so* $\bar{\mathscr{H}}^{\psi_\theta}(\tilde{\mathscr{T}}) < \infty$ *a.s.*].

Proof. (i) Fix $t > r$. For $\omega \in \Omega$, $k_0 \in \mathbb{N}$ and $\delta = 2^{-[\log k_0]}$, define

$$B^* = B^*(\omega) = \left\{\sigma \in \mathbb{N}^{\mathbb{N}_+}\colon W_{(\sigma|k)^*}(\omega) < \left(\frac{1}{t}\log\log\frac{1}{2^{-k}}\right)^\theta \right.$$
$$\left. \text{for all } k = [\log k_0],\ [\log k_0]+1,\ldots,k_0\right\} . \tag{3.17a}$$

If $\sigma \in \mathbb{N}^{\mathbb{N}} - B^*$, let $k(\sigma)$ be the smallest $k \geqq [\log k_0]$ such that

$$W_{(\sigma|k)^*}(\omega) \geqq \left(\frac{1}{t}\log\log\frac{1}{2^{-k}}\right)^\theta \tag{3.17b}$$

and set

$$\tilde{\Gamma}(k_0) = \{\sigma|k(\sigma)\colon \sigma \in \mathbb{N}^{\mathbb{N}} - B^*\} . \tag{3.17c}$$

Then $[\log k_0] \leqq k(\sigma) \leqq k_0$; $\tilde{\Gamma}(k_0)$ is an antichain and so there exists a maximal antichain $\Gamma(k_0)$ such that $\tilde{\Gamma}(k_0) \subseteq \Gamma(k_0)$. If $\sigma \in B^*$, then $\sigma|k_0 \in B^*_{k_0}$ with $B^*_{k_0} = B^*_{k_0}(\omega)$ defined in Lemma 3.3(i). Thus

$$\tilde{\mathscr{T}}(\omega) \subseteq \left(\bigcup_{\sigma\in\tilde{\Gamma}(k_0)} D_\sigma(\omega)\right) \cup \left(\bigcup_{\sigma\in B^*_{k_0}} D_\sigma(\omega)\right) , \tag{3.18a}$$

where

$$D_\sigma(\omega) = \{\tau \in \tilde{\mathscr{T}}(\omega)\colon \tau > \sigma\} = B(\sigma) \cap \tilde{\mathscr{T}}(\omega)$$

denotes the closed descendants of σ. Notice that $D_\sigma(\omega) = \emptyset$ if $\sigma \notin \mathscr{T}$ and $|D_\sigma| \leqq |B(\sigma)| = 2^{-|\sigma|}$ if $\sigma \in \mathscr{T}$, we obtain by (3.18a) that

$$\begin{aligned}\bar{\mathscr{H}}^{\phi_\theta}_\delta(\tilde{\mathscr{T}}(\omega)) &\leqq \sum_{\sigma\in\tilde{\Gamma}(k_0),\sigma\in\mathscr{T}} \phi_\theta(|D_\sigma(\omega)|) + \sum_{\sigma\in B^*_{k_0},\sigma\in\mathscr{T}} \phi_\theta(|D_\sigma(\omega)|)\\ &\leqq \sum_{\sigma\in\tilde{\Gamma}(k_0)} \ell_\sigma \left(\log\log\frac{1}{2^{-|\sigma|}}\right)^\theta + \sum_{\sigma\in B^*_{k_0}} \ell_\sigma \left(\log\log\frac{1}{2^{-|\sigma|}}\right)^\theta .\end{aligned} \tag{3.18b}$$

By Lemma 3.3(i), we can choose a sequence (k_i) of integers increasing to $+\infty$ such that

$$\lim_{i\to\infty} \mathbb{E}\left[\sum_{\sigma\in B^*_{k_i}} \ell_\sigma \left(\log\log\frac{1}{2^{-|\sigma|}}\right)^\theta\right] = 0 . \tag{3.19}$$

Using (3.18b) for $k_0 = k_i$ and $\delta = \delta_i = 2^{-[\log k_i]}$ and taking into account of (3.19), we see that

$$\liminf_{i\to\infty} \mathbb{E}[\bar{\mathscr{H}}^{\phi_\theta}_{\delta_i}(\tilde{\mathscr{T}}(\omega))] \leqq \liminf_{i\to\infty} \mathbb{E}\left[\sum_{\sigma\in\tilde{\Gamma}(k_i)} \ell_\sigma \left(\log\log\frac{1}{2^{-|\sigma|}}\right)^\theta\right]$$

$$\leqq \liminf_{i\to\infty} \mathbb{E}\left[\sum_{\sigma\in\tilde{\Gamma}(k_i)} \ell_\sigma t^\theta W_{\sigma^*}\right], \tag{3.20}$$

where the last step holds by the definition of $\tilde{\Gamma}(k_i)$ [cf. (3.17b) and (3.17c)]. First calculating conditional expectations given $\{Z^\sigma\colon |\sigma| \leqq k_i\}$, we obtain that

$$\mathbb{E}\left[\sum_{\sigma\in\tilde{\Gamma}(k_i)} \ell_\sigma W_{\sigma^*}\right] = \mathbb{E}\left[\sum_{\sigma\in\tilde{\Gamma}(k_i)} \ell_\sigma\right] = \mathbb{E}\left[\sum_{\sigma\in\tilde{\Gamma}(k_i)} \ell_\sigma W_\sigma\right]$$

$$\leqq \mathbb{E}\left[\sum_{\sigma\in\Gamma(k_i)} \ell_\sigma W_\sigma\right] = \mathbb{E}[W] = 1\,, \tag{3.21}$$

where the penultimate equality holds because, with probability 1, $W = \sum_{\sigma\in\Gamma} \ell_\sigma W_\sigma$ for any maximal antichain Γ [cf. (3.1)]. Therefore, by (3.20) and (3.21),

$$\liminf_{i\to\infty} \mathbb{E}[\bar{\mathscr{H}}^{\phi}_{\delta_i}(\tilde{\mathscr{T}}(\omega))] \leqq t^\theta\,. \tag{3.22}$$

Since $\bar{\mathscr{H}}^{\phi_\theta}_{\delta_i}(\tilde{\mathscr{T}}(\omega))$ increases as δ_i decreases, this implies that

$$\mathbb{E}[\bar{\mathscr{H}}^{\phi_\theta}(\tilde{\mathscr{T}}(\omega))] \leqq t^\theta\,.$$

As $t > r$ is arbitrary, we obtain $\mathbb{E}[\bar{\mathscr{H}}^{\phi_\theta}(\tilde{\mathscr{T}}(\omega))] \leqq r^\theta$, the desired conclusion in part (i) of the proposition.

(ii) The proof of part (ii) is similar. Assume (3.9) and let $\bar{I}^*_k(\theta)$ be defined as in Lemma 3.3(ii). Then by that lemma,

$$I \equiv I(\theta) := \liminf_{k\to\infty} \bar{I}^*_k(\theta) < +\infty\,.$$

Instead of B^*, we define, for $k_0 \in \mathbb{N}$ and $\omega \in \Omega$,

$$\bar{B}^* = \bar{B}^*(\omega) = \left\{ \sigma \in \mathbb{N}^{\mathbb{N}}\colon W_{(\sigma|k)^*}(\omega) < \left(\log\frac{1}{2^{-k}}\right)^\theta \right.$$

$$\left. \text{for all } k = [\log k_0],\ [\log k_0]+1,\ldots,k_0 \right\}\,.$$

For $\sigma \in \mathbb{N}^{\mathbb{N}} - \bar{B}^*$, let $\bar{k}(\sigma)$ be the smallest $k \geqq [\log k_0]$ such that

$$W_{(\sigma|k)^*}(\omega) \geqq \left(\log\frac{1}{2^{-k}}\right)^\theta$$

and set

$$\tilde{\Gamma}'(k_0) = \{\sigma|\bar{k}(\sigma)\colon \sigma \in \mathbb{N}^{\mathbb{N}} - \bar{B}^*\}\,.$$

Then $\tilde{\Gamma}'(k_0)$ is an antichain and there exists a maximal antichain $\Gamma'(k_0)$ such that $\tilde{\Gamma}'(k_0) \subseteq \Gamma'(k_0)$; if $\sigma \in \bar{B}^*$, then $\sigma|k_0 \in \bar{B}^*_{k_0}$, where $\bar{B}^*_{k_0} = \bar{B}^*_{k_0}(\omega)$ is defined in Lemma 3.3(ii). Let $\{k_i\colon i \geqq 1\}$ be a sequence of positive integers increasing to $+\infty$, such that

$$I = \lim_{i\to\infty} \bar{I}^*_{k_i}(\theta)\,.$$

The rest of the proof is exactly the same as in the proof of part (i): instead of (3.18a), (3.20) and (3.22) we have respectively

$$\tilde{\mathscr{T}}(\omega) \subseteq \left(\bigcup_{\sigma\in\tilde{\Gamma}'} D_\sigma(\omega)\right) \cup \left(\bigcup_{\sigma\in\bar{B}^*_{k_0}(\omega)} D_\sigma(\omega)\right),$$

$$\liminf_{i\to\infty} \mathbb{E}[\bar{\mathscr{H}}^{\psi_\theta}_{\delta_i}(\tilde{\mathscr{T}}(\omega))] \leqq \liminf_{i\to\infty} \mathbb{E}\left[\sum_{\sigma\in\tilde{\Gamma}'(k_i)} 1_\sigma W_{\sigma^*}\right] + I$$

and

$$\liminf_{i\to\infty} \mathbb{E}[\bar{\mathscr{H}}^{\psi_\theta}_{\delta_i}(\tilde{\mathscr{T}}(\omega))] \leqq 1 + I\,.$$

Therefore,

$$\mathbb{E}[\bar{\mathscr{H}}^{\psi_\theta}(\tilde{\mathscr{T}}(\omega))] \leqq 1 + I$$

and so $\bar{\mathscr{H}}^{\psi_\theta}(\tilde{\mathscr{T}}(\omega)) < +\infty$ a.s. This establishes the second assertion of part (ii). To see the first assertion, we note that if $\mathbb{E}(W^{1/\theta}) = \infty$, then by Lemma 3.3(ii),

$$I(\theta - \varepsilon) = 0 \quad \text{for all } \varepsilon > 0\,.$$

Thus by the preceding argument,

$$\bar{\mathscr{H}}^{\psi_{\theta-\varepsilon}}(\tilde{\mathscr{T}}(\omega)) < +\infty \quad \text{a.s.}$$

Since the result also holds for $\varepsilon/2$, we see that $\bar{\mathscr{H}}^{\psi_{\theta-\varepsilon}}(\tilde{\mathscr{T}}(\omega)) = 0$ a.s. So the proof is completed. □

3.4 Some proofs

Our theorems in Sect. 1 can be easily deduced from Propositions 3.2 and 3.3 and the theorems in Sect. 2. We recall that the event "$\tilde{\mathscr{T}} \neq \emptyset$" coincides with "$W > 0$" a.s. under the condition ($Z \log Z$), and $\mathscr{H}^f(\,\cdot\,) = \bar{\mathscr{H}}^f(\,\cdot\,)$ on I for any dimension function f.

Proof of Theorem 1 If $0 < r < r_{1/\theta}$, then $\mathbb{E}[e^{rW^{1/\theta}}] < \infty$ and, by Proposition 3.2(i), $\bar{\mathscr{H}}^{\phi_\theta}(\tilde{\mathscr{T}}) \geqq r^\theta W$ a.s. Hence,

$$\bar{\mathscr{H}}^{\phi_\theta}(\tilde{\mathscr{T}}) \geqq (r_{1/\theta})^\theta W \quad \text{a.s.} \tag{3.23}$$

If $r_{1/\theta} < r < \infty$, then $\mathbb{E}[e^{rW^{1/\theta}}] = \infty$ and, by Proposition 3.3(i), $\mathbb{E}[\bar{\mathscr{H}}^{\phi_\theta}(\tilde{\mathscr{T}})] \leqq r^\theta$. So

$$\mathbb{E}[\bar{\mathscr{H}}^{\phi_\theta}(\tilde{\mathscr{T}})] \leqq (r_{1/\theta})^\theta\,. \tag{3.24}$$

Therefore, if $0 < r_{1/\theta} < \infty$, then by (3.23) and (3.24), $\bar{\mathscr{H}}^{\phi_\theta}(\tilde{\mathscr{T}}) - (r_{1/\theta})^\theta W \geqq 0$ a.s. and $\mathbb{E}[\bar{\mathscr{H}}^{\phi_\theta}(\tilde{\mathscr{T}}) - (r_{1/\theta})^\theta W] \leqq 0$, so

$$\bar{\mathscr{H}}^{\phi_\theta}(\tilde{\mathscr{T}}) = (r_{1/\theta})^\theta W \quad \text{a.s ;} \tag{3.25}$$

if $r_{1/\theta} = 0$, then by (3.24), $\bar{\mathscr{H}}^{\phi_\theta}(\tilde{\mathscr{T}}) = 0$ a.s.; if $r_{1/\theta} = \infty$, then by (3.23), $\bar{\mathscr{H}}^{\phi_\theta}(\tilde{\mathscr{T}}) = \infty$ a.s. on $W > 0$. Therefore, (3.25) holds on $W > 0$ in all cases. □

Proof of Theorem 2 If $0 < \beta < 1$, then Z_1 is not a.s. a constant and, by Theorem 2.2, $0 < r_{1/\beta} < \infty$; so the result follows from Theorem 1. If $\beta = 0$, then with probability 1, $Z_1 = \mu$ and, for all $\sigma \in \mathscr{T}$, $W_\sigma = 1$ and

$$\mu_\omega[B(\sigma)] = m^{-|\sigma|} = 2^{-|\sigma|\alpha} = |B(\sigma)|^\alpha .$$

So it is easily seen that $\bar{\mathscr{H}}^\alpha(\tilde{\mathscr{T}}) = \mu_\omega(\tilde{\mathscr{T}}) = 1$ a.s. □

Theorem 3 follows immediately from Theorem 1 since, by Theorem 3.3, $r_{1/\theta} = 0$ for all $\theta < 1$ and, by Theorem 3.3, $r_1 = r(W)$ is positive, finite, or positive and finite if and only if the same is true for $r(Z)$.

Proof of Theorem 4 If $\theta > 1/(\gamma - 1)$, then $1 + 1/\theta < \gamma$, $\mathbb{E}[W^{1+1/\theta}] < \infty$ and so, by Proposition 3.2(ii), $\bar{\mathscr{H}}^{\psi_\theta}(\tilde{\mathscr{T}}) = \infty$ a.s. on $W > 0$. This gives the second conclusion of part (i). If $\gamma = \infty$, then the first conclusion of that part is interpreted as $\bar{\mathscr{H}}^{\psi_\theta}(\tilde{\mathscr{T}}) = 0$ if $\theta < 0$, which holds evidently since $\bar{\mathscr{H}}^\alpha(\tilde{\mathscr{T}}) < \infty$ [cf. Theorem 0(i)] and $\lim_{t\to 0} \psi_\theta(t)/t^\alpha = 0$ if $\theta < 0$. If $\gamma < \infty$ and $0 < \theta < 1/\gamma$, then we can choose $\theta' \in \mathbb{R}$ such that $\theta < \theta' < 1/\gamma$. So $\mathbb{E}[W^{1/\theta'}] < \infty$ and, by Proposition 3.3(ii), $\bar{\mathscr{H}}^{\psi_\theta}(\tilde{\mathscr{T}}) = 0$ a.s. This ends the proof of part (i). Parts (ii) and (iii) follow immediately from propositions 3.2(ii) and 3.3(ii) respectively. □

Remark 6 is easily seen by the proofs.

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References

1. Athreya, K.B., Ney, P.E.: Branching processes. Berlin, Heidelberg: Springer 1972
2. Besicovitch, A.S.: On the fundamental geometrical properties of linearly measurable plan sets of points. Math. Ann. **98**, 422–464 (1928)
3. Falconer, K.J.: The Geometry of Fractal Sets. Cambridge: Cambridge University Press 1985
4. Falconer, K.J.: Random fractals. Math. Proc. Camb. Phil. Soc. **100**, 559–582 (1986)
5. Falconer, K.J.: Cut set sums and tree processes. Proc. Amer. Math. Soc. **101**(2), 337–346 (1987)
6. Falconer, K.J.: Fractal Geometry. New York: Wiley 1990

7. Graf, S., Maudlin, R.D. Williams, S.C.: The exact Hausdorff dimension in random recursive constructions. Mem. Amer. Math. Soc. **71**(381) (1988)
8. Hawkes, J.: Trees generated by a simple branching process. J. London Math. Soc. **24**(2), 373–384 (1981)
9. Kahane, J.P., Peyrière, J.: Sur certaines martingales de B. Mandelbrot, Adv. Math. **22**, 131–145 (1976)
10. Liu, Q.: Sur quelques problèmes à propos des processus de branchement, des flots dans les réseaux et des mesures de Hausdorff associées. Thèse de doctorat de L'Université Paris 6, Laboratoire de Probabilités, Paris, 1993
11. Lyons, R.: Random walks and percolation on trees. Ann. Probab. **18**, 931–952 (1990)
12. Lyons, R., Pemantle, R.: Random walk in a random environment and first passage percolation on trees. Ann. Probab. **20**, 125–135 (1992)
13. Maudlin, R.D., Williams, S.C.: Random constructions, asymptotic geometric and topological properties. Trans. Amer. Math. Soc. **295**, 325–346 (1986)
14. Neveu, J.: Arbre et processus de Galton – Watson. Ann. Inst. Henri Poincaré **22**, 199–207 (1986)
15. Rogers, C.A.: Hausdorff Measures. Cambridge: Cambridge University Press 1970
16. Taylor, S.J.: The measure theory of random fractals. Math. Proc. Cambridge Philos. Soc. **100**, 383–406 (1986)