Probab. Theory Relat. Fields 104, 427–466 (1996)

Probability Theory and Related Fields

Surface order large deviations for Ising, Potts and percolation models

Agoston Pisztora★

Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, NY 10012, USA

Received: 19 December 1994/In revised form: 20 July 1995

Summary. We derive uniform surface order large deviation estimates for the block magnetization in finite volume Ising (or Potts) models with plus or free (or a combination of both) boundary conditions in the phase coexistence regime for $d \geqq 3$. The results are valid up to a limit of slab-thresholds, conjectured to agree with the critical temperature. Our arguments are based on the renormalization of the random cluster model with $q \geqq 1$ and $d \geqq 3$, and on corresponding large deviation estimates for the occurrence in a box of a largest cluster with density close to the percolation probability. The results are new even for the case of independent percolation ($q = 1$). As a byproduct of our methods, we obtain further results in the FK model concerning semicontinuity (in p and q) of the percolation probability, the second largest cluster in a box and the tail of the finite cluster size distribution.

Mathematics Subject Classification (*1991*):60F10, 60K35, 82B20, 82B43.

1 Introduction and statement of results

In this article we study the large deviation behavior at surface order of Ising, Potts and some percolation models. We derive large deviation estimates for finite volume (empirical) quantities, which converge to the order parameter of the underlying model as the volume goes to $\mathbb{Z}^d$. Our main object is to develop methods which allow the study of the large deviation behavior in the ordered (resp. percolative) phase up to (or close to) the critical point, for dimensions larger or equal to three. Preliminary versions of these results and methods can be found in [36], and a number of applications appear in [34, 5, 27].

For studying Ising and Potts models, the utility of the Fortuin–Kasteleyn (FK) representation [19] is widely acknowledged. This approach leads to the

★ *Present address*: Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA 02138, USA (e-mail: pisztora@math.harvard.edu)

analysis of an associated *dependent* percolation process, called FK percolation (or random cluster model). Since this model has features in common with ordinary percolation, like the FKG property, some of the techniques of percolation theory can be applied to it. The main point of this work is to transfer the 'slab-technology' of percolation theory to the Ising, Potts and random cluster models.

We begin with the presentation and discussion of the results for the Ising model. In the next subsection, we turn to FK and Bernoulli percolation.

1.1 Large deviations for the Ising model

Consider the d-dimensional Ising model for $d \geqq 2$ in a large box $B(n) = \mathbb{Z}^d \cap (-n/2, n/2]^d$ with ferromagnetic nearest neighbor interaction at inverse temperature β. In this model each site $x \in B(n)$ is assigned a spin variable σ_x, which can take the values $\sigma_x = \pm 1$. The energy of a spin configuration is given by

$$H(\sigma) = -\sum_{\{x,y\} \subseteq B(n)} (\delta_{\sigma_x,\sigma_y} - 1) J_{\{x,y\}}\,, \tag{1.1}$$

where $J_{\{x,y\}} = 1$ if x and y are nearest neighbors, and $J_{\{x,y\}} = 0$ otherwise. The corresponding (finite volume) Gibbs measures with *free* and *plus* boundary conditions are denoted by $\mu_{B(n)}^{f,\beta}$ and $\mu_{B(n)}^{+,\beta}$, respectively. The measures $\mu_{B(n)}^{+,\beta}$ converge weakly to the infinite volume probability measure $\mu_\infty^{+,\beta}$ as n tends to infinity. The order parameter for this model is the *spontaneous magnetization* $m^*(\beta) = \int \sigma_0 \, d\mu_\infty^{+,\beta}$. The *critical temperature*, given by $\beta_c = \inf\{\beta \geqq 0 \,|\, m^*(\beta) > 0\}$, satisfies $0 < \beta_c < \infty$ for $d \geqq 2$.

The classical large deviation theory for the Ising model (cf. [10, 15, 17, 34]) describes the asymptotic behavior of the magnetization $m_{B(n)} = n^{-d} \sum_{x \in B(n)} \sigma_x$ inside the box: the sequence $(m_{B(n)})_{n=1,2,\dots}$ satisfies a *volume order* large deviation principle with a rate function $I : [-1,1] \to \mathbb{R}$, which is zero on $[-m^*, m^*]$ and strictly positive outside this interval. Moreover, the function I is independent of boundary conditions.

In the subcritical regime ($\beta < \beta_c$), where m^* vanishes, we therefore have complete information about the large deviations of the magnetization. On the contrary, in the phase coexistence region ($\beta > \beta_c$), the classical result does not allow the control of the large deviations in the interval $(-m^*, m^*)$. In fact, it is easy to show (cf. [38, 18]) that for any a, b with $-m^* \leqq a < b \leqq m^*$, we have the following *surface order* lower bound:

$$-\infty < \varliminf_{n\to\infty} \frac{1}{n^{d-1}} \log \mu_{B(n)}(m_{B(n)} \in [a,b])\,, \tag{1.2}$$

where $\mu_{B(n)}$ denotes a finite volume Gibbs measure with arbitrary boundary conditions. Schonmann [38] gave a corresponding surface order upper bound for plus boundary conditions. He showed the existence of a threshold $0 < \beta_1(d) < \infty$ such that for any $\beta > \beta_1$ and for any a, b with $-1 \leqq a < b < m^*$,

$$\varlimsup_{n\to\infty} \frac{1}{n^{d-1}} \log \mu_{B(n)}^{+,\beta}(m_{B(n)} \in [a,b]) < 0\,. \tag{1.3}$$

He could extend this result (with $a, b \in (-m^*, m^*)$) to free boundary conditions only in two dimensions. Also, it turns out that $\beta_1(d)$ is strictly larger than $\beta_c(d)$, whenever the dimension is large enough (cf. [3]), and the same is believed to be true for $d \geqq 3$. The remaining questions were the following:

(i) Does (1.3) hold for all $\beta > \beta_c$?

(ii) Does (1.3) hold with free boundary conditions in dimensions $d \geqq 3$ for $a, b \in (-m^*, m^*)$?

(iii) Is the behavior of the magnetization in $(-m^*, m^*)$ governed by a surface order large deviation principle? If yes, what is the rate function?

The remarkable progress in the following years has been restricted to the two-dimensional Ising model. Chayes et al. [9] have shown that in two dimensions β_1 is actually equal to the critical inverse temperature β_c. An important breakthrough occurred when Dobrushin et al. [13] succeeded in establishing a surface order large deviation principle at low temperatures for $d = 2$, see also Pfister [36]. They identified the rate function as a certain expression involving the 'Wulff functional'. Recently, Ioffe [27, 28] completed the two-dimensional analysis by extending the exact large deviation bounds of [13] and [36] to the whole phase coexistence region.

Whereas these problems have been resolved in two dimension, basic questions remained open for $d \geqq 3$. One aim of this paper is to answer some of those questions, especially (i) and (ii).

In order to present our results, we need some additional notation. We first introduce a critical value $\widehat{\beta}_1 = \widehat{\beta}_1(d)$ for $d \geqq 3$, which can be regarded as a precise analogue of the 'slab-threshold' of percolation theory. The basic property of this threshold is the following: for $\beta > \widehat{\beta}_1$, we can find a number L, such that for every n large, in the finite slab $S(n, L) := [1, n]^{d-1} \times [1, L]$ long range order occurs, meaning that the spin-correlation (with respect to $\mu^{f,\beta}_{S(n,L)}$), between any two points in $S(n, L)$ 'far' away from the boundary, is bounded away from zero uniformly in n. The precise definition of $\widehat{\beta}_1$, with a discussion of its properties, is given in Sect. 3. It is generally believed that thresholds such as $\widehat{\beta}_1$ coincide with β_c for $d \geqq 3$.

Next, we would like to point out a difficulty, which arises due to the incomplete knowledge concerning the uniqueness of translation invariant Gibbs states in three or more dimensions. Let us consider the left-hand limit

$$m^{*,f}(\beta) := \lim_{\beta' \to \beta-} m^*(\beta'), \tag{1.4}$$

which we call 'free boundary spontaneous magnetization', since it turns out that $m^{*,f}$ is intimately related to free boundary conditions, as we will discuss below in the context of the random cluster model. Clearly, the limit in (1.4) exists, and $m^{*,f}$ is lower semi-continuous, since m^* is monotone increasing. For the same reason, $m^{*,f} \leq m^*$ with equality in all but the (at most countably many) discontinuity points of m^*. It is known that there are no such points for β large enough, and the same behavior is expected for every temperature. However, we must pay attention to the (hypothetical) gap between m^* and $m^{*,f}$, and in our results we must occasionally replace m^* by $m^{*,f}$.

Finally, we introduce the class of Gibbs measures with mixed plus and free boundary conditions. For $\Delta \subseteq \partial B(n)$ (the boundary of $B(n)$), we define the conditional measure

$$\mu_{B(n)}^{\Delta(+),\beta}[\,\cdot\,] = \mu_{B(n)}^{f,\beta}[\,\cdot\,|\sigma_x = +1 \text{ for every } x \in \Delta]\,, \tag{1.5}$$

and denote by $\mathscr{I}^+(\beta, B(n))$ the set of all such measures. The Gibbs measures with free and plus boundary conditions are contained in this class, since they correspond to the choice $\Delta = \emptyset$ and $\Delta = \partial B(n)$, respectively.

We are now ready to state our main result concerning the Ising model. It extends the upper bound (1.3) up to criticality (modulo the conjecture $\widehat{\beta}_1 = \beta_c$) and for a reasonably large class of boundary conditions.

Theorem 1.1 *Let $d \geqq 3$ and $\beta > \widehat{\beta}_1$. For every a, b with $-m^{*,f} < a < b < m^{*,f}$,*

$$\overline{\lim_{n\to\infty}} \frac{1}{n^{d-1}} \log \left(\sup_{\mu \in \mathscr{I}^+(\beta, B(n))} \mu[m_{B(n)} \in [a,b]] \right) < 0\,. \tag{1.6}$$

For plus boundary conditions, we have for every a, b with $-1 \leqq a < b < m^{*,f}$,

$$\overline{\lim_{n\to\infty}} \frac{1}{n^{d-1}} \log \mu_{B(n)}^{+,\beta}[m_{B(n)} \in [a,b]] < 0\,. \tag{1.7}$$

Remarks. 1). In Sect. 5, Theorem 5.4, we give the analogous result for the q-state Potts model. 2). (1.7) holds for boundary conditions more general than 'plus'. Let $\mu_{B(n)}^{\Delta_n(+),\beta}$ be a sequence of measures. It can be shown that the following property is sufficient (and also necessary) for the analogue of (1.7): $\underline{\lim}_{n\to\infty} n^{-d+1}|\Delta_n| > 0$. For reasons of space, we will not discuss this further.

1.2 Results for the random cluster process and percolation

The basic tool of our analysis is the Fortuin–Kasteleyn (FK) representation of the Ising model (cf. [20, 19] and the more recent works [4, 16, 23, 32, 33]) (see also Sect. 2). First we recall the definition of *random cluster (or FK) measures.* Consider independent bond percolation with parameter p in the box $B(n)$ and denote the corresponding probability measure by P^p. Let c stand for the (random) number of clusters of the process. For given $q \geqq 1$, the FK measure $\mathbf{\Phi}_{B(n)}^{f,p,q}$ with free boundary conditions is defined by its Radon–Nykodim density

$$\frac{d\mathbf{\Phi}_{B(n)}^{f,p,q}}{dP^p} = q^c / E^p[q^c]\,.$$

Note that for $q = 1$, this measure is just Bernoulli bond percolation. Suppose, we count the clusters in a slightly different way: for a given partition π of $\partial B(n)$, we *identify* sites in the same class, and regard their clusters to be *connected.* The (new) number of clusters in the box will be denoted by c^π, and the corresponding measure by $\mathbf{\Phi}_{B(n)}^{\pi,p,q}$. We refer to such measures as FK measures with general boundary conditions, and denote their set by $\mathscr{R}(p, q, B(n))$. Note that the free b.c. corresponds to the partition f defined to have exactly $|\partial B(n)|$ classes, each of them containing exactly one element of $\partial B(n)$. Let

w be the partition of $\partial B(n)$ with only one class (containing all elements of $\partial B(n)$); w is then called the *wired* boundary condition.

It is well-known that the finite volume FK measures with free and wired b.c.s weakly converge to infinite volume limits, which we denote by $\mathbf{\Phi}_\infty^{f,p,q}$ and $\mathbf{\Phi}_\infty^{w,p,q}$, respectively. The corresponding percolation probabilities are θ^f and θ^w. The critical parameter is defined by $p_c = p_c(q,d) := \inf\{p \geqq 0; \theta^w(p,q,d) > 0\}$. It is a direct consequence of the FK representation that by setting

$$p = 1 - e^{-\beta}\,, \tag{1.8}$$

we have the equality $m^*(\beta,d) = \theta^w(p,q=2,d)$, cf. (2.8) in [4], which implies at once that the critical values p_c and β_c are related by (1.8). It is less obvious and was only recently proved (cf. [30, 23] and the end of subsection 2.3) that $m^{*,f}(\beta,d) = \theta^f(p,q=2,d)$, except possibly at the critical point. Finally, for $d \geqq 3$, we introduce the critical value $\widehat{p}_1 = \widehat{p}_1(q,d)$ as the limit of (slab-) thresholds for uniform long range order in slab-systems with free boundary conditions; see (3.5) for the precise definition. It follows from the FK representation that $\widehat{p}_1(q=2)$ and $\widehat{\beta}_l$ satisfy the relation (1.8).

Let us now present our result concerning the random cluster model. (We will see below that Theorem 1.1 is actually a simple consequence of this result.) A cluster is called *crossing* if it intersects all $2d$ faces of $B(n)$. For $l \in \mathbb{N}$, we say that a cluster is *l-small* if its diameter does not exceed l; it is called *l-intermediate* if it is neither l-small nor has maximal size (among all clusters in $B(n)$). Let $\mathbb{S}_l$ and $\mathbb{J}_l$ stand for the set of l-small and l-intermediate clusters. Set

$$K(n,\varepsilon,l) = \Big\{ \exists!\text{ open cluster } C_m \text{ of maximal volume, } C_m \text{ is a crossing cluster,}$$
$$n^{-d}|C_m| \in (\theta^f - \varepsilon, \theta^w + \varepsilon),\ \sum_{C\in\mathbb{J}_l} |C| < \varepsilon n^d \Big\}\,.$$

Theorem 1.2 *Let $d \geqq 3$, $q \geqq 1, p > \widehat{p}_1$ and $\varepsilon \in (0, \theta^f/2)$ be fixed. Then there exists a constant $L = L(p,q,d,\varepsilon)$ such that*

$$-\infty < \varliminf_{n\to\infty} \frac{1}{n^{d-1}} \log\left(\inf_{\Phi\in\mathscr{R}(p,q,B(n))} \Phi[K(n,\varepsilon,L)^c]\right) \tag{1.9}$$

$$\leqq \varlimsup_{n\to\infty} \frac{1}{n^{d-1}} \log\left(\sup_{\Phi\in\mathscr{R}(p,q,B(n))} \Phi[K(n,\varepsilon,L)^c]\right) < 0\,. \tag{1.10}$$

Remark. Note that the theorem includes Bernoulli bond percolation ($q = 1$). The proof can easily be adapted to site percolation.

First, let us see how Theorem 1.2 can be used to prove Theorem 1.1. Suppose we want to study the magnetization in the Ising model with, e.g., free boundary conditions in $B(n)$ at some inverse temperature $\beta > \widehat{\beta}_1$. We first generate a bond configuration according to the random cluster measure

$\Phi_{B(n)}^{f,p,q=2}$, where p is given by (1.8). Typically, we get a configuration contained in $K(n,\varepsilon,L)$, since $p > \widehat{p}_{\mathrm{l}}$. In the second step we choose a spin for each cluster at random with probability 1/2 independently from the others. By the FK representation, the resulting spin configuration has the same distribution as in the Ising model. For ε small and n large enough, the total magnetization of the L-small clusters will be around zero. Thus, the absolute value of the magnetization in the whole box is around $n^{-d}|C_m|$, which is larger than $m^{*,f} - \varepsilon$. Because the number of the small clusters grows in volume order, it is easy to see that the large deviations of their magnetization are also of volume order. Clearly, they have no effect in large deviations of surface order. The remaining possibilities of having a magnetization in the interval $(-m^{*,f} + 2\varepsilon, m^{*,f} - 2\varepsilon)$ are contained in the event $K(n,\varepsilon,L)^c$. As we have seen, however, the probability of this event decays exponentially in surface order.

Next, we make some comments on the proof of Theorem 1.2. The basic technique is block renormalization, as used in percolation theory. The new feature is that we are able to control the *interaction* between the block-variables, including the neighboring ones. The key result is Proposition 4.1, which says that for large block-size, the process of 'good' (or occupied) blocks stochastically dominates independent high-density site percolation. This allows us to apply a result of Deuschel and Pisztora [11], which implies that, up to surface order large deviations, there exists a *cluster* $\mathbf{C}$ of good blocks with exceedingly high density. The events defining 'goodness' are chosen in such a way that the following properties hold:

(i) in each occupied block there is a unique 'big' cluster with an approximate density close to θ, and

(ii) if two neighboring blocks are both occupied then their 'big' clusters must be connected in the union of those blocks.

This implies that the (local) big clusters of the blocks lying in $\mathbf{C}$ build a big cluster C_m with a density not much smaller than θ. A slightly more careful argument shows that C_m has not an essentially larger density than θ and that every other cluster is either contained in exactly one block, which provides a bound on its diameter, or is contained in a region of negligible fractional volume.

We would like to emphasize that at this level of accuracy, the large deviation behavior described in Theorem 1.2 (and in Theorem 1.1) is determined by the large deviation behavior of the process of good blocks. In other words, the surface order large deviations estimate in [11] for high-density Bernoulli percolation implies (via renormalization) analogous estimates for the Ising model and (FK) percolation in the whole phase transition regime ($p > \widehat{p}_{\mathrm{l}}$).

Finally, we briefly review some related earlier results and compare them with Theorem 1.2. Consider Bernoulli percolation on the lattice $\mathbb{Z}^d$. The order parameter here is the percolation probability θ, which plays a role similar to that of the spontaneous magnetization in the Ising model. The block magnetization can be interpreted as an empirical quantity approaching m^*. An analogous quantity in percolation is the density of sites in $B(n)$ lying in an infinite cluster, i.e., $X_{B(n)} := n^{-d} \sum_{x \in B(n)} \mathbb{1}_{\{x \leftrightarrow \infty\}} = n^{-d}|C_\infty \cap B(n)|$, since the infinite cluster C_∞ is unique, whenever it exists. It is a consequence of the ergodic theorem that $X_{B(n)}$ converges to θ as n tends to infinity. As in the Ising model, the large deviations from this behavior are substantially different for the events $D_n^+ := \{X_{B(n)} \geqq \theta + \varepsilon\}$ and $D_n^- := \{X_{B(n)} \leqq \theta - \varepsilon\}$. By results of Lebowitz and Schonmann [31] and Durrett and Schonmann [14] it is known that the probabilities of D_n^+ decay exponentially with n^d in contrast to the surface order large deviations of the events D_n^-. This latter was proved for two dimensions in [14] and for higher dimensions by Gandolfi [21]. In the two-dimensional case, Alexander et al. [5] could identify the corresponding rate function by establishing a percolation version of the 'Wulff-construction'.

The basic difference between the restriction of Theorem 1.2 to Bernoulli percolation ($q = 1$), and the results in [14] and [21] is the following. The intersection of the infinite cluster with a box is not generally connected. However, intuition suggests that $C_\infty \cap B(n)$ has a unique big component and many small components that are concentrated around the boundary $\partial B(n)$. Our result says that this picture is correct (up to large deviations of surface order) and that already the big part (the largest cluster in the box) has the 'right' volume, $n^d\theta$. Theorem 1.2 contains implicitly the following (uniqueness) statement which is not treated in [14] or [21]: any two clusters in a box with a positive fractional volume must be connected to each other, up to large deviations of surface order.

1.3 Overview and organization of the paper

In the following section we introduce notation and give a summary of the FK model and FK representation. In Sect. 3, after giving the precise definition of long-range order in slabs, we study connectivity properties of FK percolation in a large box $B(n)$. We show in Theorem 3.1 the existence of a constant $\kappa > 0$, such that for each function $\phi(n)$ with $\kappa \log(n) < \phi(n) \leqq n$ the following is true: up to (large) deviations of order $\phi(n)$, there is a unique cluster in the box $B(n)$ intersecting each sub-box with diameter larger than $\phi(n)$, and the diameter of the second largest cluster does not exceed $\phi(n)$. This result will be used to establish the basic properties of the renormalized process. However, it may be of independent interest, even in the case of Bernoulli percolation. The technique we develop (which is a refinement of methods used by Kesten and Zhang [29], and Chayes et al. [8]) allows us to prove a natural conjecture concerning the equality: $\lim_{n\to\infty} \mathbf{\Phi}_{B(n)}^{f,p,q}[0 \leftrightarrow \partial B(n)] = \theta^f$. This result will play a key role in controlling the volume of the largest cluster. As a byproduct, we obtain the joint lower semicontinuity of $\theta^f(p,q)$ in the region $\{(p,q) \mid q \geqq 1, \widehat{p}_1(q) < p \leqq 1\}$. Let us remark that recently, by using completely different techniques, Grimmett [23] has proved the lower semicontinuity of the function $\theta^f(p)$ for fixed q with the possible exception of $p_c(q)$. A further application is the extension of a result of Kesten and Zhang [29] to the FK model ($q > 1$), concerning the tail of the finite cluster size distribution, see the remark after Theorem 3.1.

The core of this work is Sect. 4, establishing a comparison inequality between the renormalized process and high-density Bernoulli percolation. Section 5 finally, contains the proofs of the main theorems.

2 Preliminaries

In this section we first introduce notation. In the second part, we recall some useful properties of FK (or random cluster) measures. Finally, we give a short description of the Potts model and a summary of the FK representation.

2.1 Notation and terminology

Throughout, we will deal with stochastic processes living on graphs with vertex set $\mathbb{Z}^d$, $d \geqq 2$, or some subset of it. According to the $\mathscr{L}^1$-norm on $\mathbb{R}^d$, defined by $|r| = \sum_{i=1,\ldots,d}|r_i|$, we may turn $\mathbb{Z}^d$ into a graph with vertex set $\mathbb{Z}^d$ and edge set $\mathbb{E}^d = \{\{x,y\}; |x-y| = 1\}$. This graph is called the *d-dimensional cubic lattice*. We often think of this graph embedded in $\mathbb{R}^d$ the edges being straight lines between nearest neighbors. If x and y are nearest neighbors, we denote this relation by $x \sim y$. A *path* γ is a finite or infinite sequence $x_1, x_2 \ldots$ of distinct nearest neighbors. Any two sites x, y in a path γ are said to be joined by γ. A set $A \subseteq \mathbb{Z}^d$ is called *connected* if any two elements can be joined by a path contained in A.

The cardinality of a set A is denoted by $|A|$. For $i \in \{1,\ldots,d\}$, we set $\mathrm{diam}_i(A) = \sup\{|x_i - y_i|;\ x, y \in A\}$. The *diameter* of A is given by $\mathrm{diam}\,(A) = \max\{\mathrm{diam}_i(A);\ \ i = 1,\ldots,d\}$. For $A, B \subseteq \mathbb{Z}^d$, the distance between them is given by $\mathrm{dist}(A,B) := \inf\{|x-y|; x \in A, y \in B\}$. We denote by τ_a the shift in $\mathbb{R}^d$ by $a \in \mathbb{R}^d$, i.e., $\tau_a : x \mapsto x + a$ for $x \in \mathbb{R}^d$.

With $A \subseteq \mathbb{Z}^d$, we associate a set of edges (bonds) given by $[A]_e := \{\{x,y\};\ \ x, y \in A\}$. The graph $(A, [A]_e)$ will be often identified by its vertex set A. We define two kinds of boundary for A. The (*inner*) *vertex boundary* is given by $\partial A = \{x \in A;\ \exists y \in A^c \text{ with } x \sim y\}$, and the *edge boundary* is given by $\partial^{\mathrm{edge}} A = \{\{x,y\} \in \mathbb{E}^d;\ \ x \in A, y \in A^c\}$. For $V \supseteq A$ we also define $\partial_V A = \{x \in A;\ \exists y \in V \backslash A \text{ with } x \sim y\}$ and $\partial_V^{\mathrm{edge}} A = \{\{x,y\} \in \partial^{\mathrm{edge}} A; x \in A, y \in V \backslash A\}$.

A *box* Λ is a finite subset of $\mathbb{Z}^d$ of the form $\mathbb{Z}^d \cap \Pi_{i=1,\ldots,d}[a_i, b_i]$. For $i = \pm 1, \ldots, \pm d$, we define the ith face $\partial_i \Lambda$ of Λ by $\partial_i \Lambda = \{x \in \Lambda; x_i \text{ is maximal}\}$ for i positive, and $\partial_i \Lambda = \{x \in \Lambda;\ x_{|i|} \text{ is minimal}\}$ for i negative. For $\underline{r} \in (0,\infty)^d$, we define a box centered at the origin by

$$B(\underline{r}) = \mathbb{Z}^d \cap \prod_{i=1,\ldots,d} (-r_i/2, r_i/2]\,.$$

Note that for $\underline{n} \in (\mathbb{N}^+)^d$, we have $|B(\underline{n})| = \Pi_{i=1,\ldots,d}\, n_i$. We say that the box is symmetric, if $r_1 = \ldots = r_d =: r$, and we denote it by $B(r)$. For $t \in \mathbb{R}^+$, we define the set $\mathscr{X}_2(t) = \{\underline{r} \in \mathbb{R}^d; r_i \in [t, 2t] \text{ for } i = 1,\ldots,d\}$. The set of all boxes in $\mathbb{Z}^d$, which are congruent to a box $B(\underline{r})$ with $\underline{r} \in \mathscr{X}_2(t)$, will be denoted by $\mathscr{B}_2(t)$.

A *finite slab with thickness L* is any box in $\mathbb{Z}^d$, which is congruent to a region

$$S(\underline{r}, L) = \mathbb{Z}^d \cap \left(\prod_{i=1,\ldots,d-1} (-r_i/2, r_i/2] \right) \times (-L/2, L/2]$$

with $\underline{r} \in (0,\infty)^d$. Similarly, we write $S(r,L)$ for symmetric slabs (with respect to the first $d-1$ coordinate directions) centered at the origin.

Our setup for bond percolation is as follows. The basic sample space is given by $\Omega = \{0,1\}^{\mathbb{E}^d}$; the elements are called *configurations in* $\mathbb{Z}^d$. The natural projections are given by $\mathrm{pr}_b : \omega \in \Omega \mapsto \omega(b) \in \{0,1\}$, where $b \in \mathbb{E}^d$. A bond b is called open in the configuration ω if $\mathrm{pr}_b(\omega) = 1$, and closed otherwise.

For $E \subseteq \mathbb{E}^d$ with $E \neq \emptyset$, we write $\Omega(E)$ for the set $\{0,1\}^E$; the elements are called *configurations in* E. Note that there is a one-to-one correspondence between cylinder sets and configurations, which is given by $\eta \mapsto \{\eta\} := \{\omega \in \Omega;\ \omega(b) = \eta(b)$ for every $b \in E\}$, where $\eta \in \Omega(E)$. We will use the following convention: the set Ω is regarded to be a cylinder (set) corresponding to the 'empty configuration' (with the choice $E = \emptyset$.) We will sometimes identify cylinders with the corresponding configuration. For $A \subseteq \mathbb{Z}^d$, let Ω_A stand for the set of configurations in A: $\{0,1\}^{[A]_e}$, and Ω^A for the set configurations *outside* A: $\{0,1\}^{\mathbb{E}^d \setminus [A]_e}$. In general, for $A \subseteq B \subseteq \mathbb{Z}^d$, we set $\Omega_B^A = \{0,1\}^{[B]_e \setminus [A]_e}$. Given $\omega \in \Omega$ and $E \subseteq \mathbb{E}^d$, we denote by $\omega(E)$ the restriction of ω to $\Omega(E)$. Analogously, ω_B^A stands for the restriction of ω to the set $[B]_e \setminus [A]_e$.

Given $\eta \in \Omega(E)$, we denote by $\mathcal{O}(\eta)$ the set of open bonds in E. The connected components of the graph $(\mathbb{Z}^d, \mathcal{O}(\eta))$ are called *η-clusters*. The η-cluster of the site x is denoted by $C_x(\eta)$. The path $\gamma = (x_1, x_2, \ldots)$ is said to be η-open if all the bonds $\{x_i, x_{i+1}\}$ are contained in E and are open with respect to the configuration η. We write $\{A \leftrightarrow B\}$ for the event that there exists an open path joining some site of A with some site of B. Similarly, we denote by $\{A \leftrightarrow \infty\}$ the event that there exists $x \in A$ with $|C_x| = \infty$.

For given $E \subseteq \mathbb{E}^d$ we write $\mathscr{F}(E)$ for the σ-field generated by the finite-dimensional cylinders associated with configurations in $\Omega(E)$. Similarly, for $A \subseteq B \subseteq \mathbb{Z}^d$, we use the notation $\mathscr{F}_B^A$ for the σ-field generated by finite-dimensional cylinders associated with configurations in Ω_B^A. If $A = \emptyset$ or $B = \mathbb{Z}^d$, then (as before) we omit them from the notation $\mathscr{F}_B^A$.

There is a partial order $\preccurlyeq$ in Ω given by $\omega \preccurlyeq \omega'$ iff $\omega(b) \leqq \omega'(b)$ for every $b \in \mathbb{E}^d$. A function $f : \Omega \to \mathbb{R}$ is called *increasing* if $f(\omega) \leqq f(\omega')$, whenever $\omega \preccurlyeq \omega'$. An event is called increasing if its characteristic function is increasing. Let $\mathscr{F}$ be a σ-field of subsets of Ω. For a pair of probability measures μ and ν on $(\Omega, \mathscr{F})$, we say that μ (*stochastically*) *dominates* ν if for any $\mathscr{F}$-measurable increasing function g the expectations satisfy $\mu(g) \geqq \nu(g)$. We denote this relation with $\mu \succcurlyeq \nu$. If, in addition, for each $\mathscr{F}$-measurable cylinder Z with $\mu(Z) \wedge \nu(Z) > 0$, we have $\mu(g|Z) \geqq \nu(g|Z)$, then we say that μ *strongly dominates* ν, and denote this relation by $\mu \overset{s}{\succcurlyeq} \nu$.

Occasionally, we will deal with site percolation on the lattice $(\mathbb{Z}^d, \mathbb{E}^d)$. We use analogous notation to bond percolation indicating the difference by a subscript 'site'. For example, the sample space $\{0,1\}^{\mathbb{Z}^d}$ is denoted by Ω_{site}, and for $V \subseteq \mathbb{Z}^d$, we denote by $\mathscr{F}_{V,\text{site}}$ the σ-field generated by finite-dimensional cylinders associated with configurations in V, etc.

2.2 FK measures and FK percolation

Although the natural setting for FK percolation is a finite multi-graph, we restrict our attention to certain subgraphs of the lattice $(\mathbb{Z}^d, \mathbb{E}^d)$ for $d \geqq 2$ for notational convenience. Let $V \subseteq \mathbb{Z}^d$ be finite and set $E = [V]_e$. We first introduce *boundary conditions* as follows. Consider a partition π of the set ∂V, say $\pi = \{B_1, \dots, B_n\}$. (The sets B_i are disjoint non-empty subsets of ∂V with $\bigcup_{i=1,\dots,n} B_i = \partial V$). We say that $x, y \in \partial V$ are *π-wired*, if $x, y \in B_i$ for an $i \in \{1, \dots, n\}$, and denote this relation by $x \overset{\pi}{\sim} y$. Fix a configuration $\eta \in \Omega_V$. We want to count the η-clusters in V in such a way that π-wired sites are considered to be connected. This can be done in the following formal way. We introduce an equivalence relation on V: x and y are said to be *$\pi \cdot \eta$-wired* if they are both joined by η-open paths to (or identical with) sites $x', y' \in \partial V$ which are themselves π-wired. The new equivalence classes are called $\pi \cdot \eta$-clusters, or η-clusters in V with respect to the boundary condition π.

For fixed $p \in [0, 1]$ and $q \geqq 1$, the *FK measure with parameters* (p, q) *and boundary conditions* π is a probability measure on the σ-field $\mathscr{F}_V$, defined by the formula

$$(2.1) \qquad \mathbf{\Phi}_V^{\pi,p,q}[\{\eta\}] = \frac{1}{Z_V^{\pi,p,q}} \left(\prod_{b \in E} p^{\eta(b)} (1-p)^{1-\eta(b)} \right) q^{c^{\pi}(\eta)} ,$$

where $\eta \in \Omega_V$ and $Z_V^{\pi,p,q}$ is an appropriate normalization factor. Since $\mathscr{F}_V$ is an atomic σ-field with atoms $\{\eta\}, \eta \in \Omega_V$, (2.1) determines a unique measure on $\mathscr{F}_V$. Note that every cylinder has non-zero probability. Recall that the free boundary condition corresponds to the partition f defined to have exactly $|\partial V|$ classes, and the wired b.c. corresponds to the partition w with only one class. The set of all such measures corresponding to different b.c.s will be denoted by $\mathscr{R}(p, q, V)$, and we write $\overline{\mathscr{R}}(p, q, V)$ for its convex hull.

The stochastic process $(\mathrm{pr}_b)_{b \in \Omega_V} : \Omega \to \Omega_V$ given on the probability space $(\Omega, \mathscr{F}, \mathbf{\Phi}_V^{\pi,p,q})$ is called *FK percolation with boundary conditions* π. We will identify the law of this process with the measure $\mathbf{\Phi}_V^{\pi,p,q}$ throughout this paper.

We now will list some useful properties of FK measures. Although these properties are surely well-known and/or easy to derive, not all of them are discussed in modern reviews in the form which we will use. Especially the somewhat awkward subject of boundary conditions as well as conditioned versions of comparison inequalities seem to be neglected. Since these issues will play an important role in our considerations, we include some background material on this topic in the appendix. References or proofs of the statements below can be found there.

Perhaps the most important fact about FK measures is that every $\mathbf{\Phi} \in \mathscr{R}(p, q, V)$ has the *strong FKG property*. (Note that $q \geqq 1$). This means that for every $\mathscr{F}_V$-measurable cylinder Z, and for all $\mathscr{F}_V$-measurable increasing functions f, g, we have

$$(2.2) \qquad \mathbf{\Phi}[fg \mid Z] \geqq \mathbf{\Phi}[f \mid Z] \mathbf{\Phi}[g \mid Z] .$$

In particular, if $Z = \Omega$, we have $\mathbf{\Phi}[fg] \geqq \mathbf{\Phi}[f] \mathbf{\Phi}[g]$, which is called the FKG property.

A direct consequence is the following, sometimes useful fact. Let $Z = \{\mathrm{pr}_b = i \text{ for } b \in E_i,\ i = 0, 1\}$ and $Z' = \{\mathrm{pr}_b = i \text{ for } b \in E'_i,\ i = 0, 1\}$. If $E_0 \supseteq E'_0$ and $E_1 \subseteq E'_1$, then

$$\Phi(\cdot \mid Z) \overset{s}{\preccurlyeq} \Phi(\cdot \mid Z') . \tag{2.3}$$

Another valuable property of FK measures is a pair of stochastic domination inequalities known as *comparison principles of Fortuin*. Let π be a boundary condition. Then

$$\text{either } q' \leqq q \text{ and } p \leqq p' \text{ or } q' \geqq q \text{ and } \frac{p'/(1-p')}{p/(1-p)} \geqq \frac{q'}{q}$$

implies

$$\Phi_V^{\pi, p, q} \overset{s}{\preccurlyeq} \Phi_V^{\pi, p', q'} . \tag{2.4}$$

In some cases it is possible to compare FK measures with different b.c.s as well. There is a partial order on the set of partitions of ∂V. We say that π is *finer* than π', if for every x and y, $x \overset{\pi}{\sim} y$ implies $x \overset{\pi'}{\sim} y$. We then have

$$\Phi_V^{\pi, p, q} \overset{s}{\preccurlyeq} \Phi_V^{\pi', p, q} . \tag{2.5}$$

This implies immediately that for each $\Phi_V \in \mathscr{R}(p, q, V)$,

$$\Phi_V^{f, p, q} \overset{s}{\preccurlyeq} \Phi_V \overset{s}{\preccurlyeq} \Phi_V^{w, p, q} . \tag{2.6}$$

Next we discuss conditioning properties of FK measures. We begin with formula (2.7), which can be thought of as a kind of Markov property.

For given $U \subseteq V$ and $\omega \in \Omega$, we define a partition $W_V^U(\omega)$ of ∂U by declaring $x, y \in \partial U$ to be $W_V^U(\omega)$-wired if they are joined by an ω_V^U-open path. Fix a partition π of ∂V. We define a new partition of ∂U, denoted by $\pi \cdot W_V^U(\omega)$, by considering $x, y \in \partial U$ to be $\pi \cdot W_V^U(\omega)$-wired if they are both joined by ω_V^U-open paths to (or identical with) sites x', y', which are themselves π-wired. Then, for every $g \in \mathscr{F}_U$,

$$\Phi_V^{\pi, p, q}[g \mid \mathscr{F}_V^U](\omega) = \Phi_U^{\pi \cdot W_V^U(\omega), p, q}[g], \quad \Phi_V^{\pi, p, q}\text{-a.s.} \tag{2.7}$$

A direct consequence is the *finite-energy property*. Fix a bond $e \in E$ and denote by $\mathscr{F}_V^e$ the σ-field generated by the random variables $\{\mathrm{pr}_b; b \in E \setminus \{e\}\}$. Then

$$\Phi_V^{\pi, p, q}[e \text{ is open} \mid \mathscr{F}_V^e](\omega) = \begin{cases} p & \text{on the set } \{\text{the endpoints of } e \text{ are } \pi \cdot W_V^e\text{-wired}\}, \\ \frac{p}{p+q(1-p)} & \text{otherwise}. \end{cases} \tag{2.8}$$

A nice feature of FK measures is the *decoupling property*. We describe here only the simplest (but most useful) application. Let B be a box contained in V. Consider the cylinders

$$D = \{\mathrm{pr}_b = 0 \text{ for } b \in \partial_V^{\mathrm{edge}} B\} \quad \text{and}$$
$$S = \{\mathrm{pr}_b = 1 \text{ for } b = \{x, y\} \text{ with } x, y \in \partial B\}\,.$$

Then for $\mathbf{\Phi} \in \mathscr{R}(p, q, V)$, the σ-fields

(2.9) $\mathscr{F}_B$ and $\mathscr{F}_V^B$ are *independent* with respect to $\mathbf{\Phi}(\,\cdot\,|\,D)$ and $\mathbf{\Phi}(\,\cdot\,|\,S)$.

Note that by (2.7), we have an exact description of the conditional measures $\mathbf{\Phi}(\,\cdot\,|\,D)$ and $\mathbf{\Phi}(\,\cdot\,|\,S)$.

Let $U \subseteq V$. The combination of (2.6) with (2.7) yields that for every increasing function $g \in \mathscr{F}_U$ and $\mathbf{\Phi}_V \in \mathscr{R}(p, q, V)$,

$$\mathbf{\Phi}_U^{f,p,q}[g] \leqq \mathbf{\Phi}_V[g\,|\,\mathscr{F}_V^U] \leqq \mathbf{\Phi}_U^{w,p,q}[g] \quad \mathbf{\Phi}_V\text{-a.s.}\,, \tag{2.10}$$

and

$$\mathbf{\Phi}_U^{f,p,q}[g] \leqq \mathbf{\Phi}_V^{f,p,q}[g] \leqq \mathbf{\Phi}_V^{w,p,q}[g] \leqq \mathbf{\Phi}_U^{w,p,q}[g]\,. \tag{2.11}$$

The common interpretation of (2.11) is to say that 'free (wired) b.c. FK measures stochastically increase (decrease) with growing volume'. This guarantees the existence of weak limits of the measures $\mathbf{\Phi}_{B(n)}^{f,p,q} \to \mathbf{\Phi}_\infty^{f,p,q}$ and $\mathbf{\Phi}_{B(n)}^{w,p,q} \to \mathbf{\Phi}_\infty^{w,p,q}$ as n tends to infinity. We define $\mathscr{R}(p, q, \mathbb{Z}^d) = \{\mathbf{\Phi}_\infty^{f,p,q}, \mathbf{\Phi}_\infty^{w,p,q}\}$ and denote by $\overline{\mathscr{R}}(p, q, \mathbb{Z}^d)$ its convex hull. For technical reasons, we introduce a further class of probability measures defined on the σ-field $\mathscr{F}_V$:

$$\mathscr{R}(\succcurlyeq, p, q, V) = \{\mathbf{\Psi} \text{ prob. measure on } \mathscr{F}_V; \mathbf{\Psi} \overset{\mathrm{s}}{\succcurlyeq} \mathbf{\Phi}_V^{f,p,q}\} \tag{2.12}$$

By (2.6), $\mathscr{R}(p, q, V) \subseteq \mathscr{R}(\succcurlyeq, p, q, V)$. Strong domination can be expressed in terms of conditional expectations. Let $\mathbf{\Phi} \in \mathscr{R}(\succcurlyeq, p, q, V)$ and $E' \subseteq E$ be fixed. The σ-field $\mathscr{F}' := \mathscr{F}(E')$ is atomic with atoms $\{\eta\}, \eta \in \Omega(E')$. Thus

$$\mathbf{\Phi}(\,\cdot\,|\,\mathscr{F}')(\omega) = \sum_{\eta \in \Omega(E')} \mathbb{1}_{\{\eta\}}(\omega)\mathbf{\Phi}(\,\cdot\,|\{\eta\}) \quad \mathbf{\Phi}\text{-a.s.} \tag{2.13}$$

Therefore, we have for each increasing $\mathscr{F}_V$-measurable function g

$$\mathbf{\Phi}(g\,|\,\mathscr{F}') \geqq \mathbf{\Phi}_V^{f,p,q}(g\,|\,\mathscr{F}') \quad \mathbf{\Phi}\text{-a.s.} \tag{2.14}$$

Note that $\mathbf{\Phi}$ is absolutely continuous with respect to $\mathbf{\Phi}_V^{f,p,q}$, thus inequality (2.14) is meaningful.

Lemma 2.1. *Let U be finite and $V \supseteq U$ finite or equal to $\mathbb{Z}^d$. Fix $\mathbf{\Phi} \in \mathscr{R}(p, q, V)$, an $\mathscr{F}_V^U$-measurable cylinder Z and an increasing event $J \in \mathscr{F}_V$. Assume $\mathbf{\Phi}(J \cap Z) > 0$, and let $\mathbf{\Phi}^{J \cap Z}$ stand for the conditional measure $\mathbf{\Phi}(\,\cdot\,|\,J \cap Z)$. Denote by $(\mathbf{\Phi})_U$ and $(\mathbf{\Phi}^{J\cap Z})_U$ the restriction of $\mathbf{\Phi}$ and $\mathbf{\Phi}^{J \cap Z}$, respectively, to the σ-field $\mathscr{F}_U$. Then*

(2.15) $(\mathbf{\Phi})_U \in \overline{\mathscr{R}}(p, q, U)$. *Moreover, if V is finite*

$$(\mathbf{\Phi}^{J \cap Z})_U \in \mathscr{R}(\succcurlyeq, p, q, U)\,. \tag{2.16}$$

The proof can be found in the appendix.

2.3 Potts models and the FK representation

We begin with a short description of the Potts model and refer to [4] for more details.

Potts spin systems are generalizations of the Ising model. Whereas in Ising systems the spins take on two different values, in the q-state Potts model q distinct values are allowed (called colors), which we represent by the elements of the set $\{1,2,\ldots,q\}$. For the finite box $\Lambda \subseteq B(n)$, a color (or spin) configuration is a generic element of the set $\Sigma_\Lambda = \{1,2,\ldots,q\}^\Lambda$. The Hamiltonian has the same form as in (1.1), and the Gibbs measure with free b.c. is defined by

$$\mu_\Lambda^{f,\beta,q}[\sigma] = \exp(-H(\sigma))/Z(f,\beta,q)\,,$$

where $Z(f,\beta,q)$ is an appropriate normalizing factor. Note that the case $q=2$ coincides with the Ising model, by interpreting color 1 as spin $+1$ and color 2 as spin -1. For a given color c and $\Delta \subseteq \partial\Lambda$, we define the conditional measure

$$\mu_\Lambda^{\Delta(c),\beta,q}[\,\cdot\,] = \mu_\Lambda^{f,\beta,q}[\,\cdot \mid \sigma_x = c \text{ for every } x \in \Delta]\,, \tag{2.17}$$

and denote the class of all such measures by $\mathscr{P}^{(c)}(\beta,q,\Lambda)$. For the choice $\Delta = \partial\Lambda$, we write $\mu_\Lambda^{(c),\beta,q}$ for the measures $\mu_\Lambda^{\Delta(c),\beta,q}$. They correspond to plus (minus) b.c.s in the Ising model. The measures $\mu_{B(n)}^{(c),\beta,q}$ converge weakly $(n \to \infty)$ to infinite volume limiting measures denoted by $\mu_\infty^{(c),\beta,q}$. The order parameter of the Potts model is given by

$$m^*(\beta,q) := \frac{q}{q-1}\left(\mu_\infty^{(1),\beta,q}[\sigma_0 = 1] - \frac{1}{q}\right)\,.$$

Note that for $q=2$, this agrees with the spontaneous magnetization. The model exhibits a phase transition, more precisely, there exists $\beta_c(q) \in (0,\infty)$ such that $m^*(\beta,q) = 0$ for $\beta < \beta_c(q)$ and $m^*(\beta,q) > 0$ for $\beta > \beta_c(q)$.

We now turn to the FK representation of the Potts model. In the late 1960s, Fortuin and Kasteleyn observed that the q-state Potts model can be viewed as the following doubly-stochastic system. Fix a finite box Λ, and for given β set $p = 1 - e^{-\beta}$. In the first step we generate bond configurations according to the FK measure $\mathbf{\Phi}_\Lambda^{f,p,q}$. Given $\eta \in \Omega_\Lambda$, we equip each η-cluster in Λ with a color $c \in \{1,2,\ldots,q\}$ at random with probability $1/q$ for each color, independently from the others. Then the obtained color configuration has the same distribution as in the q-state Potts model with free boundary conditions.

Let us now discuss more general b.c.s in detail. Fix $\mathbf{\Phi}_\Lambda^{\Delta(1),\beta,q} \in \mathscr{P}^{(1)}(\beta,q,\Lambda)$. Denote by $\pi(\Delta)$ the partition of the set $\partial\Lambda$ characterized by the property that two distinct sites belong to the same class iff they are contained in Δ. For $\eta \in \Omega_\Lambda$, we set

$$s^\Delta(\eta) = \begin{cases} c^{\pi(\Delta)} - 1 & \text{if } \Delta \neq \emptyset\,, \\ c^{\pi(\Delta)} & \text{if } \Delta = \emptyset\,. \end{cases}$$

Recall that $\Delta = \emptyset$ corresponds to the free b.c. For $\sigma \in \Sigma_\Lambda$, set $\mathcal{O}(\sigma) = \{\{x,y\} \in [\Lambda]_e; \sigma_x = \sigma_y\}$. We then define a probability measure on Σ_Λ as follows:

$$P^{\eta,\Delta}[\{\sigma\}] = q^{-s^{\Delta}(\eta)} \mathbb{1}_{\{\mathcal{O}(\eta) \subseteq \mathcal{O}(\sigma)\}} \mathbb{1}_{\{\sigma=1 \text{ on } \Delta\}} \, .$$

This measure allows the following interpretation: for each $\pi(\Delta) \cdot \eta$-cluster, except for that containing Δ, we choose a uniformly distributed color at random, independently from the others. We then define the spin σ_x at site x to be 1 if x belongs to the cluster containing Δ, and to be the chosen color of the $\pi(\Delta) \cdot \eta$-cluster of x, otherwise. The joint distribution of the variables $(\sigma_x)_{x \in \Lambda}$ is then equal to $P^{\eta,\Delta}$.

The next formula describes in a precise way the FK representation of the measure $\mu_\Lambda^{\Delta(1),\beta,q}$:

$$\mu_\Lambda^{\Delta(1),\beta,q}[\,\cdot\,] = \int_{\Omega_\Lambda} P^{\eta,\Delta}[\,\cdot\,] \Phi_\Lambda^{\pi(\Delta),p,q}[d\eta] \, . \tag{2.18}$$

As a direct consequence of this representation, we have the following identities.

$$\frac{q}{q-1}\left(\mu_\Lambda^{(1),\beta,q}[\sigma_x = 1] - \frac{1}{q}\right) = \Phi_\Lambda^{w,p,q}[x \leftrightarrow \partial\Lambda] \, , \tag{2.19}$$

$$\begin{aligned} &\frac{q}{q-1}\left(\mu_\Lambda^{\Delta(1),\beta,q}[\sigma_x = \sigma_y] - \frac{1}{q}\right) \\ &\quad = \Phi_\Lambda^{\pi(\Delta),p,q}[\{x \leftrightarrow y\} \cup (\{x \leftrightarrow \Delta\} \cap \{y \leftrightarrow \Delta\})] \, , \end{aligned} \tag{2.20}$$

$$m^*(\beta,q) = \theta^w(p,q) \, , \tag{2.21}$$

$$p_c(q) = 1 - e^{-\beta_c(q)} \, . \tag{2.22}$$

For proofs, extensions and background material, the reader is referred to the works [20, 4, 33, 16] or [32]. Note that for $q = 2$, the l.h.s. of Eqs. (2.19) and (2.20) can be replaced by the more familiar expressions $\int \sigma_x d\mu_\Lambda^{(1),\beta,q}$ and $\int \sigma_x \sigma_y d\mu_\Lambda^{\Delta(1),\beta,q}$, respectively.

It was recently proved in [23], that for fixed q, $\theta^f(p,q)$ is lower semicontinuous in p with the possible exception of $p_c(q)$, see also Theorem 3.2. On the other hand, it is known (cf. [30, 23]) that for each q, $\theta^f(p)$ coincides with $\theta^w(p)$ up to at most countably many values of p. Since $m^{*,f}(\beta,q) := \lim_{\beta' \to \beta^-} m^*(\beta',q)$ is also lower semicontinuous and agrees with m^* on a dense set, we have for all $\beta \neq \beta_c(q)$,

$$\theta^f(1 - e^{-\beta}, q) = m^{*,f}(\beta,q) \, . \tag{2.23}$$

3 Long-range order in slabs and connectivity in boxes

In this section we study connectivity properties of FK percolation in a finite box. We begin with the precise definition of (uniform) long-range order in finite slabs and the corresponding critical values. Let $q \geqq 1$ and $d \geqq 3$ be fixed. Using the terminology of statistical mechanics, we say that in the slab-system $(S(n,L))_{n\geqq 1}$, with thickness $L \in \mathbb{N}^+$, at parameter p 'short long-range order' (s.l.o.) exists if we can find $\alpha \geqq 1$ such that

$$\varliminf_{n\to\infty} \inf_{x\in S(n,L)} \Phi^{f,p,q}_{S(\alpha n,L)}[0 \leftrightarrow x] > 0 \,. \tag{3.1}$$

We speak about 'long long-range order' (l.l.o.), if we can choose $\alpha = 1$ above. Using the finite-energy property (2.8), we can replace (3.1) by

$$\inf_{n\geqq 1} \inf_{x,y\in S(n,L)} \Phi^{f,p,q}_{S(\alpha n,L)}[x \leftrightarrow y] > 0 \,. \tag{3.2}$$

It is a consequence of the stochastic domination inequality (2.4) that the existence of short (or long) long-range order at p ensures the same for any $p' \geqq p$. Therefore, it is reasonable to define the thresholds

$$p_s(L) = \inf\{p \geqq 0 \,|\, \exists \alpha \geqq 1 \text{ with } \varliminf_{n\to\infty} \inf_{x\in S(n,L)} \Phi^{f,p,q}_{S(\alpha n,L)}[0 \leftrightarrow x] > 0\}\,, \tag{3.3}$$

$$p_l(L) = \inf\{p \geqq 0 \,|\, \varliminf_{n\to\infty} \inf_{x\in S(n,L)} \Phi^{f,p,q}_{S(n,L)}[0 \leftrightarrow x] > 0\}\,. \tag{3.4}$$

Using (2.11), it can be easily seen that $p_s(L) \leqq p_l(L)$. A further property of the critical values $p_s(L)$ and $p_l(L)$ is that they decay with growing L. To see this, pick any $p \gneqq p_s(L)$ and $L' \geqq L$. Then there exists $\alpha \geqq 1$ with (3.1). Set $\pi_d(x) = (x_1,\dots,x_{d-1},0)$ and $\tilde{p} = p/p + q(1-p)$. By the FKG inequality, (2.11) and (2.8), we have for any $x \in S(n,L')$

$$\begin{aligned}\Phi^{f,p,q}_{S(\alpha n,L')}[x \leftrightarrow 0] &\geqq \Phi^{f,p,q}_{S(\alpha n,L')}[\pi_d(x) \leftrightarrow 0]\cdot \Phi^{f,p,q}_{S(\alpha n,L')}[x \leftrightarrow \pi_d(x)] \\ &\geqq \Phi^{f,p,q}_{S(\alpha n,L)}[\pi_d(x) \leftrightarrow 0](\tilde{p})^{L'}\,,\end{aligned}$$

which implies s.l.o. at p in the slab-system with thickness L'. Hence, $p > p_s(L')$. The same argument works also for $p_l(L)$. Next, we set

$$\widehat{p}_s = \widehat{p}_s(q,d) = \lim_{L\to\infty} p_s(L) \quad \text{and} \quad \widehat{p}_l = \widehat{p}_l(q,d) = \lim_{L\to\infty} p_l(L)\,. \tag{3.5}$$

It is easily seen that $p_c \leqq \widehat{p}_s \leqq \widehat{p}_l$. Finally, we show that $\widehat{p}_l$ is strictly less than one. For $q = 1$, i.e., for Bernoulli percolation, this follows from the equality $p_c = \widehat{p}_l$, see [25] and Lemma 3 in [29]. For $q > 1$, we first choose p with $p_c(1,d) < p < 1$. Let $p' < 1$ be such that $p'/(1-p') > q \cdot p/(1-p)$. Then by the (second) comparison inequality of Fortuin (2.4), we have for every slab $S(n,L)$: $\Phi^{f,p',q}_{S(n,L)} \succcurlyeq \Phi^{p,q=1}_{S(n,L)}$, which implies that $p > \widehat{p}_l(q,d)$, and thus $1 > \widehat{p}_l(q,d)$.

Remark. As one of the first consequences of the renormalization procedure described in the next section, we will see that in fact the two critical values $\widehat{p}_s$ and $\widehat{p}_l$ coincide. Moreover, we conjecture that for $q > 1$, in analogy to the case of Bernoulli percolation, we have the equality $p_c = \widehat{p}_s$.

Before stating the main results of this section, Theorems 3.1 and 3.2, we first define the notion of *crossing*. Let $B \subseteq B'$ be boxes contained in a region $\Lambda \subseteq \mathbb{Z}^d$. Consider bond percolation in Λ. We say that for B an i-crossing occurs if there exists an open path $\gamma \subseteq B$ which joins the left face to the right face of the box in the ith direction, i.e., if $\gamma : \partial_{-i}(B) \leftrightarrow \partial_i(B)$ in B. A cluster C in B' is called $(i_1,\dots,i_s)$*-crossing for* B if for $j = i_1,\dots,i_s, C \cap B$ contains an open j-crossing path. A cluster C in B' is said to be a *crossing cluster for* B in B' if it is $(1,\dots,d)$-crossing. Note that the occurrence of i-crossings in all directions does not imply the existence of a crossing cluster in general.

A slab is a special kind of box, therefore, our definitions extend also to slabs. For our purpose, however, it is convenient to ignore crossings in the 'short' direction. Assume $S(\underline{n},L) \subseteq B'$. We then say that C in B' is a *crossing cluster for* $S(\underline{n},L)$ if it is $(1,\dots,d-1)$-crossing.

Finally, we define the following events. For $\alpha \geqq 1$ fixed and $\underline{n} \in \mathscr{X}_2(n)$, set $U(\underline{n}) = \{\exists!$ open crossing cluster C^* for $B(\underline{n})$ in $B(\alpha\underline{n})\}$. Consider a (fixed) monotone increasing function $g : \mathbb{N} \to [0,\infty)$ with $g(n) \leqq n$. Denote by $R^g(\underline{n})$ the event $U(\underline{n}) \cap \{$every open path $\gamma \subseteq B(\underline{n})$ with $\operatorname{diam}(\gamma) \geqq g(n)$ is contained in $C^*\}$. Finally, $O^g(\underline{n})$ stands for the event $R^g(\underline{n}) \cap \{C^*$ crosses every sub-box $Q \in \mathscr{B}_2(g(n))$ contained in $B(n)\}$.

Theorem 3.1 *Let $d \geqq 3$, $q \geqq 1$ and assume one of the following hypotheses (H): $p > \widehat{p}_s$ and $\alpha > 1$, (H'): $p > \widehat{p}_1$ and $\alpha = 1$. Then*

$$(3.6) \qquad \overline{\lim_{n\to\infty}} \frac{1}{n} \log \left(\sup_{\underline{n}\in\mathscr{X}_2(n)} \ \sup_{\Phi\in\mathscr{R}(\succcurlyeq,p,q,B(\alpha\underline{n}))} \Phi[U(\underline{n})^c] \right) < 0 \, .$$

Also, there exists a constant $\kappa = \kappa(p,q,d,\alpha) > 0$ such that $\underline{\lim}_{n\to\infty} g(n)/\log(n) > \kappa$ implies

$$(3.7) \qquad \overline{\lim_{n\to\infty}} \frac{1}{g(n)} \log \left(\sup_{\underline{n}\in\mathscr{X}_2(n)} \ \sup_{\Phi\in\mathscr{R}(\succcurlyeq,p,q,B(\alpha\underline{n}))} \Phi\left[(R^g(\underline{n}))^c\right] \right) < 0 \, .$$

Let $p > \widehat{p}_1$. There exists a constant $\kappa' = \kappa'(p,q,d) > 0$ such that $\underline{\lim}_{n\to\infty} g(n)/\log(n) > \kappa'$ implies

$$(3.8) \qquad \overline{\lim_{n\to\infty}} \frac{1}{g(n)} \log \left(\sup_{\underline{n}\in\mathscr{X}_2(n)} \ \sup_{\Phi\in\mathscr{R}(\succcurlyeq,p,q,B(\underline{n}))} \Phi[(O^g(\underline{n}))^c] \right) < 0 \, .$$

The proof of this theorem is based on Lemmas 3.3 and 3.4, and can be found after those.

Remark. Using (3.7), it is not difficult to adapt the proof of the upper bound on the sub-exponential decay of the finite cluster size distribution in Bernoulli percolation (cf. [29], Corollary 3), to prove the following extension: for $d \geqq 3$, $q \geqq 1$ and $p > \widehat{p}_1$,

$$(3.9) \qquad \overline{\lim_{n\to\infty}} \frac{1}{n^{(d-1)/d}} \log \Phi_\infty^{*,p,q}[|C_0| = n] < 0 \, ,$$

where $*$ stands for f or w. Since this result does not fit naturally into the subject of this paper, we plan to give a full proof elsewhere.

The next theorem, which is crucial in order to control the volume of the 'largest' cluster in a box, proves a natural conjecture: $\lim_{n\to\infty}\Phi^{f,p,q}_{B(n)}[0\sim\partial B(n)]=\Phi^{f,p,q}_{\infty}[0\sim\infty]=\theta^f$, whenever $p>\widehat{p}_1$. The corresponding result is well-known for wired boundary conditions (see [4] Theorem 2.3), in which case it is a simple consequence of monotonicity properties of the wired b.c. FK measures, and holds for each $p\in[0,1]$. The same method leading to this result yields also a large deviation estimate (in n) on the quantity $(\Phi^{*,p,q}_{\infty}[0\leftrightarrow\partial B(n)]-\theta^*(p))$, where $*$ stands for f or w. Remarkably, this bound is *uniform* in the parameters p and q on any compact subset of the supercritical region S, given by

$$S=\{(p,q);\ 1\leqq q<\infty,\ \widehat{p}_1(q)<p\leqq 1\}\,. \tag{3.10}$$

This enables us to establish the joint lower semicontinuity of the percolation probability $\theta^f(p,q)$ in the region S.

Theorem 3.2 *Let $d\geqq 3$, $q\geqq 1$ and $p>\widehat{p}_1$ be fixed. Then*

(i) *we have the equality* $\lim_{n\to\infty}\Phi^{f,p,q}_{B(n)}[0\leftrightarrow\partial B(n)]=\theta^f(p,q)$.

(ii) *The sequence of functions $\theta^*_n(p,q):=\Phi^{*,p,q}_{\infty}[0\leftrightarrow\partial B(n)]$, converges to $\theta^*(p,q)$, as n tends to infinity. The convergence is exponential in n and uniform on compacta in the region S.*

(iii) *The function $\theta^f(p,q)$ is jointly lower semicontinuous in S.*

The proof is deferred to the end of this section. The next two Lemmas prepare the proof of Theorem 3.1.

Lemma 3.3 *Let $d\geqq 3$, $q\geqq 1$ and $L\geqq 1$. Under each of the following hypotheses (H): $p>p_s(L)$ and $\alpha>1$, (H'): $p>p_1(L)$ and $\alpha=1$, we have*

$$\varliminf_{n\to\infty}\ \inf_{\underline{n}\in\mathscr{X}_2(n)}\ \inf_{x,y\in S(\underline{n},L)}\ \Phi^{f,p,q}_{S(\alpha\underline{n},L)}[x\leftrightarrow y]>0 \tag{3.11}$$

and

(3.12)

$$\varliminf_{n\to\infty}\ \inf_{\underline{n}\in\mathscr{X}_2(n)}\ \inf_{x\in S(\underline{n},L)}\ \Phi^{f,p,q}_{S(\alpha\underline{n},L)}[C_x \text{ is crossing for } S(\underline{n},L) \text{ in } S(\alpha\underline{n},L)]>0\,.$$

Proof. By the FKG inequality we have $\Phi^{f,p,q}_{S(\alpha\underline{n},L)}[x\leftrightarrow y]\geqq\Phi^{f,p,q}_{S(\alpha\underline{n},L)}[x\leftrightarrow 0]\,\Phi^{f,p,q}_{S(\alpha\underline{n},L)}[0\leftrightarrow y]$. Therefore, we can replace y by 0 in (3.11). Assuming hypothesis (H), it is possible to find $k_0\geqq 4$, $\beta\geqq 1$ (which can be very large) and $\delta>0$ such that for all $k\geqq k_0$

$$\inf_{x\in S(k,L)}\Phi^{f,p,q}_{S(\beta k,L)}[0\leftrightarrow x]\geqq\delta\,. \tag{3.13}$$

Set $\alpha'=(\alpha-1)/2$ and $n_0=\lfloor k_0\beta/\alpha'\rfloor+1$. Fix $n\geqq n_0$ and $\underline{n}\in\mathscr{X}_2(n)$. Set $k=k(n)=\lfloor n\alpha'/\beta\rfloor$ and denote by E the dth coordinate hyper-plane, i.e. $E=\{x\in\mathbb{Z}^d\,|\,x_d=0\}$. Note that $k\geqq k_0$, and for each $y\in S(\underline{n},L)\cap E$, the slab $S_y(\beta k,L):=\tau_yS(\beta k,L)$ is contained in $S(\alpha\underline{n},L)$. It is then possible to find a sequence $\{y^0,y^1,\dots,y^m\}\subseteq S(\underline{n},L)\cap E$ of nearest neighbors in the lattice

$[k/2]\mathbb{Z}^d$ such that $y^0 = 0$, $x \in S_{y^m}(k,L)$ and $m \leqq 6d\beta/\alpha'$. Set $y^{m+1} = x$. Using the FKG inequality and (2.11), we have

$$\text{(3.14)} \quad \Phi^{f,p,q}_{S(\alpha\underline{n},L)}[0 \leftrightarrow x] \geqq \prod_{j=0\dots m} \Phi^{f,p,q}_{S(\alpha\underline{n},L)}[y^j \leftrightarrow y^{j+1}]$$

$$\geqq \prod_{j=0\dots m} \Phi^{f,p,q}_{S_{y^j}(\beta k,L)}[y^j \leftrightarrow y^{j+1}] \geqq \delta^{m+1} \geqq \delta^{6d\beta/\alpha'+1} ,$$

which implies (3.11).

As for the proof of (3.12) (with hypothesis (H)), note first, that by (3.11) and the FKG inequality we can replace x by 0 in (3.12). Again, by (3.11), we can find $k_0 \geqq 1$ and $\delta > 0$ such that for $k \geqq k_0$ and $y \in S(k,L)$

$$\text{(3.15)} \qquad \Phi^{f,p,q}_{S((\alpha\wedge 2)k,L)}[0 \leftrightarrow y] \geqq \delta ,$$

where $a \wedge b$ denotes the minimum of a and b. Set $n_0 = 16k_0$. Let $n \geqq n_0$ and $\underline{n} \in \mathscr{X}_2(n)$ be fixed. For $i = \pm 1, \dots, \pm(d-1)$, set $A_i = \{0 \leftrightarrow \partial_i S(\underline{n},L)$ in $S(\underline{n},L)\}$. Then

(3.16)

$$\Phi^{f,p,q}_{S(\alpha\underline{n},L)}[C_0 \text{ is crossing for } S(\underline{n},L)] \geqq \Phi^{f,p,q}_{S(\alpha\underline{n},L)}\left[\bigcap_{i=\pm 1,\dots,\pm(d-1)} A_i\right]$$

$$\geqq \left(\min_{i=\pm 1,\dots,\pm(d-1)} \Phi^{f,p,q}_{S(\alpha\underline{n},L)}[A_i]\right)^{2(d-1)} .$$

We now give a lower bound for the r.h.s. of Eq. (3.16). We may assume $i = 1$. For $j \in \mathbb{N}$, set $z^j = (j\lfloor n/8\rfloor, 0, \dots, 0) \in \mathbb{Z}^d$, and denote by m the smallest natural number with $S_{z^m}(\lfloor n/4\rfloor, L) \cap \partial_1 S(\underline{n},L) \neq \emptyset$. Then $1 \leqq m \leqq n/\lfloor n/8\rfloor \leqq 16$. Set $y^j := z^j$ for $j = 0,1,\dots,m$, and $y^{m+1} := (\lfloor \underline{n}_1/2\rfloor, 0, \dots, 0) \in \partial_1 S(\underline{n},L)$. We observe that for $j = 0,\dots,m$, $y^{j+1} \in S_{y^j}(\lfloor n/4\rfloor, L)$ and

$$S_{y^j}((\alpha\wedge 2)\lfloor n/4\rfloor, L) \cap \left(\bigcup_{i=-1,\pm 2,\dots,\pm(d-1)} \partial_i S(\underline{n},L)\right) = \emptyset .$$

Therefore, any open path joining 0 with y^{m+1} within $\bigcup_{j=0,\dots,m} S_{y^j}((2\wedge\alpha)\lfloor n/4\rfloor, L)$ contains an open path joining 0 with $\partial_1 S(\underline{n},L)$ within $S(\underline{n},L)$. Note that for $j = 0,1,\dots,m$, $S_{y^j}((\alpha\wedge 2)\lfloor n/4\rfloor, L) \subseteq S(\alpha\underline{n},L)$. Thus

$$\text{(3.17)} \quad \Phi^{f,p,q}_{S(\alpha\underline{n},L)}[A_1]$$

$$\geqq \Phi^{f,p,q}_{S(\alpha\underline{n},L)}\left[\bigcap_{i=0,\dots,m} \{y^i \leftrightarrow y^{i+1} \text{ in } S_{y^i}((2\wedge\alpha)\lfloor n/4\rfloor, L)\}\right] \geqq \delta^{m+1} ,$$

which, together with (3.16) finishes the proof. The proofs with hypothesis (H′) are analogous, in fact, much easier. □

Consider a (fixed) monotone increasing function $g : \mathbb{N} \to [0, \infty)$ with $g(n) \leqq n$ for $n \in \mathbb{N}$. For given $n, \underline{n} \in \mathscr{X}_2(n)$ and $i \in \{1, \ldots, d\}$ let us define the event $A^g(i, \underline{n}) = \{\exists i\text{-crossing cluster } C \text{ for } B(\underline{n}) \text{ in } B(\alpha\underline{n}) \text{ and } \exists \text{ open path } \gamma \subseteq B(\underline{n}) \text{ with } \mathrm{diam}_i(\gamma) \geqq g(n) \text{ such that } C \cap \gamma = \emptyset\}$. Note that the event $A^g(i, \underline{n})$ is neither increasing nor decreasing.

Lemma 3.4 *Let $d \geqq 3$, $q \geqq 1$ and assume one of the following hypotheses (H): $p > \widehat{p}_s$ and $\alpha > 1$, (H'): $p > \widehat{p}_l$ and $\alpha = 1$. Then*

$$(3.18) \qquad \overline{\lim_{n\to\infty}} \frac{1}{n} \log \left(\sup_{\underline{n} \in \mathscr{X}_2(n)} \Phi^{f,p,q}_{B(\alpha\underline{n})}[\nexists \textit{ crossing for } B(\underline{n})] \right) < 0 .$$

Moreover, there exists a constant $\kappa = \kappa(p, q, d, \alpha) > 0$ such that $\underline{\lim}_{n\to\infty} g(n)/\log(n) > \kappa$ implies for each $i \in \{1, \ldots, d\}$

$$(3.19) \qquad \underline{\lim}_{n\to\infty} \frac{1}{g(n)} \log \left(\sup_{\underline{n} \in \mathscr{X}_2(n)} \sup_{\Phi \in \mathscr{R}(\succcurlyeq, p, q, B(\alpha\underline{n}))} \Phi[A^g(i, \underline{n})] \right) < 0 .$$

Proof. We will simplify the notation throughout the proof by setting $\Phi^f = \Phi^{f,p,q}_{B(\alpha\underline{n})}$. Assume (H). As $p > \widehat{p}_s$, there exists $L \in \mathbb{N}^+$ with $p > p_s(L)$. By Lemma 3.3, we can find $n_0 \in \mathbb{N}^+$ and $\delta \in (0, 1)$ such that for any $n \geqq n_0$ and $\underline{n} \in \mathscr{X}_2(n)$, we have

(3.20)

$$\left(\inf_{x, y \in S(\underline{n}, L)} \Phi^{f,p,q}_{S(\alpha\underline{n}, L)}[x \leftrightarrow y] \right) \wedge \left(\inf_{x \in S(\underline{n}, L)} \Phi^{f,p,q}_{S(\alpha\underline{n}, L)}[C_x \text{ is crossing for } S(\underline{n}, L)] \right) \geqq \delta .$$

In a first step we will show that in the box $B(\underline{n})$, up to large deviations of order n, an i-crossing occurs. We may assume $i = 1$. Fix $n \geqq n_0$ and $\underline{n} \in \mathscr{X}_2(n)$. We divide the box $B(\alpha\underline{n})$ into slabs as follows. Set $t = \min\{x_d \,|\, x \in B(\underline{n})\} - 1$ and $k_0 = \lfloor \underline{n}_d / L \rfloor$. We define for $k = 1, \ldots, k_0$

$$S_k = B(\alpha\underline{n}) \cap \{x \in \mathbb{Z}^d \,|\, t + (k-1)L < x_d \leqq t + kL\} .$$

Note that for each k, S_k and $S_k \cap B(\underline{n})$ are congruent to $S(\alpha\underline{n}, L)$ and $S(\underline{n}, L)$ respectively. Consider the decreasing event $D = \bigcap_{k=1,\ldots,k_0} \{\text{each bond in } \partial_{B(\alpha\underline{n})}(S_k) \text{ is closed}\}$. Note that conditioning on D decouples the slabs $S_1, \ldots, S_k$ in the sense of (2.9). Using (3.20), we have

(3.21)

$$\begin{aligned} \Phi^f[\nexists\, 1\text{-crossing for } B(\underline{n})] &\leqq \Phi^f \left[\bigcap_{k=1,\ldots,k_0} \{\nexists\, 1\text{-crossing for } B(\underline{n}) \cap S_k \text{ in } S_k\} \,\middle|\, D \right] \\ &= \prod_{k=1,\ldots,k_0} \Phi^{f,p,q}_{S_k}[\nexists\, 1\text{-crossing for } B(\underline{n}) \cap S_k] \\ &\leqq (1 - \delta)^{\lfloor n/L \rfloor} . \end{aligned}$$

We now turn to the proof of (3.18). By (3.21), it will be sufficient to estimate the probability of the event $F = \{\exists\, d\text{-crossing but no crossing for } B(\underline{n}) \text{ in}$

$B(\alpha\underline{n})\}$. For any site $z \in \Lambda \subseteq B(\alpha\underline{n})$, we denote by $C_z(\Lambda)$ the cluster of z in Λ. For $z \in \partial_{-d}B(\underline{n})$, we set $F_z = \{\exists$ open path $\gamma \subseteq B(\underline{n})$ with $\gamma : z \leftrightarrow \partial_d B(\underline{n})$ and $C_z(B(\alpha\underline{n}))$ is not $(1,\dots,d-1)$-crossing for $B(\underline{n})\}$. Then

$$\Phi^f[F] \leqq (2n)^{d-1} \sup_{z\in\partial_{-d}B(\underline{n})} \Phi^f[F_z] \tag{3.22}$$

To estimate the r.h.s. of (3.22), we fix $z \in \partial_{-d}B(\underline{n})$. For $k = 1,\dots,k_0$, we define the regions $T_k = S_1 \cup S_2 \cup \dots \cup S_k$ and $T_0 = \emptyset$, and the events $G_0 = \Omega$ and $G_k = \{\exists$ open path $\gamma_k \subseteq B(\underline{n})$ with $\gamma_k : z \leftrightarrow \partial_d(T_k)$ and $C_z(T_k)$ is not $(1,\dots,d-1)$-crossing for $B(\underline{n}) \cap T_k\}$. It is easy to see that $G_k \in \mathscr{F}_{T_k}$ and $G_0 \supseteq G_1 \supseteq \dots G_{k_0} \supseteq F_z$. We claim, that for $k = 0,1,\dots,k_0-1$,

$$\Phi^f[G_{k+1} \mid \mathscr{F}_{T_k}] \leqq 1 - \tilde{p}\delta \quad \Phi^f\text{-a.s.}, \tag{3.23}$$

where δ is defined by (3.20) and $\tilde{p} = p/[p + q(1-p)]$. Once (3.23) is proved, we proceed as follows:

$$\begin{aligned} \Phi^f[F_z] &\leqq \Phi^f[G_{k_0}] = \int \mathbb{1}_{G_{k_0-1}}(\omega)\Phi^f[G_{k_0} \mid \mathscr{F}_{T_{k_0-1}}](\omega)\Phi^f[d\omega] \\ &\leqq (1-\tilde{p}\delta)\Phi^f[G_{k_0-1}] \leqq \dots \leqq (1-\tilde{p}\delta)^{k_0} \leqq (1-\tilde{p}\delta)^{\lfloor n/L \rfloor} \end{aligned} \tag{3.24}$$

by recursion. Now, the statement (3.18) follows easily by putting together the inequalities (3.22) and (3.24). We still have to prove (3.23). Note first that $\Phi^f[G_{k+1} \mid \mathscr{F}_{T_k}](\omega) = 0$ Φ^f-a.s. on G_k^c. On the set G_k we can find $z_k = z_k(\omega) \in \partial_d(T_k \cap B(\underline{n}))$ such that $z \leftrightarrow z_k$ in $T_k \cap B(\underline{n})$. Let us denote by z_k' the nearest neighbor of z_k lying in S_{k+1} and by b_k the bond joining them. Using (2.13) and (2.2), (3.20) and (2.8), we have Φ-a.s. on G_k,

$$\begin{aligned} &\Phi^f[G_{k+1} \mid \mathscr{F}_{T_k}] \\ &\quad\leqq 1 - \Phi^f[C_z(T_{k+1}) \text{ is } (1,\dots,d-1)\text{-crossing for } T_{k+1} \cap B(\underline{n}) \mid \mathscr{F}_{T_k}] \\ &\quad\leqq 1 - \Phi^f[b_k \text{ is open} \mid \mathscr{F}_{T_k}]\Phi^f[C_{z_k'}(S_{k+1}) \text{ is} \\ &\qquad\qquad (1,\dots,d-1)\text{-crossing for } S_{k+1} \cap B(\underline{n}) \mid \mathscr{F}_{T_k}] \\ &\quad\leqq 1 - \tilde{p}\delta\,, \end{aligned}$$

which finishes the proof of (3.18).

The proof of (3.19) is very similar to that of (3.18), so we will merely sketch the argument. We may assume $i = d$. For fixed $x^{(1)} \in \partial_{-d}B(\underline{n})$ and $x^{(2)} \in B(\underline{n})$ with $x_d^{(2)} < \lfloor \underline{n}_d/2 \rfloor - g(n)$, we define the event $F_{x^{(1)},x^{(2)}} = \{$for $i \in \{1,2\}$, $\exists$ open paths $\gamma^{(i)} \subseteq B(\underline{n})$ with $\gamma^{(i)} : x^{(i)} \leftrightarrow \{y \in B(\underline{n}) \mid y_d = x_d^{(2)} + \lfloor g(n) \rfloor\}$ such that $C_{x^{(1)}}(B(\alpha\underline{n})) \cap \gamma^{(2)} = \emptyset\}$ Let s be the largest integer with $x^{(2)} \notin T_s$ and set $j_0 = \lfloor g(n)/L \rfloor$. For $j = 1,\dots,j_0$, consider the events $G_j' = \{$for $i \in \{1,2\}, \exists$ open paths $\gamma^{(i)} \subseteq B(\underline{n})$ with $\gamma^{(i)} : x^{(i)} \leftrightarrow \partial_d(T_{s+j}) \cap B(\underline{n})\}$. It is easily seen that $G_j' \in \mathscr{F}_{T_{s+j}}$ and $A^g(d,\underline{n}) \subseteq G_{j_0}' \subseteq G_{j_0-1}' \subseteq \dots \subseteq G_1'$. Pick $\Phi \in \mathscr{R}\,(\succcurlyeq, p, q, B(\alpha\underline{n}))$. As before, $\Phi[G_{j+1}' \mid \mathscr{F}_{T_{s+j}}] = 0$ Φ-a.s. on $(G_j')^c$. On the set G_j', we can find sites $x_j^{(i)} = x_j^{(i)}(\omega) \in \partial_d(T_{s+j}) \cap B(\underline{n}), i \in \{1,2\}$, with

$x^{(i)} \leftrightarrow x_j^{(i)}$ in $B(\underline{n}) \cap T_{s+j}$. Let us denote by $y_j^{(i)}$ their nearest neighbors in S_{s+j+1} and by $b_j^{(i)}$ the corresponding bonds. Then, by using (2.13), (2.2), (3.20) and (2.8), we have Φ-a.s. on G_j':

$$\begin{aligned}
&\Phi[G_{j+1}' \mid \mathscr{F}_{T_{s+j}}] \\
&\quad \leqq 1 - \Phi[x^{(1)} \leftrightarrow x^{(2)} \text{ in } T_{s+j+1} \mid \mathscr{F}_{T_{s+j}}] \\
&\quad \leqq 1 - \Phi_{B(\alpha\underline{n})}^{f,p,q}[x^{(1)} \leftrightarrow x^{(2)} \text{ in } T_{s+j+1} \mid \mathscr{F}_{T_{s+j}}] \\
&\quad \leqq \Phi_{B(\alpha\underline{n})}^{f,p,q}\left[\bigcap_{i=1,2}\{b_j^{(i)} \text{ is open}\} \cap \{y_j^{(1)} \leftrightarrow y_j^{(2)} \text{ in } S_{s+j+1}\} \,\middle|\, \mathscr{F}_{T_{s+j}}\right] \\
&\quad \leqq 1 - (\tilde{p})^2\delta\,.
\end{aligned}$$

By successive conditioning, as in (3.24), we then have $\Phi[F_{x^{(1)},x^{(2)}}] \leqq (1-(\tilde{p})^2\delta)^{\lfloor g(n)/L \rfloor - 1}$, which yields $\Phi[A^g(d,\underline{n})] \leqq (2n)^{2d-1}(1-(\tilde{p})^2\delta)^{\lfloor g(n)/L \rfloor - 1}$. Taking logarithm and dividing by $g(n)$, we see that by setting $\kappa = L\,(2d-1)/\log(1/[1-(\tilde{p})^2\delta])$, (3.19) holds. The proof of the Lemma with hypothesis (H′) is analogous. □

Proof of Theorem 3.1. The existence of a crossing cluster, up to large deviations of order n, was proved in (3.18). The uniqueness is an application of (3.19) by setting $g(n) = n$. The proof of (3.7) follows also immediately from Lemma 3.4. Simply observe that

$$(R^g(\underline{n}))^c \subseteq U^c(\underline{n}) \cup \left(\bigcup_{i=1,\dots,d} U(\underline{n}) \cap A^g(i,\underline{n})\right)$$

The estimates (3.6) and (3.19) finish the proof of (3.7). We now turn to (3.8). Let $\underline{n} \in \mathscr{X}_2(n)$ be fixed and pick $\Phi \in \mathscr{R}(\succcurlyeq, p, q, B(\underline{n}))$. First we observe

$$\text{(3.25)} \qquad \Phi[O^g(\underline{n})^c] \leqq \Phi[R^g(\underline{n})^c] + \binom{|B(n)|}{2} \sup_{\substack{Q \in \mathscr{B}_2(g(n)) \\ Q \subseteq B(n)}} \Phi[\nexists \text{ crossing for } Q \text{ in } Q]\,.$$

Since $\Phi \succcurlyeq \Phi_{B(\underline{n})}^{f,p,q}$, and by (2.11) and (3.6), there exist strictly positive constants c_1, c_2 such that for each Q in the expression above, we have

$$\Phi[\nexists \text{ crossing for } Q \text{ in } Q] \leqq \Phi_Q^{f,p,q}[\nexists \text{ crossing for } Q \text{ in } Q] \leqq c_1 e^{-c_2 g(n)}\,,$$

for n large enough. Putting the estimates together, we find that for $\kappa' := \kappa \vee 2d/c_2$, (3.8) holds. □

Proof of Theorem 3.2. For given $q \geqq 1$ and $p > \widehat{p}_1(q)$, we choose $L = L(p,q,d) \in \mathbb{N}^+$ and $\delta = \delta(p,q,d) > 0$ such that

$$\text{(3.26)} \qquad \inf_{n \geqq 1} \inf_{x,y \in S(n,L)} \Phi_{S(n,L)}^{f,p,q}[x \leftrightarrow y] \geqq \delta\,.$$

By using the same kind of slab argument as in the proof of Lemma 3.4 (cf. (3.22)–(3.24)), we have

(3.27)
$$\sup_{\Phi \in \mathscr{R}(\succcurlyeq, p,q,B(n))} \Phi[0 \leftrightarrow \partial B(N) \text{ and } 0 \not\leftrightarrow \partial B(n)] \leqq 2d(1-\tilde{p}\delta)^{\lfloor N/L \rfloor} \leqq c_1 e^{-c_2 N}\,,$$

where $c_1 = c_1(p,q,d) > 0$ and $c_2 = c_2(p,q,d) > 0$ are appropriate constants and $\tilde{p} = p/[p+q(1-p)]$. Pick $\Phi \in \mathscr{R}(\succcurlyeq, p,q,B(n))$. Since

$$\Phi[0 \leftrightarrow \partial B(N)] = \Phi[0 \leftrightarrow \partial B(n)] + \Phi[0 \leftrightarrow \partial B(N) \text{ and } 0 \not\leftrightarrow \partial B(n)]\,,$$

we have

$$\Phi[0 \leftrightarrow \partial B(N)] \geqq \Phi[0 \leftrightarrow \partial B(n)] \geqq \Phi[0 \leftrightarrow \partial B(N)] - c_1 e^{-c_2 N}\,. \tag{3.28}$$

Applying this to the measures $\Phi^{*,p,q}_{B(n)}$ and taking the limit $n \to \infty$, we get

(3.29)
$$\begin{aligned}\Phi^{*,p,q}_{\infty}[0 \leftrightarrow \partial B(N)] &\geqq \overline{\lim_{n\to\infty}}\, \Phi^{*,p,q}_{B(n)}[0 \leftrightarrow \partial B(n)] \geqq \underline{\lim_{n\to\infty}}\, \Phi^{*,p,q}_{B(n)}[0 \leftrightarrow \partial B(n)] \\ &\geqq \Phi^{*,p,q}_{B(n)}[0 \leftrightarrow \partial B(N)] - c_1 e^{-c_2 N}\,.\end{aligned}$$

Finally, we take the limit $N \to \infty$, which yields $\theta^*(p,q) = \lim_{n\to\infty} \Phi^{*,p,q}_{B(n)}[0 \leftrightarrow \partial B(n)]$. This proves part (i). To prove (ii), we need the following lemma.

Lemma 3.5 *Let $d \geqq 3$ be fixed. The function $\widehat{p}_1 = \widehat{p}_1(q) : [1,\infty) \to (0,1)$ is monotone increasing and Lipschitz-continuous.*

Proof. The analogous result is known for $p_c(q)$, see Theorem 3.1 in [24]. The monotonicity is a simple consequence of the comparison inequality (2.4). The proof of the continuity is based on an inequality comparing the critical parameters for different values of q, cf. (4.8) in [4]. Let $q' \geqq q \geqq 1$ and $p, p' \in (0,1)$ be given such that $q'/q \leqq (p'/(1-p'))/(p/(1-p))$. We claim that

$$p' < \widehat{p}_1(q') \quad \text{implies } p \leqq \widehat{p}_1(q)\,. \tag{3.30}$$

Once this is proved, the same arguments as in the proof of Theorem 3.1 in [24] yield the Lipschitz-continuity of $\widehat{p}_1(q)$.

As $p_l(L,q')$ is monotone decreasing in L, $p' < \widehat{p}_1(q')$ implies $p < p_l(L,q')$ for each L. This means that there is no l.l.o. at (p',q') in the slab-system with thickness L. But for each $n \geqq 1$, the measure $\Phi^{f,p',q'}_{S(n,L)}$ stochastically dominates $\Phi^{f,p,q}_{S(n,L)}$ by (2.4). Therefore, there is no l.l.o. at (p,q), which implies $p \leqq p_l(L,q)$. This is valid for each L, thus $p \leqq \widehat{p}_l(q)$. □

We observe that the region S is open with respect to the induced (Euclidean) topology in $T = \{(p,q);\ 1 \leqq q < \infty,\ 0 \leqq p \leqq 1\}$, since the function $\widehat{p}_1(q)$ is monotone and continuous. Fix $(p,q) \in S$. Then there exists $\rho > 0$ such that $(p_\rho, q_\rho) := (p-\rho, q+\rho) \in S$. For $i = 1,2$, set $c_{i,\rho} := c_i(p_\rho, q_\rho, d)$, where c_i was defined in (3.27). Note that (3.29) is valid for every $p' \geqq p_\rho$ and

$q' \leqq q_\rho$ if we replace c_1, c_2 by $c_{1,\rho}, c_{2,\rho}$, since $\Phi_{B(n)}^{*,p',q'} \in \mathscr{R}(\succcurlyeq, p_\rho, q_\rho, B(n))$. Therefore, by using the last inequality in (3.29), each point in S has a compact neighborhood (containing an open neighborhood), where we have uniformly in (p', q')

$$\theta^*(p', q') \leqq \Phi_\infty^{*,p',q'}[0 \leftrightarrow B(N)] \leqq \theta^*(p', q') + c_{1,\rho}\exp(-c_{2,\rho}N) .$$

This proves (ii). We now turn to the proof of part (iii). Note that for each n, the function $\vartheta_n^f(p,q) := \Phi_{B(n)}^{f,p,q}[0 \leftrightarrow \partial B(n)]$ is smooth in (p,q), since the volume $B(n)$ is finite. Fix $(p,q) \in S$. Then there exists a neighborhood U of (p,q) such that $U \subseteq S$, and for every $(p',q') \in U$, we have for certain positive constants c_3, c_4 and for N large enough,

$$\theta^f(p', q') \geqq \Phi_\infty^{f,p',q'}[0 \leftrightarrow \partial B(N)] - c_3 e^{-c_4 N} \geqq \vartheta_N^f(p', q') - c_3 e^{-c_4 N} .$$

The second inequality follows from (2.11). Therefore, by using part (i), we obtain

$$\varliminf_{(p',q')\to(p,q)} \theta^f(p', q') \geqq \lim_{N\to\infty} \varliminf_{(p',q')\to(p,q)} (\vartheta_N^f(p', q') - c_3 e^{-c_4 N}) = \theta^f(p,q) .$$

This proves lower semicontinuity in (p,q). □

4 Renormalization

Renormalization techniques are of great value in percolation theory. Our renormalization method has its origin in the 'block-argument' of Ref. [29], where the tail of the finite cluster size distribution was studied in supercritical independent site percolation. Our main result is a comparison inequality (Proposition 4.1) between the renormalized process (defined in (4.5)) and high-density independent site percolation. This inequality can be used to show that certain properties of high-density Bernoulli percolation extend to the FK model in the supercritical phase $p > \widehat{p}_1$. A direct application is stated in Proposition 4.2. It establishes the equality of the critical values $\widehat{p}_1$ and $\widehat{p}_s$.

4.1 The blocks

We begin with some geometrical considerations. The first step in any 'static' renormalization procedure is to re-scale the lattice (or a part of it) by introducing blocks of fixed size. Let $N \geqq 24$ be a natural number. We say that $\Lambda \subseteq \mathbb{Z}^d$ is an *N-large box*, if either $\Lambda = \mathbb{Z}^d$ or Λ is a finite box containing a symmetric box of side-length $3N$, i.e., if $\Lambda = \mathbb{Z}^d \cap \Pi_{i=1,\dots,d}(a_i, b_i]$ with $b_i - a_i \geqq 3N$ for $i = 1, \dots, d$. Given an N-large box Λ, we divide it into small boxes of an approximate size N as follows.

Assume first Λ is finite and $d = 1$, i.e., Λ is an interval I of the form $I = (a,b] \cap \mathbb{Z}$ with $b - a \geqq 3N$. For $k \in \mathbb{Z}$, set $\tilde{I}_k = (kN - N/2, kN + N/2]$ and define the rescaled interval $I^{(N)} = \{k \in \mathbb{Z} \mid \tilde{I}_k \subseteq I\}$. For $s = \min I^{(N)}$ and $l = \max I^{(N)}$, (note that $s \neq l$) we define $I_s = (a, sN + N/2]$ and $I_l = (lN - N/2, b]$. For the remaining elements of $I^{(N)}$, we just set $I_k := \tilde{I}_k, (s < k < l)$.

Clearly, the set $\{I_k \mid k \in I^{(N)}\}$ is a partition of I, moreover $N \leqq |I_k| < 2N$ for each $k \in I^{(N)}$.

Now we turn to the d-dimensional case. Assume first $|\Lambda| < \infty$. Setting $I(i) = (a_i, b_i] \cap \mathbb{Z}$, we define the *rescaled box* $\Lambda^{(N)} = \Pi_{i=1,\dots,d}[I(i)]^{(N)}$. The corresponding blocks are defined by $B_{\mathbf{k}} = \Pi_{i=1,\dots,d}[I(i)]_{\mathbf{k}_i}$ for $\mathbf{k} \in \Lambda^{(N)}$. In the case $\Lambda = \mathbb{Z}^d$, we just divide Λ into the set of disjoint boxes $B_{\mathbf{k}} = \tau_{N\mathbf{k}} B(N)$ for $\mathbf{k} \in \mathbb{Z}^d$, and set $\Lambda^{(N)} = \mathbb{Z}^d$. In both cases, the triple $(\Lambda, \Lambda^{(N)}, \{B_{\mathbf{k}} \mid \mathbf{k} \in \Lambda^{(N)}\})$ is called the *N-partition of the box* Λ. Note that for each $\mathbf{k} \in \Lambda^{(N)}, B_{\mathbf{k}} \in \mathscr{B}_2(N)$.

With this partition, we will associate a set of further boxes $\{D_{\mathbf{i},\mathbf{j}} \mid \mathbf{i},\mathbf{j} \in \Lambda^{(N)}, \mathbf{i} \sim \mathbf{j}\}$ representing 'bonds' between neighboring blocks. Let $e^{(k)}$ stand for the kth unit vector in $\mathbb{Z}^d$. For $\mathbf{i} \sim \mathbf{j}$ with $\mathbf{i}_k - \mathbf{j}_k = 1$, we first set $m(\mathbf{j},\mathbf{i}) = \tau_{N\mathbf{j}}(\lfloor N/2 \rfloor e^{(k)})$, which can be thought of as the middle-point of the kth face of the block $B_{\mathbf{j}}$. Then we define $D_{\mathbf{j},\mathbf{i}} = \tau_{m(\mathbf{i},\mathbf{j})}(B(\lfloor N/4 \rfloor))$ and $D_{\mathbf{i},\mathbf{j}} := D_{\mathbf{j},\mathbf{i}}$.

Let us fix a set of bonds $\mathscr{E}(N) \subseteq [B(\lfloor N^{1/2} \rfloor)]_e$ with cardinality $\lfloor N^{1/2} \rfloor$. Set $B'_{\mathbf{k}} = \tau_{N\mathbf{k}} B(N^{1/2})$ and $\mathscr{E}_{\mathbf{k}} = \tau_{N\mathbf{k}} \mathscr{E}(N)$. Note that for any $\mathbf{j} \sim \mathbf{k}$, the sets $B'_{\mathbf{k}}$ and $D_{\mathbf{k},\mathbf{j}}$ are disjoint.

For $\rho \in \{0, 0.1\}$, we denote by $B^{\rho}_{\mathbf{k}}$ the 'ρ-interior' of the block $B_{\mathbf{k}}$, i.e. $B^{\rho}_{\mathbf{k}} = \{x \in B_{\mathbf{k}} \mid \mathrm{dist}(x, \partial B_{\mathbf{k}}) \geqq \rho N\}$. Similarly $D^{\rho}_{\mathbf{k},\mathbf{j}}$ denotes the set of sites in $D_{\mathbf{k},\mathbf{j}}$ with a distance from $\partial D_{\mathbf{k},\mathbf{j}}$ at least $\rho \lfloor N/4 \rfloor$.

4.2 The renormalized process

We now introduce events related to our renormalization procedure. For given $N \geqq 24$ and N-large box Λ, we consider the N-partition of Λ. Let $\rho \in \{0, 0.1\}$ be fixed and set $g(N) = N^{1/2}/10$. For $\mathbf{i},\mathbf{j} \in \Lambda^{(N)}$ with $\mathbf{i} \sim \mathbf{j}$ and $|\mathbf{i}_k - \mathbf{j}_k| = 1$, we set $K_{\mathbf{i},\mathbf{j}} = \{\exists k\text{-crossing in } D^{\rho}_{\mathbf{i},\mathbf{j}}\}$. Given $\mathbf{i} \in \Lambda^{(N)}$, we define

(4.1)
$$R_{\mathbf{i}} = \{\text{(i) } \exists!\text{ crossing cluster } C^*_{\mathbf{i}} \text{ for } B^{\rho}_{\mathbf{i}} \text{ in } B_{\mathbf{i}} \text{ and}$$
$$\text{(ii) any open path } \gamma \subseteq B^{\rho}_{\mathbf{i}} \text{ with } \mathrm{diam}(\gamma) \geqq g(N) \text{ is contained in } C^*_{\mathbf{i}}\}$$

$$K_{\mathbf{i}} = \bigcap_{\mathbf{k} \sim \mathbf{i},\, \mathbf{k} \in \Lambda^{(N)}} K_{\mathbf{i},\mathbf{k}} \tag{4.2}$$

$$S_{\mathbf{i}} = \{\exists \text{ closed bond in } \mathscr{E}_{\mathbf{i}}\} \tag{4.3}$$

We consider a family of arbitrary events $\mathscr{V} = \{V_\Gamma \in \mathscr{F}_\Gamma;\ \Gamma \in \bigcup_{n \geqq 1} \mathscr{B}_2(n)\}$, with the property

$$\sup_{\Gamma \in \mathscr{B}_2(N)} \sup_{\Phi \in \mathscr{R}(p,q,\Gamma)} \Phi[(V_\Gamma)^c] =: v(N) \to 0 \quad (N \to \infty)\,. \tag{4.4}$$

In the sequel $V_{\mathbf{i}}$ stands for $V_{B_{\mathbf{i}}}$. Let us now define the 'block-variables' $X_{\mathbf{i}}$ for $\mathbf{i} \in \Lambda^{(N)}$, by setting

$$X_{\mathbf{i}} = \begin{cases} 1 & \text{if } R_{\mathbf{i}} \cap K_{\mathbf{i}} \cap S_{\mathbf{i}} \cap V_{\mathbf{i}} \text{ occurs}\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{4.5}$$

We call the (dependent) site percolation process defined by the variables $(X_{\mathbf{i}})_{\mathbf{i}\in\Lambda^{(N)}}$ as the *renormalized process with block-size* N. As usual, a block is said to be occupied if the corresponding block-variable is 1, and vacant otherwise.

An important geometrical property of this process is the following. Consider a cluster $\mathbf{C}$ of occupied blocks. Then there is a cluster C of the original (microscopic) process crossing all the blocks contained in $\mathbf{C}$ and it is unique with this property. To see this it is sufficient to observe that if two neighboring blocks $\mathbf{i}$ and $\mathbf{j}$ are occupied, then the (unique) crossing clusters $C_{\mathbf{i}}^*$ and $C_{\mathbf{j}}^*$ are connected to each other, because of the occurrence of $K_{\mathbf{i},\mathbf{j}}$ and property (ii) in the definition of $R_{\mathbf{i}}$. The uniqueness of C is also clear, because any two clusters crossing $B_{\mathbf{k}}^{\rho}$ in the occupied block $B_{\mathbf{k}}$ must be connected by definition.

By the results of Sect. 3 and (4.4), the blocks are occupied with high probability, whenever N is large enough. The basic result (Proposition 4.1 below) says that they are occupied with high probability *independently* of the state of the other blocks, including the neighboring ones. This will allow us to compare the renormalized process with high-density Bernoulli site percolation.

The somewhat artificial event $S_{\mathbf{i}}$ will play an important, though merely technical, role in establishing the comparison result, (Proposition 4.1). The reason to include $S_{\mathbf{i}}$ in the definition of $X_{\mathbf{i}}$ is roughly the following: we would like that the conditional probability of the event $K_{\mathbf{i}}$ (which has a strong influence on the state of any neighboring block) given $\{X_{\mathbf{i}} = 0\}$, is still close to one, as soon as the block-size is large. This will follow from a Bayes-type argument, since the absolute probability of $K_{\mathbf{i}}^c$ behaves like $\exp(-cN)$ whereas the probability of $S_{\mathbf{i}}^c$ behaves like $\exp(-c'N^{1/2})$. Therefore, even if $\{X_{\mathbf{i}} = 0\}$ occurs, it does, with high probability, not because of the occurrence of $K_{\mathbf{i}}^c$. For more details, see, for example, (4.18). This simple argument turns out to be powerful and convenient in order to establish comparison inequalities, moreover, it seems to be quite generally applicable.

As an example for a possible choice of the family $\mathscr{V}$, we refer to (5.7). At the same time we emphasize that *any* "typical" family can be included in this renormalization procedure.

4.3 A comparison inequality

Before stating Proposition 4.1, we need some preparation. By (3.2) and Theorem 3.1, there exist constants $N_0 \geqq 24$, $c_1 > 0$, $c_2 > 0$ and $\delta_1 \in (0,1]$, depending only on the parameters (p,q,d), with the following properties:

$$\text{(4.6)} \qquad \text{for } p > \widehat{p}_{\mathrm{l}}, \quad \inf_{n\geqq 1} \inf_{x,y\in S(n,\lfloor N_0/4\rfloor)} \Phi^{f,p,q}_{S(n,\lfloor N_0/4\rfloor)}[x \leftrightarrow y] \geqq \delta_1 \,,$$

for $p > \widehat{p}_{\mathrm{s}}$ and $\rho = 0.1$, or for $p > \widehat{p}_{\mathrm{l}}$ and $\rho = 0$, $N \geqq N_0$ implies

$$\text{(4.7)} \qquad \sup_{\Phi\in\mathscr{R}(\succcurlyeq,p,q,B_{\mathbf{i}})} \Phi[(R_{\mathbf{i}})^c] \leqq \exp(-c_1 N^{1/2}) \,,$$

$$\text{(4.8)} \qquad \sup_{\Phi\in\mathscr{R}(\succcurlyeq,p,q,B_{\mathbf{i}})} \Phi[(K_{\mathbf{i}})^c] \leqq \exp(-c_2 N) \,,$$

for each $\mathbf{i} \in \Lambda^{(N)}$. The following estimates can be easily derived from (2.8). Setting $c_3(p,q) = \log(1 + q(1-p)/p)$ and $c_4(p) = \log(1/p)/2$, we have for $p \in (\widehat{p}_1, 1)$ and for each $\mathbf{\Phi} \in \mathscr{R}(p,q,\Lambda)$,

$$\text{(4.9)} \qquad \exp(-c_3 N^{1/2}) \leqq \mathbf{\Phi}[(S_{\mathbf{i}})^c \,|\, \mathscr{F}_{\Lambda}^{B'_{\mathbf{i}}}] \leqq \exp(-c_4 N^{1/2}) \quad \mathbf{\Phi}\text{-a.s.}$$

For p, q fixed, we define the functions $\overline{q}_{\mathrm{i}} : \mathbb{N} \to [0, \infty)$ by

$$\text{(4.10)} \quad \overline{q}_1(N) = 2d \exp(-c_2 N + c_3 N^{1/2}) + \exp(-c_1 N^{1/2}) + \exp(-c_4 N^{1/2}),$$

$$\text{(4.11)} \qquad \overline{q}_2(N) = 2d\delta_1^{-2} \exp(-c_2 N + c_3 N^{1/2}) + \exp(-c_1 N^{1/2}) + \delta_1^{-(d+2)} \exp(-c_4 N^{1/2}),$$

$$\text{(4.12)} \qquad \overline{q}_3(N) = 2d \exp(-c_2 N + c_3 N^{1/2}) + \exp(-c_1 N^{1/2}) + \exp(-c_4 N^{1/2}) + \delta_1^{-d} v(N),$$

and set for $i = 1, 2, 3$,

$$\text{(4.13)} \qquad \overline{p}_i(N) = (1 - \overline{q}_i(N) \wedge 1) \mathbb{1}_{[N_0, \infty)}(N).$$

Note that the functions $\overline{p}_i : \mathbb{N} \to [0, 1)$ converge to 1 as N tends to infinity.

Finally, for given Λ and finite subsets $\Delta_1, \Delta_2 \subseteq \Lambda$, we introduce the following three hypotheses. Each of them corresponds to a certain choice of p, ρ, of the family $\mathscr{V}$ and of an increasing event $J \in \mathscr{F}_{\Lambda}$.

(H_1): $p \in (\widehat{p}_s, 1)$, $\rho = 0.1$, $V = \Omega$ for each $V \in \mathscr{V}$ and $J = \Omega$,

(H_2): $p \in (\widehat{p}_1, 1)$, $\rho = 0$, $V = \Omega$ for each $V \in \mathscr{V}$ and $J = \{\Delta_1 \leftrightarrow \Delta_2\}$,

(H_3): $p \in (\widehat{p}_1, 1)$, $\rho = 0$, $\mathscr{V}$ is an arbitrary family with property (4.4) and $J = \Omega$.

Proposition 4.1 *Let $d \geqq 3$ and $q \geqq 1$. Then for every $N \geqq 24$, every N-large box Λ and every $\mathbf{\Phi} \in \mathscr{R}(p,q,\Lambda)$ we have; assuming one of the hypotheses* H_1–H_3, *say* (H_i), *the law of the renormalized process $(X_{\mathbf{i}})_{\mathbf{i} \in \Lambda^{(N)}}$, (defined by* (4.5)) *with respect to the conditional measure $\mathbf{\Phi}^J := \mathbf{\Phi}(\cdot \,|\, J)$, stochastically dominates independent site percolation in the box $\Lambda^{(N)}$ with parameter $\overline{p}_i(N)$, meaning that for any measurable increasing event $I \in \mathscr{F}_{\Lambda^{(N)},\mathrm{site}}$*

$$\text{(4.14)} \qquad \mathbf{\Phi}^J[X_{\bullet} \in I] \geqq \mathbf{P}_{\Lambda^{(N)},\mathrm{site}}^{\overline{p}_i(N),\mathrm{indpt.}}[I].$$

Remark. Although not used in the present paper in this generality, we consider in (4.14) conditional measures $\mathbf{\Phi}^J$, since it may be useful for future applications. The other reason to do so is that it demonstrates – together with the freedom in choosing the events $V_{\mathbf{i}}$ – the possibility of *adapting* the basic renormalization procedure (H_2) to different kinds of problems. Note however that the increasing event J could be chosen more general than $\{\Delta_1 \leftrightarrow \Delta_2\}$.

Proof. Let $N \geqq N_0$ and an N-large box Λ be fixed. Suppose first $|\Lambda| < \infty$. Obviously, it is enough to prove (4.14) for $\mathbf{\Phi} \in \mathscr{R}(p,q,\Lambda)$. Pick $\mathbf{\Phi} \in \mathscr{R}(p,q,\Lambda)$. It is sufficient to show that for each $\mathbf{i} \in \Lambda^{(N)}$,

$$\mathbf{\Phi}^J[X_\mathbf{i} = 0 \mid \sigma(X_\mathbf{j}; \mathbf{j} \in \Lambda^{(N)}, \mathbf{j} \neq \mathbf{i})] \leqq \bar{q}_\mathbf{i}(N) \quad \mathbf{\Phi}^J\text{-a.s.} \tag{4.15}$$

Using the definition of $X_\mathbf{i}$, we can bound the l.h.s. of (4.15) as follows.

$$\begin{aligned}
\mathbf{\Phi}^J[X_\mathbf{i} = 0 \mid \sigma(X_\mathbf{j}; \mathbf{j} \in \Lambda^{(N)}, \mathbf{j} \neq \mathbf{i})] &\leqq \sum_{\mathbf{j} \in \Lambda^{(N)}, \mathbf{j} \sim \mathbf{i}} \mathbf{\Phi}^J[K^c_{\mathbf{i},\mathbf{j}} \mid \sigma(X_\mathbf{j}; \mathbf{j} \in \Lambda^{(N)}, \mathbf{j} \neq \mathbf{i})] \\
&\quad + \mathbf{\Phi}^J[R^c_\mathbf{i} \cap K_\mathbf{i} \mid \sigma(X_\mathbf{j}; \mathbf{j} \in \Lambda^{(N)}, \mathbf{j} \neq \mathbf{i})] \\
&\quad + \mathbf{\Phi}^J[S^c_\mathbf{i} \cap K_\mathbf{i} \mid \sigma(X_\mathbf{j}; \mathbf{j} \in \Lambda^{(N)}, \mathbf{j} \neq \mathbf{i})] \\
&\quad + \mathbf{\Phi}^J[V^c_\mathbf{i} \cap K_\mathbf{i} \mid \sigma(X_\mathbf{j}; \mathbf{j} \in \Lambda^{(N)}, \mathbf{j} \neq \mathbf{i})] .
\end{aligned} \tag{4.16}$$

We now derive estimates for each term, beginning with the first one. Let $\mathbf{j} \in \Lambda^{(N)}$ with $\mathbf{j} \sim \mathbf{i}$ be fixed. Denote by Γ the smallest box containing $D_{\mathbf{i},\mathbf{j}}$ and $B'_\mathbf{j}$ and define the σ-field $\mathscr{G} = \sigma(X_\mathbf{j}) \vee \mathscr{F}^\Gamma_\Lambda$. Note that $\mathscr{G} \supseteq \sigma(X_\mathbf{j}; \mathbf{j} \in \Lambda^{(N)}, \mathbf{j} \neq \mathbf{i})$, since for $\{\mathbf{k},\mathbf{l}\} \neq \{\mathbf{i},\mathbf{j}\}, D_{\mathbf{k},\mathbf{l}} \supseteq \Lambda \backslash \Gamma$. Set $h_1 = h_3 = 1$ and $h_2 = \delta_1^2$. (δ_1 was defined in (4.6)). We claim that under the hypothesis (H_k)

$$\mathbf{\Phi}^J[K^c_{\mathbf{i},\mathbf{j}} \mid \mathscr{G}] \leqq h_k^{-1} \exp(-c_2 N + c_3 N^{1/2}) \quad \mathbf{\Phi}^J\text{-a.s.} \tag{4.17}$$

Clearly, the same estimate is valid if $\mathscr{G}$ is replaced by $\sigma(X_\mathbf{j}; \mathbf{j} \in \Lambda^{(N)}, \mathbf{j} \neq \mathbf{i})$. To prove (4.17), we first observe that $\mathscr{G}$ is an atomic σ-algebra, with atoms of the form $\{X_\mathbf{j} = 1\} \cap \{\eta\}$ or $\{X_\mathbf{j} = 0\} \cap \{\eta\}$, where $\eta \in \Omega^\Gamma_\Lambda$. (As before, we will identify the cylinder $\{\eta\}$ with the configuration η.) Note that on the set $\{X_\mathbf{j} = 1\}$, the conditional expectation $\mathbf{\Phi}^J[K^c_{\mathbf{i},\mathbf{j}} \mid \mathscr{G}]$ vanishes (Φ^J-a.s.), since $K_{\mathbf{i},\mathbf{j}} \supseteq \{X_\mathbf{i} = 1\}$. So we have to consider only atoms $\{X_\mathbf{j} = 0\} \cap \eta$ with $\mathbf{\Phi}^J[\{X_\mathbf{j} = 0\} \cap \eta] > 0$. By using the strong FKG property of $\mathbf{\Phi}$, (2.16), (4.8) and (4.9), we have

$$\begin{aligned}
\mathbf{\Phi}^J[K^c_{\mathbf{i},\mathbf{j}} \mid \{X_\mathbf{j} = 0\} \cap \eta] &= \mathbf{\Phi}[K^c_{\mathbf{i},\mathbf{j}} \mid \{X_\mathbf{j} = 0\} \cap \eta \cap J] \\
&\leqq \frac{\mathbf{\Phi}^J[K^c_{\mathbf{i},\mathbf{j}} \mid \eta]}{\mathbf{\Phi}[\{X_\mathbf{j} = 0\} \cap J \mid \eta]} \leqq \frac{\mathbf{\Phi}^J[K^c_{\mathbf{i},\mathbf{j}} \mid \eta]}{\mathbf{\Phi}[S^c_\mathbf{j} \mid \eta] \mathbf{\Phi}[J \mid \eta]} \\
&\leqq (\mathbf{\Phi}[J \mid \eta])^{-1} \exp(-c_2 N + c_3 N^{1/2}) .
\end{aligned} \tag{4.18}$$

Under (H_1) and (H_3), $J = \Omega$, hence (4.17) is proved in these cases. We will now show that under (H_2), one has

$$\mathbf{\Phi}[J \mid \eta] \geqq \delta_1^2 , \tag{4.19}$$

which will complete the proof of (4.17). Recall $J = \{\Delta_1 \leftrightarrow \Delta_2\}$. As $\mathbf{\Phi}[J \cap \eta] > 0$, there either exists a path among the open bonds of η joining Δ_1 with Δ_2 (in which case $\mathbf{\Phi}[J \mid \eta] = 1 \geqq \delta_1^2$), or for $i \in \{1,2\}$, there exist sites $x^{(i)} \in \partial^{\text{in}}(\Gamma)$ and paths γ_i among the open edges of η with $\gamma_i : \Delta_i \leftrightarrow x^{(i)}$. In the latter case, by (2.16)

$$\mathbf{\Phi}[J \mid \eta] = \mathbf{\Phi}[x^{(1)} \leftrightarrow x^{(2)} \text{ in } \Gamma \mid \eta] \geqq \mathbf{\Phi}_{\Gamma}^{f,p,q}[x^{(1)} \leftrightarrow x^{(2)}] .$$

In order to give a lower bound for $\mathbf{\Phi}_{\Gamma}^{f,p,q}[x^{(1)} \leftrightarrow x^{(2)}]$, we will find two non-disjoint finite slabs $S_1, S_2 \subseteq \Gamma$ with thickness $L := \lfloor N_0/4 \rfloor$, such that for $i \in \{1,2\}, x^{(i)} \in S_i$. Then, (4.6) and the FKG inequality yield (4.19). Let Γ be of the form $\mathbb{Z}^d \cap \Pi_{k=1,\dots,d}(a_k, b_k]$, where $a_k, b_k \in \mathbb{Z}$. For $i = 1,2$, set $d_i = (x_i^{(i)} - L) \vee a_i$. The following slabs have all the required properties.

$$S_1 = \mathbb{Z}^d \cap (d_1, d_1 + L] \times \prod_{k=2,\dots,d} (a_k, b_k] ,$$

$$S_2 = \mathbb{Z}^d \cap (a_1, b_1] \times (d_2, d_2 + L] \times \prod_{k=3,\dots,d} (a_k, b_k] .$$

We now turn to the second term in (4.16). Again, we introduce a new σ-field: $\mathscr{A} = \mathscr{F}_{\Lambda}^{B_{\mathbf{i}}} \vee \sigma(K_{\mathbf{i},\mathbf{j}}; \mathbf{j} \sim \mathbf{i}, \mathbf{j} \in \Lambda^{(N)})$. We claim that

$$\text{(4.20)} \qquad \mathbf{\Phi}^J[R_{\mathbf{i}}^c \cap K_{\mathbf{i}} \mid \mathscr{A}] \leqq \exp(-c_1 N^{1/2}) \quad \mathbf{\Phi}^J\text{-a.s.}$$

Note that $\mathbf{\Phi}^J[R_{\mathbf{i}}^c \cap K_{\mathbf{i}} \mid \mathscr{A}]$ vanishes on the set $K_{\mathbf{i}}^c$, therefore, it is sufficient to consider only atoms of the form $K_{\mathbf{i}} \cap \eta$, $\eta \in \Omega_{\Lambda}^{B_{\mathbf{i}}}$, with $\mathbf{\Phi}^J[K_{\mathbf{i}} \cap \eta] > 0$. But $\mathbf{\Phi}^J[R_{\mathbf{i}}^c \cap K_{\mathbf{i}} \mid K_{\mathbf{i}} \cap \eta] = \mathbf{\Phi}[R_{\mathbf{i}}^c \mid K_{\mathbf{i}} \cap \eta \cap J]$ and the measure $\mathbf{\Phi}[\,\cdot\, \mid K_{\mathbf{i}} \cap \eta \cap J] \in \mathscr{R}(\succcurlyeq, p, q, B_{\mathbf{i}})$ by (2.16). Hence, (4.7) yields (4.20).

To deal with the next term on the r.h.s. of (4.16), we first set $h_1 = h_3 = 1$ and $h_2 = \delta_1^{d+2}$. We claim that under hypothesis (H_k)

$$\text{(4.21)} \qquad \mathbf{\Phi}^J[S_{\mathbf{i}}^c \cap K_{\mathbf{i}} \mid \mathscr{A}] \leqq h_k^{-1} \exp(-c_4 N^{1/2}) \quad \mathbf{\Phi}^J\text{-a.s.}$$

For the same reason as before, it is enough to consider atoms of the form $K_{\mathbf{i}} \cap \eta, \eta \in \Omega_{\Lambda}^{B_{\mathbf{i}}}$, with $\mathbf{\Phi}^J[K_{\mathbf{i}} \cap \eta] > 0$. Assuming ($\mathrm{H}_1$) or ($\mathrm{H}_3$), $J = \Omega$. Therefore, by (4.9)

$$\text{(4.22)} \qquad \mathbf{\Phi}[S_{\mathbf{i}}^c \cap K_{\mathbf{i}} \mid K_{\mathbf{i}} \cap \eta] = \mathbf{\Phi}[S_{\mathbf{i}}^c \mid K_{\mathbf{i}} \cap \eta] \leqq \exp(-c_4 N^{1/2}) ,$$

since $K_{\mathbf{i}} \cap \eta \in \mathscr{F}_{\Lambda}^{B_{\mathbf{i}}'}$. Assuming ($\mathrm{H}_2$), we proceed as follows.

(4.23)

$$\mathbf{\Phi}^J[S_{\mathbf{i}}^c \cap K_{\mathbf{i}} \mid K_{\mathbf{i}} \cap \eta] \leqq \frac{\mathbf{\Phi}[S_{\mathbf{i}}^c \mid \eta]}{\mathbf{\Phi}[K_{\mathbf{i}} \cap J \mid \eta]} \leqq \frac{\exp(-c_4 N^{1/2})}{\mathbf{\Phi}[J \mid \eta] \Pi_{\mathbf{j} \in \Lambda^{(N)}, \mathbf{j} \sim \mathbf{i}} \mathbf{\Phi}[K_{\mathbf{i},\mathbf{j}} \mid \eta]} ,$$

where we have used the strong FKG property of $\mathbf{\Phi}$. Replacing Γ by $B_{\mathbf{i}}$ in the proof of (4.19), we have $\delta_1^2 \leqq \mathbf{\Phi}[J \mid \eta]$. Similar (in fact simpler) arguments lead to $\delta_1 \leqq \mathbf{\Phi}[K_{\mathbf{i},\mathbf{j}} \mid \eta]$. (Note that η has the property $\mathbf{\Phi}[K_{\mathbf{i}} \cap \eta] > 0$.) This gives

$$\text{(4.24)} \qquad \mathbf{\Phi}[K_{\mathbf{i}} \mid \eta] \geqq \delta_1^d ,$$

which finishes the proof of (4.21).

Assuming (H_1) and (H_2), the last term in (4.16) does not appear at all, since $V_{\mathbf{i}} = \Omega$ for all $\mathbf{i} \in \Lambda^{(N)}$. So assume ($H_3$), in which case $J = \Omega$. Again, we estimate the conditional expectation with respect to the σ-field $\mathscr{A}$, and we consider only atoms of the form $K_{\mathbf{i}} \cap \eta$, with $\eta \in \Omega_\Lambda^{B_{\mathbf{i}}}$ and $\mathbf{\Phi}[K_{\mathbf{i}} \cap \eta] > 0$. By (4.4) and (4.24)

$$\mathbf{\Phi}[V_{\mathbf{i}}^c \cap K_{\mathbf{i}} \,|\, K_{\mathbf{i}} \cap \eta] \leqq \frac{\mathbf{\Phi}[V_{\mathbf{i}}^c \,|\, \eta]}{\mathbf{\Phi}[K_{\mathbf{i}} \,|\, \eta]} \leqq v(N)\delta_1^{-d} .$$

Therefore,

$$\mathbf{\Phi}[V_{\mathbf{i}}^c \cap K_{\mathbf{i}} \,|\, \mathscr{A}] \leqq v(N)\delta_1^{-d} \quad \mathbf{\Phi}\text{-a.s.} \tag{4.25}$$

Comparing the estimates (4.17), (4.20), (4.21), (4.25) with the definitions (4.10)–(4.12), we can easily verify (4.15).

We now turn to the case $\Lambda = \mathbb{Z}^d$. Instead of proving (4.15), we verify (4.14) directly by finite volume approximations. Although the steps are straightforward, some care is required, since by changing the volume, the measure $\mathbf{\Phi}$, the event J and even the renormalized process will be changed (but only at the boundary of Λ).

For $J = \{\Delta_1 \leftrightarrow \Delta_2\}$, we consider the following finite volume approximation. Since $\Delta_1, \Delta_2 \subseteq \mathbb{Z}^d$ are finite, we can find $n_0' \in \mathbb{N}$ such that $B(n_0')$ contains those sets. For $n \geqq n_0'$, we set $J_n = \{\Delta_1 \leftrightarrow \Delta_2 \text{ in } B(n)\}$. Clearly, $\bigcap_{n \geqq n_0'} J_n = J$.

In order to prove (4.14), we fix a local increasing event $I \in \mathscr{F}_{\Gamma,\text{site}}$ for $\Gamma \subseteq \mathbb{Z}^d$, $|\Gamma| < \infty$. Let n_0 have the following properties:

(i) $B(n_0)$ contains Δ_1 and Δ_2,
(ii) $B(n_0)^{(N)} \supseteq \Gamma \cup \partial^{\text{out}}(\Gamma)$.

Given $n \geqq n_0$, we denote by $\mathbf{\Phi}_n$ the restriction of $\mathbf{\Phi}$ to the σ-field $\mathscr{F}_{B(n)}$. By (2.15), $\mathbf{\Phi}_n \in \overline{\mathscr{R}}(p,q,B(n))$. Thanks to property (ii) above, the event $G := \{X_\cdot \in I\}$ does not depend on the actual value of n (although the entire process $(X_{\mathbf{i}})_{\mathbf{i} \in B(n)^{(N)}}$ does). Applying (4.14) to the box $\Lambda = B(n)$, we have under the hypothesis (H_k)

$$\mathbf{\Phi}_n^{J_n}[G] \geqq \mathbf{P}_{\Gamma,\text{site}}^{\overline{p}_k(N),\text{indpt.}}[I] . \tag{4.26}$$

On the other hand, since G is a local event and $\bigcap_{n \geqq n_0} G \cap J_n = G \cap J$,

$$\lim_{n\to\infty} \mathbf{\Phi}_n^{J_n}[G] = \lim_{n\to\infty} \frac{\mathbf{\Phi}_n[G \cap J_n]}{\mathbf{\Phi}_n[J_n]} = \lim_{n\to\infty} \frac{\mathbf{\Phi}[G \cap J_n]}{\mathbf{\Phi}[J_n]} = \mathbf{\Phi}^J[G] = \mathbf{\Phi}^J[X_\cdot \in I] .$$

Thus (4.26) implies (4.14). By standard arguments (cf. the proof of the FKG inequality in [22]), (4.14) extends to all non-local measurable increasing events. □

Proposition 4.2 *Let $d \geqq 3$ and $q \geqq 1$. Then $\widehat{p}_s = \widehat{p}_l$.*

Proof. We will show that for $p > \widehat{p}_s$ there exists $L = L(p)$ with $p \geqq p_l(L)$. Thus $p \geqq \widehat{p}_l$, which implies $\widehat{p}_s = \widehat{p}_l$ (since $\widehat{p}_s \leqq \widehat{p}_l$).

Suppose $p > \widehat{p}_s$. Choose N so large that $\overline{p}_1(N)$ (defined in (4.13)) is larger than the critical parameter of d-dimensional Bernoulli site percolation.

It is well-known (cf. [29 Lemma 3]) that there exists $L' \in \mathbb{N}$ and $\delta > 0$ such that for all $n \geqq 1$

$$(4.27) \qquad \inf_{x,y \in S(n,L')} \mathbf{P}^{\overline{p}_1(N),\text{indpt.}}_{S(n,L'),\text{site}}[x \leftrightarrow y] \geqq \delta .$$

We may assume that L' is odd. Set $L = NL'$ and consider the slab $S(n,L)$, where $n \geqq 3N$. We will now prove

$$(4.28) \qquad \inf_{x,y \in S(n,L)} \mathbf{\Phi}^{f,p,q}_{S(n,L)}[x \leftrightarrow y] \geqq \delta \cdot (\widetilde{p})^{2d(2N)^d} ,$$

where $\widetilde{p} = p/[p + q(1-p)]$. This will imply $p > p_l(L)$ and finish the proof.

Consider the N-partition of the slab $S(n,L)$. As L' is odd, the corresponding box of blocks $\mathbf{S} := S(n,L)^{(N)}$ is a slab with thickness L'. Let us fix $x, y \in S(n,L)$. Then there exist $\mathbf{i}, \mathbf{j} \in \mathbf{S}$ with $x \in B_{\mathbf{i}}$ and $y \in B_{\mathbf{j}}$. Assume $\mathbf{i} \neq \mathbf{j}$, otherwise $\mathbf{\Phi}^{f,p,q}_{S(n,L)}[x \leftrightarrow y] \geqq (\widetilde{p})^{d(2N)^d}$. (Note that $(\widetilde{p})^{d(2N)^d}$ is a lower bound for the probability that all bonds in $[B_{\mathbf{i}}]_e$ are open, since $|B_{\mathbf{i}}| \leqq (2N)^d$). Define the event $A = \{\exists \text{ open path } \gamma \subseteq S(n,L) \text{ with } \gamma : B_{\mathbf{i}} \leftrightarrow B_{\mathbf{j}}\}$. By the FKG inequality and (2.8),

$$(4.29) \qquad \mathbf{\Phi}^{f,p,q}_{S(n,L)}[x \leftrightarrow y] \geqq \mathbf{\Phi}^{f,p,q}_{S(n,L)}[A \cap \{\text{all bonds in } [B_{\mathbf{i}}]_e \cap [B_{\mathbf{j}}]_e \text{ are open}\}]$$
$$\geqq (\tilde{p})^{2d(2N)^d} \cdot \mathbf{\Phi}^{f,p,q}_{S(n,L)}[A] .$$

We will now show that $\mathbf{\Phi}^{f,p,q}_{S(n,L)}[A] \geqq \delta$. Consider the renormalized process with hypothesis (H_1) on $\mathbf{S}$ and denote by $\mathbf{C}_{\mathbf{i}}$ the cluster of occupied blocks of $\mathbf{i}$ ($\mathbf{C}_{\mathbf{i}}$ is empty, if $\mathbf{i}$ is vacant). By the discussion after (4.5), the existence of occupied blocks joining $\mathbf{i}$ with $\mathbf{j}$ implies the occurrence of an open path in the microscopic process joining the blocks $B_{\mathbf{i}}$ with $B_{\mathbf{j}}$. Therefore $\{\mathbf{j} \in \mathbf{C}_{\mathbf{i}}\} \subseteq A$. On the other hand, by Proposition 4.1, the renormalized process dominates Bernoulli percolation with parameter $\overline{p}_1(N)$. Therefore, by (4.27), we have $\mathbf{\Phi}^{f,p,q}_{S(n,L)}[\mathbf{i} \in \mathbf{C}_{\mathbf{j}}] \geqq \delta$, which yields the desired lower bound. □

5 Proofs

In order to prove Theorem 1.2, we need two preparatory results. For given $\underline{n} \in \mathscr{X}_2(n)$, we denote by $\mathbb{B}(\underline{n})$ the set of 'boundary-clusters' of $B(\underline{n})$, i.e., the clusters intersecting $\partial B(\underline{n})$.

Lemma 5.1 *Let $d \geqq 2$, $q \geqq 1$ and $p \in [0,1]$. For $\delta > 0$, we have*

(5.1)
$$\overline{\lim_{n\to\infty}} \frac{1}{n^d} \log \left(\sup_{\underline{n} \in \mathscr{X}_2(n)} \sup_{\mathbf{\Phi} \in \mathscr{R}(p,q,B(\underline{n}))} \mathbf{\Phi}\left[\sum_{C \in \mathbf{B}(\underline{n})} |C| > (\theta^w + \delta)|B(\underline{n})| \right] \right) < 0 .$$

Proof. The event occurring in the expression above is increasing. Consequently, it is sufficient to show (5.1) for wired boundary conditions. The corresponding FK measure will be denoted by $\mathbf{\Phi}^w_{B(\underline{n})}$. For $n \in \mathbb{N}^+$, set $Q(n) = \{x \in$

$B(n)$; $\operatorname{dist}(x, \partial B(n)) \geqq n^{1/2}\}$, and let $B(x,r)$ stand for the box $\tau_x(B(r))$. We have the following estimate on the expected fractional volume of the boundary clusters in $B(n)$.

$$\varlimsup_{n\to\infty} \mathbf{\Phi}^w_{B(n)}\left[n^{-d}\sum_{C\in\mathbf{B}(n)}|C|\right] = \varlimsup_{n\to\infty} n^{-d}\sum_{x\in Q(n)} \mathbf{\Phi}^w_{B(n)}[x\leftrightarrow \partial B(n)]$$

$$\leqq \varlimsup_{n\to\infty} n^{-d}\sum_{x\in Q(n)} \mathbf{\Phi}^w_{B(x,n^{1/2})}[x\leftrightarrow \partial B(x,n^{1/2})] = \theta^w\,,$$

where we have used the equality $\theta^w = \lim_{n\to\infty}\mathbf{\Phi}^w_{B(n)}[0\leftrightarrow \partial B(n)]$. Now fix $\delta>0$ and choose N so large that the expected value $\mathbf{\Phi}^w_{B(N)}[N^{-d}\sum_{C\in\mathbf{B}(N)}|C|]\leqq \theta^w+\delta/2$. For a given N-large box $B(\underline{n})$ consider its N-partition, and denote by $\mathbf{B}$ the re-scaled box $B(\underline{n})^{(N)}$. For $\mathbf{i}\in\mathbf{B}$, set $Y_{\mathbf{i}} = |\{x\in B_{\mathbf{i}};\ x\leftrightarrow\partial B_{\mathbf{i}}\}|$ and denote by $S_{\mathbf{i}}$ the increasing event that every bond joining two sites in $\partial B_{\mathbf{i}}$ is open. Setting $\overset{\circ}{\mathbf{B}} = \mathbf{B}\backslash\partial\mathbf{B}$, we have for any $\underline{n}\in\mathscr{X}_2(n)$

$$|B(\underline{n})|^{-1}\sum_{C\in\mathbf{B}(\underline{n})}|C| \leqq |B(\underline{n})|^{-1}\sum_{\mathbf{i}\in\overset{\circ}{\mathbf{B}}} Y_{\mathbf{i}} + 4Nd/n\,.$$

Thus, by using the FKG inequality,

$$\mathbf{\Phi}^w_{B(\underline{n})}\left[\sum_{C\in\mathbf{B}(\underline{n})}|C| > (\theta^w+\delta)|B(\underline{n})|\right]$$

$$\leqq \mathbf{\Phi}^w_{B(\underline{n})}\left[\sum_{\mathbf{i}\in\overset{\circ}{\mathbf{B}}} Y_{\mathbf{i}} > (\theta^w+\delta-4Nd/n)|B(\underline{n})|\ \middle|\ \bigcap_{\mathbf{i}\in\overset{\circ}{\mathbf{B}}} S_{\mathbf{i}}\right].$$

By (2.9), the (bounded) variables $(Y_{\mathbf{i}})_{\mathbf{i}\in\overset{\circ}{\mathbf{B}}}$ are i.i.d. with respect to the conditional measure $\mathbf{\Phi}^w_{B(\underline{n})}[\,\cdot\mid\bigcap_{\mathbf{i}\in\overset{\circ}{\mathbf{B}}} S_{\mathbf{i}}]$. By the choice of N, their expected value is not larger than $(\theta^w+\delta/2)N^d$. Therefore, the Theorem of Cramer applies and (5.1) follows immediately. □

In Sect. 3 we have introduced the event $U(\underline{n}) = \{\exists!$ crossing cluster C^* in $B(\underline{n})\}$ and have seen that it is a typical event up to large deviations of order n, cf. (3.6). For $\delta>0$ let us define the event $V(\underline{n},\delta) = U(\underline{n})\cap\{|C^*|\in(\theta^f-\delta,\ \theta^w+\delta)|B(\underline{n})|\}$.

Lemma 5.2 *Let* $d\geqq 3$, $q\geqq 1$ *and* $p>\widehat{p}_1$. *Then for each* $\delta>0$,

$$\text{(5.2)}\qquad \varlimsup_{n\to\infty}\frac{1}{n}\log\left(\sup_{\underline{n}\in\mathscr{X}_2(n)}\ \sup_{\mathbf{\Phi}\in\mathscr{R}(p,q,B(\underline{n}))}\mathbf{\Phi}[V(\underline{n},\delta)^c]\right) < 0\,.$$

Proof. Because $C^*\in\mathbf{B}(B(\underline{n}))$, in view of Lemma 5.1 we can replace $V(\underline{n},\delta)^c$ in the above expression by the event $U(\underline{n})\cap\{|C^*|\leqq(\theta^f-\delta)|B(\underline{n})|\}$. As before, we use an appropriate block-argument, which allows us to apply the Theorem of Cramer. Using the notation of Lemma 5.1, we start with the following estimate.

$$\lim_{n\to\infty} \Phi^f_{B(n)}\left[n^{-d} \sum_{C;\,\mathrm{diam}(C)\geqq n^{1/2}} |C|\right] \tag{5.3}$$

$$\geqq \lim_{n\to\infty} n^{-d} \sum_{x\in Q(n)} \Phi^f_{B(n)}[\mathrm{diam}(C_x) \geqq n^{1/2}]$$

$$\geqq \lim_{n\to\infty} n^{-d} \sum_{x\in Q(n)} \Phi^f_{B(x,n^{1/2})}[x \leftrightarrow \partial B(x,n^{1/2})] = \theta^f .$$

In the last line we have used part (i) of Theorem 3.2. Let us now fix $\delta > 0$. Theorem 1.1 of [11] ensures that we can choose $p_0 \in (0,1)$ such that for all $p > p_0$,

$$\overline{\lim_{m\to\infty}} \frac{1}{m^{d-1}} \log\left(\sup_{\underline{m}\in\mathscr{X}_2(m)} \mathbf{P}^{p,\,\mathrm{indpt.}}_{B(\underline{m}),\,\mathrm{site}}[\exists!\text{ crossing cluster } \tilde{C} \text{ with } |\tilde{C}| \geqq (1-\delta/2^{d+1})|B(\underline{m})|]\right) < 0 .$$

Set $v(N) = 1$ for every N in the definition of $\overline{q}_3$ in (4.12), and denote by $\overline{p}(N)$ the corresponding values of $\overline{p}_3(N)$. Choose N so large that $\overline{p}(N) \geqq p_0$ and

$$\Phi^f_{B(N)}\left[N^{-d} \sum_{C;\,\mathrm{diam}(C)\geqq N^{1/2}} |C|\right] \geqq \theta^f - \delta/4 .$$

For a given N-large box $B(\underline{n})$, consider its N-partition and the corresponding renormalized process defined in (4.5) with hypothesis (H_3), where we set $V = \Omega$ for every $V \in \mathscr{V}$. The re-scaled box will be denoted by $\mathbf{B}$. Set

$$Z(\underline{n}) = U(\underline{n}) \cap \{\exists!\text{ crossing cluster (of blocks) } \tilde{\mathbf{C}} \text{ in } \mathbf{B} \text{ with } |\tilde{\mathbf{C}}| \geqq (1-\delta/2^{d+1})|\mathbf{B}|\} .$$

By (3.6) and Proposition 4.1, we have

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log\left(\sup_{\underline{n}\in\mathscr{X}_2(n)} \sup_{\Phi\in\mathscr{R}(p,q,B(\underline{n}))} \Phi[Z(\underline{n})^c]\right) < 0 . \tag{5.4}$$

Therefore, we have only to give an upper bound on the probability of the event $W(\underline{n}) := Z(\underline{n}) \cap \{|C^*| < (\theta^f - \delta)|B(\underline{n})|\}$. On the set $Z(\underline{n})$, as explained after (4.5), there exists a (microscopic) cluster $\tilde{C}$ containing all 'small' crossing clusters $C^*(B_{\mathbf{i}})$, $\mathbf{i} \in \tilde{\mathbf{C}}$. Since $\tilde{C}$ is a crossing cluster for $B(\underline{n})$, $\tilde{C}$ and C^* must be the same. Setting $Y_{\mathbf{i}} = \sum_{C;\,\mathrm{diam}(C)\geqq N^{1/2}} |C|$, we have the following lower bound for $|C^*|$

$$|C^*| \geqq \sum_{\mathbf{i}\in\tilde{\mathbf{C}}} Y_{\mathbf{i}} \geqq \sum_{\mathbf{i}\in\mathbf{B}} Y_{\mathbf{i}} - (\delta/2)|B(\underline{n})| \geqq \sum_{\mathbf{i}\in\mathring{\mathbf{B}}} Y_{\mathbf{i}} - (\delta/2)|B(\underline{n})| .$$

Hence,

$$W(\underline{n}) \subseteq Z(\underline{n}) \cap \left\{ \frac{1}{|B(\underline{n})|} \sum_{\mathbf{i} \in \overset{\circ}{\mathbf{B}}} Y_{\mathbf{i}} - (\delta/2) < \theta^f - \delta \right\}$$
$$\subseteq \left\{ \frac{1}{|B(\underline{n})|} \sum_{\mathbf{i} \in \overset{\circ}{\mathbf{B}}} Y_{\mathbf{i}} < \theta^f - \delta/2 \right\}.$$

Denote by $E(\underline{n})$ the event, that for each $\mathbf{i} \in \overset{\circ}{\mathbf{B}}$ and for every bond in $\partial^{\text{out}} B_{\mathbf{i}}$ is closed. Observing that $\sum_{\mathbf{i} \in \mathbf{B}^\circ} Y_{\mathbf{i}}$ is an increasing function, we have for each $\mathbf{\Phi} \in \mathscr{R}(p, q, B(\underline{n}))$,

$$\mathbf{\Phi}[W(\underline{n})] \leqq \mathbf{\Phi}^f_{B(\underline{n})} \left[\left\{ \frac{1}{|B(\underline{n})|} \sum_{\mathbf{i} \in \overset{\circ}{\mathbf{B}}} Y_{\mathbf{i}} < \theta^f - \delta/2 \right\} \middle| E(\underline{n}) \right]. \tag{5.5}$$

By (2.9), the variables $(Y_{\mathbf{i}})_{\mathbf{i} \in \overset{\circ}{\mathbf{B}}}$ are i.i.d. with respect to the conditional measure $\mathbf{\Phi}^f_{B(\underline{n})}[\,\cdot \mid E(\underline{n})]$, with an expected value larger than $(\theta^f - \delta/4)|B(N)|$. The Theorem of Cramer completes the proof. □

Proof of Theorem 1.2. First we prove the upper bound. Replace the condition $n^{-d}|C_m| \in (\theta^f - \varepsilon, \theta^w + \varepsilon)$ in the definition of $K(n, \varepsilon, l)$ by the condition $n^{-d}|C_m| > (\theta^f - \varepsilon)$, and denote the new but otherwise unchanged event by $K'(n, \varepsilon, l)$. By Lemma 5.1, we can replace $K(n, \varepsilon, L)$ by $K'(n, \varepsilon, L)$ in (1.10). Theorem 1.1 in [11] allows us to choose $p_0 \in (0, 1)$, such that for all $p > p_0$, we have

$$\overline{\lim_{k \to \infty}} \frac{1}{k^{d-1}} \log \mathbf{P}^{p,\text{indpt.}}_{B(k),\text{site}}[\{\exists! \text{ crossing cluster } \tilde{C} \text{ in } B(k) \text{ with } |\tilde{C}| \geqq (1 - \varepsilon/4)k^d\}^c] < 0. \tag{5.6}$$

We want to apply Proposition 4.1 with hypothesis (H_3). First we choose the family $\mathscr{V}$ by setting

$$V_\Gamma = \left\{ \exists! \text{ cluster } C^* \text{ in } \Gamma \text{ and } \frac{1}{|\Gamma|}|C^*| \in (\theta^f - \varepsilon/4, \theta^w + \varepsilon/4) \right\}. \tag{5.7}$$

Note that by Lemma 5.2, (4.4) is satisfied. Choose N so large that $\overline{p}_3(N) \geqq p_0$ ($\overline{p}_3(N)$ was defined in (4.12)) and

$$N^{1/2} \leqq 16d/\varepsilon. \tag{5.8}$$

Set $L = 2N$. Clearly, L depends only on p, q, d, ε. Consider the N-partition of the box $B(n)$, where $n \geqq 16(2N)^d/\varepsilon$, and the corresponding renormalized process on $\mathbf{B} := B(n)^{(N)}$. Set

$$Z(n, \varepsilon, N) = \{\exists! \text{ crossing cluster } \tilde{\mathbf{C}} \text{ of occupied blocks in } \mathbf{B} \text{ with } |\tilde{\mathbf{C}}| \geqq (1 - \varepsilon/4)|\mathbf{B}|\}$$

By Proposition 4.1 and (5.6), we have

$$\overline{\lim_{n\to\infty}} \frac{1}{n^{d-1}} \log\left(\sup_{\Phi\in\mathscr{R}(p,q,B(n))} \Phi\left[Z(n,\varepsilon,N)^c\right]\right) < 0\,. \tag{5.9}$$

We claim $Z(n,\varepsilon,N) \subseteq K'(n,\varepsilon,L)$. This fact, together with (5.9), will imply (1.10). In order to prove this inclusion, we first define the following regions:

$$G_{\mathbf{i}} = \{x \in B_{\mathbf{i}} \,|\, \mathrm{dist}\,(x, \partial B_{\mathbf{i}}) \leqq N^{1/2}\}; \quad G = \bigcup_{\mathbf{i}\in\mathbf{B}} G_{\mathbf{i}}$$

$$Q_{\mathbf{i}} = B_{\mathbf{i}} \backslash G_{\mathbf{i}}$$

By (5.8), the volume of G is less than $(\varepsilon/4)n^d$. Note that $n \geqq 16d(2N)^d/\varepsilon$ implies that $n^{-d}\sum_{\mathbf{i}\in\partial\mathbf{B}}|B_{\mathbf{i}}| < \varepsilon/4$.

Consider the cluster C_m in $B(n)$ containing all the 'small' crossing clusters $C^*(B_{\mathbf{i}})$ for $\mathbf{i} \in \tilde{\mathbf{C}}$. As $\tilde{\mathbf{C}}$ is a crossing cluster, C_m is also crossing. By elementary calculations, we have $|C_m| \geqq (\theta^f - \varepsilon)n^d$. Any other cluster $C \neq C_m$ lies either in just one block (in this case $\mathrm{diam}(C) \leqq 2N = L$, thus the cluster is L-small) or in at least two different blocks. However, in the latter case C may not touch the set $\bigcup_{\mathbf{i}\in\tilde{\mathbf{C}}} Q_{\mathbf{i}}$; otherwise we would have that $\mathrm{diam}(C \cap B_{\mathbf{i}}) \geqq N^{1/2}$ for an $\mathbf{i} \in \tilde{\mathbf{C}}$, and therefore, $C = C_m$. Consequently, any cluster $C \neq C_m$, which is not L-small, lies in the set $G \cup \{\bigcup_{\mathbf{i}\in\mathbf{C}^c} B_{\mathbf{i}}\}$. The volume of this set is smaller than $(3\varepsilon/4)n^d$. Since $n^d(3\varepsilon/4) \vee L^d < (\theta^f - \varepsilon)n^d$, C_m must be a unique cluster of maximal size, and the intermediate class $\mathbb{J}_L$ has a smaller total volume than $(\varepsilon/2)n^d$. This proves that $Z(n,\varepsilon,N) \subseteq K'(n,\varepsilon,L)$ and implies finally (1.10).

The lower bound is much easier to prove. First, we think of the lattice $\mathscr{L}^d$ embedded in $\mathbb{R}^d$. The bonds can be identified with line segments between two nearest neighbors. For given n, we consider the following hyper-planes:

$$E(n,k) = \{r \in \mathbb{R}^d \,|\, r_1 = kn\theta^f/3\}; \quad k \in \mathbb{Z}\,.$$

Denote by $F(n)$ the set of bonds in $[B(n)]_e$ intersecting the union of these planes $\bigcup_{k\in\mathbb{Z}} E(n,k)$. Then $|F(n)| \leqq cn^{d-1}$ for a positive constant c. Consider the event that all bonds in $F(n)$ are closed. If this event occurs, so does $K(n,\varepsilon,L)^c$, since there cannot exist any cluster with a size larger than $(\theta^f/3)n^d$. But the probability of this event with respect to any measure $\Phi \in \mathscr{R}(p,q,B(n))$ has the lower bound $(p/p + q(1-p))^{|F(n)|}$, which implies (1.9). □

Proof of Theorem 1.1. Recall that for $\beta \neq \beta_c$, $m^* = \theta^w$ and $m^{*,f} = \theta^f$, cf. (2.21) and (2.23). To prove (1.16), it is enough to consider intervals (a,b) of the form $a = -\theta^f + \delta$ and $b = \theta^f - \delta$ for $\delta > 0$. Let $\delta \in (0, \theta^f)$ be fixed and denote by $G(\delta, n)$ the event $\{|m_{B(n)}| < \theta^f - \delta\}$. Set $\varepsilon = (\delta/4) \wedge (\theta^f/2) \wedge (1-\theta^w)/4 (> 0)$ and $p = 1 - e^{-\beta}$. Choose $L = L(p,d,\varepsilon)$ according to Theorem 1.2. For given $n \geqq 2dL/\varepsilon$ pick $\mu = \mu_{B(n)}^{\Delta(+),\beta} \in \mathscr{I}^+(\beta, B(n))$. Denote by Φ the corresponding random cluster measure $\Phi_{B(n)}^{\pi(\Delta),p,2}$ in the FK representation

(2.18). Then $\mathbf{\Phi} \in \mathscr{R}(p, 2, B(n))$ and we have

$$\mu[G(\delta,n)] = \int P^{\eta,\Delta}[G(\delta,n)]\mathbf{\Phi}[d\eta] \tag{5.10}$$
$$\leqq \mathbf{\Phi}[K(n,\varepsilon,L)^c] + \int_{K(n,\varepsilon,L)} P^{\eta,\Delta}[G(\delta,n)]\mathbf{\Phi}[d\eta].$$

We show below that there exists a constant $c = c(p,d,\delta) > 0$ such that for each $n \geqq 2dL/\varepsilon$, $\eta \in K(n,\varepsilon,L)$ and $\Delta \subseteq \partial B(n)$

$$P^{\eta,\Delta}_{B(n)}[G(\delta,n)] \leqq 2\exp(-cn^d). \tag{5.11}$$

This and Theorem 1.2 then immediately imply (1.6). We call an L-small cluster *internal* if it does not touch $\partial B(n)$, and denote by $\mathbb{S}'_L$ the set of such clusters. The total magnetization of sites lying in an internal L-small cluster will be denoted by $M_{\mathbb{S}'_L}$.

In order to prove (5.11), fix a configuration $\eta \in \Omega_{B(n)}$ with $\eta \in K(n,\varepsilon,L)$. Then we know that there exists a largest cluster C_m with $n^{-d}|C_m| \in (\theta^f - \varepsilon, \theta^w + \varepsilon)$, and the total volume of the L-intermediate class is smaller than εn^d. By choice of n, we have $|\mathbb{S}_L \backslash \mathbb{S}'_L| < \varepsilon n^d$ as well. Since $4\varepsilon \leqq \delta$, after the Δ-coloring, the absolute value of the total magnetization of sites lying either in C_m or in a cluster of the intermediate class or in an L-small cluster touching $\partial B(n)$ must be in $(n^d(\theta^f - 3\varepsilon),\ n^d(\theta^w + 3\varepsilon))$. Consequently, the occurrence of $G(\delta,n)$ implies $|M_{\mathbb{S}'_L}| > \varepsilon n^d$. Let $(X_i)_{i\geqq 1}$ be a sequence of i.i.d. variables with $\mathbf{P}[X_1 = 1] = \mathbf{P}[X_1 = -1] = 1/2$ w.r.t. a certain probability measure $\mathbf{P}$. Enumerate the internal L-small clusters in $B(n)$: $C_1, C_2, \ldots, C_{k_0}$ and set $c_i = |C_i|$ for $i = 1, \ldots, k_0$. Since $k_0 \leqq n^d$, we have

$$P^{\eta,\Delta}[G(\delta,n)] \leqq P^{\eta,\Delta}[|M_{\mathbb{S}'_L}| > \varepsilon n^d] = \mathbf{P}\left[\left|\left(\sum_{i=1,\ldots,k_0} c_i X_i\right)\right| > \varepsilon n^d\right] \tag{5.12}$$

$$\leqq 2\mathbf{P}\left[\frac{1}{k_0}\sum_{i=1,\ldots,k_0} c_i X_i > \varepsilon\right] \tag{5.13}$$

By Lemma 5.3 below, we can bound this by

$$2\exp(-k_0 \Lambda^*_{L^d \cdot X_1}(\varepsilon)), \tag{5.14}$$

where $\Lambda^*_{L^d \cdot X_1}$ denotes the Legendre-transform of the logarithmic moment-generating function of the random variable $L^d \cdot X_1$. Note that $\Lambda^*_{L^d \cdot X_1}(\varepsilon)$ is strictly positive, since $\varepsilon > 0$. We now claim that k_0 grows in volume order with the block size n uniformly in $\eta \in K(n,\varepsilon,L)$. Note first that $\sum_{C \in \mathbb{S}'_L}|C| \geqq (1 - \theta^w - 3\varepsilon)n^d$. By choice of ε, $(1 - \theta^w - 3\varepsilon) \geqq \varepsilon$. Since the volume of a small cluster is bounded by L^d, the number of small clusters k_0 must satisfy $k_0 \geqq \varepsilon(n/L)^d$. Substituting this into (5.14), we arrive at (5.11) with $c(p,d,\delta) := \varepsilon\Lambda^*_{L^d \cdot X_1}(\varepsilon)/L^d$. (Recall that ε depends only on (p,d,δ).) Finally, we give Lemma 5.3.

Lemma 5.3 *Consider a sequence* $(X_i)_{i\geqq 1}$ *of bounded i.i.d. variables with mean zero on a probability space* $(\Omega, \mathscr{F}, P)$. *For given* $K \geqq 0$ *and (deterministic) sequence* $(c_i)_{i\geqq 1}$ *with* $c_i \in [0,K]$, *set* $Y_i = c_i X_i$ *and* $\overline{Y}_n = \frac{1}{n}\sum_{i=1,\dots,n} Y_i - E[\frac{1}{n}\sum_{i=1,\dots,n} Y_i]$. *Then, we have for any* $\varepsilon > 0$,

$$P[\overline{Y}_n \geqq \varepsilon] \leqq \exp(-n\Lambda^*_{K\cdot X_1}(\varepsilon)), \tag{5.15}$$

where $\Lambda^*_{K\cdot X_1}$ *denotes the Legendre-transform of the logarithmic moment-generating function of the random variable* $K \cdot X_1$.

Proof. By a standard estimate (cf. [12, Lemma 1.2.3]), we have $P[\overline{Y}_n \geqq \varepsilon] \leqq \exp(-\Lambda^*_{\overline{Y}_n}(\varepsilon))$. Therefore, we need a lower bound for $\Lambda^*_{\overline{Y}_n}(\varepsilon)$. Using Jensen's inequality, we first have

$$\Lambda_{\overline{Y}_n}(\lambda) = \log E\left[\exp\left(\sum_{i=1,\dots,n} (\lambda/n) c_i X_i\right)\right] = \sum_{i=1,\dots,n} \log E[\{\exp((\lambda K/n) X_i)\}^{c_i/K}]$$
$$\leqq n\Lambda_{L\cdot X_1}(\lambda/n)$$

This yields $\Lambda^*_{\overline{Y}_n}(\varepsilon) \geqq n\Lambda^*_{K\cdot X_1}(\varepsilon)$. □

The proof of (1.7) is based on the observation that the sign of the spin given to C_m determines the sign of $m_{B(n)}$. Since C_m is a crossing cluster, it intersects $\partial B(n)$. By the 'coloring rules' of the FK representation (2.18) for + boundary conditions, C_m has (deterministically) spin +1. This in turn implies (1.7). □

The final part of this section is devoted to the generalization of Theorem 1.1 to the q-state Potts model. The only difficulty is that, at least from the mathematical point of view, there is no *natural* analogue of the empirical magnetization $m_{B(n)} = n^{-d}\sum_{x\in B(n)}\sigma_x$ in the Potts model. Nevertheless, we have the following empirical quantity, which, in the case $q = 2$, bears the same information as $m_{B(n)}$. Fix a color $c \in \{1,\dots,q\}$, and set $X^{(c)}_{B(n)} = n^{-d}\sum_{x\in B(n)} \mathbb{1}_{\{\sigma(x)=c\}}$ and $X_{B(n)} = \{X^{(1)}_{B(n)}, X^{(2)}_{B(n)}, \dots, X^{(q)}_{B(n)}\}$. (For $q = 2$, we have $m_{B(n)} = 2X^{(1)}_{B(n)} - 1$).

For $* = f$ or w, we set $\tau^* = \tau^*(p,q) = (1-\theta^*)/q$. Given a color r, we define the event

$$A(n,(r),\delta) = \{X^{(r)}_{B(n)} \in (\theta^f + \tau^f - \delta, \theta^w + \tau^w + \delta)\}$$
$$\cap \bigcap_{c\in\{1,2,\dots,q\}\setminus\{r\}} \{X^{(c)}_{B(n)} \in (\tau^w - \delta, \tau^f + \delta)\}\,.$$

Recall, that by (2.21) and (2.23), for every $\beta \neq \beta_c$, $m^*(\beta,q) = \theta^w(p,q)$ and $m^{*,f}(\beta,q) = \theta^f(p,q)$, where $p = 1 - e^{-\beta}$. The (surface order) large deviation behavior of X is described in the following theorem. Its proof is omitted, since it would only be a repetition of the arguments in the proof of Theorem 1.1.

Theorem 5.4 *Let $d \geqq 3$, $q \geqq 1$ and $\beta > \hat{\beta}_1(q)$. For every $\delta > 0$,*

$$(5.16)\quad \overline{\lim_{n\to\infty}} \frac{1}{n^{d-1}} \log\left(\sup_{\mu \in \mathscr{P}^{(1)}(\beta,q,B(n))} \mu\left[\left(\bigcup_{r\in\{1,2,\dots,q\}} A(n,(r),\delta)\right)^c\right]\right) < 0\,.$$

For (1) *boundary conditions, we have*

$$(5.17)\quad \overline{\lim_{n\to\infty}} \frac{1}{n^{d-1}} \log \mu_{B(n)}^{(1),\beta,q}[A(n,(1),\delta)^c] < 0\,.$$

Appendix A

Here we provide some supplementary material, references and proofs to Subsect. 2.2. We first recall some general facts about FKG-Holley inequalities. We will use the notation of Sect. 2 and write η for the cylinder $\{\eta\}$. Let $V \in \mathbb{Z}^d$ be finite and set $E = [V]_e$. A probability measure μ on $\mathscr{F}_V$ satisfies the (FKG) *lattice condition* if for every $\omega, \eta \in \Omega_V$

$$(A.1)\quad \mu(\omega \vee \eta)\mu(\omega \wedge \eta) \geqq \mu(\omega)\mu(\eta)\,,$$

where $(\omega \vee \eta)(b) = \omega(b) \vee \eta(b)$ and $(\omega \wedge \eta)(b) = \omega(b) \wedge \eta(b)$. Then μ is strong FKG (see e.g. Theorem 1 in [26]). For two probability measures μ and ν on $\mathscr{F}_V$, we say that μ *convexly dominates* ν if for all $\omega, \eta \in \Omega_V$

$$(A.2)\quad \mu(\omega \vee \eta)\nu(\omega \wedge \eta) \geqq \mu(\omega)\nu(\eta)\,.$$

Then $\mu \succcurlyeq \nu$ (see e.g. Theorem 2 in [26]). It can be directly verified that (A.2) implies for each cylinder Z, that the conditional measure $\mu(\,\cdot\,|Z)$ convexly dominates $\nu(\,\cdot\,|Z)$. Therefore, $\mu \overset{s}{\succcurlyeq} \nu$.

Let $f \in \mathscr{F}_V$ be an increasing function. For given μ, we define μ' by its Radon–Nykodim derivative $d\mu'/d\mu := f/\mu(f)$. If μ satisfies the lattice condition then it can be immediately seen that the measure μ' convexly dominates μ, and, therefore, $\mu' \overset{s}{\succcurlyeq} \mu$.

We now turn to the discussion of the individual statements of Sect. 2.2. We begin with inequality (2.2), which was proved for free and wired boundary conditions in e.g. [4], Theorem 2.1. In fact, the arguments there are valid for every $\mathbf{\Phi} = \mathbf{\Phi}_V^{\pi,p,q} \in \mathscr{R}(p,q,V)$, since c^f shares the relevant monotonicity properties with c^π.

The proof of (2.3) is as follows. Set $S_1 = \{\text{pr} \equiv 0 \text{ on } E_0' \text{ and pr} \equiv 1 \text{ on } E_1\}$, $S_2 = \{\text{pr} \equiv 0 \text{ on } E_0 \backslash E_0'\}$ and $S_3 = \{\text{pr} \equiv 1 \text{ on } E_1' \backslash E_1\}$. Note that S_2 is decreasing and S_3 is increasing. Occasionally, we denote by $\mathbf{\Phi}^Z$ the conditional measure $\mathbf{\Phi}(\cdot\,|Z)$. Then, $\mathbf{\Phi}^Z(\cdot) = \mathbf{\Phi}^{S_1}(\cdot\,|S_2)$ and $\Delta^{Z'}(\cdot) = \mathbf{\Phi}^{S_1}(\cdot\,|S_3)$. Since $\mathbf{\Phi}^{S_1}$ satisfies the lattice condition, we have for any increasing event $J \in \mathscr{F}_V$,

$$\mathbf{\Phi}^Z(J) = \mathbf{\Phi}^{S_1}(J\,|S_2) \leqq \mathbf{\Phi}^{S_1}(J) \leqq \mathbf{\Phi}^{S_1}(J\,|S_3) = \mathbf{\Phi}^{Z'}(J)\,,$$

which shows $\Phi^Z \preccurlyeq \Phi^{Z'}$. This proves $\Phi^Z \overset{s}{\preccurlyeq} \Phi^{Z'}$ as well, since two-fold conditioning on (compatible) cylinders is just a one-fold conditioning on a 'smaller' cylinder.

We now turn to (2.4). It was shown in [4], Theorem 4.1 that under both assumptions, the measure $\Phi_V^{f,p',q'}$ has an increasing density function with respect to $\Phi_V^{f,p,q}$. This implies convex domination, and therefore strong domination for free b.c.s. Again, the same arguments are valid for general b.c.s, since the relevant monotonicity properties of c^π are the same as those of c^f.

Statement (2.5) seems not to appear in modern accounts, so we give a short proof. We will directly verify inequality (A.2). Denoting by P^p the law of Bernoulli bond percolation with parameter p, we rewrite inequality (A.2) as follows

$$q^{c^{\pi'}(\omega\vee\eta)}q^{c^{\pi}(\omega\wedge\eta)}P^p[\omega\vee\eta]P^p[\omega\wedge\eta] \geqq q^{c^{\pi'}(\omega)}q^{c^{\pi}(\eta)}P^p[\omega]P^p[\eta]\,.$$

Since $P^p[\omega\vee\eta]P^p[\omega\wedge\eta] = P^p[\omega]P^p[\eta]$ (by direct computation), we have to show

$$c^{\pi'}(\omega) - c^{\pi'}(\omega\vee\eta) \leqq c^{\pi}(\omega\wedge\eta) - c^{\pi}(\eta) \tag{A.3}$$

for every $\omega,\eta\in\Omega_V$. As a byproduct of the proof of (2.2), one knows that for fixed $\eta, c^{\pi'}(\omega) - c^{\pi'}(\omega\vee\eta)$ is a decreasing function of ω. Since $\omega \succcurlyeq \omega\wedge\eta$, the r.h.s. of (A.3) is not larger than $c^{\pi'}(\omega\wedge\eta) - c^{\pi'}(\eta)$. So it is sufficient to prove

$$c^{\pi'}(\omega\wedge\eta) - c^{\pi'}(\eta) \leqq c^{\pi}(\omega\wedge\eta) - c^{\pi}(\eta)\,. \tag{A.4}$$

Using an analogous telescoping decomposition of (A.4) as in the proof of (2.2), it is easily seen that it is sufficient to show (A.4) for the case in which ω has exactly one closed bond, say $b = \{x,y\}$. Then both sides have value 0 or 1, and are equal, except in the following case:

(i) x and y are π'-wired but not π-wired
(ii) $\eta(b) = 1$ but x and y are not connected in $E\backslash\{b\}$ by an η-open path.

In this case, the l.h.s. is equal to 0, and the r.h.s. is equal to 1. This completes the proof.

The proof of the statements (2.7)–(2.9) needs nothing more than straightforward calculation, and is omitted. Finally, we prove Lemma 2.1. Denote by $\operatorname{Ext}\mathscr{R}(U)$ the set of extremal elements of $\overline{\mathscr{R}(p,q,U)}$. Obviously, $\operatorname{Ext}\mathscr{R}(U) \subseteq \mathscr{R}(p,q,U)$, and therefore, finite. Note that each element $\Phi\in\overline{\mathscr{R}(p,q,U)}$ has a unique representation of the form $\Phi = \sum_{\rho\in\operatorname{Ext}\mathscr{R}(U)} a_\rho(\Phi)\rho$, with $a_\rho(\Phi) \geqq 0$ and $\sum_\rho a_\rho(\Phi) = 1$.

We now turn to the proof of (2.15). Assume first that V is finite, and fix $\Phi_V^\pi \in \mathscr{R}(p,q,V)$. By (2.7), we have for every $g\in\mathscr{F}_U$

$$\begin{aligned}\Phi_V^\pi[g] &= \Phi_V^\pi(\Phi_V^\pi[g\,|\,\mathscr{F}_V^U])\\ &= \sum_{\phi\in\mathscr{R}(p,q,U)} \Phi_V^\pi[\Phi_U^{\pi\cdot W_V^U(\,\cdot\,)} = \phi]\phi(g) = \sum_{\rho\in\operatorname{Ext}\mathscr{R}(U)} a_\rho(\Phi_V^\pi)\rho(g)\,,\end{aligned}$$

for certain coefficients $a_\rho(\Phi_V^\pi)$ depending only on the measure Φ_V^π, which proves the claim for finite V. Assume now $V = \mathbb{Z}^d$. Let $*$ stand for f or w. Then

$$\Phi_\infty^*[g] = \lim_{n\to\infty} \Phi_{B(n)}^*[g] = \lim_{n\to\infty} \sum_{\rho\in \mathrm{Ext}\,\mathscr{R}(U)} a_\rho(\Phi_{B(n)}^*)\rho(g)\,.$$

Since $a_\rho(\Phi_{B(n)}^*)$ does not depend on g, and by compactness, it follows that for each ρ the sequence $a_\rho(\Phi_{B(n)}^*)$ converges to a value in [0,1]. This completes the proof of (2.15). To prove (2.16), we fix a cylinder $S \in \mathscr{F}_U$, and an increasing event $A \in \mathscr{F}_U$. By using the strong FKG property and (2.3), we have

$$\Phi_V^{J\cap Z}[A\,|\,S] = \Phi_V^{Z\cap S}[A\,|\,J] \geqq \Phi_V^{Z\cap S}[A] = \Phi_V^\pi[A\,|\,Z\cap S]$$

$$\geqq \Phi_V^\pi[A\,|\,S\cap\{\mathrm{pr}\equiv 0 \text{ on } [V]_e\backslash[U]_e\}] = \Phi_U^f[A\,|\,S]\,,$$

where the last equality can be verified by direct calculation.

Acknowledgements. I would like to thank J.-D. Deuschel for proposing a problem which eventually led to the results reported here and for many helpful discussions. I also would like to thank C.M. Newman for suggesting the use of the FK representation to prove Theorem 1.1 all the way up to the critical temperature. I am grateful to C.E. Pfister for valuable comments and for his encouragement.

References

1. Aizenman, M.: Rigorous studies of critical behavior II. In: Progress in Physics, vol. 10, pp. 453–481. Basel: Birkhäuser 1985
2. Aizenman, M., Bricmont, J., Lebowitz, J.L.: Percolation of the minority spins in high-dimensional Ising models. J. Statist. Phys. **49**, 859–865 (1987)
3. Aizenman, M., Chayes, J.T., Chayes, L., Newman, C.M.: Discontinuity of the magnetization in one-dimensional $1/|x-y|^2$ Ising and Potts models. J. Statist. Phys. **50**, 1–40 (1988)
4. Alexander, K., Chayes, J.T., Chayes, L.: The Wulff construction and asymptotics of the finite cluster distribution for two-dimensional Bernoulli percolation. Comm. Math. Phys. **131**, 1–50 (1990)
5. Antal, P., Pisztora, A.: On the chemical distance for supercritical Bernoulli percolation. Ann. Probab. (to appear)
6. Barsky, D.J., Grimmett, R.G., Newman, C.M.: Percolation in half-spaces: equality of critical densities and continuity of the percolation probability. Probab. Theory Relat. Fields **90**, 111–148 (1991)
7. Chayes, J.T., Chayes, L., Newman, C.M.: Bernoulli percolation above threshold: an invasion percolation analysis. Ann. Probab. **15**, 1272–1287 (1987)
8. Chayes, J.T., Chayes, L., Schonmann, R.H.: Exponential decay of connectivities in the two-dimensional Ising model. J. Statist. Phys. **49**, 433–445 (1987)
9. Comets, F.: Grandes deviations pour des champs de Gibbs sur $\mathbb{Z}^d$. C. R. Acad. Sci. Ser. I, **303**, 511–513 (1986)
10. Deuschel, J.-D., Pisztora, A.: Surface order large deviations for high-density percolation. Probab. Theory Relat. Fields (in press)
11. Deuschel, J.-D., Stroock, D.W.: Large deviations. New York: Academic Press 1989
12. Dobrushin, R.L., Kotecky, R., Shlosman, S.: Wulff construction: a global shape from local interaction. AMS Translations Series (1992)
13. Durrett, R. and Schonmann, R.H.: Large deviations for the contact process and two dimensional percolation. Probab. Theory Relat. Fields **77**, 583–603 (1988)

14. Ellis, R.S.: Entropy, Large deviations and statistical mechanics. Berlin: Springer 1986
15. Edwards, R.G., Sokal, A.D.: Generalization of the Fortuin–Kasteleyn–Swendsen–Wang representation and Monte Carlo algorithm. Phys. Rev. D **38**, 2009–2012 (1988)
16. Föllmer, H., Orey, S.: Large deviations for the empirical field of a Gibbs measure. Ann. Probab. **16**, 961–977 (1988)
17. Föllmer, H., Ort, M.: Large deviations and surface entropy for Markov fields. Asterisque 157–158, 173–190 (1988)
18. Fortuin, C.M.: On the random cluster model. II. Physica **58**, 393–418 (1972)
19. Fortuin, C.M., Kasteleyn, P.W.: On the random cluster model. I. Physica **57**, 536–564 (1972)
20. Gandolfi, A.: Thesis. (1989)
21. Grimmett, G.R.: Percolation. Berlin: Springer 1989
22. Grimmett, G.R.: The stochastic random-cluster process, and the uniqueness of random-cluster measures. Preprint (1994)
23. Grimmett, G.R.: Comparison and disjoint occurrence inequalities for random-cluster models. Preprint (1994)
24. Grimmett, G.R., Marstrand J.M.: The supercritical phase of percolation is well behaved. Proc. R. Soc. Lond. Ser. A **430**, 439–457 (1990)
25. Den Hollander, F., Keane, M.: Inequalities of FKG type. Physica **138A**, 167–182 (1986)
26. Ioffe, D.: Large deviations for the 2D Ising model: a lower bound without cluster expansions. J. Stat. Phys. **74**, 411–432 (1993)
27. Ioffe, D.: Exact large deviation bounds up to T_c for the Ising model in two dimensions. Preprint (1994)
28. Kesten, H., Zhang, Y.: The probability of a large finite cluster in supercritical Bernoulli percolation. Ann. Probab. **18**, 537–555 (1990)
29. Lebowitz, J.L.: Coexistence of phases in Ising ferromagnets. J. Statist. Phys. **16**, 463–476 (1977)
30. Lebowitz, J.L., Schonmann, R.H.: Pseudo-free energies and large deviations for non-Gibbsian FKG measures. Probab. Theory Relat. Fields. **77**, 49–64 (1988)
31. Newman, C.M.: Ising models and dependent percolation. In: Topics in Statistical Dependence. IMS Lecture Notes, **16**, 395–401 (1991)
32. Newman, C.M.: Disordered Ising systems and random cluster representations in Probability and Phase Transition. Kluwer, pp. 247–260 (1993)
33. Olla, S.: Large deviations for Gibbs random fields. Probab. Theory Relat. Fields **77**, 395–409 (1988)
34. Penrose, M.D., Pisztora, A.: Large deviations for discrete and continuous percolation. Adv. Appl. Probab. (to appear)
35. Pfister, C.E.: Large deviations and phase separation in the two-dimensional Ising model. Helv. Phys. Acta **64**, 953–1054 (1991)
36. Pisztora, A.: Thesis. (1993)
37. Schonmann, R.H.: Second order large deviation estimates for ferromagnetic systems in the phase coexistence region. Comm. Math. Phys. **112**, 409–422 (1987)