Commun. Math. Phys. 71, 159—205 (1980)

Communications in Mathematical Physics

Construction of Quantised Gauge Fields

II. Convergence of the Lattice Approximation

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I. Introduction

We continue in this paper a program initiated in [1], henceforth referred to as Paper I. One of the objectives set forth in that paper was a mathematically complete construction of a super-renormalisable continuum gauge theory. This paper contains results in this line of work.

The study of gauge theories on a lattice was originally suggested [2] as a suitable starting point for learning more about gauge theories generally, because lattice gauge theories provide a setting in which one can utilise methods of statistical mechanics: – low and high temperature expansions and correlation inequalities, etc. In addition these theories possess the two important properties of Osterwalder-Schrader positivity and gauge invariance. No other method, yet proposed, of regularizing continuum gauge theories so that they become mathematically well-defined objects possesses all these attractive features. It is therefore an important problem to verify that these theories converge in a suitable sense to continuum theories when the lattice spacing is taken to zero. The limit would then share these properties and in addition one would hope to verify that it is Euclidean invariant (unlike the lattice theories). Various consequences of the correlation inequalities which will be of interest to physicists as well as mathematicians have been outlined in [3].

Unfortunately, it is unlikely that our method of proving convergence is optimal. We have adopted a method of embedding lattice gauge theories in continuum theories which is not natural in the context of geometry. It might be rewarding to search for methods that treat the geometrical side with less than the insensitivity that we have been able to muster. In the meantime we have in this paper a number of functional analytic techniques that will extend to more singular theories, abelian and non abelian and some of them will very likely be useful in future improvements.

* Research partly supported by N.S.F. Grant PHY-78-08066
** Research supported in part by Grants from the US NSF, No. PHY-76-80958, and Centre National de la Recherche Scientifique

0010-3616/80/0071/0159/$09.20

We are here mostly concerned with two dimensional abelian gauge theories interacting with Bose matter. An analogous program for fermion matter has been started in [4]. Some of our results are valid for nonabelian gauge fields also. The major simplification in the abelian case is that the measure describing a pure gauge field is Gaussian in the continuum limit. We exploit this by noting that we may obtain a Gaussian lattice gauge field by conditioning the continuum measure. Thus given a continuum gauge field one may formally obtain a lattice gauge field, which is a function from bonds of the lattice to group elements, by integrating the gauge field along a given bond and applying the exponential map to the Lie algebra element so obtained to get an element of the group. (If the group is nonabelian, one should use an ordered exponential.)

One can then couple this lattice field to a matter field on the lattice and the resulting lattice theory is gauge invariant. The procedure may be considered as amounting to a special choice of lattice measure for the gauge field which differs from Wilson's [2] and others so far proposed, but which is also gauge invariant and has the correct continuum limit, at least formally.

This procedure is not possible in more than two dimensions because with probability one the gauge field is a distribution with insufficient regularity to be integrated along a bond. However, as pointed out in Paper I, it is possible to put in an ultraviolet cutoff, i.e., change the Gaussian measure describing the continuum gauge field to another one whose sample functions are more regular (almost surely) and still retain a type of gauge invariance. Furthermore if the ultraviolet cutoff is suitably designed (a cutoff in all but one direction in $\mathbb{R}^n$) we obtain a lattice theory with Osterwalder-Schrader positivity in one direction. This is of course not a new observation. Lastly, as discussed in Paper I, we have correlation inequalities even in the presence of an ultraviolet cutoff. They are in fact valid for any lattice Gaussian measure for the gauge field.

Even in our case of two dimensions we find it convenient to use an ultraviolet cutoff on the gauge field. This is in order to separate off the complexities of renormalisation from proving the convergence of a lattice approximation. In other words, if we did not impose an ultraviolet cutoff, we would have to insert counterterms and cancel quantities that diverge as the lattice spacing is taken to zero. We prefer to put in a cutoff and its subsequent removal (after the lattice spacing is taken to zero) will be discussed in Paper III. Finally, we also give the gauge field a mass (an infrared cutoff). This does not affect the Ward identities which express the gauge invariance of the coupling between matter and gauge fields. Correlation inequalities allow then to take this mass to zero. Full gauge invariance is impossible in the continuum limit and gauge fixing is always necessary. We really prove "gauge covariance". The zero mass limit will also be given in Paper III, and in fact we first take the infinite volume limit which is easier whilst the gauge field has an infrared cutoff and then the zero mass limit.

We now give a rough formulation of our principal results. We will supply more details and precise definitions later. It applies to a theory in a rectangle in $\mathbb{R}^2$ with a continuum gauge field with a mass and an ultraviolet cutoff interacting with a Bose field on a lattice with spacing $\varepsilon > 0$. The Bose field is allowed self interactions.

Theorem A. *Given a sequence of simple cubic lattices whose spacings tend to zero, the lattice measures which correspond to the theory described above converge in the sense of characteristic functions.*

The main results required for the proof of Theorem A can be found in Sects. III and IV. Some of the more significant ones can be summarized as follows. Let C_h^ε (C_A), denote the lattice (continuum) Green's function for the covariant finite difference (continuum) Laplacian, Δ_h^ε (Δ_A), in a lattice (continuum) gauge field, h (A). The gauge field may be non abelian. We impose either free of periodic boundary conditions at the boundary of a rectangle Λ.

Theorem B. *Let (h^ε) be a sequence of lattice gauge fields converging to a locally bounded measurable gauge field A as ε tends to zero. Then the kernel of $C_{h^\varepsilon}^\varepsilon$ converges locally in L_p, for all p with $1 \leqq p < \infty$, to C_A.*

Theorem C. *Let (h^ε) be convergent to a Hölder continuous gauge field A, then the determinant, $z_{h^\varepsilon}^\varepsilon$, defined to be*

$$\det((-\Delta^\varepsilon+m^2)^{-1/2}(-\Delta_{h^\varepsilon}^\varepsilon+m^2)(-\Delta^\varepsilon+m^2)^{-1/2})$$

with $m^2>0$, converges to its formal continuum limit as ε tends to zero. The limit is finite and strictly positive.

Our methods would also be useful in proving the appropriate analogues of Theorems B and C in three space-time dimensions.

The limiting theory obtained in Theorem A is Euclidean covariant. It is not invariant because of the boundary and also the cutoff on the gauge field. In two dimensions it is possible to identify it with a theory constructed directly in the continuum and then Euclidean covariance is obvious. However it is also possible to obtain it directly from our theorems because they are valid when limits are taken through lattices of varying orientation. We have slightly emphasized this point because it may be a superior strategy in more singular theories. Obviously Euclidean covariance is necessary if the final theory obtained by taking the infinite volume limit and removing the ultraviolet cutoff is to be Euclidean invariant. Note that Euclidean invariance and Osterwalder-Schrader positivity in one direction combine to yield positivity in all directions.

Let us now briefly outline the steps in our proof. We begin in Sect. II by collecting our notation and conventions and summarizing some useful facts about trace class ideals $(\mathscr{I}_p)$ of operators [5]. In Sect. III we prove Theorem B. One reason why this part of our work is more difficult than the corresponding parts of the lattice convergence proof in [6] for Bose fields without gauge fields is that we can no longer use the Fourier transform to diagonalize all our Euclidean propagators $C_{h^\varepsilon}^\varepsilon$ simultaneously. Instead we rely heavily on the theory of trace class ideals and analyticity. We have prefaced Sect. III by a short verbal description of these methods since they may find other applications.

In Sect. IV we prove convergence for lattice fields of bosons in an *external* Yang Mills field as $\varepsilon \searrow 0$. The Yang Mills field can be non abelian. Although we do not prove it in this paper, the limiting partition function is closely related to that investigated by Schrader [7]. The differences are as follows: (1) we include the factor $z^\varepsilon(A)$ (see IV and Theorem C) which Schrader et al. [7] refer to as the "renormalized determinant"; (2) our normal ordering of the bose self interaction is with respect to C_0^ε instead of C_A^ε. Both these features are forced on us since we are going to integrate over the gauge field (in the next section). The renormalized determinant is a considerable nuisance because it contains contributions which diverge as $\varepsilon \searrow 0$, and one must use gauge invariance in the form of a Ward identity to prove that the divergent parts cancel each other up to a remainder which is finite in the limit. (This type of phenomenon is well known to physicists.) The change in normal ordering (2) is not a simplification either. The point of Theorem 3.5 and its quite lengthy proof is to control this change of normal ordering as $\varepsilon \searrow 0$.

Our proof of convergence owes much to [6]. We also proceed by embedding all our lattice theories in one continuum theory (white noise instead of the free Euclidean field used in [6]). We find that we need to prove that the square roots $\sqrt{C_A^{(\varepsilon)}}$ converge in $\mathscr{I}_4$ and since we cannot use the Fourier transform we prove a little lemma that provides a sufficient condition that the (non linear) map $A \mapsto f(A)$ be continuous from $\mathscr{I}_p$ to $\mathscr{I}_q$.

In Sect. V we complete the proof of Theorem A, in the form of Corollary 5.2 by showing that the integral over the *abelian* gauge field, A, of the lattice external gauge field partition functions of Sect. IV converges as $\varepsilon \searrow 0$. This then is merely a matter of justifying the interchange of the $\varepsilon \searrow 0$ limit with the A integral so that we can apply the results of IV. To do this we use dominated convergence, appealing to the diamagnetic bound of Paper I, Corollary 2.4, and Theorem 4.1, to show a uniform bound on the external gauge field partition functions. We also have to show that the class of gauge fields allowed in Sects. III and IV are a set of measure one. This is a slightly fine point since the ultraviolet cutoff on the A field does not regularize the sample functions much because we wish to have Osterwalder-Schrader positivity in one direction. We appeal to a beautiful paper [8] by Garsia on the continuity properties of sample functions of Gaussian measures to settle this point.

We also discuss Osterwalder-Schrader positivity in this section (Theorem 5.5). We explain what types of cutoff on the covariance of the Gaussian measure describing the gauge field yield a continuum limit with positivity in one direction.

In our final section, VI, we provide some technical preparations for our next paper in which we will remove the ultraviolet cutoff. We discuss counterterms and define renormalized partition functions and measures for abelian gauge theories. We give the Feynman rules and in Theorem 6.1 prove an identity, the change of covariance formula, inspired by similar formulas in [9]. This formula will be used in Paper III to generate (by iteration) an expansion of the Glimm-Jaffe type [10] which will prove that the partition function, when correctly renormalized, is bounded above and below uniformly in the ultraviolet cutoff. This is the most difficult step involved in removing the ultraviolet cutoff. The formula is of the following type

$$\langle P \rangle_1 - \langle P \rangle_0 = \int_0^1 \langle KP \rangle_t \, dt$$

in which P is a polynomial in the fields, $\langle\rangle_1$, $\langle\rangle_0$, $\langle\rangle_t$ are unnormalized (but renormalized!) expectations. The subscripts 1,0 refer to different ultraviolet cutoffs; t parametrises a family of cutoffs that interpolate between 0 and 1; K is a partial differential operator in $\delta/\delta\phi$. The important point about K is that it depends only on renormalized quantities and so does not diverge in the ultraviolet limit. For this reason this formula can be made the basis of a method of removing the ultraviolet cutoff.

In an appendix we briefly sketch how to extend our results to the case where Dirichlet boundary conditions are imposed on the Bose field.

II. Preliminaries: Notation, Trace Ideals

In this section we fix notation, give some definitions and quote some theorems on trace ideals.

First we present a list of symbols followed by an explanation of their meaning

$\Lambda \subset \mathbb{R}^2$, a bounded open set,

$L \subset \mathbb{R}^2$, a simple cubic lattice, unit spacing,

$L^{(\varepsilon)}(\Lambda) \equiv \varepsilon L \cap \Lambda, L^{(\varepsilon)} \equiv \varepsilon L,$

$\mathscr{B}^\varepsilon \equiv \{\langle x, \varepsilon e_\mu \rangle : x \in L^{(\varepsilon)}, \mu = 0, 1\}.$

$\mathscr{B}^\varepsilon$ is the set of bonds considered as closed subsets of $\mathbb{R}^2$; e_μ, $\mu = 0, 1$ are the unit vectors which generate L, i.e.,

$$L = \{n_0 e_0 + n_1 e_1 : n_0, n_1 \in \mathbb{Z}\}.$$

Let $\mathscr{B}^\varepsilon(\Lambda)$ be the subset of bonds contained in Λ. We denote by ∂^ε the finite difference gradient

$$\partial^\varepsilon_\mu f(x) = \varepsilon^{-1}[f(x + \varepsilon e_\mu) - f(x)]$$

associated with L^ε; ∂^ε is defined both on functions on $L^{(\varepsilon)}$ and on continuum functions. The continuum gradient is denoted by ∂.

We now wish to introduce covariant derivatives. Let G be a compact Lie group unitarily represented on a finite dimensional Hilbert space V. Let A_μ be a *gauge field*. For $\mu = 0, 1$, A_μ is a map from $\mathbb{R}^2$ into the Lie algebra $\mathscr{L}(G)$ of G. The covariant derivative is defined on V-valued functions on $\mathbb{R}^2$ by

$$D_{A,\mu}\phi \equiv \partial_\mu \phi - ieA_\mu \phi, \tag{2.1}$$

e is a constant, the electric charge. The finite difference covariant derivative is defined only on lattice functions with values in V,

$$D^\varepsilon_{h,\mu}\phi(x) \equiv \varepsilon^{-1}[h^{\varepsilon *}_\mu(x)\phi(x + \varepsilon e_\mu) - \phi(x)], \tag{2.2}$$

where h^ε, a *lattice gauge field* is a map from bonds $\langle x, \varepsilon e_\mu \rangle$ into G.

The covariant Laplacians are defined by

$$\begin{aligned} \Delta_A &\equiv -D^*_{A,\mu} D_{A,\mu} \\ \Delta^\varepsilon_h &\equiv -D^{\varepsilon *}_{h,\mu} D^\varepsilon_{h,\mu}, \end{aligned} \tag{2.3}$$

where we use the Einstein summation convention on $\mu = 0, 1$.

Let

$$L_\infty = L_\infty(\mathbb{R}^2, \mathscr{L}(V))$$

be the space of two component measurable functions with values in linear operators $\mathscr{L}(V)$ on V, given the norm

$$\|B\| = \text{ess.}\sup_{x\in\mathbb{R}^2} (\|B_0(x)\|^2_{\mathscr{L}(V)} + \|B_1(x)\|^2_{\mathscr{L}(V)})^{1/2}, \tag{2.4}$$

where the subscripts refer to the lattice directions and $\| \ \|_{\mathscr{L}(V)}$ is the operator norm on V. We introduce this norm because it appears to be appropriate for the discussion of convergence of gauge fields in Theorem B. The derivatives in the definition of the covariant Laplacian are applied in the distribution sense. We take the gauge field A to be in L_∞.

We now introduce some notation whose purpose is to make the lattice objects resemble their continuum limits in order to facilitate the discussion of convergence. Let $B = B^\varepsilon_\mu$ be a two-component map from $L^{(\varepsilon)}$ into $\mathscr{L}(V)$. Set

$$\begin{aligned} D^\varepsilon_{B,\mu} &\equiv \partial^\varepsilon_\mu - ieB^\varepsilon_\mu \\ \Delta^\varepsilon_B &\equiv -D^{\varepsilon *}_{B,\mu} D^\varepsilon_{B,\mu}. \end{aligned} \tag{2.5}$$

These are operators on V valued functions on $L^{(\varepsilon)}$.

We will be particularly concerned with the following two choices for B,

$$B^\varepsilon_\mu = A^\varepsilon_\mu, \qquad B^\varepsilon_\mu = \mathscr{A}^\varepsilon_\mu,$$

where

$$\begin{aligned} A^\varepsilon_\mu &\equiv (ie\varepsilon)^{-1}(h^\varepsilon_\mu(x) - \mathbb{1}) \\ e^{ie\varepsilon\mathscr{A}^\varepsilon_\mu(x)} &= h^\varepsilon_\mu(x). \end{aligned} \tag{2.6}$$

The second equation defines $\mathscr{A}^\varepsilon$ in terms of h^ε, provided h is sufficiently close to the identity that the exponential map may be inverted, $\mathscr{A}^\varepsilon$ belongs to $\mathscr{L}(G)$, the Lie algebra. A^ε does not. Note that if we choose $B^\varepsilon = A^\varepsilon$,

$$\begin{aligned} D^\varepsilon_{h,\mu} &= (h^\varepsilon_\mu)^{-1} D^\varepsilon_{A,\mu} \\ \Delta^\varepsilon_h &= \Delta^\varepsilon_A. \end{aligned} \tag{2.7}$$

The Q Identification. Let f be a function on $\mathbb{R}^2$. We can obtain a function on a lattice $L^{(\varepsilon)}$, $Q^\varepsilon f$, defined by averaging, i.e.,

$$Q^\varepsilon f(x) \equiv \varepsilon^{-2} \int_{\varepsilon\Delta_x} f(y)dy,$$

where Δ_x is a unit square centred at the lattice point x. Conversely, given a function f defined on a lattice, we can obtain a continuum function $Q^{\varepsilon *}f$ which is the piecewise constant (constant inside each lattice square) function which coincides with f at lattice points. With the aid of Q, Q^*, we can obtain continuum operators from lattice operators, e.g.,

$$D^\varepsilon_A \to Q^{\varepsilon *} D^\varepsilon_A Q^\varepsilon.$$

The main reason why we like these operators is that ∂^ε and all functions of ∂^ε commute with $Q^{\varepsilon *}Q^\varepsilon$. (Recall that ∂^ε can be considered to be an operator on continuum functions.) Another way of stating the same thing is that if A is a function of ∂^ε, we can consider it either as a lattice operator or a continuum operator $\tilde{A}$. Then if f is a continuum function

$$AQ^\varepsilon f = Q^\varepsilon \tilde{A} f.$$

Thus Q gives us an embedding of lattice into continuum. We will simplify our formulas by omitting these Q operators. Therefore if the context requires it lattice functions and operators are to be identified with their continuum counterparts derived via Q.

Euclidean Propagators, Boundary Conditions. Let $\ell_2^{(\varepsilon)}(\Lambda) = \ell_2(\Lambda)$ be the space of square summable V-valued functions on $L^{(\varepsilon)}(\Lambda)$ with norm (first example of Q identification)

$$\|f\|_{L_2}^2 = \int \|f\chi_\Lambda\|_V^2 d^2x,$$

where χ_Λ is the lattice characteristic function of Λ and $\| \ \|_V$ is the norm on V. Δ_h^ε is an operator on $\ell_2(\mathbb{R}^2)$. By a form method [11] we can extend Δ_Λ to a selfadjoint unbounded operator, also denoted Δ_Λ, on $L_2(\mathbb{R}^2)$. The inverses

$$C_\Lambda \equiv (m^2 - \Delta_\Lambda)^{-1}, \qquad C_h^\varepsilon \equiv (m^2 - \Delta_h^\varepsilon)^{-1},$$

where $m^2 > 0$ are bounded operators; their norm is less than or equal to m^{-2}. Their kernels, the covariant Green's functions are henceforth called "covariances" in view of their later rôle as covariances of Gaussian measures.

If the gauge field *vanishes outside* Λ which by definition means that it is zero on all bonds not contained in Λ, in the lattice case, we say that the covariance has *free boundary conditions.* We introduce an operator C_h^F, on $\ell_2(\Lambda)$ by

$$C_h^F = \chi_\Lambda C_h \chi_\Lambda .$$

The covariant Laplacian with free boundary conditions, Δ_h^F, is defined by

$$m^2 - \Delta_h^F = (C_h^F)^{-1} . \tag{2.9}$$

A Convention for the Internal Degrees of Freedom. In order to clean up our language we are going to suppress V, $\mathscr{L}(V)$ in some of our norms and spaces, e.g., our use of ℓ_2 for V-valued functions is an instance of this.

The Interaction. The operator on ℓ_2 given by

$$\begin{aligned} W_h^\varepsilon &\equiv \Delta_h^\varepsilon - \Delta^\varepsilon \\ &= -ie A_\mu^{\varepsilon *}\partial_\mu^\varepsilon + ie\partial_\mu^{\varepsilon *} A_\mu^\varepsilon - e^2 A_\mu^{\varepsilon *} A_\mu^\varepsilon \end{aligned}$$

will be referred to as the *interaction* with the gauge field. In the case where h^ε is derived from $\mathscr{A}^\varepsilon$ (see 2.6), it may be written

$$\begin{aligned} W_h^\varepsilon &= +\varepsilon^{-2} T_\mu^{\varepsilon *}(h^\varepsilon - 1)_\mu + \varepsilon^{-2}(h^{\varepsilon *} - 1)_\mu T_h^\varepsilon \\ &= +ie\varepsilon^{-1} T_\mu^{\varepsilon *}\mathscr{A}_\mu^\varepsilon - ie\varepsilon^{-1}\mathscr{A}_\mu^\varepsilon T_\mu^\varepsilon \\ &\quad - \frac{e^2}{2}(T_\mu^{\varepsilon *}(\mathscr{A}_\mu^\varepsilon)^2 + (\mathscr{A}_\mu^\varepsilon)^2 T_\mu^\varepsilon) + O(\varepsilon e^3), \end{aligned} \tag{2.11}$$

where T^ε_μ is the operator of translation by ε in the μ direction. The term $O(\varepsilon e^3)$ is of order εe^3 in operator norm if $A \in L_\infty$.

The kernel of the Fourier transform of this operator is

$$W^\varepsilon_h(p,q) \equiv +ie\varepsilon^{-1}(e^{-i\varepsilon p_\mu} - e^{i\varepsilon q_\mu})\hat{\mathscr{A}}^\varepsilon_\mu(p-q)$$
$$-\frac{e^2}{2}(e^{-i\varepsilon p_\mu} + e^{i\varepsilon q_\mu})(\mathscr{A}^\varepsilon_\mu)^{2\wedge}(p-q) - O(\varepsilon e^3). \tag{2.12}$$

The Fourier transform is defined by

$$\hat{f}(p) \equiv \frac{1}{2\pi} \sum_{x \in L^{(\varepsilon)}} \varepsilon^2 f(x) e^{-ipx}.$$

The variable $p=(p_0, p_1)$ lies in the square $\left[-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}\right]^{\times 2}$, because the dual space for the lattice is a torus.

Trace Norms [12]. We will have frequent occasions to use the following spaces of operators. Let H be a Hilbert space. A compact operator $T: H \to H$ belongs to the class $\mathscr{I}_p$, $1 \leqq p \leqq \infty$, iff

$$\|T\|_p \equiv (\mathrm{tr}(T^*T)^{p/2})^{1/p} < \infty$$
$$\|T\|_\infty \equiv \text{operator norm} \equiv \|T\|. \tag{2.13}$$

It can be shown that $\mathscr{I}_p$ is complete, and furthermore the Hölder inequality

$$\left\|\prod_{i=1}^n T_i\right\|_p \leqq \prod_{i=1}^n \|T_i\|_{p_i}, \qquad \sum_{i=1}^n \frac{1}{p_i} = \frac{1}{p} \tag{2.14}$$

is valid. In this inequality we can drop the condition that T_i be compact if $p_i = \infty$.

Proposition. a) *For $1 \leqq p \leqq \infty$, finite rank operators are dense in $\mathscr{I}_p$.* b) *$\mathscr{I}_p$ is closed with respect to taking adjoints.*

Theorem (Grümm [13]). *Let A_n be a sequence of operators in $\mathscr{I}_p$, $1 \leqq p < \infty$. If A_n (A_n^*) converges to A (A^*) strongly and $\|A_n\|_p$ converges to $\|A\|_p$, then A_n converges to A in $\mathscr{I}_p$.*

Remark. Simon [14] shows that strong convergence can be replaced by weak convergence in the hypothesis, if $p > 1$.

III. Bounds, Analyticity, and Convergence of Covariant Lattice Green's Functions

In this section we establish some properties of our covariant Green's functions (covariances) which will be needed for the proof of convergence of the lattice approximation.

In Definition 3.2 we define a notion of convergence for a sequence of gauge fields $h^{(\varepsilon_n)}$ associated to lattices $L^{(\varepsilon_n)}$ with arbitrary orientations, $\varepsilon_1, \varepsilon_2, \ldots$ being a sequence of lattice spacings tending to zero. Given that a sequence of lattice gauge fields converges to a continuum gauge field in this sense, we show in Theorem 3.2 that the associated covariances, considered as operators on L_2 via the Q

identification of the last section, converge in a "local" Hilbert Schmidt norm. We also show that the functions obtained by restricting to the diagonal the kernels of the differences between the covariant covariances and the free covariances converge in L_p^{loc} for $1 \leqq p < \infty$. This is done in Theorem 3.3. Actually all operators we consider are finite matrices (for $\varepsilon > 0$), or finite rank operators after using the Q identification to put them on L_2, but it is useful to state results and think of them in continuum language since we are taking a continuum limit.

To prove these results we use the diamagnetic bound [15], stated here as Theorem 3.1, to obtain uniform bounds. The other main technical device is to first prove convergence when the gauge field is small and then use analyticity, as proven in Lemma 3.4, to extend the convergence to arbitrary gauge fields. We give a proof of Lemma 3.4 for the sake of being self contained, but the result is a special case of well known general theorems [16].

The notion of convergence in Definition 3.2 is sufficient for the results of this section but has to be strengthened to prove convergence of the lattice partition function in an external gauge field. The reader is referred to the next section for this.

We begin by stating the results.

Theorem 3.1 [15] *(the diamagnetic bound)*

$$\|(C_h^\varepsilon)^\alpha(x,y)\| \leqq \|(C_{\mathbb{1}}^\varepsilon)^\alpha(x,y)\| \qquad 0 \leqq \alpha < \infty .$$

$(C_h^\varepsilon)^\alpha(x,y)$ denotes the kernel of the operator C_h^ε raised to the power α in the operator sense.

This is an easy generalisation of the Nelson-Simon inequality [15]. A simple proof has been reproduced in Paper I.

Remark. The same inequality is valid for periodic, Dirichlet and Neumann boundary conditions on both sides.

Before stating the next theorem, which is the main result of this section, we need

Definition 3.2. A family of lattice gauge fields h^ε is convergent to a gauge field A as $\varepsilon \to 0$ iff A^ε, defined by

$$A_\mu^\varepsilon(x) \equiv (ie\varepsilon)^{-1}(h_\mu^\varepsilon(x) - \mathbb{1})$$

converges to A in L_∞, i.e., $\|A^\varepsilon \to A\| \to 0$.

Theorem 3.3. *If a family (h^ε) of lattice gauge fields converges to A as $\varepsilon \to 0$, then the kernel $C_{h^{(\varepsilon)}}^\varepsilon(x,y)$ of $C_{h^{(\varepsilon)}}^\varepsilon$ converges in $L_p(\Lambda \times \Lambda)$, $1 \leqq p < \infty$.*

Remark. The limit is $C_A(x,y)$.

The proof of this theorem will use Lemma 3.4 given below.

Lemma 3.4. *Let $B = B_\mu^\varepsilon$, $E = E_\mu^\varepsilon$ be bounded $\mathscr{L}(V)$ valued functions on $L^{(\varepsilon)}$. Then $C_{B+\lambda E}^\varepsilon(x,y)$ is a real analytic $\ell_2(\Lambda \times \Lambda)$ valued function in λ, which extends to a function analytic in the strip*

$$2\frac{e}{m}\operatorname{Im}\lambda\|E\| + \left(\frac{e}{m}\operatorname{Im}\lambda\right)^2\|E\|^2 \equiv \xi < 1 .$$

The extension $\tilde{C}^\varepsilon_{B+\lambda E}$ is bounded by

$$\|\tilde{C}^\varepsilon_{B+\lambda E}\|_{\ell_2(\Lambda\times\Lambda)} \leqq \|C_{B+\mathrm{Re}\,\lambda E}\|_{\ell_2(\Lambda\times\Lambda)}(1-\xi)^{-1}.$$

Remarks. 1) $C^\varepsilon_{B+\lambda E}$ is real analytic but not analytic as defined in (2.5), (2.8) because of the adjoints in (2.5).

2) The same lemma holds for the continuum covariance.

3) Periodic, Dirichlet, Neumann boundary conditions could be accomodated.

The final result of this section will be used to control Wick ordering terms. Define the operator

$$\delta C^\varepsilon_h \equiv C^\varepsilon_h - C^\varepsilon_1.$$

The kernel will be denoted $\delta C^\varepsilon_h(x,y)$.

Theorem 3.5. *Let (h^ε) be a family of gauge fields converging as ε tends to zero to a continuum gauge field A, then for $1 \leqq p < \infty$,*

$$\int_\Lambda |\mathrm{tr}\,\delta C^\varepsilon_{h^\varepsilon}(x,x) - \mathrm{tr}\,\delta C^{\varepsilon'}_{h^{\varepsilon'}}(x,x)|^p \to 0, \qquad \varepsilon, \varepsilon' \to 0,$$

where the tr denotes a sum (trace) over internal indices.

Remark. The theorem asserts that δC^ε is a Cauchy sequence. In fact the limit is the continuum expression

$$\delta C_A(x,x) = (C_A - C_0)(x,x).$$

It can be shown that δC has a kernel which is continuous in x and y so that the restriction to the diagonal is well defined.

Proof of Lemma 3.4. We will compress the notation by suppressing ε, μ. Let F, G be bounded $\mathscr{L}(V)$ valued function on $L^{(\varepsilon)}$. Then

$$D_{F+G} = D_F - ieG.$$

Therefore

$$\begin{aligned}\Delta_{F+G} &= \Delta_F - ie(G^*D_F - D_F^*G) - e^2G^*G \\ &\equiv \Delta_F + W_{F,G}.\end{aligned}$$

Let χ_Λ be the characteristic function of Λ. We show that the Neumann series for the resolvent

$$\chi_\Lambda C_{F+G}\chi_\Lambda = \chi_\Lambda C_F^{1/2} \sum_{n=0}^{\infty} (C_F^{1/2} W_{F,G} C_F^{1/2})^n C_F^{1/2}\chi_\Lambda \tag{3.1}$$

is convergent in Hilbert-Schmidt norm ($=\mathscr{I}_2$ norm $=$ norm of kernel considered as a function in $\ell^2(\mathbb{R}^2\times\mathbb{R}^2)$) provided $\|G\|$ is sufficiently small. By Hölder's inequality for $\mathscr{I}_p$ spaces

$$\begin{aligned}\|\chi_\Lambda C_F^{1/2}(C_F^{1/2}W_{F,G}C_F^{1/2})^n C_F^{1/2}\chi_\Lambda\|_2 &\leqq \|\chi_\Lambda C_F^{1/2}\|_4^2 \, \|C_F^{1/2}W_{F,G}C_F^{1/2}\|^n \\ &\leqq \|\chi_\Lambda C_F\chi_\Lambda\|_2 (e\|C_F^{1/2}G^*D_FC_F^{1/2}\| + e\|C_F^{1/2}D_F^*GC_F^{1/2}\| + e^2\|C_F^{1/2}G^*GC_F^{1/2}\|)^n \\ &\leqq \|C_F\|_2\left(2\frac{e}{m}\|G\| + \frac{e^2}{m^2}\|G\|^2\right)^n.\end{aligned} \tag{3.2}$$

The last bound is obtained by applying the easy bounds

$$\begin{aligned}&\|C_F^{1/2}\| \leqq \frac{1}{m}\\ &\|C_F^{1/2}G^*D_F C_F^{1/2}\| \leqq \|C_F^{1/2}\| \, \|G^*\| \, \|C_F^{1/2}D_F^* D_F C_F^{1/2}\| \leqq \|C_F^{1/2}\| \, \|G\|\\ &\|C_F^{1/2}D_F^* G C_F^{1/2}\| \leqq \|C_F^{1/2}\| \, \|G\| .\end{aligned} \tag{3.3}$$

The bound (3.2) shows that (3.1) is convergent if

$$\left(2\frac{e}{m}\|G\| + \frac{e^2}{m^2}\|G\|^2\right) \equiv \xi' < 1 . \tag{3.4}$$

By taking norms under the sum in (3.1)

$$\|C_{F+G}\|_2 \leqq \|C_F\|_2 \frac{1}{1-\xi'} .$$

To prove the lemma, take $F = B + (\mathrm{Re}\,\lambda)E$, $G = \mathrm{im}\,\lambda E$. This completes the proof of Lemma 3.4.

Remark. In the proof of Theorem 3.5 we will use the fact that the argument above is trivially adapted to show that $\chi_\Lambda C^{\varepsilon}_{B+\lambda E}\partial^{\varepsilon *}$ is $\mathscr{I}_4$ real analytic and bounded in a strip.

Proof of Theorem 3.3. We begin by assembling some simple lemmas which will be used in the proof.

Lemma 3.6. *Let A_n be a sequence of operators in $\mathscr{I}_p$, $1 \leqq p < \infty$, which converge in $\mathscr{I}_p$ to A. Let B_n be a sequence of operators which are uniformly bounded in operator norm and $B_n \to B$, $B_n^* \to B^*$ as $n \to \infty$ in the strong operator topology. Then $A_n B_n \to AB$ in $\mathscr{I}_p$ as $n \to \infty$.*

Remark. A related result was an important ingredient in the lattice convergence proof of [6].

Proof

$$\begin{aligned}\|AB - A_n B_n\|_p &\leqq \|(A-A_n)B\|_p + \|A_n(B-B_n)\|_p \leqq \|A-A_n\|_p \, \|B\| + \|A(B-B_n)\|_p\\ &\quad + \|(A-A_n)\|_p \left(\|B\| + \sup_n \|B_n\|\right).\end{aligned}$$

The first and final terms tend to zero. Let $C_n = B_n - B$. We are reduced to showing AC_n tends to zero in $\mathscr{I}_p$. Approximate A by a finite rank operator $\tilde{A}$ so that

$$\|A - \tilde{A}\|_p \leqq \delta$$

for a given $\delta > 0$. It is enough to show that $\tilde{A}C_n$ tends to zero in $\mathscr{I}_p$. Equivalently, one can show that $C_n^* A^*$ tends to zero, i.e.,

$$\mathrm{tr}(\tilde{A}C_n C_n^* \tilde{A}^*)^{p/2} \xrightarrow[n\to\infty]{} 0 .$$

Since this is a finite rank operator, it is sufficient that C_n, C_n^* tend to zero strongly because the uniform operator norm bound then implies $C_n C_n^*$ tends to zero strongly.

Lemma 3.7. *Let* $C^\varepsilon = C^\varepsilon_{\mathbb{1}}$, $x, y \in L^{(\varepsilon)}$.

$$C^\varepsilon(x,y) = \frac{1}{2\pi} \int_{[\frac{\pi}{\varepsilon},\frac{\pi}{\varepsilon}]^2} d^2k \hat{C}^{(\varepsilon)}(k) e^{ik(x-y)},$$

$$\hat{C}^\varepsilon(k) \equiv \frac{1}{2\pi}\left(2\varepsilon^{-2} \sum_{\mu=0,1} (1-\cos\varepsilon k_\mu) + m^2\right)^{-1} \otimes \mathbb{1}_V.$$

Proof. Easy consequence of definitions and Fourier series (see [6]).

Lemma 3.8. *Let* $U \subset \mathbb{R}^2$ *be bounded and measurable.*

$$\|\chi_U C^\varepsilon_h \chi_U\|_{L_p} \leqq \text{const}$$

uniformly in ε, $h^\varepsilon \cdot 1 \leqq p < \infty$.

Proof. Theorem 3.1 reduces these statements to the special case $h^\varepsilon = \mathbb{1}$ for which they are well known. A simple proof can be based on Lemma 3.7 and the Hausdorff-Young inequality.

Lemma 3.9. *Let* $U \subset \mathbb{R}^2$ *be bounded and measurable, then*

1) $\chi_U (C^\varepsilon)^\alpha \xrightarrow[\varepsilon\to 0]{} \chi_U C^\alpha$ in $\mathscr{I}_4$, $\alpha > 1/4$.

2) $\partial^\varepsilon (C^\varepsilon)^{1/2} \xrightarrow[\varepsilon\to 0]{} \partial C^{1/2}$

3) $(C^\varepsilon)^{1/2} \partial^{\varepsilon *} \xrightarrow[\varepsilon\to 0]{} C^{1/2}\partial^*$

(2) and 3)) *in strong operator topology.*

Proof. 1) To begin with it is sufficient to take U to be a rectangle in $\mathbb{R}^2$. To see this let $D^\varepsilon(x,y)$ be the kernel of

$$((C^{(\varepsilon)})^\alpha - C^\alpha)^2$$

then

$$\|\chi_U((C^\varepsilon)^\alpha - C^\alpha)\|_4^4 = \int_U \int_U |D^\varepsilon(x,y)|^2$$

so that the norm is increasing in U. Next, by Grümm's theorem (Sect. II), it is enough to prove that

a) $\|\chi_U (C^\varepsilon)^\alpha\|_4 \to \|\chi_U C^\alpha\|_4$.
b) $(C^\varepsilon)^\alpha \to C^\alpha$ in strong operator topology.

For a), by Lemma 3.7

$$\|\chi_U (C^\varepsilon)^\alpha\|_4^4 = \sum \int d^2k \int d^2k' |\widehat{Q^\varepsilon \chi}(k+k')|^2 \, |\hat{C}(k)\hat{C}(k')|^{2\alpha},$$

where the range of integration is $\left[-\frac{\pi}{\varepsilon'}\frac{\pi}{\varepsilon}\right]^2$ for k and k'. The dominated convergence theorem completes Part a).

Part b), 2) and 3) are all similar. We discuss 2). An easy argument with Q^ε shows that it is enough to show that the Fourier transform

$$\varepsilon^{-1}(e^{i\varepsilon k_\mu} - 1)(\hat{C}^\varepsilon(k))^{1/2} \xrightarrow[\varepsilon\to 0]{} \frac{1}{\sqrt{2\pi}} k_\mu (k^2+m^2)^{-1/2} \otimes \mathbb{1}_V$$

in strong operator topology as an operator on $L_2(\mathbb{R}^2)$. This is easy.

The proof of Theorem 3.3 is a series of reductions.

1) We claim that it is enough to show convergence in $L_2(\Lambda \times \Lambda)$. We know that $C_h^\varepsilon \in L_p(\Lambda \times \Lambda)$ uniformly in ε by Lemma 3.8. Combining this with Hölder's inequality and L_2 convergence proves L_p convergence by an easy argument.

2) It is enough to prove L_2-convergence in the special case that

$$\|A\| \ll 1 .$$

Proof. If ε' is sufficiently small, the definition of convergence of h^ε implies h^ε is in a small neighborhood of $\mathbb{1}$, uniformly in the bonds in $\mathscr{B}^\varepsilon$ and $\varepsilon \leqq \varepsilon'$. Therefore we may define $\mathscr{A}^\varepsilon$, a Lie algebra valued function on bonds (with two components) by

$$h^\varepsilon = e^{ie\mathscr{A}^\varepsilon}$$

and then, given $\lambda \in \mathbb{R}$, set

$$h^\varepsilon(\lambda) \equiv e^{ie\lambda\mathscr{A}^\varepsilon} .$$

It is then easy to verify that $h^\varepsilon(\lambda)$ converges in the sense of Definition 3.2 to λA. Furthermore, by Lemma 3.1 the covariance $C_{h(\lambda)}^\varepsilon$ is real analytic in λ. It extends to a function which is analytic in a strip of width *independent* of $\varepsilon \leqq \varepsilon'$. Lemma 3.4 combined with Theorem 3.1 shows the extensions $C_{h(\lambda)}^\varepsilon$ are bounded uniformly in $\varepsilon \leqq \varepsilon'$, $\lambda \in \mathbb{R}$. Therefore a form of Vitali's theorem (see the remark below) tells us that convergence for all λ is guaranteed by convergence for λ in a neighboorhood of zero. This completes the proof of Part 2) because we may replace A by λA with $|\lambda| \ll 1$.

3) We will now assume $\|A\|$, ε' are sufficiently small so that the resolvent expansion

$$\chi_\Lambda C_{h^\varepsilon}^\varepsilon \chi_\Lambda = \chi_\Lambda C^{\varepsilon 1/2} \sum_{n=0}^{\infty} (C^{\varepsilon 1/2} W^\varepsilon C^{\varepsilon 1/2})^n C^{\varepsilon 1/2} \chi_\Lambda$$

is convergent in $L_2(\mathbb{R}^2 \times \mathbb{R}^2)$ uniformly in $\varepsilon \leqq \varepsilon'$. To see this we refer to the proof of Lemma 3.4. Recall

$$W^{(\varepsilon)} \equiv -ie(A^{\varepsilon *} \partial^\varepsilon - \partial^{\varepsilon *} A^\varepsilon) - e^2 A^{\varepsilon *} A^\varepsilon .$$

By virtue of the uniformity, we can prove $\chi_\Lambda C_{h^\varepsilon}^\varepsilon \chi_\Lambda$ is convergent as ε tends to zero by proving

$$\chi_\Lambda C^{\varepsilon 1/2} ((C^\varepsilon)^{1/2} W^\varepsilon C^{\varepsilon 1/2})^n (C^\varepsilon)^{1/2} \chi_\Lambda , \qquad n = 0, 1, \ldots$$

is convergent in $L_2(\mathbb{R}^2 \times \mathbb{R}^2)$ as ε tends to zero. The operator in brackets raised to the power of n is strongly convergent by virtue of Lemma 3.9, Parts 2) and 3) and the fact that $(C^\varepsilon)\partial^{\varepsilon *}$ and its adjoint are bounded uniformly in operator norm. The factors $\chi_\Lambda C^{\varepsilon 1/2}$ are convergent in $\mathscr{I}_4$ by Lemma 3.9, Part 1). The proof is completed by Lemma 3.6 with $p = 4$, together with: $A_n \to A$, $B_n \to B$ in $\mathscr{I}_4 \Rightarrow A_n B_n \to AB$ in $\mathscr{I}_2$, which is a simple consequence of Hölder's inequality.

A Remark on Vitali's Theorem. Vitali's theorem [17] does not in its usual formulation hold for operator valued normal families. However if a normal family $\mathscr{F}$ of operator valued functions, analytic in a region Ω, is known to contain a subsequence convergent in some open set U in Ω, then that subsequence converges throughout Ω. A simple proof may be constructed by exhausting Ω by a set of overlapping open discs. The power series expansions associated with each disc are convergent uniformly in $\mathscr{F}$, so it is enough to prove termwise convergence, i.e., convergence of all derivatives at the centre points of the open discs. This is already given for any disc whose centre is in U. Any point in Ω may be reached by passing along a suitable chain of discs.

Remark 3.10. In the proof of Theorem 3.5 we will use the fact that the argument given above can easily be adapted to show that $\chi_\Lambda C^{\varepsilon}_{h^\varepsilon}\partial^{\varepsilon*}$ is Cauchy in $\mathscr{I}_4$.

Proof of Theorem 3.5. We begin by proving a lemma based on Corollary 4.8 of [14].

Define the following norm on functions on $\mathbb{R}^2$,

$$\|f\|^2_{2,\delta}=\int|f(x)(1+x^2)^\delta|^2\,dx\,.$$

Lemma 3.11. *For p, δ, α satisfying*

$$1\leqq p\leqq 2\,,\qquad \delta>0\,,\qquad \alpha>1/2+\delta\,,\qquad p>\frac{2}{1+2\delta}$$

$$\|(C^\varepsilon_h)^\alpha f\|_{\mathscr{I}_p}\leqq C_{\alpha,p,\delta}\|f\|_{2,\delta}$$

uniformly in ε, h.

Proof. Define $z\in[0,1]$ by

$$p=[z+1/2(1-z)]^{-1}\,.$$

Define γ, $\beta>1/2$ by

$$\delta=z\gamma\,,\qquad \alpha=\beta+z\gamma$$

and let K_z be the operator with kernel

$$K_z(x,y)\equiv(C^\varepsilon_h)^{\beta+z\gamma}f(y)(1+y^2)^{-z\gamma}\,.$$

The lemma is equivalent to proving

$$\|K_z\|_{\mathscr{I}_p}\leqq C_{\beta,\gamma}\|f\|_2\,.$$

By interpolation, [18], it is sufficient to prove this for $z=0$, $z=1$. When $z=0$, $p=2$; $z=1$, $p=1$. By the diamagnetic bound, Theorem 3.1,

$$\|K_0\|^2_{\mathscr{I}_2}\leqq\int|f(x)|^2(C^\varepsilon_1)^{2\beta}(x-x)\,dx\,dy\,.$$

We have omitted internal indices which are to be summed over. By the Fourier transform Lemma 3.7, the right hand side is bounded by a constant times $\|f\|^2_2$ which completes the $z=0$ case. For the $z=1$ case we write

$$K_1=AB\,,\qquad \|K_1\|_{\mathscr{I}_1}\leqq\|A\|_{\mathscr{I}_2}\|B\|_{\mathscr{I}_2}$$

and choose A, B to have kernels

$$A(x,y)=C^\beta_h(x,y)(\Lambda+y^2)^{-\gamma}$$

$$B(x,y)=(1+x^2)^\gamma C^\gamma_h(x,y)f(y)(1+y^2)^{-\gamma}\,.$$

We have omitted and will omit ε's to simplify the formulas.

The techniques used in the $z=0$ case can be applied to show that $\|A\|_{\mathscr{I}_2}$ is bounded by a constant depending on γ, β, because $(1+y^2)^{-\gamma}$ belongs to L_2. The $\mathscr{I}_2$ norm of B is equal to

$$\iint |f(x)(1+x^2)^{-\gamma}|^2 |C_1^\gamma(x-y)^2(1+y^2)^{2\gamma}dxdy.$$

We show this is less than a constant times $\|f\|_2$ by using

$$(1+y^2)^{2\gamma} \leqq (1+x^2)^{2\gamma} + (1+(x-y)^2)^{2\gamma}$$

together with

$$|C_1^\gamma(x-y)| \leqq ce^{-n\|x-y\|}$$

which follows from the analyticity of the Fourier transform of C_1^γ.

We now return to the proof of Theorem 3.5. We wish to show that $\delta C_{h^\varepsilon}^\varepsilon$ is Cauchy in L_p when restricted to the diagonal. We first show that $\delta C_{h^\varepsilon}^\varepsilon(x,x)$ is in L_p uniformly in ε. Thus

$$\left(\int_\Lambda |\mathrm{tr}_V \delta C_{h^\varepsilon}^\varepsilon(x,x)^p dx\right)^{1/p} \leqq \sup_f \int_\Lambda \mathrm{tr}_V(f(x)\delta C_{h^\varepsilon}^\varepsilon(x,x))dx, \tag{3.5}$$

where f is a function whose values are scalar multiples of the identity in $\mathscr{L}(V)$. Internal indices have been omitted, they are summed to form the trace (tr_V) on V. The supremum is taken over f such that

$$\left(\int_\Lambda |\mathrm{tr}_V f|^{p'} dx\right)^{1/p'} = 1, \qquad \frac{1}{p} + \frac{1}{p'} = 1.$$

The right hand side of the inequality (3.5) can be written as a trace, i.e.

$$\sup_f \mathrm{tr}(\delta C_h^\varepsilon f). \tag{3.6}$$

We are omitting ε's to simplify the notation. Define $h(\lambda)$ as in the proof of Theorem 3.3,

$$h = e^{i\varepsilon\mathscr{A}}, \qquad h(\lambda) \equiv e^{i\varepsilon\lambda\mathscr{A}}.$$

(3.6) can be written

$$\sup \int_0^1 d\lambda \frac{d}{d\lambda} \mathrm{tr}(\delta C_\lambda f), \tag{3.7}$$

where $\delta C_\lambda = \delta C_{h(\lambda)}$. Expand using

$$\frac{d}{d\lambda}\delta C_\lambda = -C_\lambda\left(\frac{d}{d\lambda}W\right)C_\lambda,$$

$$W = ie[\partial^* A_\lambda - A_\lambda^* \partial] - e^2 A_\lambda^* A_\lambda,$$

$$A_\lambda = (ie\varepsilon)^{-1}(h(\lambda) - \mathbb{1}).$$

We are as usual suppressing μ's. Therefore

$$\frac{d}{d\lambda}\mathrm{tr}(\delta C_\lambda f) = ie\,\mathrm{tr}(C_\lambda \partial^* A'_\lambda C_\lambda f) - ie\,\mathrm{tr}(C_\lambda A_\lambda^{*\prime} \partial C_\lambda f) - e^2\,\mathrm{tr}(C_\lambda (A^*A)' C_\lambda f). \tag{3.8}$$

The prime indicates differentiation with respect to λ. The integral over λ of this is less than the supremum over $\lambda \in [0,1]$. We now will show how to bound the first term in (3.8) by a constant times the $L_{p'}$ norm, $\|f\|_{p'}$, of f which is one. Similar steps yield the same bound for the second term and the third term is easier so we will not dicuss these further. Thus this bound will show that the L_p norm of $\delta C^{\varepsilon}_{h^\varepsilon}$ is bounded uniformly in ε. From this point we will drop the tr_V. A sum over internal indices is to be understood.

We bound the first term in (3.8) using Hölder's inequality,

$$|\mathrm{tr}(C\partial^* A' C f)| \leqq \|C^\alpha \partial^* A' C^\alpha\|_2 \, \|C^\beta f C^\beta\|_2 \,, \tag{3.9}$$

where $\alpha+\beta=1$. We are now suppressing λ also. The cyclicity of the trace was used to move a factor C^β. The second $\mathscr{I}_2$ norm equals

$$\left(\int_\Lambda \int_\Lambda \bar{f}(x)\, |C^{2\beta}_{h(\lambda)}(x,y)|^2 f(y)\, dx dy\right)^{1/2}. \tag{3.10}$$

By Hölder's inequality and Theorem 3.1, the diamagnetic bound, this is less than a constant times

$$\left(\int_\Lambda \int_\Lambda \|C^{2\beta}_{\mathbb{1}}(x,y)\|^{2p}_{\mathscr{L}(V)}\right)^{1/2p} \|f\|_{p'} \,. \tag{3.11}$$

The first factor is bounded uniformly in ε provided

$$2p(1-2\beta) < 1 \tag{3.12}$$

because homogeneity considerations applied to the Fourier transform of $C^{2\beta}$ show that

$$|C^{2\beta}(x,y)| \leqq c|x-y|^{-2(1-2\beta)} \tag{3.13}$$

uniformly in ε. Our choice of β is constrained by (3.12). Our proof that δC is uniformly in L_p will be complete if we can show that $\alpha = 1-\beta$ can be picked consistent with (3.12) so that the first $\mathscr{I}_2$ norm in (3.9) is bounded uniformly in ε. We have

$$\|C^\alpha \partial^* A' C^\alpha\|_2 \leqq \|C^\alpha \partial^*\| \, \|A' C^\alpha\|_2 \,. \tag{3.14}$$

The second norm is bounded uniformly in ε if $\alpha > 1/2$ by an argument like that used to bound (3.10). One has to use the fact that A'_λ is bounded in L_∞ norm uniformly in λ, ε. We claim that if $\alpha > 1/2$, the first norm is also bounded uniformly in ε. Thus by the triangle inequality and the definition of $D_{h(\lambda)}$,

$$\|C^\alpha_\lambda \partial^*\| \leqq \|C^\alpha_\lambda D^*_{h(\lambda)}\| + e\|C^\alpha A^*_\lambda\| \,. \tag{3.15}$$

The second norm is bounded uniformly in ε because $\|A_\lambda\|$ is bounded and $\|C^\alpha\|$ is less than $(m^2)^{-\alpha}$. We bound the first norm by

$$\|C^{\alpha-1/2}_\lambda\| \, \|C^{1/2}_\lambda D^*_{h(\lambda)}\| \leqq \left(\frac{1}{m^2}\right)^{\alpha-1/2} \|C^{1/2}_\lambda D^*_{h(\lambda)} D_{h(\lambda)} C^{1/2}_\lambda\|^{1/2} \leqq \left(\frac{1}{m^2}\right)^{\alpha-1/2} \tag{3.16}$$

as was used in the proof of Lemma 3.4. We have now proved that the L_p norm of δC is bounded uniformly in ε.

We now combine this result with Lemma 3.11 to complete the proof of Theorem 3.5. By Hölder's inequality, it is enough to prove δC is L_1-Cauchy. If $A(x,y)$ is the kernel of an operator $A \in \mathscr{I}_1$

$$\int_{\Lambda} |A(x,x)|\,dx \leqq \sup_f \operatorname{tr}(f\chi_\Lambda A\chi_\Lambda) \leqq \|\chi_\Lambda A\chi_\Lambda\|_{\mathscr{I}_1},$$

where the supremum is over f with $\|f\|_\infty = 1$ and χ_V is the characteristic function of Λ; To make then the left hand side unambiguous one should of course think of A being factorized into two Hilbert-Schmidt operators. By this inequality it follows that we may prove our theorem by showing that δC is convergent in $\mathscr{I}_1$.

Since

$$\delta C = C_h W_h C_{\mathbb{1}} = ieC_h\partial^* AC_{\mathbb{1}} - ieC_hA^*\partial C_{\mathbb{1}} + e^2 C_h A^* AC_{\mathbb{1}}$$

(where subscripts μ, ε have been suppressed) it is enough to show that

a) $\chi_\Lambda C_h \partial^*$ Cauchy in $\mathscr{I}_4$,

b) $AC_h\chi_\Lambda$ Cauchy in $\mathscr{I}_{4/3}$,

e.g. the first term in the expansion for $\chi_\Lambda \delta C \chi_\Lambda$ is $\mathscr{I}_1$ Cauchy because we may take $h=1$ in b) and combine a) and b) by Hölder's inequality. A similar argument involving the adjoints of the operators in a) and b) (which converge because taking the adjoint is a continuous map from $\mathscr{I}_p$ to $\mathscr{I}_p$) suffices for the second term. The third term is Cauchy in $\mathscr{I}_1$ because b) implies $\chi_\Lambda C_h A^*$ and $AC_1\chi_\Lambda$ are each Cauchy in $\mathscr{I}_2$.

As has already been remarked, the proof of a) can be accomplished along the same lines as the proof of Theorem 3.2. To prove b) observe that by Lemma 3.11 it follows that $C_h^\varepsilon \chi_\Lambda$ is in $\mathscr{I}_p$ for $2 \geqq p > 1$ uniformly in ε. By Theorem 3.3 it is convergent in $\mathscr{I}_2$. Hölder's inequality implies b). The proof of Theorem 3.5 is complete.

IV. Convergence of the Lattice Approximation in an External Yang Mills Field

In this section we prove that the partition function and its associated finite volume expectation, for the case in which the Yang Mills field is external, converge as the lattice spacing tends to zero. We allow the orientation of the lattice to vary as the limit is taken, in order to be able to conclude Euclidean covariance of the limit. For simplicity we consider a lattice theory with just one boson field. Extra boson fields would not be a serious complication.

We begin by some changes in notation and normalisation of the partition function described in Sect. 2.3 in Paper I. These are necessary for a convenient description of the continuum limit. We factor the partition function into a renormalised determinant $z_\Lambda^\varepsilon(h^\varepsilon)$ and a partition function $Z_\Lambda^\varepsilon(h^\varepsilon)$ of the type considered by Schrader [7], but on a lattice; it differs also in that the boson self interaction V_Λ^ε is normal ordered with respect to $(m^2 - \Delta^\varepsilon)^{-1}$. We show convergence for these two factors separately in Theorems 4.2 and 4.1 respectively. The convergence proof for $Z_\Lambda^\varepsilon(h^\varepsilon)$ is based in spirit if not in body on [6]. One difference which appears to help in this case is that we embed our lattice Gaussian processes in white noise. The diamagnetic bound, Corollary 2.4 of Paper I, is an important ingredient.

The convergence proof for $\varkappa_\Lambda^\varepsilon(h^\varepsilon)$ involves a study of some divergent (as $\varepsilon \downarrow 0$) contributions to the vacuum polarisation, $\Pi_{\mu\nu}^\varepsilon$, which cancel up to a finite transverse part by a Ward identity, or gauge invariance. This work is rather grubby and is postponed to Appendix A.

In Paper I we defined partition functions for matter in external Yang Mills fields (see for example Sect. 2.3 in Paper I). We now specialise to Bose matter in $\mathbb{R}^2$ with free boundary conditions. We will also be making some normalisation changes to obtain partition functions which will converge as $\varepsilon \to 0$.

Let ϕ be a function from $L^{(\varepsilon)}$ to V represented in components by $(\phi_{x,i})$, $x \in L^{(\varepsilon)}$, $i = 1, \ldots, \dim V$. Define

$$\begin{aligned} D\phi &\equiv \prod_{x,i} d\,\mathrm{Re}\,\phi_{x,i} d\,\mathrm{Im}\,\phi_{x,i}\,, \\ \tilde{Z}_\Lambda^\varepsilon(h) &\equiv \int D\phi e^{-A_\Lambda^{M\varepsilon}(\phi,h)}\,, \\ A_\Lambda^{M\varepsilon}(\phi,h) &\equiv -1/2(\phi,(m^2-\Delta_h^{\varepsilon F})\phi)_\Lambda + V_\Lambda^\varepsilon(\phi)\,, \\ V_\Lambda^\varepsilon(\phi) &\equiv \sum_x \varepsilon^2 \sum_{i=1}^p :P_i(\phi_x):_\varepsilon (Q^\varepsilon g_i)(x)\,. \end{aligned} \tag{4.1}$$

The tilde on the Z is there because we wish to reserve Z for another partition function. Sums and products over x run over $L^{(\varepsilon)}(\Lambda)$. A_Λ^M is the matter action, hence the M superscript. V_Λ is the Bose self interaction. $:P_i:_\varepsilon$ is a monomial normal ordered with respect to $C_{\mathbb{1}}^\varepsilon \cdot g_i \in C^\infty(\mathbb{R}^2)$. We assume that V is bounded below as a polynomial in ϕ when the normal ordering is dropped. At this stage V does not have to be gauge invariant.

Since $\tilde{Z}_\Lambda^\varepsilon(h)$ diverges as ε decreases to zero, we renormalise by dividing by $\zeta_\Lambda^\varepsilon(\mathbb{1})$ where

$$\zeta_\Lambda^\varepsilon(h) \equiv \int D\phi e^{-1/2(\phi,(m^2-\Delta_h^F)\phi)}\,. \tag{4.2}$$

Thus let

$$\begin{aligned} Z_\Lambda^\varepsilon(h) &\equiv \tilde{Z}_\Lambda^\varepsilon(h)/\zeta_\Lambda^\varepsilon(\mathbb{1}) \\ &= \frac{\tilde{Z}_\Lambda^\varepsilon(h)}{\zeta_\Lambda^\varepsilon(h)} \frac{\zeta_\Lambda^\varepsilon(h)}{\zeta_\Lambda^\varepsilon(\mathbb{1})} \\ &= \left(\int dv_h^\varepsilon e^{-V_\Lambda^\varepsilon}\right) \varkappa_\Lambda^\varepsilon(h)\,, \end{aligned} \tag{4.3}$$

where $dv_h^\varepsilon(\phi)$ is the normalised Gaussian measure with mean zero and covariance C_h^ε. (The F on the covariance can be dropped because V depends on fields supported inside Λ.) $\zeta_\Lambda^\varepsilon(h)$ is different from zero by explicit Gaussian integration

$$\varkappa_\Lambda^\varepsilon(h) \equiv \zeta_\Lambda^\varepsilon(h)/\zeta_\Lambda^\varepsilon(\mathbb{1})\,. \tag{4.4}$$

We can now state our first theorem for this section.

Theorem 4.1. *If (h^ε) is a family of lattice gauge fields converging in the sense of Definition 3.2 to a continuum field A and $\Lambda \subset \mathbb{R}^2$ is bounded, then*

$$\int dv_{h^\varepsilon}^\varepsilon e^{-\lambda V_\Lambda^\varepsilon}$$

is convergent to a non zero limit dependent only on A for all $\lambda \geqq 0$.

Remark. In particular the limit does not depend on the orientations of the lattices $L^{(\varepsilon)}$.

The convergence of $\varkappa_A^\varepsilon(h)$ requires a stronger topology. We will now define a norm which seems to be as convenient as any. Given $\alpha>0$, set [cf. (2.4)]

$$\|A\|_{\infty,\alpha}\equiv\|A\|+\left(\int_{A\times A} dxdy\left\|\frac{(A(x)-A(y))(A(x)-A(y))^*}{|x-y|^{2+\alpha}}\right\|_{\mathscr{L}(V)}\right)^{1/2}. \tag{4.5}$$

This norm is chosen so that $\Pi_{\mu\nu}^\varepsilon$, the second order vacuum polarisation graphs, converges as ε tends to zero (see Theorem 4.3 and the Appendix).

Definition. 4.2. A family (h^ε) of lattice gauge fields is convergent to A in the (∞,α) sense if

$$A_\mu^\varepsilon(x)\equiv(ie\varepsilon)^{-1}(h^\varepsilon(x)-\mathbb{1}_V)$$

converges to A_μ in the sense $\|A^\varepsilon-A\|_{\infty,\alpha}\to 0$ as $\varepsilon\to 0$.

For our next theorems we assume Λ is a *bounded rectangle.* We also require that our gauge fields h be supported inside Λ.

Theorem 4.3. *If a family (h^ε) of gauge fields is convergent to a continuum gauge field A in the (∞,α) sense, then $\varkappa_A^\varepsilon(h^\varepsilon)$ is convergent to a non zero limit.*

Define the unnormalised measure

$$d\omega_h^\varepsilon=\varkappa_A^\varepsilon(h)dv_h^\varepsilon e^{-V_A^\varepsilon}. \tag{4.6}$$

In Paper I we showed that $Z_A^\varepsilon(h)$ is non zero. Therefore we can divide through and thus define the corresponding normalised measure $d\omega_h^{\varepsilon'}$.

We now wish to examine the limit as ε tends to zero of these measures. The limiting continuum measures will be defined on $\mathscr{S}'(\mathbb{R}^2)$, the Schwartz distribution space.

Corollary 4.4. *Let (h^ε) be convergent as in Theorem 4.2. $d\omega_{h^\varepsilon}^{\varepsilon'}$ converges as ε tends to zero to a limit $d\omega'_A$. The convergence is in the sense of convergence of characteristic functions. All moments converge also, i.e.,*

$$\int d\omega_{h^\varepsilon}^{\varepsilon'}e^{i\phi(Q^\varepsilon f)}\to\int d\omega'_A e^{i\phi(f)},$$

$$\int d\omega_{h^\varepsilon}^{\varepsilon'}\prod_{i=1}^q\phi(Q^\varepsilon f_i)\to\int d\omega'_A\prod_{i=1}^q\phi(f_i),$$

where $f,f_i\in C_0^\infty(\Lambda)$.

We now begin the proof of Theorem 4.1. We will need the following lemma.

Lemma 4.5. *Let $f:\mathbb{R}^+\to\mathbb{R}$ be a continuous function on the positive real line. Let $\mathscr{I}_p^+$ denote the cone of positive self adjoint operators in $\mathscr{I}_p$. We assume that f satisfies*

$$\|f(A)\|_p\leqq F(\|A\|_q)\qquad\forall A\in\mathscr{I}_p^+,$$

where $F(t)$ is a positive continuous function on $\mathbb{R}^+$ decreasing to zero as $t\searrow 0$. Then the map $A\to f(A)$ is continuous from $\mathscr{I}_q^+$ to $\mathscr{I}_p^+$.

Proof of Lemma 4.5. We will use the following standard facts: if A_n is a sequence of positive compact operators converging in operator norm to an operator A so that the spectra are discrete and of finite multiplicity, then the eigenvalues of A_n converge, the spectral projections $P^{(n)}_{[a,b]}$, $a<b\leqq\infty$, $a, b\notin\sigma(A)$ converge in operator norm (see for example [5], Vol. I, Theorem VIII. 23).

From this we conclude that $f(P^{(n)}_{(a,\infty)}A_n)$ converges in $\mathscr{I}_p$ for all p provided $a>0$ is not an eigenvalue of A. Choose a so small that for a given $\varepsilon>0$,

$$\|f(P_{[-a,a]}A)\|_p\leqq F(\|P_{[-a,a]}A\|_q)<\varepsilon/2\,. \tag{4.7}$$

By the triangle inequality

$$\begin{aligned}\|f(A_n)-f(A)\|_p\leqq \|f(P^{(n)}_{[-a,a]}A_n)\|_p+\|f(P_{[-a,a]}A)\|_p\\ +\|f(P^{(n)}_{(a,\infty)}A_n)-f(P_{(a,\infty)}A)\|_p\,.\end{aligned}$$

The third term converges to zero by the remarks above. The second term is less than $\varepsilon/2$ by (4.7). To bound the first term note that

$$P^{(n)}_{[-a,a]}A_n\to P_{[-a,a]}A \quad \text{in} \quad \mathscr{I}_q$$

because $A_n\to A$ in $\mathscr{I}_q$ and the projections converge in operator norm.

Thus

$$\begin{aligned}\limsup_{n\to\infty}\|f(P^{(n)}_{[-a,a]}A_n)\|_p&\leqq \limsup_{n\to\infty}F(\|P^{(n)}_{[-a,a]}A_n\|_q)\\ &=F(\|P_{[-a,a]}A\|_q)<\varepsilon/2\,.\end{aligned}$$

Proof of Theorem 4.1 (assuming Theorem 4.3). It suffices to consider $\lambda=1$. To begin with, we embed all the lattice path spaces in the space for white noise. Let $dw(\psi)$ be the white noise measure, i.e., the Gaussian process of mean zero and covariance equal to the identity operator. Define

$$\begin{aligned}&E^\varepsilon\equiv(Q^{\varepsilon *}\chi_\Lambda C^\varepsilon_{h^\varepsilon}\chi_\Lambda Q^\varepsilon)^{1/2}\,,\\ &\phi^\varepsilon\equiv E^\varepsilon\psi\,.\end{aligned}$$

E^ε is an operator on L_2. Then

$$\begin{aligned}\int dv^\varepsilon_{h^\varepsilon}e^{-V^\varepsilon}&=\int dw(\psi)e^{-V^\varepsilon(E^\varepsilon\psi)}\\ &(\equiv\int dwe^{-V^\varepsilon})\,.\end{aligned}$$

Therefore as in [6], (II.24), we can show convergence by

$$\begin{aligned}|\int dwe^{-V^\varepsilon}-\int dwe^{-V^{\varepsilon'}}|&\leqq\int dw|V^\varepsilon-V^{\varepsilon'}|\\ &\quad\cdot(e^{-V^\varepsilon}+e^{-V^{\varepsilon'}})\\ &\leqq(\int dw|V^\varepsilon-V^{\varepsilon'}|^2)^{1/2}\\ &\quad\cdot\{(\int dwe^{-2V^\varepsilon})^{1/2}+(dwe^{-2V^{\varepsilon'}})^{1/2}\}\,.\end{aligned}$$

The second inequality is simply Cauchy-Schwarz together with $\sqrt{(x+y)}\leqq\sqrt{x}+\sqrt{y}$.

The integrals in curly brackets may be bounded uniformly in $\varepsilon, \varepsilon'$ by the diamagnetic bound, Theorem 4.1 of Paper I,

$$\begin{aligned}\int dw\, e^{-2V^\varepsilon} &= (\varkappa_A^\varepsilon)^{-1} \varkappa_A^\varepsilon \int dv_{h^\varepsilon}^\varepsilon\, e^{-2V^\varepsilon} \\ &\leqq (\varkappa_A^\varepsilon)^{-1} \int dv_1^\varepsilon e^{-2V^\varepsilon} .\end{aligned}$$

By Theorem 4.3 the first term converges as ε tends to zero to a finite number. The second factor is bounded uniformly in ε by Nelson's boundedness below proof for $P(\phi_2)$ (see [19]).

To complete the proof it now remains to show that

$$\int dw |V^\varepsilon - V^{\varepsilon'}|^2 \to 0 \quad \text{as} \quad \varepsilon, \varepsilon' \to 0 . \tag{4.8}$$

We may without losing any generality assume that for some positive integer N

$$V^\varepsilon = \int_A :(\phi^\varepsilon)^N(x):_\varepsilon dx$$

because in general V is a sum of such monomials. By virtue of the change of normal ordering formula [29], p. 11 (internal indices suppressed)

$$:\phi^N:_{C^\varepsilon} = \sum_{j=0}^{[N/2]} d_j (\delta C_{h^\varepsilon}^\varepsilon(x,x))^j : \phi^{N-2j}(x) :_{C_{h^\varepsilon}^\varepsilon} ,$$

where $d_1, \ldots$ are universal constants and $[N/2]$ is the largest integer less than or equal to $N/2$, we may without loss take

$$V^\varepsilon = \int_A (\delta C_{h^\varepsilon}^\varepsilon(x,x))^j :(\phi^\varepsilon)^{N-2j}(x):_{C_{h^\varepsilon}^\varepsilon} dx .$$

With V^ε of this form we prove (4.8) by showing that

$$\int dw V^\varepsilon (V^\varepsilon - V^{\varepsilon'}) \to 0 \qquad \varepsilon, \varepsilon' \to 0 .$$

By the standard methods [20] for evaluating Gaussian integrals, this is equivalent to

$$\int_A \int_A \delta C^j(x,x) (\delta C^{N-2j}(y,y) C^j(x,y) - \delta C'^j(y,y) (EE')^{N-2j}(x,y)) \to 0 \tag{4.9}$$

as $\varepsilon, \varepsilon' \to 0$. We have suppressed $h^\varepsilon, \varepsilon, \varepsilon'$ in favour of primes. EE' is the operator product i.e.,

$$\int E(x,z) E'(z,y) dz .$$

We know by Theorem 3.5 that δC converges in L_p for all $1 \leqq p \leqq \infty$. Theorem 3.3 and Lemma 4.5 [with $f(x) = \sqrt{x}$] imply that E^2 converges in $\mathscr{I}_4$, therefore $E^\varepsilon E^{\varepsilon'}$ converges in $\mathscr{I}_2$ which is the same as convergence in $L_2(A \times A)$. Recall that $C_{h^\varepsilon}^\varepsilon$ is in $L_p(A \times A)$ uniformly in ε for $1 \leqq p < \infty$ by the diamagnetic inequality, Theorem 3.1. A judicious assortment of triangle inequalities and Hölder inequalities yields (4.9). This proves that

$$\int dv_{h^\varepsilon}^\varepsilon e^{-V^\varepsilon}$$

is a Cauchy sequence.

The proof of Theorem 4.1 is complete once we show the limit is not zero. Therefore, by Jensen's inequality

$$\int dv^{\varepsilon}_{h\varepsilon} e^{-V^{\varepsilon}_{\Lambda}} \geqq \exp\left(-\int dv^{\varepsilon}_{h\varepsilon} V^{\varepsilon}_{\Lambda}\right).$$

The integral in the exponent is not infinite in the limit ε tends to zero. If one does the integral by explicit Gaussian integration, the result is a sum of L_p norms of δC which by Theorem 3.5 converge as ε tends to zero.

Proof of Corollary 4.4. Since $\varkappa^{\varepsilon}_{\Lambda}$ and $Z^{\varepsilon}_{\Lambda}$ converge (we are assuming Theorems 4.1 and 4.3) as ε tends to zero, it suffices to prove that

$$\int dv^{\varepsilon}_{h\varepsilon} F(Q^{\varepsilon}\phi) e^{-V^{\varepsilon}} = \int dw e^{-V^{\varepsilon}} F(Q^{\varepsilon}\phi^{\varepsilon})$$

converges. F is a polynomial or exponential. This follows from L_2 convergence of e^{-V} (see the proof of Theorem 4.1) and of F [see (4.8)]. These are standard arguments (see [6]).

Before beginning the proof of Theorem 4.3, we rewrite $\varkappa^{\varepsilon}_{\Lambda}$ in a more convenient form, namely

$$\varkappa^{\varepsilon}_{\Lambda}(h^{\varepsilon}) = \det{}^{-1/2}(1 - C^{\varepsilon}W^{\varepsilon}), \tag{4.10}$$

where as usual

$$W^{\varepsilon} = -ieA^{\varepsilon *}_{h\varepsilon}\partial^{\varepsilon} + ie\partial^{\varepsilon *}A^{\varepsilon}_{h\varepsilon} - e^2 A^{\varepsilon *}_{h\varepsilon}A^{\varepsilon}_{h\varepsilon}. \tag{4.11}$$

To simplify notation subscripts μ have been omitted. We will also suppress ε in the equations below. To obtain (4.10), first explicitly integrate the Gaussian integrals in $\varkappa_{\Lambda}$

$$\begin{aligned}\varkappa_{\Lambda}(h) &= \det{}^{-1/2}(m^2 - \Delta^F_h)\det{}^{1/2}(m^2 - \Delta^F)\\ &= \det{}^{-1/2}(1 + C^F(\Delta^F - \Delta^F_h)).\end{aligned}$$

This coincides with (4.10) once we argue that the F denoting free boundary conditions can be dropped. Since C^F and C coincide when their kernels are restricted to $\Lambda \times \Lambda$ we need to show that

$$\chi_{\Lambda}(\Delta^F_h - \Delta^F)\chi_{\Lambda} = \Delta_h - \Delta.$$

This in turn follows from the following facts

(1) $\Delta_h - \Delta = \chi_{\Lambda}(\Delta_h - \Delta)\chi_{\Lambda}$.

This is easily verified using the definitions. Recall that h is supported inside Λ.
(2) The kernels of $\Delta^{\varepsilon F}_h$ and Δ^{ε}_h coincide when restricted to $\Lambda \times \Lambda$ except at the lattice points on the boundary. At these points the difference is independent of h. This second fact may easily be proved by going through the proof of Theorem IV.7 in [6] with Δ replaced by Δ_h.

We now introduce the following standard notation [21]. Given $K \in \mathscr{I}_1$, define renormalised determinants, $n = 2, 3, \ldots$

$$\det{}_n(1+K) = \det(1+K)\exp\left[\sum_{j=1}^{n-1} \frac{(-1)^j}{j}\operatorname{tr} K^j\right].$$

Then

$$\det{}^{-1/2}(1-C^\varepsilon W^\varepsilon)=\det{}_4^{-1/2}(1-C^\varepsilon W^\varepsilon)\cdot\exp\left[\sum_{j=1}^{3}\frac{1}{j}\frac{1}{2}\operatorname{tr}(C^\varepsilon W^\varepsilon)^j\right]. \tag{4.12}$$

Proof of Theorem 4.3 (using Appendix A). We see by (4.12) that it is enough to show that

1) $\det_4(1+K^\varepsilon)$ is convergent as $\varepsilon\searrow 0$.
2) $\operatorname{tr}(K^\varepsilon)^3$ is convergent as $\varepsilon\searrow 0$.
3) $-\frac{1}{2}\operatorname{tr}(K^\varepsilon)+\frac{1}{4}\operatorname{tr}(K^\varepsilon)^2$ is convergent as $\varepsilon\searrow 0$.
4) $|\varkappa_A^\varepsilon(h^\varepsilon)|\leqq 1$,

where

$$K^\varepsilon\equiv -C^\varepsilon W^\varepsilon\,. \tag{4.13}$$

First note that 4) is the diamagnetic bound of R. Schrader, R. Seiler. A proof is also given in Paper I, Sect. 3.3.

Proof of 1). We suppress ε's. Set

$$H\equiv -C^{1/2}WC^{1/2}$$

and note that since W is finite rank,

$$\det{}_4(1+K)=\det{}_4(1+H)\,.$$

We now appeal to the well known fact [21 a, e, f] that $\det_n$ is Lipschitz continuous on $\mathscr{I}_n$. Then 1) follows if we show that H is Cauchy in $\mathscr{I}_4$. To prove this, expand W using (4.11) and factor each term in the sum into products of

$$C^{1/2}\partial,\ A_h,\ \chi_A C^{1/2} \tag{4.14}$$

and their adjoints. The factor χ_A can be skipped by using the condition on the support of h. The first operator converges strongly, the second in operator norm, the third in $\mathscr{I}_4$ by Lemmas 3.7 and 3.11. Each term in the sum contains at least one of the third kind, thus using Lemma 3.6 one obtains 1).

Proof of 2). This is essentially the same as 1). Expand K^3. Write each term as a product of operators as in (4.14) and their adjoints. Each term contains at least two factors converging in $\mathscr{I}_4$. This is sufficient to prove 2).

The proof of 3) is more subtle and is the only place where we need the stronger notion of (∞,α) convergence. The problem is that the individual traces in 3) diverge as ε tends to zero. There is a cancellation between them due to a Ward identity (gauge invariance). For the proof of 3), see Appendix A. □

Remark. We conclude this section by sketching some constructive, uniform upper and lower bounds for $\varkappa^\varepsilon(h)$, valid for all A^ε with $|\operatorname{Im}A^\varepsilon|\leqq$ const, uniformly in ε.

Suppose that $A^\varepsilon \to A$, as $\varepsilon \to 0$, in the (∞,α) sense [see (4.5) and Sect. V]. We require that

$$A = A_1 + iA_2 , \qquad \|A_1\|_{\infty,\alpha} < \infty , \qquad \|A_2\|_{\infty,\alpha} < \xi , \tag{4.15}$$

where A_1, A_2 are real and ξ will be chosen below. The norm $\| \ \|_{\infty,\alpha}$ is defined in (4.5).

Choose a positive integer N so that

$$N^{-1}\|A_1\|_{\infty,\alpha} < \xi . \tag{4.16}$$

Recall the definition of $\mathscr{A}^\varepsilon$ given in Eq. (2.6). We decompose $\mathscr{A}^\varepsilon$ into its real and imaginary parts:

$$\mathscr{A}^\varepsilon = \mathscr{A}_1^\varepsilon + \mathscr{A}_2^\varepsilon . \tag{4.17}$$

For each ε we define a sequence of gauge fields by

$$h_m^\varepsilon = e^{i\frac{m}{N}e\varepsilon\mathscr{A}_1^\varepsilon} . \tag{4.18}$$

Our bounds are based on the trivial identity

$$\varkappa(h) = \left\{ \prod_{m=1}^{N} \frac{\varkappa(h_m)}{\varkappa(h_{m-1})} \right\} \frac{\varkappa(h)}{\varkappa(h_N)} . \tag{4.19}$$

We have suppressed ε's. The idea is to obtain a uniform (in ε, ε small) upper and lower bound on each factor using direct methods, in particular the loop expansion.

Let W_m be defined by

$$\Delta_m = \Delta_{m-1} + W_m , \tag{4.20}$$

where $\Delta_m = \Delta_{h_m}$. Set $h = h_{N+1}$ and $C_m = C_{h_m}$. Then

$$\begin{aligned} \frac{\varkappa(h_m)}{\varkappa(h_{m-1})} &= \det{}^{-1/2}(1 + C_{m-1}^{1/2} W_m C_{m-1}^{1/2}) \\ &\equiv \det{}_4^{-1/2}(1 + C_{m-1}^{1/2} W_m C_{m-1}^{1/2}) g_m . \end{aligned} \tag{4.21}$$

(This defines g_m.) Since A^ε converges to A and $\mathscr{A}^\varepsilon$, A^ε only differ by terms of order ε, it is easy to show that $\mathscr{A}^\varepsilon$ converges to A in the (∞,α) norm. We in fact show this in the next section. Next, by choosing ξ small we show that the loop expansion for $\det_4$ in (4.21) converges absolutely and uniformly in m and ε, for $\varepsilon \leqq \varepsilon_0$ for some $\varepsilon_0 > 0$. This is done by using the diamagnetic bound, Theorem 3.1, and $\mathscr{I}_2$ estimates of the type established in the proof of Lemma 3.4 [see (3.1)–(3.4)] and is not difficult. From this we obtain

$$c_1 < |\det{}_4^{-1/2}(1 + C_{m-1}^{1/2} W_m C_{m-1}^{1/2})| < c_2 \tag{4.22}$$

for some constants c_1, c_2 independent of ε and m.

The factor g_m is the exponential of all terms of order 1, 2, and 3 in W_m arising in the loop expansion of $\det^{-1/2}(1 + C_{m-1}^{1/2} W_m C_{m-1}^{1/2})$. These more singular terms are estimated by expanding C_{m-1} in a partial Neumann series. The leading terms give the contribution $\Pi_{\mu\nu}$ analysed in Appendix A. The remainders are estimated by methods resembling those in the proof of Theorem 3.5. The details are tedious but straightforward and are omitted.

The conclusion is

$$c_1' < |g_m| < c_2' \,, \tag{4.23}$$

where c_1', c_2' are constants depending only on ξ and $\|A\|_{\infty,\alpha}$. We collect (4.19), (4.22), and (4.23) to obtain

$$(c_1 c_1')^{N+1} \leqq |\varkappa^\varepsilon(h)| \leqq (c_2 c_2')^{N+1} \,. \tag{4.24}$$

Note that if A is real valued along with $\mathscr{A}^\varepsilon$ for all ε, then $\varkappa^\varepsilon$ is real and positive because by (4.4) it is the ratio of two positive integrals, therefore (4.24) is strenghthened to

$$(c_1 c_1')^N \leqq \varkappa^\varepsilon(h) \leqq 1 \,, \tag{4.25}$$

where the right hand bound is the diamagnetic bound, Theorems 2.3, 4.1, and Sect. 3, Paper I. N is determined by $\|A\|_{\infty,\alpha} < N\xi$.

V. Convergence of the Partition Function for Yang Mills and Matter Fields (Yang Mills Fields with a Cutoff)

V. 1. In this section we specialise to abelian Yang-Mills fields. This is implicit in our use of a Gaussian measure for the pure Yang-Mills field, which is incompatible with gauge invariance if the gauge group is not abelian.

Given a real measurable abelian gauge field A and a lattice $L^{(\varepsilon)}$, let A_μ be the components of A relative to the unit vectors generating $L^{(\varepsilon)}$. Given a bond b in the μth direction let

$$\begin{aligned} h^\varepsilon(b) &\equiv e^{ie\int_b A^\mu(x)dx} \quad \text{if} \quad b \in \Lambda \\ &\equiv 0 \qquad\qquad\quad \text{if} \quad b \notin \Lambda \,. \end{aligned} \tag{5.1}$$

This defines a lattice gauge field h^ε on $L^{(\varepsilon)}(\Lambda)$. Throughout this section, all lattice gauge fields will be derived from a continuum gauge field in this way. We will therefore regard the partition function $Z^\varepsilon_\Lambda(h^\varepsilon)$ of the last section as a function $Z^\varepsilon_\Lambda(A)$ of A. The ϕ field is complex.

The full Yang Mills and matter partition function, denoted $\mathcal{Z}^\varepsilon_\Lambda$ has the form

$$\mathcal{Z}^\varepsilon_\Lambda \equiv \int d\mu_D(A) Z^\varepsilon_\Lambda(A) \,, \tag{5.2}$$

where $d\mu_D(A)$ is a Gaussian measure, mean zero, covariance $D = D_{\mu\nu}(x, y)$.

In this section, we will assume that the covariance D is such that with probability one, the sample functions $A_\mu(x)$ are essentially uniformly Hölder continuous with modulus $\alpha < \frac{1}{2}$, (E.U.H.C.), which means that there exists a constant c_A, finite for almost all A, such that

$$|A_\mu(x) - A_\mu(y)| \leqq c_A |x-y|^\alpha \,, \qquad x, y \in \Lambda \sim E_A \,, \qquad \mu = 0, 1 \,, \tag{5.3}$$

where E_A is a set of Lebesgue measure zero, dependent on A. A sufficient condition on the covariance D for (5.3) to hold for almost all sample functions A_μ is given in

Sect. 5.2. The condition (5.3) excludes the covariances we are ultimately interested in and this is why we refer to such covariances as "cutoff". The cutoff has to be removed by taking a limit *outside* the A integral. This limit is more difficult because it involves renormalisation. It will be discussed in Paper III.

Theorem 5.1. *If Λ is a bounded rectangle and $L^{(\varepsilon)}$ is a family of lattices, $\varepsilon>0$, then* $\lim_{\varepsilon\to 0} Z^{\varepsilon}_{\Lambda}$ exists, is non zero and is unique.

Define

$$Z_{\Lambda}\equiv \lim_{\varepsilon\to 0} Z^{\varepsilon}_{\Lambda},$$

$$\langle F\rangle \equiv Z^{-1}\int d\mu_D(A)\varkappa^{\varepsilon}_{\Lambda}(A)\cdot \int d\nu^{\varepsilon}_{\Lambda} e^{-V^{\varepsilon}_{\Lambda}}F,$$

where $F\in L_p(d\mu_D\times d\nu_{\Lambda})$ for $1\leqq p<\infty$.

Corollary 5.2. *The measures $\langle\,\rangle^{\varepsilon}_{\Lambda}$ converge as $\varepsilon\to 0$ in the sense of convergence of generating functions. All moments converge.*

Proof. Essentially identical to the proof of Corollary 4.4.

Define

$$\langle\,\rangle_{\Lambda}\equiv \lim_{\varepsilon\to 0}\langle\,\rangle^{\varepsilon}_{\Lambda}. \tag{5.5}$$

Proof of Theorem 5.1. We begin by showing that if A_{μ} satisfies (5.3), then h^{ε} as defined by (5.1) converges as $\varepsilon\to 0$ in the (∞,α') sense for $\alpha'<\alpha$. By (5.3) $A^{\varepsilon}_{\mu}(x)$ is in L_{∞}. By expanding the exponential in $A^{\varepsilon}_{\mu}(x)$.

$$\|A^{\varepsilon}_{\mu}-A_{\mu}\chi_{\Lambda}\|\leqq \operatorname{ess.sup}_{b,\xi}\left|\varepsilon^{-1}\int_b A_{\mu}(x')dx'_{\mu}-A_{\mu}(\xi)\right|+O(\varepsilon),$$

where the essential supremum is taken over all $\xi\in\Lambda$ within distance $\varepsilon/2$ of a given bond b, and then over all bonds b in Λ. The first term tends to zero by (5.3). Next define

$$B^{\varepsilon}_{\mu}(x)=A^{\varepsilon}_{\mu}(x)-A_{\mu}(x)\chi_{\Lambda}(x). \tag{5.6}$$

The proof of (∞,α') convergence is complete once we show that the seminorm

$$\left(\int_{\Lambda} dx\int_{\Lambda} dy\frac{|B^{\varepsilon}_{\mu}(x)-B^{\varepsilon}_{\mu}(y)|^2}{|x-y|^{2+\alpha'}}\right)^{1/2}\equiv \|B^{\varepsilon}_{\mu}\|_{\alpha'}\to 0 \tag{5.7}$$

for $\mu=0,1$.

The following easy inequality, valid for $0<\gamma\leqq 1$,

$$\|B^{\varepsilon}_{\mu}\|_{\alpha'}\leqq 2\|B^{\varepsilon}_{\mu}\|^{1-\gamma}\|B^{\varepsilon}_{\mu}\|_{\frac{2+\alpha'}{\gamma}-2}|\Lambda|^{-\gamma+1} \tag{5.8}$$

follows from Hölder's inequality. Choose γ so that $\frac{2+\alpha'}{\gamma}-2=\alpha''<\alpha$. Since we have just shown that $\|B^{\varepsilon}_{\mu}\|$ tends to zero, it is enough to obtain a uniform bound on $\|B^{\varepsilon}_{\mu}\|_{\alpha''}$. This is easy to obtain by expanding the exponential in B^{ε} and applying (5.3). This completes the proof of (∞,α') convergence.

Now we will establish that the limit, assuming it exists, is not zero. By its definition as the ratio of two positive integrals and the diamagnetic bound of Schrader and Seiler [7], also see Paper I, Sect. 3.3,

$$0 \leqq \varkappa(A) \leqq 1 .$$

Furthermore by the convergence of h^ε just proven and Theorem 4.3 the limit of $\varkappa^\varepsilon(A)$ exists and is non zero almost surely in A see (4.25). Denote the limit by $\varkappa(A)$. Jensen's inequality implies

$$Z_A^\varepsilon \geqq \int d\mu_D(A) \varkappa_A^\varepsilon(A) [\exp - \int dv_A^\varepsilon V_A^\varepsilon] .$$

The exponent is a real valued polynomial in δC_A^ε which we know by convergence of h^ε and Theorem 3.5 is convergent as $\varepsilon \searrow 0$. Let $P(\delta C_A)$ denote the limit. Fatou's lemma implies

$$Z_A \geqq \int d\mu_D(A) \varkappa(A) e^{-P(\delta C_A)} > 0 .$$

End of proof that $Z_A \neq 0$.

By Theorems 4.1 and 4.3 and the (∞, α') convergence just established, we now have obtained convergence of $Z_A^\varepsilon(A)$ almost surely, as ε tends to zero. The proof of Theorem 5.1 is completed by combining this with the Lebesgue dominated convergence theorem and the diamagnetic bound, Theorem 4.1, Paper I:

$$|Z_A^\varepsilon(A)| \leqq Z_A^\varepsilon(0) .$$

The right-hand side is bounded uniformly in ε by Nelson's boundedness below proof [19].

V.2. Continuity of Gaussian Processes

Theorem 5.3 (A. M. Garsia). *Let $\Phi(x)$ be a Gaussian process on a bounded region Λ. A sufficient condition for Φ to satisfy* (5.3), (E.U.H.C.) *with modulus α, is that at $u=0$*

$$p(u) \equiv \sup_{|x-y| \leqq |u|/\sqrt{2}} [E((\Phi(x) - \Phi(y))^2)]^{1/2} \tag{5.9}$$

be Hölder continuous with modulus $\beta > \alpha$.

For a proof of this theorem, see the beautiful article by Garsia [8]. The condition in Theorem 5.3 follows from the condition in his Theorem 2 by integration by parts. To help the reader we indicate the basic idea in [8]. The assumption (5.9) on $p(u)$ implies that the expectation

$$E\left(\exp\left\{c\left(\frac{\Phi(x) - \Phi(y)}{|x-y|^\beta}\right)^2\right\}\right)$$

is bounded uniformly in $x, y \in \Lambda$ for a suitable $c > 0$. This implies

$$\int_\Lambda \int_\Lambda \exp\left\{c\left(\frac{\Phi(x) - \Phi(y)}{|x-y|^\beta}\right)^2\right\} dx dy < \infty$$

with probability one. This condition is evidently tantamount to some form of continuity for Φ. Garsia has proved a very clever real variable lemma (Lemma 1 of

[8]), which shows that this condition implies $\Phi(x)$ is E.U.H.C. with index α for all $\alpha<\beta$.

In the case at hand, we infer from Theorem 5.3 that A_μ is (E.U.H.C.) for $\mu=0,1$ if at $u=0$

$$p_\mu(u)\equiv \sup_{|x-y|\leqq|u|/\sqrt{2}} (D_{\mu\mu}(x,x)+D_{\mu\mu}(y,y)-2D_{\mu\mu}(x,y))^{1/2} \tag{5.10}$$

is β Hölder continuous, $\beta>\alpha$. If we specialise to the case of A_μ real and translation invariant then (5.10) is implied by: for some constant c,

$$(D_{\mu\mu}(0,0)-D_{\mu\mu}(0,x))^{1/2}\leqq c|x|^\beta, \qquad \beta>0. \tag{5.11}$$

We can transform this into a simple condition on the Fourier transform of $D_{\mu\mu}(x-y)$, denoted $\hat{D}_{\mu\mu}(k)$, by noting that the supremum norm of

$$|x|^{-2\beta}(D_{\mu\mu}(0)-D_{\mu\mu}(x))$$

is less than the L_1 norm of its Fourier transform. The Fourier transform of $|x|^{-2\beta}$ is, for $\beta<1$, $c_\beta|k|^{-2+2\beta}$ by homogeneity, therefore the L_1 norm of the Fourier transform is less than a constant times

$$\int dk_1 dk_2 \left|\frac{1}{|k_1-k_2|^{2-2\beta}}-\frac{1}{|k_1|^{2-2\beta}}\right|\hat{D}_{\mu\mu}(k_2)$$

which is finite provided $\beta<1/2$ and

$$\int dk\hat{D}_{\mu\mu}(k)\,|k|^{2\beta}<\infty\,. \tag{5.12}$$

Therefore we have proved.

Corollary 5.4. *A Gaussian process $A_\mu(x)$ with covariance $D_{\mu\nu}(x-y)$ has sample functions which are* (E.U.H.C.) *with modulus α provided condition* (5.12) *holds for some $\beta>\alpha$.*

V.3. Osterwalder-Schrader Positivity

We assume that Λ is symmetric with respect to reflection about some hyperplane π.

Let Λ_+,Λ_- denote the open subsets of Λ on either side of π. We now define Σ_+^G,Σ^G, which intuitively are the algebras of gauge invariant functions of fields supported in Λ_+,Λ_- respectively. Σ_+^G is the algebra of functions measurable with respect to the σ field generated by all functions of the form

$$\begin{aligned}
&B(f)\equiv\int \operatorname{curl}A(x)f(x)dx\,, && f\in C_0^\infty(\Lambda_+),\\
&:\bar\phi\phi(f):\equiv\int :\bar\phi\phi(x):f(x)dx, && f\in C_0^\infty(\Lambda_+),\\
&\int\bar\phi(x)e^{ie\int A}\phi(y)g(x,y)dxdy\,, && g\in C_0^\infty(\Lambda_+\times\Lambda_+).
\end{aligned}$$

In the last expression A is integrated along a contour inside $\Lambda_+\Sigma_-$ is defined by replacing Λ_+ by Λ_-. Reflection about π induces a map Θ

$$\Theta:\Sigma_+\to\Sigma\,.$$

in an obvious way (see Sect. 2.1 of Paper I).

In this section we wish to show that if the boson self interaction V is gauge invariant, i.e.,

$$V(\phi)=V(|\phi|)$$

and the covariance D is suitably chosen, then we have Osterwalder-Schrader positivity in one direction, i.e.,

$$\langle F\Theta(F)\rangle_{\Lambda}\geqq 0 \qquad (O.S)$$

for all F in $L_1\cap\Sigma_+$.

We choose covariances D of the following type

$$\hat{D}_{\mu\nu}(k)=\frac{\delta_{\mu\nu}-k_\mu k_\nu F(k^2)}{k^2+m^2}g(k_1)\frac{1}{2\pi} \qquad (5.13)$$

where $k^2=k_\mu k_\mu$, $k=(k_0,k_1)$,

$$|F(k^2)|\leqq\frac{c}{k^2}$$

and g is positive, continuous with

$$\int|g(k_1)|\,|k_1|^\beta dk_1<\infty \qquad (5.14)$$

for some $\beta>0$. Note that Corollary 5.4 implies that the Gaussian process with covariance D has (E.U.H.C.) sample functions.

Theorem 5.5. *The expectation $\langle\,\rangle_\Lambda$ is Osterwalder-Schrader positive for π parallel to the* 1-*direction if V is gauge invariant and D is of the form* (5.13).

Proof. Approximate F in (O.S) by a polynomial in the gauge invariant fields

$$B(f),\ :\bar{\phi}\phi(f):,\ \int\bar{\phi}e^{ie\int A}\phi g.$$

By Corollary 4.4 the expectation $\langle\,\rangle_\Lambda^\varepsilon$ of such a polynomial converges as $\varepsilon\searrow 0$. Therefore it is enough to prove (O.S) for $\langle\,\rangle_\Lambda$ replaced by $\langle\,\rangle_\Lambda^\varepsilon$. We now put the A field on a lattice also: consider the lattice Gaussian process with covariance $D_{\mu\nu}^{\varepsilon'}$ given by the kernel of the operator

$$\frac{\delta_{\mu\nu}-\partial_\mu^{\varepsilon'*}\partial_\nu^{\varepsilon'}F(\partial^{\varepsilon'*}\partial^{\varepsilon'})}{m^2+\partial^{\varepsilon'*}\partial^{\varepsilon'}},$$

where ∂^ε is the finite difference gradient and $\partial^{\varepsilon'*}\partial^{\varepsilon'}=\partial_\mu^{\varepsilon'*}\partial_\mu^{\varepsilon'}$. Choose $\varepsilon'=\varepsilon/N$ where N is an integer and arrange the ε' lattice so that it is a "refinement" of the ε lattice. By diagonalising the covariances $D^{\varepsilon'}$ using the Fourier transform it is easy to show that as $\varepsilon'\searrow 0$ the Gaussian measures converge, i.e.,

$$d\mu_{D^{\varepsilon'}}\to d\mu_D \qquad (5.15)$$

in the sense of convergence of moments and characteristic functions. We claim that this implies that the expectations $\langle\,\rangle_\Lambda^{\varepsilon,\varepsilon'}$ associated with this double lattice approximation converge to $\langle\,\rangle_\Lambda^\varepsilon$ as $\varepsilon'\searrow 0$ in the sense of convergence of moments. This is so because the partition function $Z_\Lambda^\varepsilon(A)$ for bosons in an external gauge field can be expanded in a convergent Fourier series in exponentials of the finite number of Gaussian variables

$$\left\{\int_b A_\mu dx_\mu : b\in\mathscr{B}^\varepsilon(\Lambda)\right\},$$

where b is a bond in the ε lattice and the contour integral along b is really a "contour sum" on the bonds of the ε' lattice. Approximate $Z^\varepsilon_\Lambda(A)$ by truncating the Fourier series and use (5.15). Thus it suffices to prove (O.S) for $\langle\rangle_\Lambda$ replaced by $\langle\rangle^{\varepsilon,\varepsilon'}_\Lambda$. This is a lattice theory and we may prove (O.S) for it in complete analogy with Theorem 5.3 and Corollary 5.4 in Paper I. The presence of two lattices, one for the A field and another for ϕ causes no additional problems.

VI. Feynman Rules, Counterterms, and the Change of A Covariance Formula

VI. 1

This section is a technical preparation for the ultraviolet limit, i.e., the removal of the condition (5.12) on the A covariance. This will be done by taking a limit *outside* the integrals over A and ϕ. To control this limit we will need a formula which we call the change of covariance formula in honour of (22). This identity expresses the difference between two partition functions with different A covariances in a form which is amenable to estimates.

The ultraviolet limit will only exist (conventional wisdom based on perturbation theory) and be non trivial if one alters the interaction V by adding in some terms known as counterterms which will be infinite in the limit. Since one of the most convenient ways of discussing the rather complex formulas which arise is the Feynman graph notation we will also spend some time explaining this. We have introduced some graphical notations which are not standard.

In this section we continue to assume that lattice gauge fields are abelian and derived from continuum gauge fields as in (5.1). We also assume that the photon propagators are translation invariant and satisfy (5.12). The ϕ field is complex.

We begin with some notation including the Feynman graph formalism. We present formulas first and explanations afterwards.

$$\begin{aligned} F(x) &= \frac{1}{2\pi}\int \hat{F}(p)e^{ip\cdot x}dp\,, \\ \hat{F}(p) &= \frac{1}{2\pi}\int F(x)e^{-ip\cdot x}dx\,, \end{aligned} \tag{6.1}$$

where p, x are in $\mathbb{R}^2$. The Fourier transform of ∂^ε is

$$i\varrho^\varepsilon_\mu(\mathrm{p}) \equiv \varepsilon^{-1}(\mathrm{e}^{i\varepsilon p_\mu}-1)\,. \tag{6.2}$$

The lattice photon propagator is defined in terms of the continuum propagator by

$$D^\varepsilon_{\mu\nu}(x-y) \equiv \int d\mu_D(A)\mathscr{A}^\varepsilon_\mu(x)\mathscr{A}^\varepsilon_\mu(y)\,, \tag{6.3}$$

where $x, y\in L^{(\varepsilon)}$ and

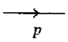
$$\mathscr{A}^\varepsilon_\mu(x) \equiv \frac{1}{\varepsilon}\int_{b_\mu(x)} A_\mu(x)\,dx_\mu \quad \text{if} \quad b_\mu(x)\subset\Lambda, =0 \quad \text{otherwise} \tag{6.4}$$

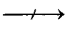
with $b_\mu(x)$ denoting the bond at x pointing in the direction e_μ.

The quantities $\mathscr{A}_\mu^\varepsilon$ are Gaussian random variables, but A_μ^ε, $A_\mu^{\varepsilon *}$ are not. Formally

$$A_\mu^\varepsilon(x) = \frac{1}{ie\varepsilon} \sum_{n=1}^{\infty} \frac{1}{n!} (ie\varepsilon \mathscr{A}_\mu^\varepsilon)^n . \tag{6.5}$$

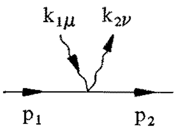
Feynman Rules (Momentum Space)

(1a)  $2\hat{C}(p)$, $2\hat{C}^\varepsilon(p)$.

(1b)  $(\sqrt{2})\hat{C}^{1/2}(p)$, $(\sqrt{2})\hat{C}^{\varepsilon 1/2}(p)$.

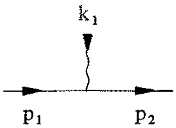
(2a)  $\hat{D}_{\mu\nu}(k)$, $\hat{D}_{\mu\nu}^\varepsilon(k)$, $\hat{D}(k)$.

(2b)  $(\sqrt{\hat{D}})_{\mu\nu}(k)$, $(\sqrt{D^\varepsilon})_{\mu\nu}(k)$.

(3) 

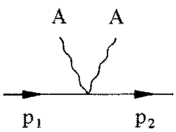
$$-\frac{e^2}{2\pi} \hat{\chi}_\Lambda(p_1 - p_2 + k_1 - k_2)\delta_{\mu\nu} .$$

$$-\frac{e^2}{2\pi} \hat{\chi}_\Lambda^\varepsilon(p_1 - p_2 + k_1 - k_2)\delta_{\mu\nu} .$$

(4) 

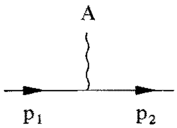
$$\frac{e}{(\sqrt{2\pi})} \hat{\chi}_\Lambda(p_1 - p_2 + k_1)(p_{1\mu} + p_{2\mu})$$

$$\frac{e}{(\sqrt{2\pi})} \hat{\chi}_\Lambda^\varepsilon(p_1 - p_2 + k_1)(\varrho_{1\mu}^{\varepsilon *} + \varrho_{2\mu}^\varepsilon) .$$

(5) 

$$-\tfrac{1}{2}e^2((\chi_\Lambda A_\mu)^2)\hat{}\,(p_1 - p_2) .$$

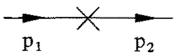
$$-\tfrac{1}{2}e^2((\chi_\Lambda^\varepsilon \mathscr{A}_\mu^\varepsilon)^2)\hat{}\,(p_1 - p_2)\, \tfrac{1}{2}(e^{-i\varepsilon p_1} + e^{i\varepsilon p_2}) + O(\varepsilon)$$

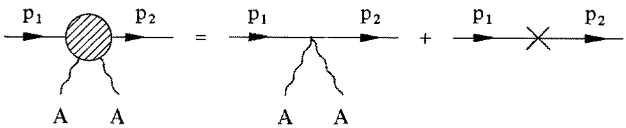
(6) 

$$\tfrac{1}{2}e(A_\mu \chi_\Lambda)\hat{}\,(p_1 - p_2)(p_{1\mu} + p_{2\mu})$$

$$\tfrac{1}{2}e(\mathscr{A}_\mu^\varepsilon \chi_\Lambda^\varepsilon)\hat{}\,(p_1 - p_2)(\varrho_{1\mu}^{\varepsilon *} + \varrho_{2\mu}^\varepsilon) + O(\varepsilon)$$

(7)  $f\sqrt{2\pi}\hat{f}(p)$.

(8)  $\delta m_D^2 \hat{\chi}_\Lambda(p_1 - p_2)$, $(\delta m_D^\varepsilon)^2 \hat{\chi}_\Lambda^\varepsilon(p_1 - p_2)$.



Feynman Rules (Configuration space)

(1a) $2C(x-y)$, $2C^{\varepsilon}(x-y)$.

(1b) $(2C)^{1/2}(x-y)$, $(2C^{\varepsilon})^{1/2}(x-y)$.

(2a) $D_{\mu\nu}(x-y)$, $D^{\varepsilon}_{\mu\nu}(x-y)$.

(2b) $(\sqrt{D})_{\mu\nu}(x-y)$, $(\sqrt{D^{\varepsilon}})_{\mu\nu}(x-y)$.

(5) $-1/2e^2 A_\mu^2(x)\chi_\Lambda(x)$, $-1/2e^2(A_\mu^{\varepsilon*}A_\mu^{\varepsilon})(x)$.

(6) $-1/2ie(A_\mu\chi_\Lambda\partial_\mu+\partial_\mu\chi_\Lambda A_\mu)(x)$, $-1/2ie(A_\mu^{\varepsilon*}(x)\partial_\mu^{\varepsilon}-\partial_\mu^{\varepsilon*}A(x))$.

(7) $f(x)$.

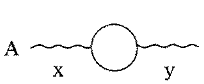
(8) $\delta m_D^2\chi_\Lambda(x)$.

Associated with each graphical symbol is a continuum kernel, written first, and a lattice kernel written second. By the Fourier transform, the kernels listed under the heading configuration space are unitarily equivalent (as operators) to the kernels listed opposite the same numbers under momentum space. The various factors of χ_Λ occur because we are using free boundary conditions. Similar formulas hold for periodic boundary conditions. Note that a factor χ_Λ is included in the definition of $A^{(\varepsilon)}$ associated with (5.1).

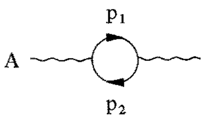
Since we are now specialising to the case of ϕ complex

$$\hat{C}(p)\equiv\frac{1}{(2\pi)}\frac{1}{p^2+m^2}.$$

To each graph that can be constructed by joining the vertices (3)–(7) by lines (1) and (2) is associated a polynomial in ϕ and A obtained by integrating over all the p's and k's. This is a standard notation in field theory so we will not explain it in detail but simply give an example which has been cropping up continuously in this paper. Let ${}_\Lambda A_\mu=\chi_\Lambda A_\mu$,



$$\equiv(ie)^2\int dxdy({}_\Lambda A_\mu\partial_\mu+\partial_{\mu\Lambda}A_\mu)(x)\cdot C(x-y)({}_\Lambda A_\mu\partial_\mu+\partial_{\mu\Lambda}A)(y)C(y-x),\tag{6.6}$$

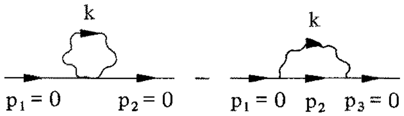


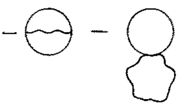
$$\equiv e^2\int dp_1 dp_{2\Lambda}\hat{A}_\mu(p_1-p_2)(p_{1\mu}+p_{2\mu})\cdot\hat{C}(p_2)_\Lambda\hat{A}_\nu(p_2-p_1)(p_{2\nu}+p_{1\nu})\hat{C}(p_1).$$

Both these integrals happen to diverge. If they were interpreted according to the lattice kernels they would not diverge and they would be equal by the Plancherel identity.

VI.2. Counterterms, Renormalised Partition Functions, Measures

Let

$\delta m_D^2 \equiv -$ 

$E_D \equiv \lim_{\epsilon \to 0} E_D^\epsilon \equiv \lim_{\epsilon \to 0} (-$  $)(\epsilon)$ (6.7)

$$\left(= -2\int dk \hat{D}_{\mu\nu}(k) \Pi_{\mu\nu}(k)\right),$$

where $\Pi_{\mu\nu}$ is the limit of the quantity $\Pi_{\mu\nu}^{\varepsilon}$ defined in Eq. (A.1), Appendix A. δm_D^2 is a continuum quantity. We will have occasion to use the corresponding lattice quantity $(\delta m_D^{\varepsilon})^2$. The existence of the limit in the definition of E_D is established in Appendix A. It requires that $D_{\mu\nu}$ satisfy (5.12). Both δm_D^2 and E_D are infinite if (5.12) does not hold, i.e., these counterterms are inserted to cancel divergences in the ultraviolet limit.

We now define the counterterms

$$U_{A,D} = 1/2\delta m_D^2 \int_A dx :\phi^2(x): + E_D, \tag{6.8}$$

where the normal ordering is with respect to C. Define U_{AD}^{ε} by substituting the corresponding lattice definitions.

The *renormalised partition functions* are, by definition,

$$\begin{aligned} Z_D^{\varepsilon}(A) &\equiv \varkappa^{\varepsilon}(A) \int dv_A^{\varepsilon} e^{-V^{\varepsilon} - U_D^{\varepsilon}} \\ Z_D^{\varepsilon} &\equiv \int d\mu_D(A) Z_D^{\varepsilon}(A) \end{aligned} \tag{6.9}$$

cf. (4.3) and (5.2). We are dropping the A subscripts everywhere from this section because A will be fixed. Instead we make D dependences explicit because the dependence on D will be of interest.

Since for a fixed ultraviolet cutoff on the gauge field the renormalisation constants $(\delta m_D^{\varepsilon})^2, E_D^{\varepsilon}$ converge as ε tends to zero, our previous convergence proof for Theorem 4.1 is easily adapted to prove that the limit as ε tends to zero of $Z_D^{\varepsilon}(A)$ exists almost everywhere. We denote the continuum limits $Z_D(A)$ and Z_D. We can take the limit past the $d\mu_D(A)$ integral because Lebesgue dominated convergence can still be justified by the diamagnetic bound, cf. the proof of Theorem 5.1.

We will use the subscript D to indicate that V is replaced by $V + U_D$ in previous definitions. For example the renormalised Bose matter action is

$$A_D^{M\varepsilon} \equiv -1/2(\phi, [m^2 - \Delta_A^{\varepsilon E}]\phi) + V^{\varepsilon} + U_D^{\varepsilon}$$

cf. (5.4). We apologise for the confusing use of A for both the Yang-Mills field and the action.

VI.3. Change of A-Covariance Formula

Let D_0, D_1 be two covariances for the gauge field. The associated independent Gaussian processes are denoted A_0, A_1. For $t \in [0, 1]$, set

$$\begin{aligned} A_t &\equiv \sqrt{(1-t)}\,A_0 + \sqrt{t}\,A_1\,, \\ D_t &\equiv (1-t)D_0 + tD_1\,. \end{aligned} \tag{6.11}$$

Note that A_t is a Gaussian process with covariance D_t. Let P be a polynomial in $\phi, \bar{\phi} \equiv \overset{(-)}{\phi}$ of the form

$$P \equiv \int dx_1 \ldots dx_q\, g(x_1, \ldots, x_q)\, \overset{(-)}{\phi}(x_1) \ldots \overset{(-)}{\phi}(x_q)\,, \tag{6.12}$$

where $g \in C_\infty$. We are interested in studying

$$Z_1 \langle P \rangle_1 - Z_0 \langle P \rangle_0\,. \tag{6.13}$$

The subscripts 1, 0 and later t replace the subscripts D_1, D_0, D_t in order to simplify our formulas.

We study (6.13) by using the fundamental theorem of calculus to write it as the integral of a t derivative. The t derivative and the $d\mu(A)$ integrals can be interchanged because the second derivative of the integrand may be controlled by the methods we are about to apply to the first derivative. Thus (6.13) becomes

$$\int_0^1 dt \int d\mu_0(A_0) d\mu_1(A_1) \frac{d}{dt}\left(\int d\omega_t(\phi) P\right). \tag{6.14}$$

The measure $d\omega_t$ is given by

$$\begin{aligned} d\omega_t(\phi) &\equiv \lim_{\varepsilon \searrow 0} \varkappa^\varepsilon(A_t) dv^\varepsilon_{A_t} e^{-V^\varepsilon - U^\varepsilon_t} \\ &= \lim_{\varepsilon \searrow 0} D\phi e^{-A^{M\varepsilon}_t} / \zeta^\varepsilon(\mathbb{1})\,. \end{aligned} \tag{6.15}$$

The limit is as usual in the sense of characteristic functions, or convergence of moments. Existence follows from the results of Sect. IV. We now show that

$$\frac{d}{dt} \int D\phi e^{-A^{M\varepsilon}_t} P = \int D\phi e^{-A^{M\varepsilon}_t} (K^\varepsilon_t P)\,, \tag{6.16}$$

where K^ε_t is a linear operator defined on the space of polynomials in ϕ. It will be defined below. By dividing through by $\zeta^\varepsilon(\mathbb{1})$ and taking the limit $\varepsilon \searrow 0$ we will obtain an identity for the t derivative in (6.14). By doing the t derivative:

$$\frac{d}{dt} \int D\phi e^{-A^{M\varepsilon}_t} P = \int D\phi (-A^{M\varepsilon}_t)' e^{-A^{M\varepsilon}_t} P\,. \tag{6.17}$$

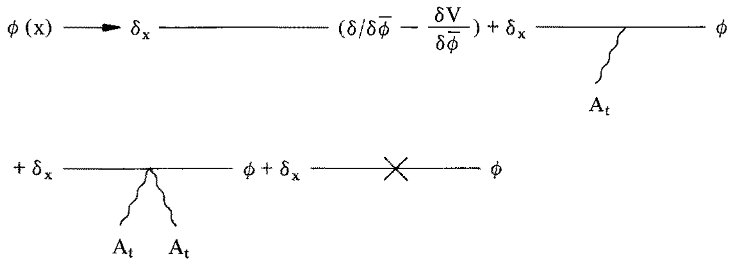
We use primes here and hereafter to denote t derivatives. The factor $\phi(x)$ in $(-A^{M\varepsilon}_t)'$ is integrated by parts. This simply amounts to replacing it in (6.17) by

$$2 \int dy C^\varepsilon(x-y) \left(\delta/\delta\bar{\phi}(y) - (\delta/\delta\bar{\phi} G_t)\right), \tag{6.18}$$

where the integral is really $\sum_y \varepsilon^2$ and

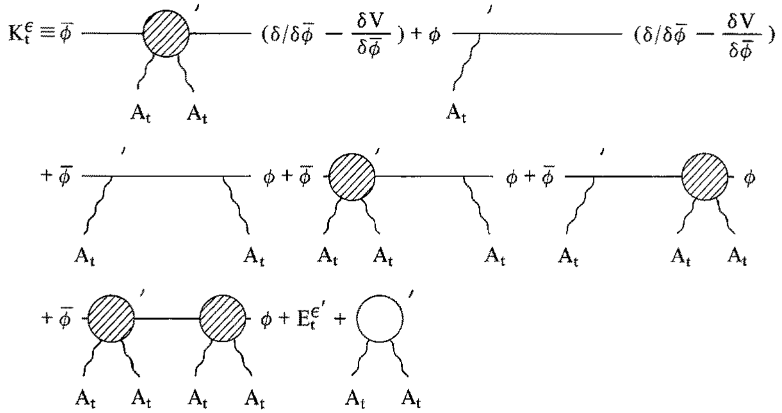
$$G_t \equiv A^{M\varepsilon}_t - 1/2(\phi, [m^2 - \Delta^{\varepsilon F}]\phi)\,. \tag{6.19}$$

These formulas are easy to derive since we are working on a finite lattice. The easiest way to manipulate integration by parts is via the graphical representation



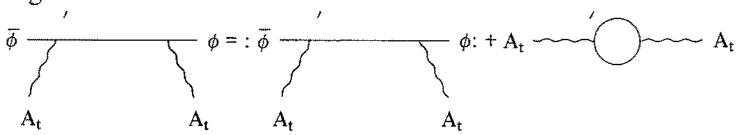
(6.20)

The conclusion obtained from integration by parts applied to (6.17) is of the form (6.16) with K_t^ε equal to

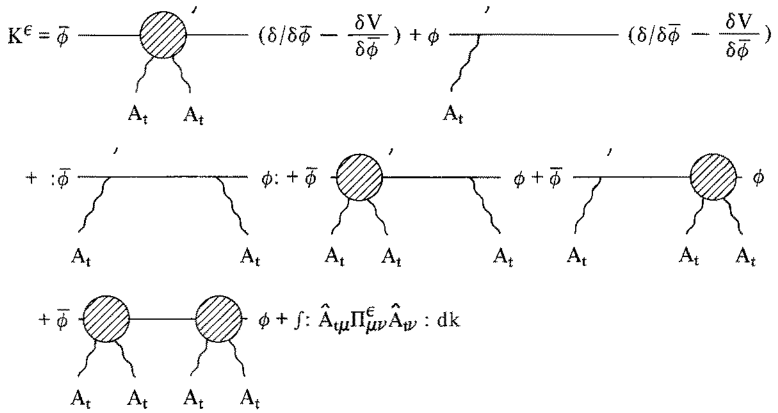


(6.21)

We now exhibit a cancellation between the third term and the last in (6.21), by writing



so that K^ε can be put in the form



(6.22)

The $E_t^{\varepsilon'}$ in (6.21) cancelled when the last term in (6.22) was normal ordered. Π^ε was defined in (6.7). We define K as in (6.22) but with diagrams interpreted by continuum Feynman rules and Π^ε replaced by Π.

The true merits of (6.22) will be more readily appreciated in the context of the stability expansion in Paper III. The main point is that the diagrams in K remain finite in the ultraviolet limit provided A is in a gauge which is approximately transverse.

Having identified the operator K_t^ε appearing in (6.16), we divide both sides by $\zeta^\varepsilon(\mathbb{1})$ and take the limit ε goes to zero. The result, after some work which is discussed below, will be

$$\frac{d}{dt}\int d\omega_t(\phi)P=\int d\omega_t(\phi)(K_tP). \tag{6.23}$$

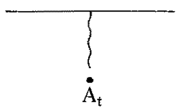
The Limit $\varepsilon\searrow 0$. The main difficulty is to show that the right hand side of (6.16) converges as $\varepsilon\searrow 0$. There is no difficulty in interchanging the limit and the t derivative because the left hand side can easily be shown to be bounded uniformly in ε by the diamagnetic bound of Paper I and the Cauchy-Schwarz inequality. Our previous results, Theorems 4.1 and 4.3 imply that the quantity under the t derivative on the left hand side converges as $\varepsilon\searrow 0$.

We use the notation introduced in the proof of Theorem 4.1. We will only sketch a proof that the right hand side of (6.16) converges because the method is similar to techniques we have already explained in proving Theorems 4.1 and 4.3. Recall that we are still working with a cutoff gauge field, A, that is (E.U.H.C.) with modulus $\alpha<1/2$.

By an argument as in the proof of Theorem 4.1, it is enough to show that

$$\int dw\,|(K_t^\varepsilon P)(\phi^\varepsilon)-(K_t^{\varepsilon'}P)(\phi^{\varepsilon'})|^2\to 0 \tag{6.24}$$

pointwise in t as $\varepsilon,\varepsilon'\to 0$. We first show this in the case that K_t, K_t' are replaced by $\dot{K}_t, \dot{K}_T'$ which are obtained from K_t, K_t' by replacing all factors $\chi_\Lambda A_t$ occuring in their definition except those in the last term in (6.22) by a C^∞ gauge field $\dot{A}_t$ compactly supported in Λ. We then gain the freedom to move all the derivatives occuring on external lines in



type vertices past the $\dot{A}_t$ by Leibniz rule onto the internal lines. It is now not difficult to prove (6.24) in this case using the methods of the proof of Theorems 4.3 and 3.3. It is now necessary to show that for any $\delta>0$ we can approximate $\chi_\Lambda A$ by $\dot{A}$ so that

$$\int dw\,|(\dot{K}^\varepsilon-K^{\varepsilon'})P(\phi^\varepsilon)|^2=\int dv_{A_t}|(\dot{K}^\varepsilon-K^{\varepsilon'})P|^2\leqq\delta$$

uniformly in ε. This follows easily from the fact that $\dot{A}_\mu$ can be chosen so that $\dot{A}_\mu^\varepsilon C^{\varepsilon 1/2}$ and its adjoint approximate $A_\mu C^{1/2}$ and its adjoint arbitrarity closely in $\mathscr{I}_4$ uniformly in ε. This concludes our discussion of $\varepsilon\searrow 0$.

We now combine (6.23) and (6.14) to obtain

Theorem 6.1. *If P is of the form* (6.12), *D_0 and D_1 are two covariances for the gauge field*

$$Z_{D_1}\langle P\rangle_{D_1} - Z_{D_0}\langle P\rangle_{D_0} = \int_0^1 dt \int d\mu_{D_0}(A_0)\, d\mu_{D_1}(A_1) \cdot \int d\omega_t(\phi)\,(K_t P)\,,$$

K_t was defined below (6.22).

Appendix A

In this appendix we study the vacuum polarisation

$$1/2\,\mathrm{tr}\,C^\varepsilon W^\varepsilon + 1/4\,\mathrm{tr}(C^\varepsilon W^\varepsilon)^2 \tag{A.1}$$

which was encountered in the proof of Theorem 4.2 and also in Sect. 6. We will specialise to the case in which ϕ is a complex scalar field and A is real. The calculations given below are not significantly changed if one combs them through with A nonabelian.

We begin by rewriting (A.1) in the form

$$\int d^2k\, \mathscr{A}^\varepsilon(k)\, \Pi^\varepsilon_{\mu\nu}(k)\, \overline{\mathscr{A}^\varepsilon_\nu}(k) + O(e^3 \varepsilon \log^2 \varepsilon)\,,$$

where by a calculation using (2.12)

$$\Pi^\varepsilon_{\mu\nu}(k) \equiv \tfrac{1}{2}\left(\frac{e}{2\pi}\right)^2 \int (m^2+\varrho_-^2)^{-1}(m^2+\varrho_+^2)^{-1} \cdot (e^{-i\varepsilon p_-} - e^{i\varepsilon p_+})_\mu (i\varepsilon)^{-1} (e^{-i\varepsilon p_+} - e^{i\varepsilon p_-})_\nu (i\varepsilon)^{-1} d^2p - \tfrac{1}{4}\left(\frac{e}{2\pi}\right)^2 \int (m^2+\varrho^2)^{-1} \sum_\lambda (e^{-i\varepsilon p} + e^{i\varepsilon p})_\lambda d^2p\, \delta_{\mu\nu}\,.$$

A subscript μ (or ν) on a bracket indicates that all p's, ϱ's inside are p_μ's, ϱ_μ's.

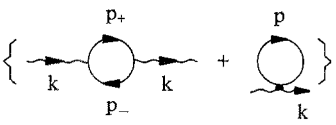
$$\varrho_\mu \equiv \varrho^\varepsilon_\mu(p) \equiv (i\varepsilon)^{-1}(e^{i\varepsilon p} - 1)_\mu\,.$$

Subscripts $+, -$ indicate that p is to be replaced by p_+, p_- in the appropriate definitions.

$$p_{+\mu} = p_\mu + \tfrac{1}{2}k_\mu\,, \qquad p_{-\mu} = p_\mu - \tfrac{1}{2}k_\mu\,.$$

ϱ^2 stands for $\varrho_\mu \bar{\varrho}_\mu$. All integrals are over $\left[-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}\right]^2$.

By using the Feynman rules in Sect. 6 it may be verified that

$\Pi^\varepsilon_{\mu\nu}(k) = \pi$  (A.2)

There are no D propagators on the external lines. This observation will become relevant when we prove that E^ε_D converges as promised in Sect. 6.

Transversality

We first show that Π^ε is transverse, which by definition means

$$\Pi^\varepsilon_{\mu\nu}(k)\varrho_\nu(k)=0. \tag{A.3}$$

We set $\varepsilon=1$ and omit ε superscripts throughout the proof of transversality. Transversality can be shown directly by shifting the variable of integration in the left hand side of (A.3) as is done in physics text books. See [23] to get the general idea. However it is really a consequence of gauge invariance. Let

$$h_\alpha=e^{ie\mathscr{A}}\,; \qquad \mathscr{A}=\mathscr{A}_0+\alpha\partial h\,.$$

By gauge invariance, see for example Paper I, Theorem 2.6, $\varkappa_\Lambda(h_\alpha)$, defined in Sect. 4, is independent of α. Therefore

$$\frac{d^2}{de^2}\log\varkappa_\Lambda(h_\alpha)|_{e=0}=2\frac{d^2}{de^2}\int d^2k\hat{\mathscr{A}}_\mu\Pi_{\mu\nu}\bar{\hat{\mathscr{A}}}_\nu$$

is independent of α. Differentiation with respect to α and setting $\alpha=0$ yields (A.3). □

Since Π is transverse it must satisfy

$$\Pi_{\mu\nu}=\Pi_{\lambda\lambda}[\delta_{\mu\nu}-\varrho_\mu\varrho_\nu/\varrho^2] \tag{A.4}$$

because the quantity in brackets is the projection onto the transverse component of a gauge field as can be checked by verifying that it vanishes on longtitudinal functions $\varrho_\nu f(k)$. The projection is rank one. (A.4) follows by taking traces.

The (Pointwise) Limit as $\varepsilon\searrow 0$ of $\Pi^\varepsilon_{\mu\nu}$

We will now show that the limit $\varepsilon\searrow 0$ of $\Pi^\varepsilon_{\mu\mu}$ exists pointwise in k and give an expression for it. We have

$$\begin{aligned}\Pi^\varepsilon_{\mu\mu}=&\tfrac{1}{2}\left(\frac{e}{2\pi}\right)^2\varepsilon^{-2}\int(m^2+\varrho_+^2)^{-1}(m^2+\varrho_-^2)^{-1}\sum_\mu|e^{-i\varepsilon p}-e^{i\varepsilon p}|^2_\mu d^2p\\&-\tfrac{1}{2}\left(\frac{e}{2\pi}\right)^2\int(m^2+\varrho^2)^{-1}\sum_\mu(e^{i\varepsilon p}+e^{-i\varepsilon p})_\mu\,.\end{aligned} \tag{A.5}$$

Substitute in (A.5) using the identity

$$\begin{aligned}\frac{|e^{-ip}-e^{ip}|^2_\mu}{(\varrho_+^2+m^2)(\varrho_-^2+m^2)}=&2([\varrho_+^2+m^2]^{-1}+[\varrho_-^2+m^2]^{-1})\\&+(|e^{-ip}-e^{ip}|^2-2|e^{ip_-}-1|^2_\mu-2|e^{ip_+}-1|^2_\mu-4m^2)\\&\cdot(m^2+\varrho_+^2)^{-1}(m^2+\varrho_-^2)^{-1}\end{aligned}$$

and note that the numerator in the second term may be written in the form:

$$-16\sin^4\frac{p_\mu}{2}+8\cos p_\mu\left(\cos\frac{k_\mu}{2}-1\right)-4m^2\,.$$

All μ's are to be summed over. We have set $\varepsilon=1$ to simplify the formulas. The result after shifting integration variables $p_+\to p$ and $p_-\to p$ is

$$\begin{aligned}&\tfrac{1}{2}\left(\frac{e}{2\pi}\right)^2\int(m^2+\varrho^2)^{-1}(2-e^{i\varepsilon p}-e^{-i\varepsilon p})_\mu d^2p\\&+\tfrac{1}{2}\left(\frac{e}{2\pi}\right)^2\int(m^2+\varrho_+^2)^{-1}(m^2+\varrho_-^2)^{-1}(-16\varepsilon^{-2}\sin^4(\varepsilon p_\mu/2)d^2p)\\&+\tfrac{1}{2}\left(\frac{e}{2\pi}\right)^2\int(m^2+\varrho_+^2)^{-1}(m^2+\varrho_-^2)^{-1}\left\{8\cos\varepsilon p_\mu\cdot\varepsilon^{-2}\left(\cos\frac{\varepsilon k_\mu}{2}-1\right)-4m^2\right\}d^2p.\end{aligned}\tag{A.6}$$

As usual all μ's are to be summed over. The range of integration is $\left[\frac{-\pi}{\varepsilon},\frac{\pi}{\varepsilon}\right]$ for each component of p.

We prove that the limit of the first two integrals exists and evaluate it by scaling $\varepsilon p\to p$. The result is

$$\begin{aligned}&\tfrac{1}{2}\left(\frac{e}{2\pi}\right)^2\int_{-\pi}^{\pi}(\varrho^2)^{-1}(2-e^{ip}-e^{ip})d^2p\\&-16\,1/2\left(\frac{e}{2\pi}\right)^2\int_{-\pi}^{\pi}(\varrho^2)^{-2}\sin^4\frac{p_\mu}{2}d^2p\equiv -J_0,\end{aligned}\tag{A.7}$$

where $\varrho_\mu=\varrho_\mu^{(1)}(p)$.

Since

$$(m^2+\varrho^2)^{-1}(m^2+p^2)\tag{A.8}$$

is bounded both above and below uniformly in p and ε for $p\in\left[-\frac{\pi}{\varepsilon},\frac{\pi}{\varepsilon}\right]^2$, we may take the limit $\varepsilon\searrow 0$ under the integral sign in the final integral in (A.6) by the dominated convergence theorem. The result is

$$\tfrac{1}{2}\left(\frac{e}{2\pi}\right)^2\int_{-\infty}^{\infty}(m^2+p_+^2)^{-1}(m^2+p_-^2)^{-1}(-k_\mu^2-4m^2)d^2p.\tag{A.9}$$

Let us call this integral $J(k)$, then we have shown that pointwise in k

$$\Pi_{\mu\nu}^\varepsilon(k)\xrightarrow[\varepsilon\searrow 0]{}(-J_0+J(k))(\delta_{\mu\nu}-k_\mu k_\nu/k^2).\tag{A.10}$$

Furthermore we can show that $J_0=J(0)$ by the following argument: $\Pi_{\mu\nu}^\varepsilon(k)$ is analytic in k near $k=0$, the transverse projection is not, therefore $\Pi_{\mu\nu}^\varepsilon(0)=0$. Pointwise convergence then implies that $J_0=J(0)$.

Remark. $J(0)$ is independent of m by a scaling argument. Thus setting $m=1$ gives

$$J(0)=-2\left(\frac{e}{2\pi}\right)^2\int\frac{d^2p}{(1+p^2)^2}=-\frac{e^2}{2\pi}.$$

Pauli-Villars regularisation of the continuum expressions gives the same result as (A.10).

By combining the upper and lower bounds on (A.8) with the arguments given above it is not difficult to prove first that for all $\alpha>0$

$$|\Pi^{\varepsilon}_{\mu\nu}(k)| \leq C_{\alpha}(1+k^2)^{\alpha}$$

and then obtain:

Lemma A.1. *For all* $\alpha>0$

$$(1+k^2)^{-\alpha}\Pi^{\varepsilon}_{\mu\nu}(k)$$

converges in $L_{\infty}(d^2k)$ *as* $\varepsilon\searrow 0$.

Proof of Statement 3) in the Proof of Theorem 4.3. $\hat{\mathscr{A}}^{\varepsilon}_{\mu}(k)$ is the Fourier transform of a function on a lattice [see below Eq. (2.12)]. Let

$$H^{\varepsilon}(k) \equiv \frac{\sin\varepsilon\frac{k_1}{2}\sin\varepsilon\frac{k_2}{2}}{\varepsilon\frac{k_1}{2}\,\varepsilon\frac{k_2}{2}}.$$

By an easy computation $H^{\varepsilon}\hat{\mathscr{A}}^{\varepsilon}_{\mu} \equiv \tilde{\mathscr{A}}^{\varepsilon}_{\mu}$ is the Fourier transform of $\mathscr{A}^{\varepsilon}_{\mu}(x)$ considered as a piecewise constant function on $\mathbb{R}^2$ via the Q identification. Therefore, omitting ε's

$$-\operatorname{tr} K + 1/2 \operatorname{tr} K^2 = \int_{-\pi/\varepsilon}^{\pi/\varepsilon} \tilde{\mathscr{A}}_{\mu}\Pi_{\mu\nu}H^{-2}\overline{\tilde{\mathscr{A}}_{\nu}}d^2k. \qquad \text{(A.12)}$$

H is bounded both above and below on the range of integration. As $\varepsilon\searrow 0$ it converges uniformly on compact subsets of $\mathbb{R}^2$. Hence by Lemma A.1

$$(1+k^2)^{-\alpha}\Pi_{\mu\nu}H^{-2}\chi_{[-\frac{\pi}{\varepsilon},\frac{\pi}{\varepsilon}]\times 2}$$

converges in $L_{\infty}(\mathbb{R}^2, d^2k)$ as $\varepsilon\searrow 0$ for all $\alpha>0$. Therefore it is sufficient to show that the $L_2(\mathbb{R}^2, d^2k)$ norm

$$\|k^{\alpha/2}\tilde{\mathscr{A}}\|_2^2 = \int \mathscr{A}(x)(k^{\alpha})\hat{}\,(x-y)\mathscr{A}(y)d^2xd^2y \qquad \text{(A.13)}$$

converges as $\varepsilon\searrow 0$. The right hand side of this equality comes from the Plancherel identity, $k=\sqrt{k_1^2+k_2^2}$.

Lemma A.2. *Let* f *be in Schwartz space. The Fourier transform of* $k^{\alpha}\hat{f}(k)$ *is a constant,* C_{α}, *times*

$$\int d^2y(f(y)-f(x))|x-y|^{-2-\alpha}.$$

For a detailed proof see [24]. It is not difficult and proceeds by exploiting the homogeneity of k. An easy argument shows that we can also use this form if f is $\tilde{\mathscr{A}}$. Thus the right hand side of (A.13) may be written as

$$\frac{c_{\alpha}}{2}\int|\mathscr{A}(x)-\mathscr{A}(y)|^2|x-y|^{-2-\alpha}d^2xd^2y. \qquad \text{(A.14)}$$

Since $\mathscr{A}$ vanishes outside Λ, a bounded rectangle, (∞, α) convergence of $\mathscr{A}_\mu^\varepsilon$ implies convergence of (A.14). This in turn is implied by (∞, α) convergence of A_μ^ε by expanding the exponent and making some simple estimates relying on the fact that $\mathscr{A}_\mu^\varepsilon$ and A_μ^ε are piecewise constant. A_μ^ε is (∞, α) convergent by hypothesis. □

Proof of Convergence of Counterterms (VI.2). The propagator defined in (6.3), $D_{\mu\nu}^\varepsilon(x)$ is a function on the lattice $L^{(\varepsilon)}$ and

$$\hat{D}_{\mu\nu}^\varepsilon(k) = \sum_x \varepsilon^2 D_{\mu\nu}^\varepsilon(x) e^{-ikx} \frac{1}{2\pi}.$$

As above $\tilde{D} = \hat{D}H$ is the Fourier transform of D considered as a piecewise constant function on $\mathbb{R}^2$ via the Q identification. Since

$$E_D^\varepsilon = 2 \int_{-\pi/\varepsilon}^{\pi/\varepsilon} \Pi_{\mu\nu}^\varepsilon \hat{D}_{\mu\nu}^\varepsilon d^2k$$

we may argue as above that convergence of E_D^ε is implied by convergence of

$$\int_{-\infty}^{\infty} k^\alpha \tilde{D}_{\mu\nu}^\varepsilon d^2k = c_\alpha \int (D_{\mu\nu}^\varepsilon(x) - D_{\mu\nu}^\varepsilon(0)) |x|^{-2-\alpha} d^2x$$

for some $\alpha > 0$. The right hand side is derived by noting that the integral on the left is equal to the Fourier transform of the integrand evaluated at zero and using Lemma A.2. $D_{\mu\nu}^\varepsilon(x)$ is now to be understood as a piecewise constant function on $\mathbb{R}^2$. Convergence of the right hand side may be easily shown using the Hölder continuity (5.11) of $D_{\mu\nu}$ and arguments analogous to those in the proof of Theorem 5.1. This concludes the proof of convergence of E_D^ε.

A very similar argument which we omit proves the convergence of $\delta m_D^{\varepsilon 2}$.

Appendix B

Convergence of the Lattice Approximation for Periodic and (Half-) Dirichlet Boundary Conditions

We want to sketch how the proofs for convergence of the lattice approximation given in this paper can be adapted to periodic, P and Dirichlet, D (or Half-Dirichlet, HD) boundary conditions, for a rectangle Λ. In the case of D or HD boundary conditions, the orientation of Λ with respect to the lattice may be arbitrary. This will be needed in Paper III for proving Euclidean invariance. Half-Dirichlet means here that we use Wick ordering with respect to the free covariance in the selfinteraction of the matter field; we use Dirichlet boundary conditions for the covariance of the matter field and free boundary conditions for the gauge field.

In the main body of this paper we reduced existence of the continuum limit for X boundary conditions to the following three convergence statements:

A) $\chi_\Lambda C_X^\varepsilon \to \chi_\Lambda C_X$ in $\mathscr{I}_\alpha$, for $\alpha > 1$.

B) $\partial^\varepsilon (C_X^\varepsilon)^{1/2} \to \partial C_X^{1/2}$, in the strong operator topology, and likewise for the adjoints.

C) $(A_\mu^\varepsilon, \Pi_{\mu\nu}^{\varepsilon, X} A_\nu^\varepsilon) \to (A_\mu, \Pi_{\mu\nu}^X A_\nu)$, whenever A_μ^ε converges to A_μ in the (∞, α) sense

Although we only considered free boundary conditions, $X=F$, our arguments show that A)–C) suffice for more general boundary conditions, in particular $X=\mathrm{P},\mathrm{D}$.

If $\Lambda\equiv\left\{(x,y)\in\mathbb{R}^2\,\middle|\,|x|<\frac{|a|}{2},\ |y|<\frac{|b|}{2}\right\}$ with a,b multiples of ε, the periodic covariance is

$$C_p^{(\varepsilon)}(x,y)=\sum_{n,m=-\infty}^{\infty}C^{(\varepsilon)}(x+ma,y+nb). \tag{B.1}$$

This representation shows that statements A)–C) remain true if C^ε, C are replaced by C_p^ε, C_p, since the series in (B.1) converges absolutely and uniformly, because of the exponential decay of C^ε, C.

So we only have to prove A)–C) for Dirichlet boundary conditions, $X=\mathrm{D}$. We will make use of the work of Guerra et al. [6].

Let p_ε be the projection, orthogonal with respect to the scalar product $(\cdot, C^\varepsilon\cdot)$, onto functions on $L^{(\varepsilon)}$ supported in $L^{(\varepsilon)}(\sim\Lambda)$; similarly p, for the continuum. Define

$$P_\varepsilon\equiv(C^\varepsilon)^{1/2}p_\varepsilon(C^\varepsilon)^{-1/2}, \tag{B.2}$$

$$P\equiv C^{1/2}pC^{-1/2}. \tag{B.3}$$

Using the imbedding $Q^{\varepsilon*}:\ell^2(L^{(\varepsilon)})\to L^2(\mathbb{R}^2)$ (see Sect. II), we obtain the orthogonal projections in $L^2(\mathbb{R}^2)$

$$\tilde{P}_\varepsilon\equiv Q^{\varepsilon*}P_\varepsilon Q^\varepsilon. \tag{B.4}$$

The crucial fact is

Lemma B.1. $s\text{-}\lim_{\varepsilon\to 0}\tilde{P}_\varepsilon=P$.

Remark. This is very similar to Lemma (VIII.9) in [26] and Lemma IV.11 in [6]. It is not identical, however, because these references use a different imbedding of $\ell_2(L^{(\varepsilon)})$ into $L_2(\mathbb{R}^2)$. This necessitates some modification in the proof.

Proof. I) We claim that for

$$g\in\operatorname{Ran}P\cap\operatorname{Ran}C^{1/2},$$
$$\|\tilde{P}_\varepsilon g-g\|\to 0.$$

Proof. By Bessel's inequality we have $\inf_h\|\tilde{P}_\varepsilon h-g\|=\|\tilde{P}_\varepsilon g-g\|$. Thus

$$\|\tilde{P}_\varepsilon g-g\|\leq\|\tilde{P}_\varepsilon Q^{\varepsilon*}(C^\varepsilon)^{1/2}Q^\varepsilon C^{-1/2}g-g\|\,;$$

$$\begin{aligned}\tilde{P}_\varepsilon Q^{\varepsilon*}&(C^\varepsilon)^{1/2}Q^\varepsilon C^{-1/2}g\\ &=(Q^{\varepsilon*}(C^\varepsilon)^{1/2}p_\varepsilon)(Q^\varepsilon C^{-1/2}g)\\ &=Q^{\varepsilon*}(C^\varepsilon)^{1/2}Q^\varepsilon C^{-1/2}g\to g\end{aligned}$$

by statement A), for $X=F$ (free); we used the fact that $Q^\varepsilon C^{-1/2}g$ is supported outside Λ.

II) If $g \in \operatorname{Ran} P$ we still have $\|\tilde{P}_\varepsilon g - g\| \to 0$ because $\operatorname{Ran} P \cap \operatorname{Ran} C^{1/2}$ is dense in $\operatorname{Ran} P$ (i.e., Λ is "regular" in the terminology of [6]).

III) Let $g \in L^2(\mathbb{R}^2)$, f a weak limit point of the bounded set $\{\tilde{P}_\varepsilon g \mid 0 < \varepsilon < 1\}$. We claim:

$$f = Pg. \tag{B.5}$$

a) Let $C^{1/2}h \in C_0^\infty(\Lambda)$:

$$(h, \tilde{P}_\varepsilon g)_{L_2} = (((C^\varepsilon)^{1/2} Q^\varepsilon - Q^\varepsilon C^{1/2})h, \, p_\varepsilon (C^\varepsilon)^{-1/2} Q^\varepsilon g)_{L_2} \tag{B.6}$$

(the second term is zero because of support properties). (B.6) converges to 0, (see note added in proof), which shows that $f \in \operatorname{Ran} P$.

b) Let

$$h \in L^2(\mathbb{R}^2) : (h, f) = (Ph, f) = \lim_{n \to \infty} (Ph, \tilde{P}_{\varepsilon_n} g)$$
$$= \lim_{n \to \infty} (\tilde{P}_{\varepsilon_n} Ph, g) = (Ph, g)$$

by Part II) of the proof; this establishes (B.5).

IV) (B.5) shows that $\tilde{P}_\varepsilon g$ converges weakly to Pg; because $\tilde{P}_\varepsilon$ are projections this implies strong convergence. (End of proof of Lemma B.1.)

As discussed in [6], we can define the Dirichlet covariances by

$$C_D^\varepsilon \equiv C^\varepsilon(1 - p_\varepsilon) = (C^\varepsilon)^{1/2}(1 - P_\varepsilon)(C^\varepsilon)^{1/2}, \tag{B.7}$$

$$C_D \equiv C(1 - p) = C^{1/2}(1 - P)C^{1/2}. \tag{B.8}$$

Statements A) and B) with C^ε, C replaced by C_D, C_D are now consequences of (B.7) and (B.8), using Lemmas 3.6, 4.5 and B.1 (see note added in proof).

Statement C) is a little more subtle.

Obviously it suffices to consider the difference

$$(A_\mu^\varepsilon, (\Pi_{\mu\nu}^{\varepsilon, D} - \Pi_{\mu\nu}^\varepsilon) A_\nu^\varepsilon) = e^2 (A_\mu^\varepsilon, [\tfrac{1}{2}\{(\partial_\mu^\varepsilon - \partial_\mu^{\varepsilon *})(C_D^\varepsilon - C^\varepsilon)\} \{(\partial_\nu^\varepsilon - \partial_\nu^{\varepsilon *})(C_D^\varepsilon + C^\varepsilon)\}] A_\nu^\varepsilon) + e^2 \int_\Lambda (A_\mu^\varepsilon)^2(x)(C_D^\varepsilon - C^\varepsilon)(x, x)\, dx. \tag{B.9}$$

(We assume A to be transverse; non-transverse components drop out.)

Because of the Hölder continuity of A_μ we can bound $|A_\mu^\varepsilon - \mathscr{A}_\mu^\varepsilon|$ uniformly in Λ, and using the Q-imbedding also $|\mathscr{A}_\mu^\varepsilon - A_\mu|$; therefore we only have to show L^1-convergence of $(\partial_\mu^\#(C_D^\varepsilon - C^\varepsilon))(\partial_\nu^\#(C_D^\varepsilon + C^\varepsilon))$, and convergence of the second term in (B.9).

Here $\partial_\mu^\#$ is either ∂_μ or ∂_μ^*. What we need is contained in

Lemma B.2.

1) $\partial_\mu^{\varepsilon\#}(C^\varepsilon - C_D^\varepsilon) \xrightarrow{\varepsilon \to 0} \partial_\mu^\#(C - C_D)$ in $L^2(\Lambda \times \Lambda)$.

2) $(\partial_\mu^{\varepsilon\#}(C^\varepsilon - C_D^\varepsilon))(\partial_\nu^{\varepsilon\#} C^\varepsilon) \xrightarrow{\varepsilon \to 0} (\partial_\mu^\#(C - C_D))(\partial_\nu^\# C)$ in $L^1(\Lambda \times \Lambda)$.

3) $(C^\varepsilon - C_D^\varepsilon)(x, x) \xrightarrow{\varepsilon \to 0} (C - C_D)(x, x)$ in $L^p(\Lambda)$, $1 \leqq p < \infty$.

Proof. The proof proceeds by the dominated convergence theorem. For the uniform upper bound we need

Proposition B.3. *For* $x, y \in \Lambda$

1) $$|C^\varepsilon(x-y)| \leqq \text{const}\left|\log\frac{|x-y|}{4(a+b)}\right|.$$

2) $$|\partial^\varepsilon_{\mu,x} C^\varepsilon(x-y)| \leqq \text{const}\frac{1}{|x-y|}.$$

3) $$|(C^\varepsilon - C^\varepsilon_D)(x,y)| \leqq \text{const}\left|\log\frac{\text{dist}(x,\partial\Lambda)+\text{dist}(y,\partial\Lambda)}{4(a+b)}\right|.$$

4) $$|\partial_{\mu,x}(C^\varepsilon - C^\varepsilon_D)(x,y)| \leqq \frac{\text{const}}{\text{dist}(x,\partial\Lambda)+\text{dist}(y,\partial\Lambda)}.$$

Proof. 1) follows from 2) by integration.

2) Follows by some work with the explicit Fourier representation of C^ε:

$$\partial^\varepsilon_\mu C^\varepsilon(x) = \int_{\substack{\varepsilon|k_1|\leqq\pi \\ \varepsilon|k_2|\leqq\pi}} \frac{\varepsilon^{-1}(e^{i\varepsilon k_\mu}-1)e^{ikx}}{2\varepsilon^{-2}(2-\cos\varepsilon k_1 - \cos\varepsilon k_2)+m^2}\, d^2k.$$

We cut the integration into a part where $|k| \leqq \Delta$ and a rest. The "inner" part is

$$\int_{|k|\leqq\Delta\varepsilon} \frac{\varepsilon^{-1}(e^{ik_\mu}-1)e^{ikx\varepsilon^{-1}}}{2(2-\cos k_1 - \cos k_2)+\varepsilon^2 m^2}\, d^2k$$

which is bounded by

$$\int_{|p|\leqq\Delta\varepsilon} \frac{\varepsilon^{-1}|\sin(k_\mu/2)|}{2-\cos k_1 - \cos k_2}\, d^2k \leqq \text{const}.$$

The outer part is bounded by $\frac{\text{const}}{|x|}$ as can be seen by doing an integration by parts with respect to the variable $|x|$.

3) Can be seen as follows:

$$(-\Delta^\varepsilon + m^2)(C^\varepsilon - C^\varepsilon_D)(x,y) = \sigma^\varepsilon_y(x), \quad (x \in \Lambda) \tag{B.10}$$

where $\sigma^\varepsilon_y(x)$ has support on $\partial\Lambda^\varepsilon$ which is the set of points in $L^{(\varepsilon)}$ which are endpoints of a lattice bond that intersects $\partial\Lambda$ or are in $\partial\Lambda$ themselves. It is not hard to see that

$$\sigma^\varepsilon_y(x) \geqq 0, \tag{B.11}$$

$$\sum_x \varepsilon^2 \sigma^\varepsilon_y(x) \leqq 1. \tag{B.12}$$

(B.11) follows from the fact that $C^\varepsilon_D \geqq 0$ and $C^\varepsilon_D = 0$ if one of its arguments is outside Λ; (B.12) follows by Gauss's theorem for the lattice:

$$0 = \sum_x \varepsilon^2 (\Delta_x C^\varepsilon_D)(x,y) = -1 + m^2 \sum_x \varepsilon^2 C^\varepsilon_D(x,y) + \sum_x \varepsilon^2 \sigma^\varepsilon_y(x).$$

From (B.10) it can be seen that

$$(C^\varepsilon - C^\varepsilon_D)(x,y) = \sum_{x'} \varepsilon^2 C^\varepsilon(x-x')\sigma_y(x'). \tag{B.13}$$

Using 2) and (B.11), (B.12), it follows that

$$|\partial^{\varepsilon}_{\mu,x}(C^{\varepsilon}-C^{\varepsilon}_{D})(x,y)| \leqq \frac{\text{const}}{\text{dist}(x,\partial\Lambda)}.$$

Since the left side of this equation is symmetric in x and y, (4) follows. 3) is similar. □

Returning to the proof of Lemma B.2 we notice that

$$\partial^{\varepsilon}_{\mu}(C^{\varepsilon}-C^{\varepsilon}_{D}) = \partial^{\varepsilon}_{\mu}(C^{\varepsilon})^{1/2} P_{\varepsilon}(C^{\varepsilon})^{1/2}.$$

L^2 convergence of this then follows from statement B), Lemma 3.6 and $\mathscr{I}_2$ convergence of $P_{\varepsilon}(C^{\varepsilon})^{1/2}$ which we now prove. By the Grümm-Simon theorem (see Sect. II), we only need to show convergence of the $\mathscr{I}_2$ norms of $P_{\varepsilon}(C^{\varepsilon})^{1/2}$, which means we have to show that

$$\mathrm{Tr}(C^{\varepsilon})^{1/2} P_{\varepsilon}(C^{\varepsilon})^{1/2} = \sum_{x\in\Lambda} \varepsilon^2 (C^{\varepsilon}-C^{\varepsilon}_{D})(x,x) \tag{B.14}$$

converges. Since Proposition B.3, 3) gives an L^p upper bound, we are reduced to showing pointwise convergence of $(C^{\varepsilon}-C^{\varepsilon}_{D})(x,x)$ to establish Lemma B.2, 1).

From Proposition B.3 we also get the following bound on the expression appearing in Lemma B.2, 2):

$$|\partial^{\varepsilon}_{\mu}(C^{\varepsilon}-C^{\varepsilon}_{D})\partial^{\varepsilon}_{\nu}C^{\varepsilon}| \leqq \text{const} \times \frac{1}{\text{dist}(x,\partial\Lambda)+\text{dist}(y,\partial\Lambda)} \frac{1}{|x-y|}. \tag{B.15}$$

This bound is in $L^1(\Lambda\times\Lambda)$ as can be seen by cutting up the region of integration into a suitable sequence of bonds parallel to the boundary.

So all that remains to be shown to complete the proof of Lemma B.2 is

Proposition B.4. *$(C^{\varepsilon}-C^{\varepsilon}_{D})(x,y)$ and $\partial^{\varepsilon}_{\mu}(C^{\varepsilon}-C^{\varepsilon}_{D})(x,y)$ converge pointwise in $\Lambda\times\Lambda$.*

Proof. Since $C^{\varepsilon}-C^{\varepsilon}_{D}$ *converges in* L^2,

$$F_{\varepsilon,\delta}(x,y) \equiv \frac{1}{\pi^2\delta^4}\int \chi_{\delta}(x-x')\chi_{\delta}(y-y')(C^{\varepsilon}-C^{\varepsilon}_{D})(x',y')dx'dy'$$

converges pointwise as $\varepsilon\to 0$, where χ_{δ} is the characteristic function of a ball of radius δ. On the other hand we can for each $(x,y)\in\Lambda\times\Lambda$ choose δ so small that $|F_{\varepsilon,\delta}(x,y)-(C^{\varepsilon}-C^{\varepsilon}_{D})(x,y)|<\eta$ (uniformly in ε) because we have a uniform bound on the "derivatives" of $C^{\varepsilon}-C^{\varepsilon}_{D}$. By a 2η argument pointwise convergence of $C^{\varepsilon}-C^{\varepsilon}_{D}$ follows.

For $\partial^{\varepsilon}_{\mu}(C^{\varepsilon}-C^{\varepsilon}_{D})$ we use the same trick: We just established L^2-convergence; a uniform (in ε) bound on the second "derivatives" in a neighborhood of any point in the interior of Λ can easily be obtained from (B.13) and we just have to repeat the argument given before.

This completes the proof of Lemma B.2 and this appendix.

Acknowledgements. David Brydges would like to thank Loren Pitt and Larry Thomas for many helpful conversations. Erhard Seiler would like to express his thanks for the kind hospitality extended to him at the I.H.E.S.

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Communicated by K. Osterwalder

Received August 6, 1979

Note Added in Proof

To obtain B) note that $\partial^\varepsilon C_D^{\varepsilon 1/2}$ is bounded uniformly in ε in operator norm so that it suffices to prove that

$$\partial^\varepsilon C_D^{\varepsilon 1/2} = [\partial^\varepsilon C^{\varepsilon 1/2}] C^{\varepsilon 1/2} C_D^{\varepsilon 1/2} C_D^{\varepsilon - 1}$$

converges strongly on the dense set $C_0^\infty(\Lambda)$.

To obtain (B.6) we prove that

$$Q^{\varepsilon *} C^{\varepsilon - 1/2} Q^\varepsilon C^{1/2} = (Q^{\varepsilon *} Q^\varepsilon)(C^{\varepsilon - 1/2} C^{1/2})$$

(see Sect. II) converges strongly because both factors on the left hand side do.