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A New Example of 'Independence' and 'White Noise'

Roland Speicher
Institut für Angewandte Mathematik, Universität Heidelberg,
Im Neuenheimer Feld 294, D-6900 Heidelberg, Federal Republic of Germany

Summary. We examine the notion of 'free independence' according to Voiculescu. This form of independence is used for defining 'free white noise' or 'process with stationary and freely independent increments'. We prove a general limit theorem giving the combinatorics of infinitely freely divisible states and thus of free white noises with the help of 'admissible' partitions. We realize the free analogues of the Wiener process and of the Poisson process as processes on the full Fock space of $L^2(\mathbb{R})$.

1. Introduction

In quantum stochastics (non-commutative probability theory) one tries to develop a probability theory for quantum systems. The main problem consists in giving good mathematical models for the description of a quantum system coupled to a quantum Markovian random generator. In classical probability theory the analogous problem of a Brownian particle is modeled by assuming a white noise as random generator and coupling this via stochastic differential equations to the particle.

This is imitated in quantum stochastics. There the classical concepts of random variables and probability measures are replaced by the functional analytic concepts of operator algebras and states. Thus we would like to describe the stochastic influence of the operator algebra for the random generator on the operator algebra for the quantum system. In analogy to the classical situation we want to give meaning to Langevin equations like

$$d a(t) = [H(t), a(t)]\, dt + d\omega(t),$$

where the operator $a(t)$ of the quantum system (in the Heisenberg picture) does not only evolve deterministic according to the Schrödinger equation, but also feels the influence of some random operator $d\omega(t)$ which we want to characterize as white noise.

We will only make assertions about the moments of the white noise. But similar to the classical case, where one usually chooses a concrete realization of the Wiener process on continuous paths, we want to find 'simple' explicit models for white noises, i.e. operator algebras and states, which give exactly the wanted moments.

So the aim is clear: Find an operator algebra with a state which deserves to be called a white noise and develop a quantum stochastic calculus for it in order to give meaning to Langevin equations with respect to this state.

Until now only two white noises are known, namely the bosonic white noise of the CCR-algebra and the fermionic white noise of the CAR-algebra. The corresponding stochastic calculi were developed by Hudson and Parthasarathy [HuP] and Applebaum and Hudson [ApH], respectively (but compare also [BSW1, 2]).

Kümmerer [Küm 1, 2] has given an axiomatic definition of 'white noise' and 'coupling to white noise'. This general frame allows the development of a stochastic integration theory. Of course the results of this abstract frame are not so considerable like the ones of the two concrete models.

We will present in this note a new concrete example of a white noise, which allows as far reaching investigations as the bosonic and fermionic models. This white noise is the Cuntz algebra, i.e. the C^*-algebra of creation and annihilation operators on the full Fock space of $L^2(\mathbb{R})$, with the vacuum expectation as state. The stochastic calculus for the Cuntz algebra was developed in [Spe] and will be published elsewhere [KSp].

This white noise is connected with a form of independence ('free' independence) introduced by Voiculescu [Voi 1], but compare also [Avi].

We will give limit theorems for this kind of independence and recognize in this way analogues of Gaussian and Poisson distributions.

The paper is organized as follows. In Sect. 2 we deal with the problem of 'independence' in quantum stochastics. In Sect. 3 the special form of independence according to Voiculescu ('free' independence) is introduced. In Sect. 4 we prove some limit theorems for the free independence and obtain especially the combinatorics of the free analogues of Gaussian and Poisson distribution. In Sect. 5 we present the full Fock space and the Cuntz algebra and show that the free analogues of the Wiener process and Poisson process may be realized in the full Fock space.

2. The Concept of 'Independence' in Quantum Stochastics

Let $(\Omega, \Sigma, P, (X_t^1, \ldots, X_t^n)_{t\in\mathbb{R}})$ be a n-dimensional classical process with stationary and independent increments. For $I=[t_1, t_2)$ let $X_I^i := X_{t_2}^i - X_{t_1}^i$ be the increment of the i-th coordinate of this process. Let $\mathfrak{R}$ be the ring generated by all semi-closed intervals I of the above form. Then the definition of X_I^i extends to $I \in \mathfrak{R}$ such that the mapping $I \mapsto (X_I^1, \ldots, X_I^n)$ is finitely additive. The distribution of $(X_I^1, \ldots, X_I^n)$ depends only on $\lambda(I)$, the Lebesgue-measure of I. The mapping $I \mapsto (X_I^1, \ldots, X_I^n)$ will be called a 'n-dimensional white noise' with respect to the probability measure P.

It is clear that $(X_I^1, \ldots, X_I^n)$ can be written as a sum of arbitrarily many independent and identically distributed random variables. Therefore the distribution of $(X_I^1, \ldots, X_I^n)$ is for all $I \in \mathfrak{R}$ an infinitely divisible one closely connected to limit theorems.

We will now deal with non-commutative versions of white noises and develop the analogues of the above statements.

In quantum probability the random variables $X_I^1, \ldots, X_I^n$ and the probability measure P are replaced by n operators $c_I^1, \ldots, c_I^n$ and a state ρ on the algebra generated by all these operators. In the classical case the random variables $X_I^1, \ldots, X_I^n$ may be viewed as one new random variable $X_I := (X_I^1, \ldots, X_I^n)$ with values in $\mathbb{R}^n$, i.e. the difference between the general case and the case $n=1$ is mainly one in notation. In the non-commutative case the operators $c_I^1, \ldots, c_I^n$ may be non-commuting and such a reduction is not possible.

The problem in defining 'white noise' is now the meaning of 'independence'. Whereas in classical probability theory there is only one possible definition of 'independence', the situation in non-commutative probability theory is not so simple. Independence of the operators $c_{I_1}^i$ and $c_{I_2}^j$ $(I_1, I_2 \in \mathfrak{R}, I_1 \cap I_2 = \emptyset)$ shall be understood as usual as the independence of the algebras $\mathscr{C}_{I_1} = \langle c_{I_1}^i \mid i=1, \ldots, n\rangle$ and $\mathscr{C}_{I_2} = \langle c_{I_2}^i \mid i=1, \ldots, n\rangle$ with respect to the states $\rho_{I_1} = \rho/\mathscr{C}_{I_1}$ and $\rho_{I_2} = \rho/\mathscr{C}_{I_2}$. There $\langle R\rangle$ denotes the algebra generated by all operators $r \in R$. Thus we have to define the independence of subalgebras $\mathscr{C}_k$ of an algebra $\mathscr{C}$ with respect to a state ρ. Of course we are led by the classical situation and demand some form of factorization.

In his axiomatic theory of 'white noise' Kümmerer [Küm2] only demands the factorizing of time ordered products, i.e.

$$\rho(a_1 \ldots a_m) = \rho_{I_1}(a_1) \ldots \rho_{I_m}(a_m) \quad \text{if } a_i \in \mathscr{C}_{I_i} \quad \text{and} \quad I_1 < I_2 < \ldots < I_m,$$

where $I_1 < I_2$ means: for all $t_1 \in I_1$ and $t_2 \in I_2$ we have $t_1 < t_2$. For other products no rule is prescribed.

Example. Let $a \in \mathscr{C}_{I_1}$, $b \in \mathscr{C}_{I_2}$ with $I_1 < I_2$. Then

$$\rho(aabb) = \rho_{I_1}(aa)\, \rho_{I_2}(bb),$$

but no formula for $\rho(abab)$ is given.

So we have different possibilities for adding rules for calculating products which are not time-ordered. These different rules lead to different forms of 'independence'.

Usually one demands that independent operators commute ('tensor case'), i.e. for $I_1, I_2 \in \mathfrak{R}$, $I_1 \cap I_2 = \emptyset$:

$$\langle \mathscr{C}_{I_1}, \mathscr{C}_{I_2}\rangle = \mathscr{C}_{I_1} \otimes \mathscr{C}_{I_2}, \qquad \rho/\langle \mathscr{C}_{I_1}, \mathscr{C}_{I_2}\rangle = \rho_{I_1} \otimes \rho_{I_2}.$$

This means that the algebra $\mathscr{C} = \langle c_I^i \mid i=1, \ldots, n; I \in \mathfrak{R}\rangle$ is built together as a 'commuting' sum of the non-commutative subalgebras $\mathscr{C}_{I_i}$ with respect to ρ. Using this form of independence for the definition of white noise leads to the bosonic white noise of the CCR-algebra, which contains the classical white noise as a special case.

If the $\mathscr{C}_I$ are graded one may also use the antisymmetric tensor product for the definition of independence. This leads to the fermionic white noise of the CAR-algebra.

We will replace these concepts by a more non-commutative one which takes instead of the tensor product the reduced free product, i.e. a 'maximal non-commuting' sum of the $\mathscr{C}_{I_i}$. This concept is due to Voiculescu and will be explained in the next section.

3. The Free Independence According to Voiculescu

Let J be an index set. For all $i \in J$ let $\mathscr{A}_i$ be a C^*-algebra and φ_i a state on $\mathscr{A}_i$. In [Voi 1] Voiculescu defined the reduced free product $(\hat{\mathscr{A}}, \hat{\varphi}) = *_{i \in J}(\mathscr{A}_i, \varphi_i)$ of the $(\mathscr{A}_i, \varphi_i)$. We will use the following characterization.

Definition. For all $i \in J$ let $\mathscr{A}_i$ be a C^*-algebra with 1 and φ_i a state on $\mathscr{A}_i$. Furthermore let a C^*-algebra $\hat{\mathscr{A}}$ with 1 and a state $\hat{\varphi}$ on $\hat{\mathscr{A}}$ be given. Then $(\hat{\mathscr{A}}, \hat{\varphi})$ is called *reduced free product* of the $(\mathscr{A}_i, \varphi_i)$ if we have:

(i) There exist unital *-homomorphism $j_i\colon \mathscr{A}_i \to \hat{\mathscr{A}}$, such that $\hat{\mathscr{A}}$ is generated by $\bigcup_{i \in J} j_i(\mathscr{A}_i)$.

(ii) $\hat{\varphi} \circ j_i = \varphi_i$ for all $i \in J$.

(iii) For $m \in \mathbb{N}$ and $k_i \in J$ with $k_1 \neq k_2 \neq \ldots \neq k_m$ (consecutive indices are distinct) and $a_i \in \mathscr{A}_{k_i}$ and $\varphi_{k_i}(a_i) = 0$ we have

$$\hat{\varphi}(j_{k_1}(a_1) \ldots j_{k_m}(a_m)) = 0 \quad \text{(independence).}$$

(iv) The GNS construction applied to $(\hat{\mathscr{A}}, \hat{\varphi})$ yields a faithful representation of $\hat{\mathscr{A}}$.

Theorem 1. *For all $(\mathscr{A}_i, \varphi_i)$ there exists a reduced free product $(\hat{\mathscr{A}}, \hat{\varphi})$ and it is unique up to isomorphism.*

Proof. See [Voi 1]. $\square$

Therefore we can speak of *the* reduced free product and we will denote it by $*_{i \in J}(\mathscr{A}_i, \varphi_i) = (*_{i \in J} \mathscr{A}_i, *_{i \in J} \varphi_i)$.

In the following we will sometimes identify $\mathscr{A}_i$ with $j_i(\mathscr{A}_i)$.

We will only use part (iii) of the definition. It says that the subalgebras $\mathscr{A}_i$ are independent in the sense of Kümmerer [Küm 1]. We will call this form of independence according to Voiculescu *free independence*. Part (iii) of the definition allows us the calculation of all moments for $a_1, \ldots, a_m$ with respect to $\hat{\varphi}$ if all moments of a_i with resepct to φ_{k_i} (for all i) are known. This calculation is done according the following recursive procedure: Let $k_1 \neq k_2 \neq \cdots \neq k_m$ and $a_i \in \mathscr{A}_{k_i}$ be given. Then define (note $\hat{\varphi}(a_i) = \varphi_{k_i}(a_i)$)

$$a_i^0 := a_i - \hat{\varphi}(a_i) \cdot 1,$$

thus $a_i^0 \in \mathscr{A}_{k_i}$ and $\hat{\varphi}(a_i^0)=0$. We have

$$\begin{aligned}\hat{\varphi}(a_1 \dots a_m) &= \hat{\varphi}[(a_1^0+\hat{\varphi}(a_1)\cdot 1)\dots(a_m^0+\hat{\varphi}(a_m)\cdot 1)]\\ &= \sum_{\pi,\sigma} \hat{\varphi}(a_{\pi(1)})\dots\hat{\varphi}(a_{\pi(n)})\,\hat{\varphi}(a^0_{\sigma(1)}\dots a^0_{\sigma(m-n)}),\end{aligned}$$

where the sum runs over all partitions of $\{1, \dots, m\}$ into two ordered sets $(\pi(1), \dots, \pi(n))$ and $(\sigma(1), \dots, \sigma(m-n))$ with $n \geqq 1$. Because of part (iii) of the definition the term for $n=0$ vanishes. After combining neighbouring elements from the same algebra the terms $a^0_{\sigma(1)}\dots a^0_{\sigma(m-n)}$ can be written again as a product $b_1 \dots b_{m'}$ with $b_i \in \mathscr{A}_{l_i}$ and $l_1 \neq l_2 \neq \dots \neq l_{m'}$. We can now repeat this procedure and because of $m' < m$ we will come to an end after finitely many steps.

Example. Let $a \in \mathscr{A}_1$, $b \in \mathscr{A}_2$. Then we have

$$\begin{aligned}\hat{\varphi}(aba) &= \hat{\varphi}[(a^0+\varphi_1(a)\cdot 1)(b^0+\varphi_2(b)\cdot 1)(a^0+\varphi_1(a)\cdot 1)]\\ &= \varphi_1(a)\,\hat{\varphi}(b^0a^0)+\varphi_2(b)\,\hat{\varphi}(a^0a^0)+\varphi_1(a)\,\hat{\varphi}(a^0b^0)+\varphi_1(a)\,\varphi_2(b)\,\hat{\varphi}(a^0)\\ &\quad+\varphi_1(a)\,\varphi_1(a)\,\hat{\varphi}(b^0)+\varphi_2(b)\,\varphi_1(a)\,\hat{\varphi}(a^0)+\varphi_1(a)\,\varphi_2(b)\,\varphi_1(a)\\ &= \varphi_2(b)\,\hat{\varphi}(a^0a^0)+\varphi_1(a)\,\varphi_2(b)\,\varphi_1(a)\\ &= \varphi_1(aa)\,\varphi_2(b).\end{aligned}$$

In the following lemmas we will give some simple properties of this calculation procedure. We will set $J=\mathbb{N}$.

Lemma 1. *Consider a product* $a=a_1 \dots a_m \in *_{i=1}^{\infty} \mathscr{A}_i$ *with*

(i) $a_i \in \mathscr{A}_{k_i}$ *for all i.*

(ii) *There exists a j with* $k_j \neq k_i$ *for all* $i \neq j$, *i.e. there is only one factor from the algebra* $\mathscr{A}_{k_j}$.

(iii) $\varphi_{k_j}(a_j)=0$.

Then $\hat{\varphi}(a)=0$.

Proof. Use the above recursive formual for $\hat{\varphi}(a_1 \dots a_m)$. All summands are either zero because they contain a factor $\varphi_{k_j}(a_j)$ or they contain a factor $b_1 \dots b_{m'}$, which fulfils again the assumptions of the lemma, but with $m' < m$. So the assertion follows by induction. □

Lemma 2. *Consider a product* $a=a_1 \dots a_m \in *_{i=1}^{\infty} \mathscr{A}_i$ *with*

(i) $a_i \in \mathscr{A}_{k_i}$ *for all i.*

(ii) *There is one algebra which appears only once, i.e. there exists a j with* $k_j \neq k_i$ *for all* $i \neq j$.

Then $\hat{\varphi}(a)=\varphi_{k_j}(a_j)\,\hat{\varphi}(a_1 \dots a_{j-1}\, a_{j+1} \dots a_m)$.

Proof. Write $a_j=a_j^0+\varphi_{k_j}(a_j)\cdot 1$. Then

$$\hat{\varphi}(a)=\varphi_{k_j}(a_j)\,\hat{\varphi}(a_1 \dots a_{j-1}\, a_{j+1} \dots a_m)+\hat{\varphi}(a_1 \dots a_{j-1}\, a_j^0\, a_{j+1} \dots a_m).$$

The second summand vanishes because of Lemma 1. □

If $a_1 \dots a_{j-1} a_{j+1} \dots a_m$ fulfils again the requirements of the lemma, we may repeat the procedure. If a product has such a form that this can be repeated until $m=1$ we will call this product *admissible*, otherwise *non-admissible*.

Example. Let $a \in \mathscr{A}_1$, $b \in \mathscr{A}_2$, $c \in \mathscr{A}_3$, $d \in \mathscr{A}_4$. Then $cbaabcdc$ is an admissible product:

$$\begin{aligned}\hat{\varphi}(cbaabcdc) &= \varphi_1(aa)\,\hat{\varphi}(cbbcdc)\\ &= \varphi_1(aa)\,\varphi_4(d)\,\hat{\varphi}(cbbcc)\\ &= \varphi_1(aa)\,\varphi_4(d)\,\varphi_2(bb)\,\hat{\varphi}(ccc)\\ &= \varphi_1(aa)\,\varphi_4(d)\,\varphi_2(bb)\,\varphi_3(ccc).\end{aligned}$$

We see that for admissible products the calculation of $\hat{\varphi}$ is the same as in the tensor case.

Given elements $a_i \in \mathscr{A}_{k_i}$ we will call an expectation of the form $\varphi_j(a_{i(1)} \dots a_{i(r)})$ ($j \in \mathbb{N}$) with $k_{i(1)} = k_{i(2)} = \dots = k_{i(r)} = j$, i.e. $a_{i(1)}, \dots, a_{i(r)} \in \mathscr{A}_j$, an *elementary moment* of the a_i.

Lemma 3. *Consider a product $a = a_1 \dots a_m \in *_{i=1}^{\infty} \mathscr{A}_i$ with $a_i \in \mathscr{A}_{k_i}$ for all i and $k_1 \neq k_2 \neq \dots \neq k_m$. Let s be the number of different algebras occurring in a, i.e. $s := \# \{k_1, \dots, k_m\}$. Then $\hat{\varphi}(a)$ can be written as a sum of products of elementary moments of the a_i, where each summand contains at least s factors.*

Proof. We will do this by induction on m. For $m = 1(\hat{\varphi}(a_1) = \varphi_{k_1}(a_1))$ and $m = 2(\hat{\varphi}(a_1 a_2) = \varphi_{k_1}(a_1)\,\varphi_{k_2}(a_2))$ the assertion is clear.
Let $m > 2$: Write $a_i = a_i^0 + \hat{\varphi}(a_i) \cdot 1$, thus $a_i^0 \in \mathscr{A}_{k_i}$ and $\hat{\varphi}(a_i^0) = 0$. Then

$$\hat{\varphi}(a) = \sum_{\pi, \sigma} \hat{\varphi}(a_{\pi(1)}) \dots \hat{\varphi}(a_{\pi(n)})\, \hat{\varphi}(a^0_{\sigma(1)} \dots a^0_{\sigma(m-n)}),$$

where the sum runs over all partitions of $\{1, \dots, m\}$ in two ordered sets $\pi = (\pi(1), \dots, \pi(n))$ and $\sigma = (\sigma(1), \dots, \sigma(m-n))$ (with $n \geqq 1$). The term $a^0_{\sigma(1)} \dots a^0_{\sigma(m-n)}$ contains at least $s-n$ elements from different algebras $\mathscr{A}_i$, thus it is according to the induction hypothesis the sum of products of elementary moments of the a_i^0, each product containing at least $s-n$ factors. But each moment of the a_i^0 can be written as a sum of products of moments of the a_i. Together with the n factors $\hat{\varphi}(a_{\pi(1)}) \dots \hat{\varphi}(a_{\pi(n)})$ this gives the assertion. □

Lemma 2 shows that admissible products have a representation with exactly s factors. The next lemma shows that for non-admissible products there is a representation with at least $s+1$ factors for each summand.

Lemma 4. *Consider a non-admissible product $a = a_1 \dots a_m \in *_{i=1}^{\infty} \mathscr{A}_i$ with $a_i \in \mathscr{A}_{k_i}$ for all i and $k_1 \neq k_2 \neq \dots \neq k_m$. Then $\hat{\varphi}(a)$ can be written as a sum of products of elementary moments of the a_i, where each summand contains at least $s+1$ factors, where $s := \# \{k_1, \dots, k_m\}$,*

Proof. It is sufficient to consider $a = a_1 \dots a_m$, $a_i \in \mathscr{A}_{k_i}$ with each k_i occurring at least twice. Consider now the factorization

$$\hat{\varphi}(a) = \sum_{\pi, \sigma} \hat{\varphi}(a_{\pi(1)}) \dots \hat{\varphi}(a_{\pi(n)})\, \hat{\varphi}(a^0_{\sigma(1)} \dots a^0_{\sigma(m-n)}).$$

Regard one fixed term of the sum: Then either all the $k_{\pi(i)}$ are distinct and $\#\{k_{\sigma(1)}, \ldots, k_{\sigma(m-n)}\} = s$ or $k_{\pi(i)} = k_{\pi(j)}$ for some pair (i,j) and $\#\{k_{\sigma(1)}, \ldots, k_{\sigma(m-n)}\} \geq s-(n-1)$. Because of Lemma 3 we have a representation of $\hat{\varphi}(a^0_{\sigma(1)} \ldots a^0_{\sigma(m-n)})$ with summands containing at least $\#\{k_{\sigma(1)}, \ldots, k_{\sigma(m-n)}\}$ factors. This gives the assertion. □

Example. Let $a \in \mathscr{A}_1$, $b \in \mathscr{A}_2$. Then

$$\begin{aligned}\hat{\varphi}(abab) &= \varphi_2(b)\, \hat{\varphi}(a^0 a^0 b^0) + \varphi_1(a)\, \hat{\varphi}(a^0 b^0 b^0) + \varphi_1(a)\, \varphi_1(a)\, \hat{\varphi}(b^0 b^0) \\ &\quad + \varphi_2(b)\, \varphi_2(b)\, \hat{\varphi}(a^0 a^0) + \varphi_1(a)\, \varphi_2(b)\, \varphi_1(a)\, \varphi_2(b) \\ &= \varphi_1(a)\, \varphi_1(a)\, \varphi_2(bb) + \varphi_1(aa)\, \varphi_2(b)\, \varphi_2(b) - \varphi_1(a)\, \varphi_1(a)\, \varphi_2(b)\, \varphi_2(b)\end{aligned}$$

Lemma 2 and Lemma 4 will be used decisively in the proof of our limit theorem in the next section.

4. Limit Theorems

We can now make precise the concept of free white noise.

Definition. A *n-dimensional free white noise* $(\mathscr{C}, \rho, (c^1_I, \ldots, c^n_I)_{I \in \mathfrak{R}})$ consists of
(i) a C^*-algebra $\mathscr{C}$ with 1,
(ii) a state ρ on $\mathscr{C}$,
(iii) a finitely additive mapping $\mathfrak{R} \to \mathscr{C}^n$, $I \mapsto (c^1_I, \ldots, c^n_I)$
such that with the following notation

$$\mathscr{C}_I := C^*(1, c^i_I \mid i = 1, \ldots, n)$$
$$\rho_I := \rho / \mathscr{C}_I$$

we have:
(i) $(\mathscr{C}_{I_1}, \rho_{I_1}), \ldots, (\mathscr{C}_{I_r}, \rho_{I_r})$ are freely independent for all $r \in \mathbb{N}$ and disjoint $I_1, \ldots, I_r \in \mathfrak{R}$,
(ii) the distribution ρ_I depends only on $\lambda(I)$.

We present now a limit theorem for freely independent, identically distributed random variables. Later we will specialize this to a central limit theorem for free Gaussian distribution and a limit theorem for free Poisson distribution. Afterwards we will see that free white noises necessarily possess (under some continuity assumption) distributions treated in our limit theorem.

We are only interested in the moments of the distributions. We call the rule for calculating them also the combinatorics of the distribution.

In the tensor case the calculation of moments in the limit is carried out with the help of partitions of sets (see, e.g., [GvW]). This is similar here, but in contrast to the tensor case, we will have to consider only special partitions, called admissible.

They are defined in the following way.

Definition. Let $\mathscr{V}=\{V_1, \ldots, V_s\}$ be a partition of the set $\{1, \ldots, r\}$, i.e. the V_i are ordered and disjoint sets and $\{1, \ldots, r\}=\bigcup_{i=1}^{s} V_i$. Then $\mathscr{V}$ is called *admissible* if for all $i, j=1, \ldots, s$ with $V_i=(v_1, \ldots, v_n)$ $(v_1<\ldots<v_n)$ and $V_j=(w_1, \ldots, w_m)$ $(w_1<\ldots<w_m)$ we have

$$w_k<v_1<w_{k+1} \Leftrightarrow w_k<v_n<w_{k+1} \qquad (k=1, \ldots, m-1).$$

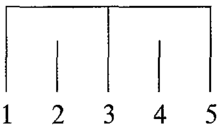
Otherwise the partition is called *non-admissible.*

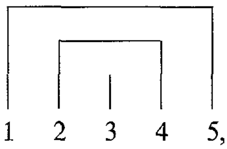
We will denote the set of all partitions of $\{1, \ldots, r\}$ by $\mathscr{P}(1, \ldots, r)$ and the set of all admissible partitions by $\mathscr{P}_a(1, \ldots, r)$.

We can reformulate the definition of 'admissible' in a recursive way: The partition $\mathscr{V}=\{V_1, \ldots, V_s\}$ is admissible if at least one of the V_i is a segment of $(1, \ldots, r)$, i.e. it has the form $V_i=(k, k+1, k+2, \ldots, k+m)$ and $\{V_1, \ldots, V_{i-1}, V_{i+1}, \ldots, V_s\}$ is an admissible partition of $\{1, \ldots, r\} \backslash V_i$ (interpreted in a canonical way).

In a more pictorial language: If we build bridges by connecting in $1\,2\,3 \ldots r$ the numbers belonging to the same V_i, then a partition is admissible if it is possible to build the corresponding bridge in such a way that the lines do not cross.

Example. $\{(1,3,5),(2),(4)\}$ and $\{(1,5),(2,4),(3)\}$ are admissible partitions of $\{1,2,3,4,5\}$, non-admissible are $\{(1,3),(2,4,5)\}$ and $\{(1,4),(2),(3,5)\}$. The respective pictures are





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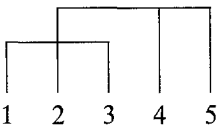

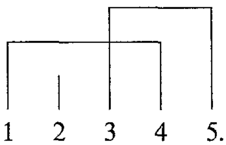

(Admissible) products are connected with (admissible) partitions in the following way: Let a product $a=a_1 \ldots a_m$ be given with $a_i \in \mathscr{A}_{k_i}$. Without restriction let $\{k_1, \ldots, k_m\}=\{1, \ldots, s\}$. Then define $V_r:=\{i \mid k_i=r\}$.

The product a is admissible if and only if the partition $\mathscr{V}=\{V_1, \ldots, V_s\}$ is an admissible partition of $\{1, \ldots, m\}$.

Example. Let $a \in \mathscr{A}_1$, $b \in \mathscr{A}_2$, $c \in \mathscr{A}_3$. The admissible product $b\,a\,a\,b\,c\,b$ corresponds to the admissible partition $\{(1,4,6),(2,3),(5)\}$. The non-admissible product $a\,b\,c\,a\,c$ corresponds to the non-admissible partition $\{(1,4),(2),(3,5)\}$.

The general setting for our limit theorem is the following: Let the pair $(\mathscr{B}, \varphi)$ consist of a C^*-algebra $\mathscr{B}$ with 1 and a state φ on $\mathscr{B}$ and for each $i \in \mathbb{N}$ let $(\mathscr{B}_i, \varphi_i) := (\mathscr{B}, \varphi)$. Let $(\hat{\mathscr{B}}, \hat{\varphi}) := *_{i=1}^{\infty}(\mathscr{B}_i, \varphi_i)$ be the reduced free product of the pairs $(\mathscr{B}_i, \varphi_i)$ with the canonical embeddings $j_i \colon \mathscr{B}_i \to \hat{\mathscr{B}}$.

Theorem 2 (limit theorem). *For each $N \in \mathbb{N}$ let n elements $b_N^1, \ldots, b_N^n \in \mathscr{B}$ be given. If for all $r \in \mathbb{N}$ and $k(1), \ldots, k(r) \in \{1, \ldots, n\}$*

$$Q(k(1), \ldots, k(r)) := \lim_{N \to \infty} N \cdot \varphi(b_N^{k(1)} \ldots b_N^{k(r)})$$

exists then we have for the sums

$$S_N^k := j_1(b_N^k) + \ldots + j_N(b_N^k)$$

for all $r \in \mathbb{N}$ and $k(1), \ldots, k(r) \in \{1, \ldots, n\}$:

$$\lim_{N \to \infty} \hat{\varphi}(S_N^{k(1)} \ldots S_N^{k(r)}) = \sum_{p=1}^{r} \sum_{\{V_1, \ldots, V_p\} \in \mathscr{P}_a(1, \ldots, r)} Q(V_1) \ldots Q(V_p),$$

where the sum runs over all admissible partitions $\{V_1, \ldots, V_p\}$ of the set $\{1, \ldots, r\}$ $(1 \leqq p \leqq r)$. There $Q(V)$ stands for $Q(k(v_1), \ldots, k(v_m))$ if $V = (v_1, \ldots, v_m)$.

Proof. In the following r and $k(1), \ldots, k(r)$ are fixed. We have to calculate the following expression for $N \to \infty$:

$$\begin{aligned} M_N &:= \hat{\varphi}(S_N^{k(1)} \ldots S_N^{k(r)}) \\ &= \hat{\varphi}[(j_1(b_N^{k(1)}) + \ldots + j_N(b_N^{k(r)})) \ldots (j_1(b_N^{k(r)}) + \ldots + j_N(b_N^{k(r)}))] \\ &= \sum_{i(1), \ldots, i(r)=1}^{N} \hat{\varphi}(j_{i(1)}(b_N^{k(1)}) \ldots j_{i(r)}(b_N^{k(r)})). \end{aligned}$$

We now make use of our correspondence between products and partitions and collect these summands which belong to the same partition in the same class. Thus we will get an equivalence relation on the set of the r-tuple $(i(1), \ldots, i(r))$:

$$(i(1), \ldots, i(r)) \sim (j(1), \ldots, j(r)) \Leftrightarrow \{i(k) = i(l) \Leftrightarrow j(k) = j(l)\}$$

The equivalence classes are in a one to one correspondence to the partitions of $\{1, \ldots, r\}$. Because of the invariance of the reduced free product under permutations the expression $\hat{\varphi}(j_{i(1)}(b_N^{k(1)}) \ldots j_{i(r)}(b_N^{k(r)}))$ has the same value for all $(i(1), \ldots, i(r))$ that belong to the same equivalence class. Thus this value depends only on the corresponding partition $\mathscr{V} = \{V_1, \ldots, V_p\}$ and on N and may be denoted by $\hat{\varphi}(\mathscr{V}; N) = \hat{\varphi}(V_1, \ldots, V_p; N)$. (example: $\hat{\varphi}((1,2),(3),(4); N) = \hat{\varphi}(j_1(b_N^{k(1)}) j_1(b_N^{k(2)}) j_3(b_N^{k(3)}) j_2(b_N^{k(4)})) = \hat{\varphi}(j_7(b_N^{k(1)}) j_7(b_N^{k(2)}) j_2(b_N^{k(3)}) j_3(b_N^{k(4)})))$. The equivalence class corresponding to $\mathscr{V}=\{V_1, \ldots, V_p\}$ contains exactly $A_{p,N}:=N(N-1)\ldots(N-p+1)$ elements. So we have

$$M_N=\sum_{p=1}^{r} A_{p,N} \sum_{\{V_1,\ldots,V_p\}\in\mathscr{P}(1,\ldots,r)} \hat{\varphi}(V_1,\ldots,V_p;N).$$

We claim that in the limit $N\to\infty$ only the admissible partitions survive and that these give the asserted expressions.

Let $\mathscr{V}=\{V_1,\ldots,V_p\}$ be a non-admissible partition. Then we have

$$\hat{\varphi}(\mathscr{V};N)=\hat{\varphi}(j_{i(1)}(b_N^{k(1)})\ldots j_{i(r)}(b_N^{k(r)})),$$

with $j_{i(1)}(b_N^{k(1)})\ldots j_{i(r)}(b_N^{k(r)})$ being a non-admissible product. According to Lemma 4 we can write $\hat{\varphi}(\mathscr{V};N)$ as a sum of products of moments of the $b_N^{k(s)}$ with respect to φ, where each product contains at least $p+1$ factors. Then $A_{p,N}\cdot\hat{\varphi}(\mathscr{V};N)$ goes to zero for $N\to\infty$ because of $A_{p,N}=N\ldots(N-p+1)$ and because of the assumption $Q(k(1),\ldots,k(r))=\lim_{N\to\infty} N\cdot\varphi(b_N^{k(1)}\ldots b_N^{k(r)})$. Thus

$$\lim_{N\to\infty} M_N=\lim_{N\to\infty}\sum_{p=1}^{n} A_{p,N}\sum_{\{V_1,\ldots,V_p\}\in\mathscr{P}_a(1,\ldots,r)}\hat{\varphi}(V_1,\ldots,V_p;N).$$

For an admissible partition $\mathscr{V}=\{V_1,\ldots,V_p\}$ the corresponding representative $j_{i(1)}(b_N^{k(1)})\ldots j_{i(r)}(b_N^{k(r)})$ is an admissible product. So we can calculate $\hat{\varphi}(V_1,\ldots,V_p;N)$ according to Lemma 2 as $\varphi(V_1;N)\ldots\varphi(V_p;N)$. The expression $A_{p,N}\cdot\varphi(V_1;N)\ldots\varphi(V_p;N)$ converges for $N\to\infty$ to $Q(V_1)\ldots Q(V_p)$. □

It is easy to see, that an analogous limit theorem is true in the tensor case; the only difference is that the sum over all admissible partitions is replaced by a sum over all partitions. The combinatorial part of the proof is the same and we can stop at

$$M_N=\sum_{p=1}^{r} A_{p,N}\sum_{\{V_1,\ldots,V_p\}\in\mathscr{P}(1,\ldots,r)}\hat{\varphi}(V_1,\ldots,V_p;N),$$

because in the tensor case $\hat{\varphi}(V_1,\ldots,V_p;N)$ is equal to $\varphi(V_1;N)\ldots\varphi(V_p;N)$ for all partitions.

The combinatorics of the distribution in the limit $N\to\infty$ does only depend on Q: $V\mapsto Q(V)$. Therefore it is justified to call Q the *generator* of the limit distribution.

Remark. For the relation between $\hat{\varphi}$ and Q we will write symbolically

$$\lim_{N\to\infty}\hat{\varphi}_{\{S_N\}}=\exp_* Q.$$

This is justified by the validity of the formal equation

$$(\exp_* Q_1)*(\exp_* Q_2)=\exp_*(Q_1+Q_2).$$

For an exact formulation of this statement we need an extended frame: Let $(\mathscr{B}, \varphi)$ be as before and $(\tilde{\mathscr{B}}, \tilde{\varphi})=(\mathscr{B}, \varphi)*(\mathscr{B}, \varphi)$ with the canonical embeddings i_1 and i_2. We take $(\tilde{\mathscr{B}}_i, \tilde{\varphi}_i)=(\tilde{\mathscr{B}}, \tilde{\varphi})$ for all i and $(\hat{\mathscr{B}}, \hat{\varphi})=*_{i=1}^{\infty}(\tilde{\mathscr{B}}_i, \tilde{\varphi}_i)$ with the canonical embeddings j_k: $\tilde{\mathscr{B}}_k \to \hat{\mathscr{B}}$. For each $N \in \mathbb{N}$ let again n elements $b_N^1, \ldots, b_N^n \in \mathscr{B}$ be given. Define

$$S_N^k := j_1(i_1(b_N^k)) + j_2(i_1(b_N^k)) + \ldots + j_N(i_1(b_N^k))$$
$$T_N^k := j_1(i_2(b_N^k)) + j_2(i_2(b_N^k)) + \ldots + j_N(i_2(b_N^k))$$

and

$$V_N^k := S_N^k + T_N^k.$$

Assume that

$$\lim_{N \to \infty} \hat{\varphi}_{\{S_N\}} = \exp_* Q_1 \quad \text{and} \quad \lim_{N \to \infty} \hat{\varphi}_{\{T_N\}} = \exp_* Q_2.$$

Then

$$(\exp_* Q_1)*(\exp_* Q_2) = \lim_{N \to \infty} \hat{\varphi}_{\{V_N\}} = \exp_*(Q_1 + Q_2).$$

The validity of the last equality sign is easily seen with the help of Lemma 3.

For $n=1$ the generator Q was also introduced by Voiculescu in a more abstract way. Our $Q(\underbrace{1, \ldots, 1}_{r \text{ times}})$ corresponds to $R_r(\hat{\varphi})$ in [Voi 1].

We will now specialize our limit theorem to analogues of a central limit theorem and a limit theorem for Poisson distribution.

Theorem 3 (free central limit theorem). *Consider n elements $b^1, \ldots, b^n \in \mathscr{B}$ with $\varphi(b^k)=0$ for all $k=1, \ldots, n$. Then we have for the sums*

$$S_N^k := \frac{j_1(b^k) + \cdots + j_N(b^k)}{\sqrt{N}}$$

for all $r \in \mathbb{N}$ and all $k(1), \ldots, k(r) \in \{1, \ldots, n\}$:

$$\lim_{N \to \infty} \hat{\varphi}(S_N^{k(1)} \ldots S_N^{k(r)})$$
$$= \begin{cases} 0, & r \text{ odd} \\ \displaystyle\sum_{\substack{\{(e_1, z_1), \ldots, (e_{r/2}, z_{r/2})\} \\ \in \mathscr{P}_\alpha(1, \ldots, r)}} \varphi(b^{k(e_1)} b^{k(z_1)}) \ldots \varphi(b^{k(e_{r/2})} b^{k(z_{r/2})}), & r \text{ even} \end{cases}$$

Proof. Consider $b_N^k := \frac{1}{\sqrt{N}} \cdot b^k$. Then

$$\lim_{N\to\infty} N\cdot\varphi(b_N^k)=0$$

$$\lim_{N\to\infty} N\cdot\varphi(b_N^{k(1)}\,b_N^{k(2)})=\varphi(b^{k(1)}b^{k(2)})$$

$$\lim_{N\to\infty} N\cdot\varphi(b_N^{k(1)}\dots b_N^{k(r)})=0 \qquad (r\geqq 3). \quad \square$$

The analogous situation for the tensor case was treated by Giri and von Waldenfels [GvW], compare also [CuH, CoH, Heg]. We want to show the combinatorial difference between the tensor case and the free case by an example: The moment of $S_N^{k(1)}\,S_N^{k(2)}\,S_N^{k(3)}\,S_N^{k(4)}$ in the limit $N\to\infty$ in the free case is given by

$$m_V=Q(i(1),i(2))\,Q(i(3),i(4))+Q(i(1),i(4))\,Q(i(2),i(3))$$

(corresponding to the admissible partitions $\{(1,2),(3,4)\}$ and $\{(1,4),(2,3)\}$), whereas in the tensor case it is given by

$$m_{GvW}=m_V+Q(i(1),i(3))\,Q(i(2),i(4))$$

(with the additional non-admissible partition $\{(1,3),(2,4)\}$).

For $n=1$ and $b=b^1$ selfadjoint the tensor case reduces to the classical central limit theorem in the following weak form: the moments of S_N^1 converge to the corresponding moments of the Gaussian distribution.

For $n=1$ and $b=b^1$ selfadjoint the 'free Gaussian' case was treated by Voiculescu [Voi 1]. It is easy to see that the moments do belong to 'Wigners semicircle distribution' $\mu=f\lambda$ with density

$$f(x)=\frac{2}{\pi\sigma}\sqrt{1-(x/\sigma)^2}\,\chi_{(-\sigma,\sigma)}(x).$$

It may be interesting to note that this Wigner distribution appears also as the normed distribution of the eigenvalues of large symmetric random matrices (compare e.g., [Wig]; see also [Voi 2]).

The moments of the Wigner distribution can be easily calculated explicitly:

$$E[X^n]=\begin{cases}0, & n \text{ odd}\\ \dfrac{1}{k+1}\dbinom{2k}{k}\sigma^n=\dfrac{2^k}{(k+1)!}\,1\cdot 3\cdot 5\cdot 7\dots(n-1)\cdot\sigma^n, & n=2k\end{cases}$$

For comparison: The moments of the classical Gaussian are given by

$$E[X^n]=\begin{cases}0, & n \text{ odd}\\ 1\cdot 3\cdot 5\cdot 7\dots(n-1)\cdot\sigma^n, & n \text{ even}\end{cases}$$

Theorem 4 (limit theorem for the free Poisson distribution). *Consider* $b_N \in \mathscr{B}\,(N \in \mathbb{N})$ *with*

$$\lim_{N\to\infty} N\cdot\varphi(\underbrace{b_N\dots b_N}_{r\text{ times}})=\alpha$$

independent of r for all $r \in \mathbb{N}$. Then we have for the sums

$$S_N := j_1(b_N)+\cdots+j_N(b_N)$$

for all r:

$$\lim_{N\to\infty}\hat{\varphi}(\underbrace{S_N\dots S_N}_{r\text{ times}})=\sum_{p=1}^{r}\ \sum_{\{V_1,\dots,V_p\}\in\mathscr{P}_a(1,\dots,r)}\alpha^p.$$

Proof. We have for all $r \in \mathbb{N}$:

$$Q(\underbrace{1,\dots,1}_{r\text{ times}})=\alpha. \quad \square$$

In the tensor case the analogous theorem gives a limit theorem for the Poisson distribution. Because of this analogy we call a distribution with the above combinatorics a free Poisson distribution. For illustration we compare the first few moments of the Poisson and the free Poisson distribution:

Classical Poisson	Free Poisson
α	α
$\alpha^2+\alpha$	$\alpha^2+\alpha$
$\alpha^3+3\alpha^2+\alpha$	$\alpha^3+3\alpha^2+\alpha$
$\alpha^4+6\alpha^3+7\alpha^2+\alpha$	$\alpha^4+6\alpha^3+6\alpha^2+\alpha$
$\alpha^5+10\alpha^4+25\alpha^3+15\alpha^2+\alpha$	$\alpha^5+10\alpha^4+20\alpha^3+10\alpha^2+\alpha$
$\alpha^6+15\alpha^5+65\alpha^4+90\alpha^3+31\alpha^2+\alpha$	$\alpha^6+15\alpha^5+50\alpha^4+50\alpha^3+15\alpha^2+\alpha$

Remark. Let X_σ be a random variable distributed according the Wigner distribution with variance σ^2 and Y_α a random variable distributed according a free Poisson distribution with expectation α. Then we have $E[Y_1^r]=E[X_1^{2r}]$. This allows the calculation of the free Poisson distribution ν_1 of Y_1:

$$\nu_1=g\cdot\lambda \quad \text{with density} \quad g(x)=\frac{1}{2\pi}\sqrt{4/x-1}\,\chi_{(0,4)}(x).$$

For $\alpha\neq 1$ the calculation is more tedious. It was done by Bozejko and Leinert [BoL] and the measure has the form

$$\nu_\alpha=a_\alpha\cdot\delta_0+\bar{\nu}_\alpha,$$

where $\bar{\nu}_\alpha$ is absolutely continuous with respect to the Lebesgue-measure λ. The same measure ν_α also appears as the distribution of the eigenvalues of the square of rectangular matrices (compare, e.g., [Voi 2], Sect. 3.6.).

We will now examine the combinatorics of free white noises.

Theorem 5. *Let $(\mathscr{C}, \rho, (c_I^1, \ldots, c_I^n)_{I\in\mathfrak{R}})$ be a n-dimensional free white noise and define $c_t^k := c_{[0,t)}^k$ for all $k=1, \ldots, n$. Assume that*

$$\lim_{t\to 0} \rho(c_t^{k(1)} \ldots c_t^{k(r)}) \to 0$$

for all $r\in\mathbb{N}$ and all $k(1), \ldots, k(r)\in\{1, \ldots, n\}$. Then there exist generators Q_t, such that for all $r\in\mathbb{N}$ and all $k(1), \ldots, k(r)\in\{1, \ldots, n\}$ we have:

$$\rho(c_t^{k(1)} \ldots c_t^{k(r)}) = \sum_{p=1}^{r} \sum_{\{V_1, \ldots, V_p\}\in\mathscr{P}_a(1, \ldots, r)} Q_t(V_1) \ldots Q_t(V_p),$$

where the sum runs over all admissible partitions of the set $\{1, \ldots, r\}$ and $Q_t(V)$ denotes $Q_t(k(v_1), \ldots, k(v_m))$ if $V=(v_1, \ldots, v_m)$. Furthermore Q_t is linear in t, i.e.

$$Q_t = t\cdot Q_1.$$

Thus the combinatorics of the free white noise is completely described by the generator Q_1.

Proof. We first note that the continuity of all moments at $t_0=0$ implies the continuity at all t_0.

We will now show that c_t^k can be identified for all $N\in\mathbb{N}$ with a sum S_N^k of our limit theorem: Write

$$c_t^k = \sum_{l=0}^{N-1} c_{I_l}^k \quad \text{with} \quad I_l = [l\cdot t/N, (l+1)\cdot t/N)$$

and note, that $c_{I_l}^k$ and $c_{I_m}^k$ have the same distribution and are freely independent for $l\neq m$, i.e. $c_{I_l}^k$ can be identified with $j_l(c_{t/N}^k)$ in the notation of Theorem 2 (with $\mathscr{B} = C^*(c_{t/N}^k \mid k=1, \ldots, n)$).

Therefore the first assertion follows, if we show the existence of the limit

$$Q_t(k(1), \ldots, k(r)) := \lim_{N\to\infty} N\cdot\rho(c_{t/N}^{k(1)} \ldots c_{t/N}^{k(r)}).$$

This is done by induction on r.

$r=1$: We have for all N

$$\rho(c_t^{k(1)}) = \rho\left(\sum_{l=0}^{N-1} c_{[l\cdot t/N, (l+1)\cdot t/N)}^{k(1)}\right) = N\cdot\rho(c_{t/N}^{k(1)}).$$

Thus $Q_t(k(1)) = \rho(c_t^{k(1)})$.

$r>1$: We will only treat an example, because this saves us a lot of indices and illustrates the procedure sufficiently. Let $r=2$.

$$\rho(c_t^{k(1)}\, c_t^{k(2)}) = \rho\left(\sum_{l=0}^{N-1} c_{[l\cdot t/N,\,(l+1)\cdot t/N)}^{k(1)} \cdot \sum_{l'=0}^{N-1} c_{[l'\cdot t/N,\,(l'+1)\cdot t/N)}^{k(2)}\right)$$
$$= N(N-1)\,\rho(c_{t/N}^{k(1)})\,\rho(c_{t/N}^{k(2)}) + N\,\rho(c_{t/N}^{k(1)}\, c_{t/N}^{k(2)}).$$

Taking the limit $N\to\infty$ and regarding our knowledge about ρ for less than r arguments we get:

$$Q_t(k(1), k(2)) = \rho(c_t^{k(1)}\, c_t^{k(2)}) - Q_t(k(1))\, Q_t(k(2)).$$

From the remark after our limit theorem we get

$$Q_{t+s} = Q_t + Q_s.$$

This gives the linearity in t, because the continuity of all moments implies also the continuity of $t\to Q_t$. $\square$

Remark. According to our earlier remark we can write

$$\rho_{\{c_t\}} = \exp_*(tQ_1).$$

The classifying of all free white noises or equivalently of all processes with stationary and freely independent increments is thus reduced to the classifying of all generators Q_1. It remains to decide, which conditions Q_1 has to fulfil in order to be a possible generator, especially under which conditions the functional $\exp_*(tQ_1)$ is positive for all t. This will be done in a forthcoming publication [GSS].

Like in the tensor case, we will now look for 'simple' explicit models of the free Gaussian and the free Poisson distribution, i.e. we look for operators $c^1, \ldots, c^n$ and a state on the operator algebra generated by these operators, which gives the combinatorics of Theorems 3 and 4. Furthermore, having processes in mind, we will look for such operators $c_I^1, \ldots, c_I^n$ for each $I\in\mathfrak{R}$ such that $c_{I_1}^k$ and $c_{I_2}^l$ are freely independent for $I_1\cap I_2=\emptyset$. All these demands can be fulfilled in the full Fock space which we present in the next section.

To recall the corresponding situation for the tensor case: The combinatorics of the tensor case is given by creation and annihilation operator (Gaussian distribution) and by $P_t = \Lambda_t + a_t + a_t^+ + t\cdot\mathrm{id}$ (Poisson distribution) in the symmetric Fock space $\mathscr{F}_s(L^2(\mathbb{R}))$ (compare [HuP]). Furthermore, operator algebras to disjoint time intervals are built together as tensor products, i.e. they are independent in the classical sense. We will see that we have the analogous results for the free independence if we replace all operators by their counterparts on the full Fock space.

5. The Full Fock Space and the Cuntz Algebra

Let $\mathscr{H}_0$ be a Hilbert space. Then the non-symmetrized or full Fock space of $\mathscr{H}_0$ is the Hilbert space

$$\mathscr{F}(\mathscr{H}_0) := \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} \mathscr{H}_0^{\otimes n}$$

with scalar product ($f_i, g_i \in \mathscr{H}_0$)

$$\langle f_1 \otimes \ldots \otimes f_n, g_1 \otimes \ldots \otimes g_m \rangle = \delta_{nm} \langle f_1, g_1 \rangle \cdots \langle f_n, g_n \rangle$$
$$\langle \Omega, \Omega \rangle = 1.$$

The vacuum expectation state is given by

$$\rho\colon B(\mathscr{F}) \to \mathbb{C}$$
$$X \mapsto \langle \Omega, X\Omega \rangle.$$

For each $f \in \mathscr{H}_0$ we define the left annihilation operator $l(f)$ and the left creation operator $l^*(f)$ by

$$l(f) f_1 \otimes \ldots \otimes f_n = \langle f, f_1 \rangle f_2 \otimes \ldots \otimes f_n$$
$$l^*(f) f_1 \otimes \ldots \otimes f_n = f \otimes f_1 \otimes \ldots \otimes f_n.$$

The operators $l(f)$ and $l^*(f)$ are bounded and mutually adjoint. Furthermore

$$\|l(f)\| = \|l^*(f)\| = \|f\|_{\mathscr{H}_0}.$$

We now take $\mathscr{H}_0 = L^2(I)$ for $I \in \mathfrak{R}$ and define

$$O(I) := C^*(l(f) \mid f \in L^2(I)) \subset B(\mathscr{F})$$

as the C^*-algebra generated by all annihilation operators adapted to I. Then $O(I)$ is as a C^*-algebra isomorphic to the Cuntz algebra O_∞ [Cun, Eva]. It is characterized by the relations

$$l(f)\, l^*(g) = \langle f, g \rangle 1 \qquad \text{for } f, g \in L^2(I)$$

and

$$\sum_{i=1}^{\infty} l^*(e_i)\, l(e_i) < 1 \qquad \text{for each CONS } \{e_i\} \text{ in } L^2(I).$$

Furthermore we define the analogues $p(T)$ of the gauge operators for $T \in B(L^2(\mathbb{R}))$ by

$$p(T)\, \Omega = 0$$
$$p(T)(f_1 \otimes f_2 \otimes \ldots \otimes f_n) = (Tf_1) \otimes f_2 \otimes \ldots \otimes f_n.$$

They are bounded: $\|p(T)\|_{B(\mathscr{F})} = \|T\|_{B(L^2(\mathbb{R}))}$. We will only need $p(h) := p(T_h)$ for multiplication operators T_h with $h \in L^\infty(\mathbb{R})$: $T_h(f) = hf$. Thus we define

$$\hat{O}(I) := C^*(p(h), l(f) \mid f \in L^2(I), h \in L^\infty(I)).$$

We will work on $O(\mathbb{R})$ and $\hat{O}(\mathbb{R})$. The subalgebras $O(I)$ and $\hat{O}(I)$ may be regarded as a filtration.

That these operator algebras are the right objects for our studies is justified by the statement, that $O(I_1)$ and $O(I_2)$ are freely independent with respect to the vacuum expectation for $I_1 \cap I_2 = \emptyset$: Let $\rho_{O(I)}$ be the vacuum expectation restricted to $O(I)$. Then we have for disjoint I_1, I_2:

$$(O(I_1 \cup I_2), \rho_{O(I_1 \cup I_2)}) = (O(I_1), \rho_{O(I_1)}) * (O(I_2), \rho_{O(I_2)})$$

and the same for $\hat{O}$ (compare [Voi 1]).

Therefore processes which are defined as linear combinations of the basic processes $l_t := l(\chi_{(0,t)})$, $l_t^* := l^*(\chi_{(0,t)})$ and $p_t := p(\chi_{(0,t)})$ have freely independent and stationary increments. Thus their increments are free white noises and have the combinatorics of Theorem 5, i.e. they are determined by a generator Q_1.

We now try to identify free Gaussian noise and free Poisson noise.

Theorem 6. *Let $c_t := l(\chi_{(0,t)})$. Then c_t and c_t^* have with respect to the vacuum expectation ρ the combinatorics of Theorem 3, i.e. (with $c_t^1 = c_t$ and $c_t^2 = c_t^*$)*

$$\rho(c_t^{k(1)} \dots c_t^{k(r)}) = \begin{cases} 0, & r \text{ odd} \\ \sum\limits_{\{V_1, \dots, V_{r/2}\} \in \mathscr{P}_a(1, \dots, r)} Q_t(V_1) \dots Q_t(V_{r/2}), & r \text{ even.} \end{cases}$$

The matrix Q_t is given by

$$Q_t = \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}.$$

Proof. According to Theorem 5 the combinatorics of c_t, c_t^* is given by a generator Q_t because $(O(\mathbb{R}), \rho, (l(\chi_I), l^*(\chi_I))_{I \in \mathfrak{R}})$ is a free white noise. We only have to calculate

$$Q_t(k(1), \dots, k(r)) = \lim_{N \to \infty} N \cdot \rho(c_{t/N}^{k(1)} \dots c_{t/N}^{k(r)}).$$

It is easy to see that the limit is only different from zero for $r = 2$ and that it gives for $r = 2$ the asserted matrix. □

Remark. It is also not very hard to see the following more general assertion: Let $c := l(f) + l^*(g)$. Then c and c^* have with respect to the vacuum expectation ρ the combinatorics of Theorem 3, i.e. (with $c^1 = c$ and $c^2 = c^*$)

$$\rho(c^{k(1)} \dots c^{k(r)}) = \begin{cases} 0, & r \text{ odd} \\ \sum\limits_{\{V_1, \dots, V_{r/2}\} \in \mathscr{P}_a(1, \dots, r)} Q(V_1) \dots Q(V_{r/2}), & r \text{ even.} \end{cases}$$

The matrix Q is given by

$$Q = \begin{pmatrix} \langle f, g \rangle & \langle f, f \rangle \\ \langle g, g \rangle & \langle g, f \rangle \end{pmatrix}.$$

Example.

$$\begin{aligned}\rho(ccc^*c) &= \langle \Omega, (l(f)+l^*(g))\,(l(f)+l^*(g))\,(l^*(f)+l(g))\,(l(f)+l^*(g))\,\Omega\rangle \\ &= \langle \Omega, l(f)\,l^*(g)\,l(g)\,l^*(g)\,\Omega\rangle + \langle \Omega, l(f)\,l(f)\,l^*(f)\,l^*(g)\,\Omega\rangle \\ &= \langle f,g\rangle\langle g,g\rangle + \langle f,f\rangle\langle f,g\rangle \\ &= \rho(cc)\,\rho(c^*c) + \rho(cc^*)\,\rho(cc).\end{aligned}$$

Theorem 7. *Let $c_t := p_t + l_t + l_t^* + t\cdot\mathrm{id}$. Then c_t has with respect to the vacuum expectation ρ the combinatorics of Theorem 4, i.e.*

$$\rho(\underbrace{c_t \dots c_t}_{r\text{ times}}) = \sum_{p=1}^{r} \sum_{\{V_1,\dots,V_p\}\in\mathscr{P}_a(1,\dots,r)} t^p.$$

Proof. We already know the existence of Q_t according to Theorem 5 because $(\hat{O}(\mathbb{R}), \rho, (p(\chi_I)+l(\chi_I)+l^*(\chi_I)+\lambda(I)\cdot\mathrm{id})_{I\in\mathfrak{R}})$ is a free white noise. It is easy to see that

$$Q_t(\underbrace{1,\dots,1}_{r\text{ times}}) = \lim_{N\to\infty} N\cdot\rho(\underbrace{c_{t/N}\dots c_{t/N}}_{r\text{ times}}) = t$$

for all $r\in\mathbb{N}$. $\square$

These two theorems show that the processes l_t, l_t^* and $p_t+l_t+l_t^*+t\cdot\mathrm{id}$ are the free analogues of the corresponding processes a_t, a_t^+ and $\Lambda_t+a_t+a_t^+ + t\cdot\mathrm{id}$ on the symmetric Fock space. Especially, $l_t+l_t^*$ is a model for the free analogue of the Wiener process and $p_t+l_t+l_t^*+t\cdot\mathrm{id}$ is a model for the free Poisson process.

In [GSS] we will show that all free white noises can be modelized as a sum of creation, annihilation and gauge processes on a more general version of the full Fock space. This is analogue to the situation for the tensor case (compare [Sch]).

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