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$\mathcal{L}^2$-perturbations of periodic equilibria of reaction diffusion systems

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I. Introduction

In this paper we consider reaction diffusion systems $(u_j)_t = \tau_j \triangle u_j + P_j(u_1, \ldots, u_n)$ where P_j is a polynomial and where $u_j = u_j(x_1, \ldots, x_m)$ are functions on $\mathbf{R}^m$. We assume that a smooth equilibrium solution $v = (v_1, \ldots, v_n)$ is given, which is L_j-periodic with respect to x_j ($j = 1, \ldots, m$), ie. such that $\tau_j \triangle v_j + P_j(v_1, \ldots, v_n) = 0$. We then investigate the stability of this equilibrium, not with respect to periodic perturbations with the same period but with respect to smooth perturbations $\varphi \in \mathcal{L}^2(\mathbf{R}^m)$. A motivation to study this problem comes from remarks in D.H.Sattinger [18], pg.182 and [19], pg.803 where it is suggested to investigate a Hill-type theory for elliptic operators with doubly periodic coefficients. In fact, usual stability investigations of periodic equilibria give only stability results with respect to perturbations in the same periodicity class. This question does not seem to have found much attention so far. In this paper, a more restricted problem is discussed, namely that of smooth $\mathcal{L}^2$-perturbations of the periodic equilibrium. However, even this restricted form of the above problem has not been considered up to now, as far as our knowledge goes. It has to be stressed that the candidates for a Hill-type analysis in [18], [19] are the well known problems of fluid mechanics which admit periodic cell type equilibrium patterns (Bénard-, Taylor problem) rather than reaction diffusion systems. The reason for concentrating on the simpler reaction diffusion systems is that we wanted to avoid in a first step a mixture of two different types of complications: those stemming from fluid mechanics (eliminations of pressure via Stokes operator etc.) and those arising in connection with a Hill type theory (direct integrals of operator families, spectral questions, etc.).

Next we come to explain the more technical side of the paper. To this end set $D = (\delta_{ij}\tau_j)$ $(i, j = 1, \ldots, n)$ and $F(u) = (P_1(u), \ldots, P_n(u))$ (where $u = (u_1, \ldots, u_n)$). Let $v = (v_1, \ldots, v_n)$ be a smooth $L_1, \ldots, L_m$-periodic equilibrium solution of the elliptic system

$$D\triangle u + F(u) = 0 \tag{1.1}$$

In order to discuss the stability of v with respect to smooth $\mathcal{L}^2$-perturbations we are led after some preliminary investigations to study the spectrum of $D\triangle + d_u F(v)$ as an unbounded operator on $(\mathcal{L}^2(\mathbf{R}^m))^n$; let $\sigma_{\mathcal{L}^2}$ denote this spectrum. Following the lines of reasoning in M.Reed-B.Simon [16], R.Eastham [6] one proceeds by an intermediate technical step which amounts to study a family of auxiliary boundary value problems

$$D\triangle w + d_u F(v) w = 0 \tag{1.2}$$

with w subject to a Floquet (or θ-) periodic boundary condition

$$w(x_1, \ldots, x_j + L_j, \ldots, x_m) = e^{i\theta_j} w(x_1, \ldots, x_m) \qquad (j = 1, \ldots, m) \tag{1.3}$$

for every $\theta = (\theta_1, \ldots, \theta_m) \in [0, 2\pi]^m$. By considering this θ-periodic boundary value problem on an appropriate function space setting, there is a spectrum σ_θ associated with $D\triangle + d_u F(v)$. The fundamental relationship between $\sigma_{\mathcal{L}^2}$ and the spectra σ_θ, $\theta \in [0, 2\pi]^m$ is described by

$$\sigma_{\mathcal{L}^2} = \bigcup_\theta \sigma_\theta, \qquad \theta \in [0, 2\pi]^m \tag{1.4}$$

In the particular case when F is a gradient, ie. $P_j = \frac{\partial Q}{\partial u_j}$ for some Q, we are in the selfadjoint case, whence (1.4) follows from the results in [16], Thm. XIII, 85. In case of arbitrary reaction diffusion systems however, F is in general not a gradient. As a consequence, the proof of Thm. XIII, 85 in [16] no longer works, since it makes essential use of the spectral decomposition of selfadjoint operators, a tool not available in the nongradient case. This forces us to prove (1.4) by alternative methods. In fact, the presentation of the proof of (1.4) for arbitrary polynomials F constitutes the major part of the paper.

A remark concerning the stability question has to be added. For a stability analysis it does not suffice to have a qualititative description of the spectrum. One must also have principles of linearized stability and instability at disposal. Now while principles of linearized stability are relatively easy to prove (at least in the case of parabolic equations) the principle of linearized instability is considerably more delicate: (*) to prove that if a $\lambda \in \sigma_{\mathcal{L}^2}$ with $\mathrm{Re}(\lambda) > 0$ exists, then the equilibrium solution under consideration is indeed Ljapounov unstable with respect to smooth $\mathcal{L}^2$-perturbations. The principle (*) has been proved in other contexts under various assumptions on the spectrum (see Kielhöfer [10], [11], D.Henry [8] chapter 5 for the case where the part of the spectrum with $\mathrm{Re}(\lambda) > 0$ is a compact spectral set and [8] for the arbitrary selfadjoint case).

However, these assumptions are in general not valid in the present case. Nevertheless it can be shown that the principle (*) applies without any restriction to the evolution equations considered here. The proof of $(*)$ is quite delicate and lenghty; for reasons of space it will be presented seperately in a subsequent paper. In this paper we tacitly assume $(*)$ to be valid.

The paper is organized as follows. In section (II) the preliminaries are fixed and a precise formulation of the above problem is given. In (III) the basic facts about direct integrals needed here are summarized, in (IV), (V) the main sections, formula (1.4) is proved. In (VI), the stability-instability problem is briefly reconsidered, while in (VII) some applications are given which show that $\mathcal{L}^2$-stability and periodic stability may be two different things. To avoid interruption the arguments we have relegated a few technical points into the appendix.

II. Notations and preliminaries

(A) First we fix some notation. **R**, **C** denote the real and complex numbers resp.. For $\mathcal{X}$, $\mathcal{Y}$ Banachspaces, $\| \ \|_{\mathcal{X}}$, $\| \ \|_{\mathcal{Y}}$ denote their respective norms and for $U \subseteq \mathcal{X}$ an open set, $C^p(U, \mathcal{Y})$ is the set of p times continuously differentiable mappings from U to $\mathcal{Y}$. For $F \in C^1(U, \mathcal{Y})$, $dF(u)$ is the derivative of F at $u \in U$. If the underlying space $\mathcal{X}$ is clear from the context we write $\| \ \|$ instead of $\| \ \|_{\mathcal{X}}$. $L(\mathcal{X}, \mathcal{Y})$ is the set of bounded linear operators T from $\mathcal{X}$ to $\mathcal{Y}$, and $\|T\|_\infty$ or even $\|T\|$ denotes the usual operator norm. For any Ω, $H^p(\Omega)$ is the Sobolev space of functions having square integrable derivatives up to order p. For any multiindex $\alpha = (\alpha_1, \ldots, \alpha_m)$ we set $|\alpha| = \sum \alpha_j$ and $D^\alpha = \partial_1^{\alpha_1} \ldots \partial_m^{\alpha_m}$ where ∂_j is the derivative with respect to x_j. Finally, $(\ , \ )_p$ is the scalar product in $H^p(\Omega)$ given by

$$(u, v)_p = \sum_{|\alpha| \leq p} (D^\alpha u, D^\alpha v)_0$$

where $(u, v)_0 = \int\limits_\Omega u(x)\bar{v}(x)dx$. We set $\mathcal{L}^2(\Omega) = H^0(\Omega)$ and write $\| \ \|_{H^p}$ instead of $\| \ \|_{H^p(\Omega)}$ if no danger of confusion arises. Henceforth we impose a restriction on the dimension m of space, ie. we assume

$$m \leq 3. \tag{2.1}$$

The reason for this restriction is that $H^p(\Omega)$ now becomes a Banach algebra if $p \geq 2$ (R.Adams) [1], pg. 115).

(B) As seen in the introduction we have to consider functions which satisfy Floquet-type periodicity conditions. This forces us to introduce suitable Sobolev spaces of such functions. To this end we fix periods $L_1, \ldots, L_m > 0$, set $Q_L = \prod\limits_j (0, L_j)$ and let $T_p(Q_L)$ be the set of finite trigonometric polynomials

$$t(x) = \sum a_k e^{i2\pi kx/L} \tag{2.2}$$

where $kx/L = \sum k_j x_j L_j^{-1}$ and $k = (k_1, \ldots, k_m) \in \mathbf{Z}^m$.

By $H^p_{per}(Q_L)$ we denote the closure of $T_p(Q_L)$ with respect to $\|\ \|_{H^p}$, $H^p = H^p(Q_L)$. Every $f \in H^p_{per}(Q_L)$ has a unique extension f', defined on $\mathbf{R}^m$, which is $L_1, \ldots, L_m$-periodic and which coincides with f on Q_L; henceforth we identify f with f'. $H^p_{per}(Q_L)$ is also obtained if we replace $T_p(Q_L)$ by the set $C^p_{per}(Q_L)$ which have continous derivatives up to order p on all of $\mathbf{R}^m$ and which are $L_1, \ldots, L_m$ periodic. Clearly $H^0_{per}(Q_L) = \mathcal{L}^2(Q_L)$. Next fix $\theta = (\theta_1, \ldots, \theta_m) \in [0, 2\pi]^m$. Let $H^p_\theta(Q_L)$ be the closure of the set of trigonometric polynomials of the form

$$\mathrm{e}^{i\theta x/L} t(x), \qquad t(x) \in T_p(Q_L) \tag{2.3}$$

with respect to $\|\ \|_{H^p}$; here $\theta x/L = \sum x_j \theta_j L_j^{-1}$. Every $f \in H^p_\theta(Q_L)$ has a unique extension f', defined on $\mathbf{R}^m$, which coincides with f on Q_L and which satisfies $f'(x_1, \ldots, x_j + L_j, \ldots, x_m) = \mathrm{e}^{i\theta_j} f(x_1, \ldots, x_m)$; henceforth we identify f' with f. The following is easy to show:

$$f \in H^p_\theta(Q_L) \quad \text{iff} \quad \mathrm{e}^{-i\theta x/L} f \in H^p_{per}(Q_L). \tag{2.4}$$

For notational simplicity we assume henceforth $L_1 = \ldots = L_m = L$; however all arguments below carry over in a verbatim way to the case of arbitrary $L_1, \ldots, L_m$.

(C) We now put the problem into precise form. Set $D = (\delta_{ij}, \tau_j)$, $(i, j = 1, \ldots, n$ $\tau_j > 0)$ and $F(u) = (P_1(u), \ldots, P_n(u))$, $(u = (u_1, \ldots, u_n))$. We assume that for some fixed $L > 0$ there is an equilibrium solution $v = (v_1, \ldots, v_n) \in (H^4_{per}(Q_L))^n$ of the parabolic system

$$u_t = D\triangle u + F(u) \tag{2.5}$$

By setting $A = D\triangle$ on $\mathrm{dom}(A) = (H^4_{per}(Q_L))^n$, A is selfadjoint if considered on the Hilbert space $((H^2_{per}(Q_L))^n$ endowed with the scalar product $\langle u, w\rangle_2 = \sum (u_j, w_j)_2$. On the other hand F is polynomial and $H^2_{per}(Q_L)$ a Banachalgebra whence it follows that F is a smooth mapping of $((H^2_{per}(Q_L))^n$ into itself. In this setting, and since $\sigma(A) \subseteq (-\infty, 0]$, (2.5) becomes a semilinear evolution equation in the sense of Pazy [13] having v as equilibrium solution. Equilibrium solutions of (2.5) are usually obtained by bifurcation techniques, if a suitable bifurcation parameter is present (see eg. C.Alexander-G.Auchmuty [2] or [21] for such considerations). Stability is then investigated in $(H^2_{per}(Q_L))^n$ or even in narrow subspaces thereof, exhibiting strong symmetry properties. What is essential in this connection is knowledge of the spectrum $\sigma(A)$, to be denoted for the moment by σ^2_{per}.

In contrast to this program we study "smooth" $\mathcal{L}^2$-perturbations of v, ie. we consider $v \in (H^4(Q_L))^n$ as an equilibrium of (2.5), defined on all of $\mathbf{R}^m$, and investigate the evolution equation

$$(v + \varphi)_t = D\triangle(v + \varphi) + F(v + \varphi) \tag{2.6}$$

where $\varphi = (\varphi_1, \ldots, \varphi_m) \in (H^2(\mathbf{R}^m))^n$. From (2.6) we get

$$\varphi_t = D\triangle\varphi + dF(v)\varphi + R(\varphi) \tag{2.7}$$

where $R(\varphi)$ is polynomial in φ_j, $j \le n$, with coefficients which are polynomial in v_j, $j \le n$. Since $v_j \in H^4_{per}(Q_L)$ it follows from [1], pg. 98 that $v_j \in C^2(\mathbf{R}^m, \mathbf{R})$ with all derivatives uniformly bounded. This implies that (2.7) is a semilinear evolution equation, now on $(H^2(\mathbf{R}^m))^n$ as basic Hilbert space. Our aim is to investigate the stability of the trivial solution $\varphi = 0$. Since $R(\varphi)$ starts with quadratic terms in φ_j, $j \le n$ we have to study the spectrum of $D\triangle + dF(v)$, now with domain $(H^4(\mathbf{R}^m))^n$ and $(H^2(\mathbf{R}^m))^n$ as underlying space; let $\sigma^2_{\mathcal{L}^2}$ denote this spectrum. To interprete "smooth" $\mathcal{L}^2$-perturbations as membership in $(H^2(\mathbf{R}^m))^n$ seems somewhat artificial. However, by the polynomial character of $R(\varphi)$ we have to exploit the Banach algebra property of $H^2(\mathbf{R}^m)$. In addition it is shown in [23], sect. V that evolution equations like (2.7) have a very natural interpretation in terms of some function spaces of continuous functions; we refer to [23] for details. The main question now is that of the relationship between the stability of v in $(H^2_{per}(Q_L))^n$ and the stability of v in $(H^2(\mathbf{R}^m))^n$. One of our main results is

$$\sigma^2_{per} \subseteq \sigma^2_{\mathcal{L}^2} \tag{2.8}$$

As to be explained in section V, $\sigma^2_{\mathcal{L}^2} \cap \{Re(\lambda) > 0\} \ne \emptyset$ indeed implies Ljapounov instability of the equilibrium solution $\varphi = 0$ of (2.7). Thus if $\sigma^2_{per} \cap \{Re(\lambda) > 0\} \ne \emptyset$, then v is unstable against smooth $\mathcal{L}^2$-perturbations, or more imprecisely, periodic instability implies $\mathcal{L}^2$-instability. This immediately leads to the question, whether there are equilibrium solutions $v \in (H^2_{per}(Q_L))^n$ of (2.5) which are periodically stable but $\mathcal{L}^2$-unstable. Examples, showing exactly this behaviour will be given in section VI.

III. Direct integrals

(A) We come to the more technical part, aiming among others at a proof of (2.8). This forces us to proceed in a first step along the lines of chapter XIII in [16]. First we need the notion of direct integral. To this end set $\mathcal{H}' = \mathcal{L}^2(Q_L)$ and $M = [0, 2\pi]^m$; μ is Lebesgue measure on M. The direct integral $\mathcal{H} = \int_M \mathcal{H}' d\mu$ is a Hilbert space, whose elements are the measurable mappings Φ from M into $\mathcal{H}'$, ([15], pg.115) which are defined for ae. $\theta \in M$ and which satisfy

$$\int_M \|\Phi(\theta)\|^2_{\mathcal{H}'} d\mu < \infty \tag{3.1}$$

and whose scalar product $\langle\ ,\ \rangle_{\mathcal{H}}$ is given by

$$\langle \Phi, \psi \rangle_{\mathcal{H}} = \int_M \langle \Phi(\theta), \psi(\theta) \rangle_{\mathcal{H}'} d\mu, \tag{3.2}$$

with $\langle\ ,\ \rangle_{\mathcal{H}'}$ the scalar product on $\mathcal{H}'$. Due to Fubini we can identify $\mathcal{H}$ with $\mathcal{L}^2(M \times Q_L)$. The image of $\psi \in \mathcal{H}$ for ae. $\theta \in M$ is $\psi(\theta, x)$, which, as a function of

x is a member of $\mathcal{L}^2(Q_L)$. Following [16] one constructs a unitary mapping V from $\mathcal{L}^2(\mathbf{R}^m)$ onto $\mathcal{H}$ which is described as follows. With $f \in C_0^\infty(\mathbf{R}^m)$ one associates the trigonometric polynomials

$$T_N(\theta, x) = (2\pi)^{-\frac{m}{2}} \sum_{|n_k| \leq N} \mathrm{e}^{-in\theta} f(x_1 + n_1 L, \ldots, x_m + n_m L) \tag{3.3}$$

where $n\theta = \sum n_j \theta_j$, $x = (x_1, \ldots, x_m) \in Q_L$. Computation yields

$$\int\int_{M \times Q_L} |T_N|^2 dx d\theta = \int\int_{Q_L(N)} |f|^2 \tag{3.4}$$

where $Q_L(N) = (-(N+1)L, (N+1)L)^m$. The sequence $\{T_N\}$ is easily recognized as a Cauchy sequence in $\mathcal{H}$, converging against an element $\psi_f \in \mathcal{H}$, which by virtue of (3.4) satisfies

$$\int\int_{M \times Q_L} |\psi_f|^2 dx d\theta = \int_{\mathbf{R}^m} |f(x)|^2 dx \tag{3.5}$$

The mapping $f \to \psi_f$ is linear and can, by virtue of (3.5), be extended into an isometry V from $\mathcal{L}^2(\mathbf{R}^m)$ into $\mathcal{H}$. That V is onto can be seen via the Fourier-expansion of $\psi \in \mathcal{H}$, ie.

$$\psi = \sum \mathrm{e}^{-in\theta} a_n(x), \qquad \text{ae.} \quad x \in Q_L, \tag{3.6}$$

where $n \in \mathbf{Z}^m$, $\theta \in M$. By setting $f(x) = a_n(x)$ for $x \in \Pi_j(n_j L, (n_j + 1)L)$ we have that $Vf = \psi_f$. Thus V is given by

$$Vf = \psi_f \tag{3.7}$$

is a unitary mapping from $\mathcal{L}^2(\mathbf{R}^m)$ onto $\mathcal{H} = \mathcal{L}^2(M \times Q_L)$. The next lemma is important:

Lemma 1 *Let $f \in H^2(\mathbf{R}^m)$. Then there is a set $E \subseteq M$ with $\mu(M - E) = 0$ such that $\theta \in E$ implies:*

(1) $(Vf)(\theta, \cdot) \in H_\theta^2(Q_L)$,
(2) $(V\partial_j f)(\theta, \cdot) \in H_\theta^1(Q_L)$, $\quad (V\partial_j f)(\theta, x) = (\partial_j Vf)(\theta, x)$ *ae.* x
(3) $(V\partial_{jk} f)(\theta, \cdot) \in \mathcal{L}^2(Q_L)$, $\quad (V\partial_{jk} f)(\theta, x)$ *ae.* x

Remark $(Vf)(\theta, \cdot)$ means $Vf(\theta, x)$ as a function of x, with θ fixed; ∂_{jk} is short for $\partial_j \partial_k$. A similar statement holds in case $f \in H^1(\mathbf{R}^m)$. A proof is given in chapter XIII, 16 of [16]; due to its importance we give an outline.

PROOF: Assume $f \in H^2(\mathbf{R}^m)$, let $\varphi_n \in C_0^\infty(\mathbf{R}^m)$, g_j, $h_{jk} \in \mathcal{L}^2(\mathbf{R}^m)$ be such that

$$\|f - \varphi_n\|_{\mathcal{L}^2}, \qquad \|g_j - \partial_j\varphi_n\|_{\mathcal{L}^2}, \qquad \|h_{jk} - \partial_{jk}\varphi_n\|_{\mathcal{L}^2} \qquad (\mathcal{L}^2 = \mathcal{L}^2(\mathbf{R}^m))$$

all tend to zero as $n \to \infty$. Abbreviating $\mathcal{L}^2(M \times Q_L)$ by $\bar{\mathcal{L}}^2$, it follows from the unitarity of V that

$$(a) \qquad \|Vf - V\varphi_n\|_{\bar{\mathcal{L}}^2}, \qquad \|Vg - V\partial_j\varphi_n\|_{\bar{\mathcal{L}}^2}, \qquad \|Vh_{jk} - V\partial_{jk}\varphi_n\|_{\bar{\mathcal{L}}^2}$$

all tend to zero as $n \to \infty$. Now let φ denote any of the φ_n. Since $\varphi \in C_0^\infty(\mathbf{R}^m)$, there is for any $x^0 \in \mathbf{R}^m$ and any spherical neighbourhood $U_\varepsilon(x^0)$ of x^0 (ε small) an N such that for any $\theta \in M$ and $x \in U_\varepsilon(x^0)$

$$(b) \qquad \sum e^{-in\theta}\varphi(x + nL) = \sum_{|n| \leq N} e^{-in\theta}\varphi(x + nL).$$

where on the left summation is over $n \in \mathbf{Z}^m$. Thus $(V\varphi)(\theta, \cdot) \in C^\infty(\mathbf{R}^m)$ and moreover

$$(c) \qquad (V\partial_{jk}\varphi)(\theta, x) = (\partial_{jk}V\varphi)(\theta, x), \quad \text{likewise with} \quad \partial_j.$$

In addition

$$(d) \qquad (V\varphi)(\theta, x + nL) = e^{in\theta}(V\varphi)(\theta, x).$$

Recalling $\varphi \in \{\varphi_n\}$ it follows from (a), (c) that

$$(e) \qquad \|Vh_{jk} - \partial_{jk}V\varphi_n\|_{\bar{\mathcal{L}}^2}, \qquad \|Vg_j - \partial_j V\varphi_n\|_{\bar{\mathcal{L}}^2}$$

tend to zero as $n \to \infty$. By virtue of (a), (e) and Fubini there is a set $E \subseteq M$ with $\mu(M - E) = 0$ and a subsequence $\{n_p\}$ such that for $\theta \in E$

$$(f) \qquad \begin{array}{l} \|(Vf)(\theta, \cdot) - (V\varphi_{n_p})(\theta, \cdot)\|_{\hat{\mathcal{L}}^2}, \\ \|(Vg_j)(\theta, \cdot) - (\partial_j V\varphi_{n_p})(\theta, \cdot)\|_{\hat{\mathcal{L}}^2} \\ \|(Vh_{jk})(\theta, \cdot) - (\partial_{jk}V\varphi_{n_p})(\theta, \cdot)\|_{\hat{\mathcal{L}}^2} \end{array}$$

all tend to zero as $p \to \infty$; thereby we have set $\hat{\mathcal{L}}^2 = \mathcal{L}^2(Q_L)$. According to the remarks in sect. II, (B), the clauses (1) to (3) in the Lemma now follow. ■

The lemma below is proved in the same way; we omit its easy proof.

Lemma 2 *Let $h \in C^0(\mathbf{R}^m)$ be L-periodic with respect to all its arguments. Assume $f \in \mathcal{L}^2(\mathbf{R}^m)$. Then there is a set $E \subseteq M$ with $\mu(M - E) = 0$ such that $\theta \in E$ implies: $(*)$ $(Vf)(\theta, \cdot)$, $(Vhf)(\theta, \cdot) \in \mathcal{L}^2(Q_L)$, and $h(x)(Vf)(\theta, x) = (Vhf)(\theta, x)$ ae. x.*

The above setting extends straightforwardly to the vector case. As "fiber" space we take $\mathcal{H}'' = (\mathcal{H}')^n = (\mathcal{L}^2(Q_L))^n$. The direct integral $\mathcal{H}^* = \int_M \mathcal{H}'' d\mu$ is now the Hilbert space whose elements are the measurable mappings Φ which map ae. $\theta \in M$ into an element

$$\Phi(\theta) = (\Phi_1(\theta), \dots, \Phi_n(\theta)) \in (\mathcal{L}^2(Q_L))^n = (\mathcal{H}')^n = \mathcal{H}''$$

such that

$$\sum \int_M \|\Phi_j(\theta)\|^2_{\mathcal{H}'} \, d\mu < \infty,$$

and provided with the scalar product

$$\langle \Phi, \varphi \rangle_{\mathcal{H}^*} = \int_M \langle \Phi(\theta), \varphi(\theta) \rangle_{\mathcal{H}''} d\mu = \sum \int_M \langle \Phi_j(\theta), \varphi_j(\theta) \rangle_{\mathcal{H}'} d\mu.$$

As in the scalar case it is easy to see that $\mathcal{H}^*$ can be identified with $(\mathcal{L}^2(M \times Q_L))^n$, ie. an element $\psi \in \mathcal{H}^*$ is now a vector $(\psi_1, \dots, \psi_n)$ with $\psi_j \in \mathcal{L}^2(M \times Q_L)$, and the value $\psi(\theta)$ for ae. $\theta \in M$ is given by

$$\psi(\theta) = (\psi_1(\theta, \cdot), \dots, \psi_n(\theta, \cdot)) \in (\mathcal{L}^2(Q_L))^n = \mathcal{H}''. \tag{3.8}$$

A unitary mapping U from $(\mathcal{L}^2(\mathbf{R}^m))^n$ onto $\mathcal{H}^*$ is then defined which maps an element $f = (f_1, \dots, f_n) \in (\mathcal{L}^2(\mathbf{R}^m))^n$ into

$$Uf = (Vf_1, \dots, Vf_n) \in \mathcal{H}^* \tag{3.9}$$

with V given by (3.7). If we define the action of ∂_j, ∂_{jk} and the multiplication with h (Lemma 2) componentwise, we obtain obvious extensions of Lemmas 1,2 whose formulation we omit. Rather we stress an immediate consequence which is important in the sequel. To this end, let $D = (\delta_{ij}\tau_j)$ be as in II.(C) and let $B = (b_{jk})$ be an $n \times n$ matrix with entries $b_{jk} \in C^0(\mathbf{R}^m)$ which are L-periodic in all arguments.

Lemma 3 *Let $f \in (H^2(\mathbf{R}^m))^n$, $g \in (\mathcal{L}^2(\mathbf{R}^m))^n$. Then there is a set $E \subseteq M$ with $\mu(M - E) = 0$ such that $\theta \in E$ implies:*

(1) $(Uf)(\theta, \cdot) \in (H^2_\theta(Q_L))^n$,
(2) $(UD\triangle f)(\theta, \cdot) \in (\mathcal{L}^2(Q_L))^n$ *and* $D\triangle(Uf)(\theta, x) = (UD\triangle f)(\theta, x)$ *ae.* x,
(3) $(UBg)(\theta, \cdot), (Ug)(\theta, \cdot) \in (\mathcal{L}^2(Q_L))^n$ *and* $B(x)(Ug)(\theta, x) = (UBg)(\theta, x)$ *ae.* x.

The proof follows from Lemmas 1,2 and the above remarks.

(B) Next we come to the concept of measurable operator valued function. Again we rely on [16], XIII and also on Dixmier [5] in particular as far as bounded operators are considered. In case of unbounded operators we deviate somewhat from [16], where only the selfadjoint case is considered. To start with, let $B(\theta) \in L(\mathcal{H}'', \mathcal{H}'')$, $\theta \in M$ be a family of bounded operators such that $\langle f, B(\theta)g \rangle_{\mathcal{H}''}$,

$\theta \in M$ is measurable for any $f, g \in \mathcal{H}''$. Let there be C such that $\|B(\theta)\|_\infty \leq C$, $\theta \in M$, holds. A bounded operator $\tilde{B} \in L(\mathcal{H}^*, \mathcal{H}^*)$ is then defined according to

$$(3.10) \qquad (\tilde{B}\varphi)(\theta) = B(\theta)\varphi(\theta) \quad \text{ae.} \quad \theta \in M, \quad \text{for} \quad \varphi \in \mathcal{H}^*.$$

That $\tilde{B}$ has the required properties is shown in [16], pg. 281, [15],II, paragraph2. The notation for $\tilde{B}$ is $\tilde{B} = \int_M B(\theta)d\mu$. Now let $A(\theta)$, $\theta \in M$ be a family of closed, densely defined operators on $\mathcal{H}''$. An unbounded operator $\tilde{A}$ on $\mathcal{H}^*$ is introduced according to

Definition 1 *$\varphi \in dom(\tilde{A})$ iff: (1) $\varphi(\theta) \in dom(A(\theta))$ for ae. $\theta \in M$, (2) the mapping $\theta \to A(\theta)\varphi(\theta)$ is measurable, (3) $\int_M \|A(\theta)\varphi(\theta)\|^2_{\mathcal{H}''} d\mu < \infty$. For $\varphi \in dom(\tilde{A})$ we set $(\tilde{A}\varphi)(\theta) = A(\theta)\varphi(\theta)$ ae. $\varphi \in M$.*

The notation for $\tilde{A}$ is again $\tilde{A} = \int_M A(\theta)d\mu$. In this generality def. 1 is not very usefull. More can be said under the following assumptions which will automatically be satisfied in our situation:

$$(3.11) \qquad \text{there exist} \quad \lambda_0 \quad \text{and} \quad k_0 \quad \text{such that} \quad \lambda_0 \in \rho(A(\theta)) \\ \text{for all} \quad \theta \in M \quad \text{and} \quad \|(A(\theta) - \lambda_0)^{-1}\|_\infty \leq k.$$

Remark: For simplicity we write from now on $\int$ for $\int_M$ and $\| \quad \|$ for $\| \quad \|_{\mathcal{H}''}$ if it is clear from the context that the argument refers to $\mathcal{H}''$ either as an element of $\mathcal{H}''$ or as an operator acting on $\mathcal{H}''$.

Lemma 4 *Let $\tilde{A} = \int A(\theta)d\mu$ satisfy (3.11). Assume that $(A(\theta) - \lambda_0)^{-1}$, $\theta \in M$ is measurable. Then $\lambda_0 \in \rho(\tilde{A})$ and $(\tilde{A} - \lambda_0)^{-1} = \int (A(\theta) - \lambda_0)^{-1} d\mu$.*

PROOF: Set $R = \int (A(\theta) - \lambda_0)^{-1} d\mu$. That $R \in L(\mathcal{H}^*, \mathcal{H}^*)$ follows from (3.11) and the above remarks. Assume first $\varphi \in \text{rg}(R)$, ie. $\varphi = R\psi$ for some $\psi \in \mathcal{H}^*$. Thus $\varphi(\theta) = (A(\theta) - \lambda_0)^{-1}\psi(\theta)$ for ae. $\theta \in M$, whence $\varphi(\theta) \in \text{dom}(A(\theta))$ for ae. $\theta \in M$. Since $A(\theta)\varphi(\theta) = \psi(\theta) + \lambda_0\varphi(\theta)$ ae. $\theta \in M$ we have that the mapping $\theta \in M \to A(\theta)\varphi(\theta) \in \mathcal{H}''$ is measurable and $\int \|A(\theta)\varphi(\theta)\|^2 d\mu < \infty$, whence $\varphi \in \text{dom}(\tilde{A})$ by def. 1. Conversely, let $\varphi \in \text{dom}(\tilde{A})$. Then $\psi(\theta) = (A(\theta) - \lambda_0)\varphi(\theta)$ is measurable as a function of θ, and $\int \|\psi(\theta)\|^2 d\mu < \infty$. We then have $\varphi(\theta) = (A(\theta) - \lambda_0)^{-1}\psi(\theta)$ for ae. $\theta \in M$, ie. $\varphi \in \text{rg}(R)$, whence $\text{rg}(R) = \text{dom}(\tilde{A})$. That $(\tilde{A} - \lambda_0)R = 1$ and $R(\tilde{A} - \lambda_0) = 1$ on $\text{dom}(\tilde{A})$ now follows by straightforward computation via the integral representations. ∎

We now specialize the above setting to the situation considered in this paper. To this end let $D = (\delta_{jk}\tau_k)$ is as in II.(C) and $B = (b_{jk})$ an $n \times n$ matrix with entries $b_{jk} \in C^0(\mathbf{R}^m)$, L-periodic in all arguments. We now consider the formal operator $D\triangle + B$ on various Hilbert spaces. In order to distinguish between the interpretations it is advantageous to characterize the various meanings by different symbols. To this effect we introduce operators $A_0, \hat{A}_0, \tilde{A}_0$ and $A_0(\theta)$, $\theta \in M$ as follows. A_0 acts on $(\mathcal{L}^2(\mathbf{R}^m))^n$, and $A_0 = D\triangle$ on $\operatorname{dom}(A_0) = (H^2(\mathbf{R}^m))^n$. $\hat{A}_0$ acts on $\mathcal{H}^*$, $\operatorname{dom}(\hat{A}_0) = U(\operatorname{dom}(A_0))$ and $\hat{A}_0 Uf = UA_0 f$ for $f \in \operatorname{dom}(A_0)$. $A_0(\theta)$ acts on $\mathcal{L}^2(Q_L)$ and $A_0(\theta) = D\triangle$ on $\operatorname{dom}(A_0(\theta)) = (H^2_\theta(Q_L))^n$. Finally we set $\tilde{A}_0 = \int A_0(\theta)d\mu$ in the sense of definition 1.

Lemma 5 *A_0, $\hat{A}_0$ and $\tilde{A}_0$ are selfadjoint, and $\hat{A}_0 = \tilde{A}_0$.*

PROOF: That A_0 is selfadjoint is well known (see eg. [22]), and since $\hat{A}_0$ is unitarily equivalent to A_0, it is selfadjoint too. As to $\tilde{A}_0$ we note that $A_0(\theta)$, $\theta \in M$ is selfadjoint, with spectrum $\sigma(A_0(\theta)) \subseteq (-\infty, 0]$. Moreover $(A_0(\theta)+i)^{-1}$, $\theta \in M$ is measurable, as is easily verified, but will also be shown in the next section. It then follows from [16], thm. XIII. 85 that $\tilde{A}$ is selfadjoint. Since $\hat{A}_0$, $\tilde{A}_0$ both are selfadjoint $\hat{A}_0 = \tilde{A}_0$ is proved if we can show $\hat{A}_0 \subseteq \tilde{A}_0$, due to well known maximality properties of selfadjoint operators. In order to show $\operatorname{dom}(\hat{A}_0) \subseteq \operatorname{dom}(\tilde{A}_0)$, consider any $f \in \operatorname{dom}(A_0)$. By Lemma 3, expressed in terms of the present notation, there is a set $E \subseteq M$ with $\mu(M-E) = 0$ such that $\theta \in E$ implies:

(a) $(Uf)(\theta,\cdot) \in \operatorname{dom}(A_0(\theta))$ and $(UA_0f)(\theta,\cdot) \in \mathcal{H}''$,

(b) $(UA_0f)(\theta,\cdot) = A_0(\theta)(Uf)(\theta,\cdot)$.

By clauses (a),(b), the element $\varphi \in Uf$ satisfies indeed the assumptions of definition 1, whence $\varphi \in \operatorname{dom}(\tilde{A}_0)$. As a consequence of (b) on the other hand we infer

(c) $(\hat{A}_0\varphi)(\theta,\cdot) = (\tilde{A}_0\varphi)(\theta,\cdot) \quad$ for ae. $\quad \theta \in M$,

whence $\hat{A}_0\varphi = \tilde{A}_0\varphi$ follows. Since $\varphi = Uf$ ranges over $\operatorname{dom}(\hat{A}_0)$ as f ranges over $\operatorname{dom}(A_0)$, the desired inclusion $\hat{A}_0 \subseteq \tilde{A}_0$ follows. ■

Next we consider the matrix operator $B = (b_{jk})$ and its action on different spaces. As before we distinguish between the various interpretations by different notations. On $(\mathcal{L}^2(\mathbf{R}^m))^n$ and $(\mathcal{L}^2(Q_L))^n$ we let B act in the obvious was as a matrix multiplication operator, but denote it by B_0 and $B(\theta)$ respectively. $\hat{B}$ on $\mathcal{H}^*$ is defined by $\hat{B}Uf = UB_0f$, $f \in (\mathcal{L}^2(\mathbf{R}^m))^n$, and $\tilde{B} = \int B(\theta)d\mu$ is a well defined bounded operator on $\mathcal{H}^*$, according to the remarks, related to (3.10).

Lemma 6 $\hat{B} = \tilde{B}$

PROOF: For $f \in (\mathcal{L}^2(\mathbf{R}^m)^n$, we have $\hat{B}Uf = UB_0f$ by definition. By Lemma 3, expressed in terms of the present notation, there is $E \subseteq M$ with $\mu(M-E) = 0$ such that $\theta \in E$ implies:

$$(a) \qquad \begin{aligned} &(UB_0f)(\theta,\cdot), \quad (Uf)(\theta,\cdot) \in \mathcal{H}'' \quad \text{and} \\ &(UB_0f)(\theta,\cdot) = B(\theta)(Uf)(\theta,\cdot) = (\tilde{B}Uf)(\theta,\cdot). \end{aligned}$$

Thus $(\tilde{B}Uf)(\theta,\cdot) = (\hat{B}Uf)(\theta,\cdot)$ ae. $\theta \in M$ whence $\tilde{B}Uf = \hat{B}Uf$ follow. Since Uf ranges over $\mathcal{H}^*$ as f varies over $(\mathcal{L}^2(\mathbf{R}^m))^n)$, we get $\hat{B} = \tilde{B}$. ■

Corollary: *(1)* $\tilde{A}_0 + \tilde{B}_0 = \int (A_0(\theta) + B(\theta))d\mu$, *(2)* $\hat{A}_0 + \hat{B} = U(A_0 + B_0)U^{-1}$, *(3)* $\hat{A}_0 + \hat{B} = \tilde{A}_0 + \tilde{B}$.

PROOF: (1) and (2) follow directly from the definitions (see also [16] thm. XIII 85 in case of (1)), while (3) is a consequence of Lemmas 5,6. ∎

As a consequence of the corollary, we have that $\sigma(A_0 + B_0) = \sigma(\tilde{A}_0 + \tilde{B}_0)$. Formula (1.4) in the introduction is thus proved if we can show

$$(3.12) \qquad \sigma(\tilde{A}_0 + \tilde{B}) = \bigcup \sigma(A_0(\theta) + B(\theta)), \quad \theta \in M.$$

The verification of (3.12) is the purpose of the next section.

IV. Spectral considerations

In order to prove (3.12) and hence (2.8) we disregard for the moment the special structure of $\tilde{A}_0 + \tilde{B}$, $A_0(\theta) + B(\theta)$, set $\tilde{A} = \tilde{A}_0 + \tilde{B}$, $A(\theta) = A_0(\theta) + B(\theta)$ and rewrite (3.12) as

$$(4.1) \qquad \sigma(\tilde{A}) = \bigcup_\theta \sigma(A(\theta)) \quad \text{ie.} \quad \rho(\tilde{A}) = \bigcap_\theta \rho(A(\theta)).$$

The proof of (4.1) is based on three lemmas. Two of these have lengthy proofs which require seperate considerations; these proofs are relegated to the next section. The present proof is more eleborate than the corresponding one for selfadjoint operators in [16] (thm. XIII. 85) in that the latter relies heavily on the spectral theorem, a tool not available here.

Lemma 7 *Let here be $\lambda \in \mathbf{C}$, $k > 0$ such that $\left\|(\tilde{A} - \lambda)\varphi\right\| \geq k\|\varphi\|$ for all $\varphi \in dom(\tilde{A})$. For any $\varphi \in dom(\tilde{A})$ we then have $\left\|(\tilde{A}(\theta) - \lambda)\varphi(\theta)\right\| \geq k\|\varphi(\theta)\|$ for ae. $\theta \in M$.*

PROOF: Let $\lambda \in \mathbf{C}$, $k > 0$ have the required properties, and pick $\varphi \in \text{dom}(\tilde{A})$. By assumption and definition we have that

$$(a) \qquad \int \left\{ \|(A(\theta) - \lambda)\,\varphi(\theta)\|^2 - k^2 \|\varphi(\theta)\|^2 \right\} d\mu \geq 0.$$

Next set

$$(b) \qquad E = \left\{ \theta / \, \|(A(\theta) - \lambda)\,\varphi(\theta)\|^2 - k^2 \|\varphi(\theta)\|^2 < 0 \right\}.$$

Since $\psi \in \mathcal{H}^*$ implies that $\|\psi(\theta)\|$, $\theta \in M$ is measurable, ([5], II, paragraph1) the set E is measurable; let χ be its characteristic function. It is then easily seen that

$\chi\varphi \in \mathrm{dom}(\tilde{A})$, whence (a), but with $\chi\varphi$ in place of φ, implies:

$$(c) \qquad \int_E \left\{ \|(A\theta) - \lambda)\,\varphi(\theta)\|^2 - k^2 \|\varphi(\theta)\|^2 \right\} d\mu \geq 0.$$

But on E the integrand in (c) is < 0. If $\mu(E) > 0$ then the integral in (c) would be < 0 contradicting (c), whence $\mu(E) = 0$ follows, proving the Lemma. ■

Next however we need that $\lambda \in \rho(\tilde{A})$ implies $\lambda \in \rho(A(\theta))$ for all $\theta \in M$. In order to prove this stronger statement we need a lemma whose proof, based on arguments from perturbation theory, is relegated to the next section.

Lemma 8 *Assume $\lambda_0 \in \sigma(A(\theta_0))$ for some $\lambda_0 \in \mathbf{C}$, $\theta_0 \in M$. Then there exists a relatively open neighbourhood $\mathcal{U} \subseteq M$ of θ_0, a mapping φ from $\theta \in \mathcal{U}$ into dom$(A(\theta))$, a function $\lambda \in C^0(\mathcal{U}, \mathbf{C})$ and constants $a, b > 0$ such that: (1) φ is measurable (on $\mathcal{U}$), (2) $(A(\theta) - \lambda(\theta))\varphi(\theta) = 0$ for $\theta \in \mathcal{U}$, (3) $\lambda(\theta_0) = \lambda_0$, (4) $0 < a \leq \|\varphi(\theta)\| \leq b$ ae. $\theta \in \mathcal{U}$.*

Remark: "measurable on $\mathcal{U}$" means measurability of $\langle f, \varphi(\theta) g\rangle$ as a function of $\theta \in \mathcal{U}$, for any $f, g \in \mathcal{H}''$.

Theorem 1 *Let $\lambda_0 \in \mathbf{C}$, $k > 0$ be such that $\left\|(\tilde{A} - \lambda_0)\varphi\right\| \geq k\|\varphi\|$ for all $\varphi \in$ dom$(\tilde{A})$. Then $\lambda_0 \in \rho(A(\theta))$ for all $\theta \in M$.*

Proof: Assume the contrary: $\lambda_0 \in \sigma(A(\theta_0))$ for some $\theta_0 \in M$. By Lemma 8, there is a relative open neighbourhood $\mathcal{U} \subseteq M$ of θ_0, a continuous mapping $\lambda : \mathcal{U} \to \mathbf{C}$, a measurable mapping $\varphi : \mathcal{U} \to \mathcal{H}''$ and constants a, b which satisfy (1) - (4) of Lemma 8. Next set $\tilde{\varphi}(\theta) = \varphi(\theta)$ for $\theta \in \mathcal{U}$ and $\tilde{\varphi}(\theta) = 0$ for $\theta \notin \mathcal{U}$. Evidently $\tilde{\varphi}$ is measurable on M and

$$(a) \qquad a^2\mu(\mathcal{U}) \leq \int \|\tilde{\varphi}(\theta)\|^2 d\mu \leq b^2\mu(\mathcal{U}).$$

Without loss of generality we may assume λ to be bounded on $\mathcal{U}$; otherwise we would let shrink $\mathcal{U}$ slightly. Since $(\tilde{A}\tilde{\varphi})(\theta) = \lambda(\theta)\varphi(\theta)$ for $\theta \in \mathcal{U}$ and $= 0$ otherwise we have that $\tilde{\varphi} \in \mathrm{dom}(\tilde{A})$. By Lemma 8, our assumptions and since $\tilde{\varphi} = \varphi$ on $\mathcal{U}$ we then have

$$(b) \qquad \|(A(\theta) - \lambda_0)\,\varphi(\theta)\| \geq k\,\|\varphi(\theta)\| \geq ka \quad \text{ae.} \quad \theta \in \mathcal{U}.$$

By (3) of Lemma 8 on the other hand we have that

$$(c) \qquad 0 \geq \|(A(\theta) - \lambda_0)\,\varphi(\theta)\| - |\lambda(\theta) - \lambda_0|\,\|\varphi(\theta)\|, \quad \theta \in \mathcal{U}.$$

By combining (b), (c) with (4) of Lemma 8 we get

$$(d) \qquad 0 \geq ka - |\lambda(\theta) - \lambda_0| b \quad \text{ae.} \quad \theta \in \mathcal{U}.$$

Since $\mathcal{U}$ is (relatively) open, the set of $\theta \in \mathcal{U}$ for which (d) holds is dense in $\mathcal{U}$. Since $\theta_0 \in \mathcal{U}$, there is a sequence $\{\theta_k\} \subseteq \mathcal{U}$ for which (d) holds and such that $\lim \theta_k = \theta_0$. We now insert θ_k for θ in (d) and let $k \to \infty$. By (3) of Lemma 8 we obtain $0 \geq ka$, ie. a contradiction, proving the theorem. ∎

In order to prove the converse of Theorem 1 we need a further lemma of perturbation theoretic type, whose proof will be presented in the next section.

Lemma 9 *Assume* $\lambda_0 \in \rho(A(\theta_0))$ *for some* $\lambda_0 \in \mathbf{C}$, $\theta_0 \in M$. *For every* $\varepsilon > 0$ *there is a* $\beta > 0$ *as follows: if* $|\theta - \theta_0| < \beta$ *then* $\lambda_0 \in \rho(A(\theta))$ *and*

$$(*) \qquad \left\| (A(\theta) - \lambda_0)^{-1} - (A(\theta_0) - \lambda_0)^{-1} \right\|_\infty \leq \varepsilon.$$

Theorem 2 *Assume* $\lambda_0 \in \rho(A(\theta))$ *for all* $\theta \in M$. *Then* $\lambda_0 \in \rho(\tilde{A})$.

PROOF: For any $\theta \in M$, set $\varepsilon_\theta = \frac{1}{2} \left\| (A(\theta) - \lambda_0)^{-1} \right\|_\infty$. By Lemma 9 there is $\beta_\theta > 0$ such that $|\theta' - \theta| < \beta_\theta$ implies:

$$(a) \qquad \left\| (A(\theta') - \lambda_0)^{-1} \right\|_\infty \leq \frac{3}{2} \left\| (A(\theta) - \lambda_0)^{-1} \right\|_\infty = \frac{3}{2} \varepsilon_\theta$$

Set $\mathcal{U}_\theta = \{\theta' / \theta' \in M \quad \& \quad |\theta - \theta'| < \beta_\theta\}$. Evidently $M \subseteq \bigcup \mathcal{U}_\theta$, $\theta \in M$. Since M is compact, we find a finite covering

$$(b) \qquad M = \bigcup_{j=1}^{n} \mathcal{U}_{\theta_j}$$

Thus given any $\theta \in M$ we find $\theta \in \mathcal{U}_{\theta_j}$ for some j. From (a) we then infer

$$(c) \qquad \left\| (A(\theta) - \lambda_0)^{-1} \right\|_\infty \leq \frac{3}{2} \varepsilon_j \leq \frac{3}{2} \max_j \varepsilon_j .$$

By Lemma 9 on the other hand, $\langle f, (A(\theta) - \lambda_0)^{-1} g \rangle$ is continous in $\theta \in M$ for any $f, g \in \mathcal{H}''$; thus $(A(\theta) - \lambda_0)^{-1}$, $\theta \in M$ is measurable. Hence all assumptions of Lemma 4 are satisfied, whence $\lambda_0 \in \rho(\tilde{A})$ follows. ∎

Corollary 1 $\lambda \in \rho(\tilde{A})$ *iff* $\lambda \in \rho(A(\theta))$ *for all* $\theta \in M$.

PROOF: If $\lambda \in \rho(\tilde{A})$ then $\left\| (\tilde{A} - \lambda) \varphi \right\| \geq k \|\varphi\|$ for some $k > 0$ and all $\varphi \in \mathrm{dom}(\tilde{A})$. By Theorem 1, $\theta \in \rho(A(\theta))$ for all $\theta \in M$. The converse is just a restatement of Theorem 2. ∎

Corollary 2 $\sigma(A_0 + B_0) = \bigcup \sigma(A_0(\theta) + B(\theta))$, $\theta \in M$.

PROOF: By the corollary to Lemma 6, $A_0 + B_0$ and $\tilde{A} = \tilde{A}_0 + \tilde{B}$ are unitarily equivalent. The corollary is then just a restatement of corollary 1. ∎

V. Results from perturbation theory

We now come to the proofs of Lemmas 8,9. From these Lemma 9 is an immediate consequence of general principles; the proof of Lemma 8 requires slightly more care. Our arguments are based on well known facts from the perturbation theory of closed operators with discrete spectrum, as presented in Rellich [17], chapter II and Kato [9], chapter VII. We will therefore content us with indications in cases where the arguments are either familiar or proceed along established lines. The statements which lead to the proofs of Lemmas 8,9 will be termed as propositions. To start with, we recall the operators $A(\theta) = A_0(\theta) + B(\theta)$, on $\mathcal{H}''$, where $A_0(\theta)$ is $D\triangle$ on $\mathrm{dom}(A_0(\theta)) = (H^2_\theta(Q_L))^n$, while $B(\theta)$ is just the matrix multiplication operator $B = (b_{jk})$ acting on $\mathcal{H}''$. In order to get rid of the variable domain of definition $\mathrm{dom}(A(\theta)) = (H^2_\theta(Q_L))^2$, we make use of (2.4) and introduce the family $U(\theta)$, $\theta \in M$ of unitary operators, defined by

$$U(\theta)f = \mathrm{e}^{-\frac{i\theta x}{L}} f, \qquad f \in \mathcal{H}'', \tag{5.1}$$

(where $\theta x = \sum \theta_j x_j$). $U(\theta)$ transforms $A(\theta)$ into another unbounded operator $H(\theta)$, related to $A(\theta)$ by

$$H(\theta)f = U(\theta)A(\theta)U(\theta)^* f \tag{5.2}$$

whose domain is given by $U(\theta)\mathrm{dom}(A(\theta))$, and where $U(\theta)^* = U(\theta)^{-1}$. By (2.4), $\mathrm{dom}(H(\theta))$ is just $(H^2_{per}(Q_L))^n$, ie. $\mathrm{dom}(H(\theta))$ is independent of θ, what makes it advantageous to work with $H(\theta)$. This is confirmed by

Proposition 1 *Lemmas 8,9 hold for $A(\theta)$, $\theta \in M$ iff they hold for $H(\theta)$, $\theta \in M$.*

The proof easily follows from the fact that $U(\theta)$ leaves (2), (3), (4) of Lemma 8 invariant and transforms a measurable mapping $\tilde{\varphi} : \mathcal{U} \to \mathcal{H}''$ into a measurable mapping $\varphi = U(\cdot)^*\tilde{\varphi}$ from $\mathcal{U}$ into $\mathcal{H}''$. Hence we concentrate henceforth on the family $H(\theta)$, $\theta \in M$. Straightforward computation shows that $H(\theta)$ maps an element $\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathcal{H}''$ into an element $H\varphi = (\psi_1, \ldots, \psi_n) \in \mathcal{H}''$ according to the rule

$$\psi_k = \tau_k \left(\triangle\varphi_k - \frac{2i}{L} \sum \theta_j \frac{\partial \varphi_k}{\partial x_j} - L^{-2} \left(\sum \theta_j^2 \right) \varphi_k \right) + \sum b_{kj}\varphi_j. \tag{5.3}$$

Clause (5.3) allows us to define $H(\theta)$ for arbitrary complex θ, by keeping $(H^2_{per}(Q_L))^n$ as fixed domain of definition. For our purposes it is suitable to consider $H(\theta)$ on a fixed neighbourhood $\mathcal{D}$ of M given by

$$\mathcal{D} = \{z/z \in \mathbf{C}^m \quad \& \quad \mathrm{dist}(z, M) < \varepsilon_0\} \tag{5.4}$$

for some fixed $\varepsilon_0 > 0$. Insight into the structure of the family $H(\theta)$, $\theta \in \mathcal{D}$ is provided by

Proposition 2 *The operators $H(\theta)$, $\theta \in \mathcal{D}$ are closed and there exists $\gamma_0 > 0$ as follows: if $\gamma \geq \gamma_0$ then $\gamma \in \rho(H(\theta))$ for all $\theta \in \mathcal{D}$ and $\left\|(H(\theta)-\gamma)^{-1}\right\|_\infty \leq k_\gamma$ for some k_γ and all $\theta \in \mathcal{D}$. If $\lambda \in \rho(H(\theta))$ for some $\theta \in \mathcal{D}$ then $(H(\theta)-\lambda)^{-1}$ is compact.*

HINT OF PROOF: One way to proceed is to take advantage of the decomposition

$$H(\theta) = U(\theta)A_0(\theta)U(\theta)^* + B.$$

Here B is the matrix multiplication part, while $H(\theta) = U(\theta)A_0(\theta)U(\theta)^*$ is the differential operator which maps $(\varphi_1, \ldots, \varphi_n)$ into an element $(\psi_1, \ldots, \psi_n)$, with ψ_k given by (5.3), but with $b_{jk} = 0$. One then first proves the proposition for $U(\theta)A_0(\theta)U(\theta)^*$ by straightforward Fourier analysis and then for $H(\theta)$ by treating $H(\theta)$ as a bounded perturbation of $U(\theta)A_0(\theta)U(\theta)^*$. Of course the γ_0 might thereby increase. A somewhat different way to proceed consists in repeating the arguments in the proof of Lemma 1 in [21], which in turn depend on Lemmas 2.1,2.3 in [10], which eventually take care of the situation. ■

The family $H(\theta)$, $\theta \in \mathcal{D}$ is a strongly holomorphic family of closed operators: for any f in $\operatorname{dom}(H(\theta)) = (H^2_{per}(Q_L))^n$, $H(\theta)f$ is a holomorphic function on $\theta \in \mathcal{D}$ with values in $\mathcal{H}''$. Such families have been extensively treated in Rellich [17], chapter II and Kato [9], chapter VII. While the treatment in [17],[9] is mainly for one variable, the arguments therein extend in a verbatim way to the case of several variables. We therefore content us to summarize the facts about such holomorphic families. One assumes $\lambda_0 \in \rho(H(\theta_0))$ for some fixed $\lambda_0 \in \mathbf{C}$, $\theta_0 \in \mathcal{D}$ and considers the family

$$T(\theta, \lambda) = (H(\theta) - \lambda)(H(\theta_0) - \lambda_0)^{-1} \tag{5.5}$$

of closed, everywhere defined operators on the domain

$$\mathcal{D}' = \mathcal{D} \times \{\lambda / |\lambda - \lambda_0| < 1 \quad \& \quad \lambda \in \mathbf{C}\} \tag{5.6}$$

Necessarily, $T(\theta, \lambda) \in L(\mathcal{H}'', \mathcal{H}'')$. By classical arguments, based among others on the "resonance" theorem (Yosida [24], pg.69) one proves that $T(\theta, \lambda)$, $(\theta, \lambda) \in \mathcal{D}'$ is a uniformly holomorphic family, ie. holomorphic as a mapping from D' into the Banach space $L(\mathcal{H}'', \mathcal{H}'')$, $\| \quad \|_\infty$. By exploiting this fact and observing that $T(\theta_0, \lambda_0) = Id$ one then straightforwardly finds:

Proposition 3 *Assume $\lambda_0 \in \rho(H(\theta_0))$ for some $\lambda_0 \in \mathbf{C}, \theta_0 \in \mathcal{D}$. There are complex neighbourhoods $\mathcal{U} \subseteq \mathcal{D}$ of θ_0 and $\mathcal{V} \subseteq \mathbf{C}$ of λ_0 such that: (a) $(\theta, \lambda) \in \mathcal{U} \times \mathcal{V}$ implies $\lambda \in \rho(H(\theta))$, (b) the mapping $(\theta, \lambda) \in \mathcal{U} \times \mathcal{V} \rightarrow (H(\theta) - \lambda)^{-1}$ is uniformly holomorphic.*

Proof of Lemma 9: Let $\lambda_0 \in \rho(H(\theta_0))$ for some $\lambda_0 \in C$, $\theta_0 \in M$. By (b) of proposition 3 we have in particular that

$$(*) \qquad \lim_{\theta \to \theta_0} \left\| (H(\theta) - \lambda_0)^{-1} - (H(\theta_0) - \lambda_0)^{-1} \right\|_\infty = 0.$$

But $(*)$ is just a restatement of Lemma 9 for the family $H(\theta)$, $\theta \in M$. By unitary aequivalence (prop.1) it follows that Lemma 9 is valid for the family $A(\theta)$, $\theta \in M$. ■

We now come to the proof of Lemma 8. By Proposition 1 it suffices to prove Lemma 8 for the family $H(\theta)$, $\theta \in M$. Again we take advantage of the fact that $H(\theta)$ may be considered on the complex neighbourhood $\mathcal{D}$ of M. To start with, we have to digress into the theory of perturbations of isolated eigenvalues, as presented in [9], [17]. To this end assume $\lambda_0 \in \sigma(H(\theta_0))$ for some fixed $\lambda_0 \in \mathbf{C}$, $\theta_0 \in M$. In accordance with Proposition 2 we fix a $\gamma > 0$ such that $\gamma \in \rho(H(\theta))$, $\theta \in \mathcal{D}$ and set $R(\theta) = (H(\theta) - \gamma)^{-1}$. In order to discuss the eigenvalue problem $H(\theta) = \lambda\varphi$ in a complex neighbourhood of θ_0, λ_0 respectively, we replace this problem by the equivalent one

$$(5.7) \qquad R(\theta)\varphi = (\mu_0 + \delta)\varphi$$

whereby $\mu_0 = (\lambda_0 - \gamma)^{-1}$, $\mu_0 + \delta = (\lambda - \gamma)^{-1}$. Since $R(\theta_0)$ is compact by Proposition 2, $\mathcal{H}''$ admits a splitting into a direct sum $\mathcal{H}'' = \mathcal{L} + \mathcal{N}$ of closed subspaces $\mathcal{L}$, $\mathcal{N}$, such that: (a) $\dim(\mathcal{L}) = N < \infty$, (b) both $\mathcal{L}$, $\mathcal{N}$ are invariant against $R(\theta_0)$, (c) $R(\theta_0) - \mu_0$ is an isomorphism of $\mathcal{N}$ onto itself and has a bounded inverse $G \in L(\mathcal{N}, \mathcal{N})$ on $\mathcal{N}$. Moreover there are bounded projections P, K onto $\mathcal{L}$, $\mathcal{N}$ respectively which commute with $R(\theta_0)$ such that $P + K = Id$. Let furthermore $e_1, \ldots, e_N$ be a basis of $\mathcal{L}$. According to the general theory there is a dual list $e_1^*, \ldots, e_N^* \in \mathcal{H}''$ such that $\det(\langle e_j e_k^* \rangle) \neq 0$ and with the property: (d) $f \in \mathcal{N}$ iff $\langle e_k^*, f \rangle = 0$ for $k = 1, \ldots, N$. For a detailed exposition of the above compilation we refer eg. to Sattinger [20] chapter II or Friedmann [7] chapter V. For notational simplicity we abbreviate $\sum \zeta_j e_j$ $(j \leq N)$ by ζe for any $\zeta = (\zeta_1, \ldots, \zeta_N) \in \mathbf{C}^N$. In order to solve (5.7) we seek solutions $\varphi \in \mathcal{H}''$ of the form $\varphi = \zeta e + g$ with $g \in \mathcal{N}$. It is thereby convenient to set $\varepsilon = \theta - \theta_0$, where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m) \in \mathbf{C}^m$ is treated as a small parameter, and to define $D(\varepsilon) = R(\theta_0 + \varepsilon) - R(\theta_0)$. Clearly, $D(\varepsilon)$ is a uniformly holomorphic family of compact operators, defined on a complex neighbourhood of $\varepsilon = 0$. In these terms, (5.7) can be rewritten as

$$(5.8) \qquad (R(\theta_0) + D(\varepsilon))(\zeta e + g) = (\mu_0 + \delta)(\zeta e + g), \qquad g \in \mathcal{N}.$$

To (5.8) we apply the projector K. By taking the commutativity into account and the fact that $D(0) = 0$ one then finds by a straightforward computation an $r_0 > 0$ such that $|\delta|$, $|\varepsilon| < r_0$ (with $|\varepsilon| = \max_j |\varepsilon_j|$) implies

$$(5.9) \qquad g = g(\delta, \varepsilon, \zeta) = -(1 - GK(\delta - D))^{-1} GKD\zeta e,$$

where we have set $D = D(\varepsilon)$. The inverse in (5.9) may be represented as a Neumann series which converges uniformly, ie. in the operator norm $\| \ \|_\infty$ of $L(\mathcal{H}'', \mathcal{H}'')$. In order to find $\zeta \in \mathbf{C}^N$ and $\delta \in \mathbf{C}$ such that $\zeta e + g$, with g given by (5.9), is indeed a solution of (5.8), one finds as nessassary and sufficient condition the "orthogonality" relation

$$(5.10) \qquad (\delta E + M(\delta, \varepsilon))\zeta = 0,$$

where $E = (\delta_{jk})$ is identity on $\mathcal{L}$, while $M(\delta, \varepsilon) = (m_{jk})$ is the $N \times N$ matrix whose entries m_{jk} are given by

$$(5.11) \quad m_{jk} = \langle e_j^*, (\mu_0 - R(\theta_0)) - De_k + D(1 - GK(\delta - D))^{-1} GKDe_k \rangle,$$

Without loss of generality we may assume that $\langle e_j^*, (R(\theta_0) - \mu_0)e_k \rangle = 0$ for $j \leq k$. Again $D = D(\varepsilon)$. The equivalence of (5.9)+(5.10) with (5.8) is given by

Proposition 4 *Assume $|\delta|$, $|\varepsilon| < r_0$. Then $\varphi = \zeta e + g$ with $g \in \mathcal{N}$ is a solution of* (5.8) *if and only if $g = g(\delta, \varepsilon, \zeta)$ is given by* (5.9) *and with the orthogonality relation* (5.10) *satisfied.*

After these preparations we can proceed to the

PROOF OF LEMMA 8: We tacitly assume $|\delta|, |\varepsilon| < r_0$. For nontrivial solutions ζ of (5.10) to exists we must have

$$(a) \qquad F(\delta, \varepsilon) = \det(\delta E + M(\delta, \varepsilon)) = 0.$$

Now $F(\delta, \varepsilon)$ is holomorphic in the polydisc $|\delta| < r_2$, $|\varepsilon| = \max_j |\varepsilon_j| < r_0$; moreover $F(\delta, 0) = \delta^N$. By the Weierstrass preparation theorem (Osgood [12]) there is a factorisation $F(\delta, \varepsilon) = P(\delta, \varepsilon)\Pi(\delta, \varepsilon)$ and a constant $r_1 \leq r_0$ with

$$P(\delta, \varepsilon) = \delta^p + A_1(\varepsilon)\delta^{p-1} + \ldots + A_{p-1}(\varepsilon)\delta + A_p(\varepsilon), \qquad p > 0$$

such that $A_j(\varepsilon)$, $(j \leq p)$ and $\Pi(\delta, \varepsilon)$ are holomorphic in the polydisc $|\delta|, |\varepsilon| < r_1$, $A_j(0) = 0$ for $j \leq p$ and $\Pi(\delta, \varepsilon) \neq 0$ for $|\delta|, |\varepsilon| < r_1$. Thus on $|\delta|, |\varepsilon| < r_1$ the zeroes of $F(\delta, \varepsilon)$ coincide with those of $P(\delta, \varepsilon)$. By elementary arguments we then find an $r_2 \leq r_1$ with the property: (a) if $|\varepsilon| < r_2$ and $P(\delta, \varepsilon) = 0$ then $|\delta| < r_1$. With $|\varepsilon| < r_2$ we now associate a well determined root $\delta(\varepsilon)$ of $P(\delta, \varepsilon) = 0$: $\delta(\varepsilon)$ is that root of $P(\delta, \varepsilon) = 0$ whose real part $\mathrm{re}(\delta(\varepsilon))$ is leftmost (ie. $P(\delta', \varepsilon) = 0$ entails $\mathrm{re}(\delta(\varepsilon)) \leq \mathrm{re}(\delta')$) and whose imaginary part is topmost (ie. $P(\delta', \varepsilon) = 0$ and $\mathrm{re}(\delta') = \mathrm{re}(\delta(\varepsilon))$ implies $\mathrm{im}(\delta') \leq \mathrm{im}(\delta(\varepsilon))$. Elementary reasoning, based on the logarithmic residuum, shows

$$(b) \qquad \delta(\varepsilon), \qquad |\varepsilon| < r_2 \quad \text{is continuous and} \quad \delta(0) = 0.$$

By inserting $\delta(\varepsilon)$ into (a) we find that for every $|\varepsilon| < r_2$ there is a solution $\zeta \neq 0$ of

$$(\delta(\varepsilon)E + M(\delta(\varepsilon), \varepsilon))\zeta = 0.$$

In order to construct the neighbourhood $\mathcal{U}$ required by Lemma 8 we set

$$\mathcal{U} = \{\theta/\theta \in M \quad \& \quad |\theta - \theta_0| < r_2\}$$

where $\theta = (\theta_1, \ldots, \theta_m)$, $\theta_0 = (\theta_1^0, \ldots, \theta_m^0)$ and $|\theta - \theta_0| = \max_j |\theta_j - \theta_j^0|$. Clearly $\mu(\mathcal{U}) > 0$. Moreover we set

$$Q(\theta) = \delta(\theta - \theta_0)E + M(\delta(\theta - \theta_0), \theta - \theta_0).$$

The matrix $Q(\theta) = (q_{jk}(\theta))$ $(j, k \leq N)$ is continuous in $\theta \in \mathcal{U}$ and has rank $r = r(\theta) < N$. It is then easy to see that $\mathcal{U}$ is the union of pairwise disjoint Borel sets, $\mathcal{U} = \bigcup_j \mathcal{U}_j$ $(j < N)$ (some of which may be empty) such that $\theta \in \mathcal{U}_r$ iff $Q(\theta)$ has rank r. Next we observe that if $Q(\theta)$ has rank r, and if $Q_r(\theta)$ is an $r \times r$ submatrix of $Q(\theta)$ (an r-minorant) such that $\det(Q_r(\theta)) \neq 0$ then a solution $\zeta(\theta) = (\zeta_1, \ldots, \zeta_N)$ of $Q(\theta)\zeta(\theta) = 0$ such that $\sum |\zeta_j|^2 = 1$ can effectively be written down in terms of $Q_r(\theta)$ according to the rules of linear algebra. Now let for any sequences $j_1 < \ldots < j_r \leq N$ and $k_1 < \ldots < k_r \leq N$ (abbreviated by j, k respectively) $Q(j, k/\theta)$ be the r-minorant of $Q(\theta)$ given by $(q_{j_s k_t}(\theta))$, $s, t \leq r$. By elementary logical reasoning we can decompose each set $\mathcal{U}_r$ further into a union of pairwise disjoint Borel sets

$$\mathcal{U}_r = \mathcal{U}_1^r \cup \ldots \cup \mathcal{U}_{P_r}^r$$

and associate with each set $\mathcal{U}_l^r$ sequences j, k, ie. $j_1 < \ldots < j_r \leq N$ and $k_1 < \ldots < k_r \leq N$, such that the associated r-minorant $Q(j, k/\theta)$ satisfies $\det(Q(j, k/\theta)) \neq 0$, provided that $\theta \in \mathcal{U}_l^r$. By combining the above remarks we find that there is a measurable mapping which associates with $\theta \in \mathcal{U}$ a vector $\zeta(\theta) = (\zeta_1, \ldots, \zeta_N) \in \mathbf{C}^N$ such that $Q(\theta)\zeta(\theta) = 0$ and $\sum |\zeta_j|^2 = 1$. From this one infers that the mapping $\theta \in \mathcal{U} \to \varphi(\theta) = \zeta(\theta)e + g(\delta(\theta - \theta_0), \theta - \theta_0, \zeta(\theta)) \in \mathcal{H}''$ is measurable. That $\varphi(\theta), \theta \in \mathcal{U}$ is the mapping required by Lemma 8 is established if we can find bounds a ,b which satisfy (4) of Lemma 8. In order to find b we note that we may assume without loss of generality that the constant r_0 involved in the definition of $g(\delta, \varepsilon, \zeta)$ ((5.9)) is so small that

$$\left\| (1 - GK(\delta - D(\varepsilon)))^{-1} GKD(\varepsilon) \right\|_\infty \leq c_0 < \infty$$

for some c_0 and all $|\delta|, |\varepsilon| < r_0$. This gives an upper bound

$$\begin{aligned} \|\varphi(\theta)\| &\leq \|\zeta(\theta)e\| + \|g(\delta(\theta - \theta_0), \theta - \theta_0, \zeta(\theta))\| \leq \\ &\leq \sum \|e_j\| + c_0 \sum \|e_j\|. \end{aligned}$$

Thus we may set $b = (1 + c_0) \sum \|e_j\|$. Next we observe that there exists a $c_1 > 0$ such that $c_1 \leq \|\sum \zeta_j e_j\|$ if $\sum (\zeta_j)^2 = 1$, and that $\|P\|_\infty \geq 1$, with P the projector onto $\mathcal{L} = \mathrm{span}(e_1, \ldots, e_N)$. We then have

$$\begin{aligned} c_1 &\leq \|\zeta(\theta)e\| = \|P(\zeta(\theta)e + g(\delta(\theta - \theta_0), \theta - \theta_0, \zeta(\theta))\| \\ &\leq \|P\|_\infty \|(\zeta(\theta)e + g(\delta(\theta - \theta_0), \theta - \theta_0, \zeta(\theta))\|, \end{aligned}$$

that is, $c_1 \|P\|_\infty^{-1} \le \|\varphi(\theta)\|$. Thus we may set $a = c_1 \|P\|_\infty^{-1}$. Since $\lambda(\theta) = \gamma + (\mu_0 + \delta(\theta - \theta_0))^{-1}$, $\theta \in \mathcal{U}$ is continuous, satisfies $\lambda(\theta_0) = \lambda_0$ and $H(\theta)\varphi(\theta) = \lambda(\theta)\varphi(\theta)$, $\theta \in \mathcal{U}$ the Lemma 8 is proved for the family $H(\theta)$, $\theta \in M$ and hence for $A(\theta), \theta \in M$ by Proposition 1. ∎

Remarks: Formulas (5.9), (5.10) are of course well known and appear in many contexts in various forms; we have taken them from [21], [22] where they appear in the context of Ljapounov-Schmidt bifurcation theory. An analysis of the proof of theorem 1 shows that it would suffice to know that $\lambda(\theta)$ is continuous at θ_0 and $\lambda(\theta_0) = \lambda_0$. For $\delta(\varepsilon)$ in the above proof this means that we would have to show that $\delta(\varepsilon)$ is measurable and continuous at $\varepsilon = 0$, ie. $\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0$. The measurability is needed because $\delta(\theta - \theta_0)$ is involved in the definition of $\varphi(\theta)$, required to be measurable. The measurability of $\delta(\varepsilon)$ is settled by recognizing $\delta(\varepsilon)$ as continuous. For a very general setting of the logarithmic residuum argument see Demailly [4]. Whether the solution $\zeta(\theta)$ of $Q(\theta)\zeta(\theta) = 0$ can be chosen to be continuous in θ is not known to us. If so, the construction must be more sophisticated that the one presented in the above proof.

VI. Stability reconsidered

If we compare corollary 2 in section III with (2.8) then we see that it does not exactly yield what is claimed in (2.8). In fact, let as in sect. II, $v \in (H^4_{per}(Q_L))^n$ be an equilibrium solution $u_t = D\triangle u + F(u)$; set $B = d_uF(v)$. Clause (2.8) clames $\sigma^2_{per} \subseteq \sigma^2_{\mathcal{L}^2}$, where σ^2_{per} is the spectrum of $D\triangle + B$ on $(H^2_{per}(Q_L))^n$ as basic space with $\mathrm{dom}(D\triangle + B) = (H^4_{per}(Q_L))^n$, while $\sigma^2_{\mathcal{L}^2}$ is the spectrum of $D\triangle + B$ on $(H^2(\mathbf{R}^m))^n$ as basic space, with $\mathrm{dom}(D\triangle + B) = (H^4(\mathbf{R}^m))^n$. Corollary 2 of theorems 1,2 on the other hand merely implies that $\sigma_{per} \subseteq \sigma_{\mathcal{L}^2}$, where σ_{per} denotes for the moment the spectrum of $D\triangle + B$ on $(\mathcal{L}^2(Q_L))^n$ as basic space, with $\mathrm{dom}(D\triangle + B) = (H^2_{per}(Q_L))^n$,, while $\sigma_{\mathcal{L}^2}$ is the spectrum of $D\triangle + B$ on $(\mathcal{L}^2(\mathbf{R}^m))^n$, with $\mathrm{dom}(D\triangle + B) = (H^2(\mathbf{R}^m))^n$. Clause (2.8) is a consequence of

Lemma 10 *(a)* $\sigma_{per} = \sigma^2_{per}$, *(b)* $\sigma_{\mathcal{L}^2} = \sigma^2_{\mathcal{L}^2}$.

Clause (a) easily follows from the fact that both σ_{per}, σ^2_{per} are pure point spectra, consisting of isolated eigenvalues of finite multiplicity, accumulating only at infinity. Part (b) is slightly more delicate. We split its proof into two propositions. Let, to this end, T denote $D\triangle + B$, acting on $(H^2(\mathbf{R}^m))^n$, with $\mathrm{dom}(T) = (H^4(\mathbf{R}^m))^n$. Let T_0 be $D\triangle + B$, acting on $(\mathcal{L}^2(\mathbf{R}^m))^n$, with $\mathrm{dom}(T_0) = (H^2(\mathbf{R}^m))^n$. Thus $\sigma(T_0) = \sigma_{\mathcal{L}^2}$, $\sigma(T) = \sigma^2_{\mathcal{L}^2}$. Moreover $T_0 = A_0 + B_0$ in the notation of section III. In order to prove (b), we first note that T, T_0 are semigroup generators on their respective spaces what entails that $\zeta \in \rho(T_0) \cap \rho(T)$ for all $\zeta \gg 0$

Proposition 5 $\rho(T) \subseteq \rho(T_0)$.

PROOF: First pick $\zeta \gg 0$, $\zeta \in \rho(T) \cap \rho(T_0)$ and observe that the Sobolev norm $\| \quad \|_{H^2}$ on $(H^2(\mathbf{R}^m))^n$ may be replaced by an equivalent one, $\| \quad \|_T$, given by

$$(a) \qquad \|f\|_T = \|(T_0 - \zeta)f\|_{\mathcal{L}^2}, \qquad f \in (H^2(\mathbf{R}^m))^n \quad (= \mathrm{dom}(T_0)),$$

with $\| \quad \|_{\mathcal{L}^2}$ the norm on $(\mathcal{L}^2(\mathbf{R}^m))^n$ (see [22] for a discussion of this point). Now assume $\lambda \in \rho(T)$. Then $T - \lambda$ maps $\mathrm{dom}(T)$ in a 1-1 way onto $(H^2(\mathbf{R}^m))^n$. By the above remarks there exists a $k > 0$ such that

$$(b) \qquad \|(T - \lambda)g\|_T \geq k\,\|g\|_T \quad \text{for all} \quad g \in \mathrm{dom}(T).$$

Taking care of the definition of $\| \quad \|_T$, (b) may be rewritten as

$$(c) \qquad \|(T_0 - \lambda)(T - \zeta)g\|_{\mathcal{L}^2} \geq k\,\|(T_0 - \zeta)g\|_{\mathcal{L}^2}, \quad g \in \mathrm{dom}(T).$$

Since $\zeta \in \rho(T)$, $T - \zeta$ maps $\mathrm{dom}(T)$ onto $\mathrm{dom}(T_0)$. Moreover, $T = T_0$ on $\mathrm{dom}(T_0)$. Thus (c) can be rewritten as

$$(d) \qquad \|(T_0 - \lambda)f\|_{\mathcal{L}^2} \geq k\,\|f\|_{\mathcal{L}^2}, \qquad f \in \mathrm{dom}(T_0).$$

Now $T_0 = A_0 + B_0$, with $A_0 + B_0$ unitarily equivalent to $\tilde{A} = \tilde{A}_0 + \tilde{B}_0$ (corollary to Lemma 6). By (d) and this unitary equivalence we thus have

$$(e) \qquad \left\|(\tilde{A} - \lambda)\varphi\right\|_{\mathcal{H}} \geq k\,\|\varphi\|_{\mathcal{H}} \quad \text{for all} \quad \varphi \in \mathrm{dom}(\tilde{A}).$$

However (e) combined with theorems 1,2 implies $\lambda \in \rho(\tilde{A})$ and hence $\lambda \in \rho(T_0)$. ■

Proposition 6 $\rho(T_0) \subseteq \rho(T)$.

PROOF: Let $\lambda \in \rho(T_0)$. Thus $(T_0 - \lambda)g = f$ has precisely one solution $g \in \mathrm{dom}(T_0) = (H^2(\mathbf{R}^m))^n$ for any f in $(\mathcal{L}^2(\mathbf{R}^m))^n$. Now let $f \in \mathrm{dom}(T_0) = (H^2(\mathbf{R}^m))^n$. Then $g \in \mathrm{dom}(T_0^2) = \mathrm{dom}(T) = (H^4(\mathbf{R}^m))^n$ (see [22] for this point, based on a regularity argument). Thus for $f \in \mathrm{dom}(T_0)$ there is exactly one $g \in \mathrm{dom}(T)$ with $(T - \lambda)g = f$. Using the fact that T, as a semigroup generator on $\mathrm{dom}(T_0)$ as basic space is closed, $\lambda \in \rho(T)$ easily follows. ■

Remarks: As pointed out earlier, our spectral considerations have to be supplied by a proof of the principle of linearized instability for the evolution equation (2.7), ie. by a proof that $\sigma^2_{\mathcal{L}^2} \cap \{\lambda/\mathrm{im}(\lambda) > 0\} \neq \emptyset$ implies Ljapounov instability of the zero solution $\varphi_0 = 0$ against small perturbations $\varphi \in (H^2(\mathbf{R}^m))^n$. As mentioned, such a proof can indeed be given; however since it is quite lengthy it will be presented separately. Nevertheless, once this principle is accepted, the relationships expressed by corollary 2 to theorems 1,2 and Lemma 10 reduce the stability question to a discussion of the θ-periodic spectra $\sigma(A_0(\theta) + B(\theta))$, $\theta \in M$, and in simpler cases to a discussion of the periodic spectra σ_{per}; examples will be given in the next section.

We briefly digress on possible generalisations. As already mentioned, our arguments and hence theorems 1,2 extend in a verbatim way to the case where the single period L is replaced by a sequence of periods $L_1, \ldots, L_m$. The arguments still go through if we replace the rectangular lattice by a more general periodic lattice generated by vectors $q_1, \ldots, q_m$ (see [16] or Alexander-Auchmuty [2] for the algebraic setting). The only point of change occurs in formula (5.3) which now assumes a slightly more complicated form. Another, straightforward generalisation concerns the occurence of the spatial variables $x_1, \ldots, x_m$. In our system (1.1), $F(u)$ was assumed to be a polynomial nonlinearity, ie. of the form $(P_1, \ldots, P_n)$ with P_j a polynomial in $u_1, \ldots, u_n$. An inspection of our arguments shows that they are not affected at all if we admit F to depend explicitely on the variables $x_1, \ldots, x_m$, ie. to let the P_j be polynomials in $u_1, \ldots, u_n$ which have coefficients $a(x)$ which are $L_1, \ldots, L_m$-periodic with respect to $x_1, \ldots, x_m$ and which are sufficiently smooth.

The arguments in (III)-(V) should still go through if one adds dissipative terms to the basic system (1.1), ie. terms of the form $\sum a_{kjp}\partial_j u_p$. The necessary prerequisites should be provided by Kielhöfer [10], Kato [9] about the persistence of semigroup properties under relatively bounded perturbations.

A higher degree of difficulty appears in the case of hydrodynamic problems such as the Bénard problem. Here two difficulties have to be overcome, that of the boundary conditions in the bounded space direction, and that of the elimination of pressure. How to handle these cases is open.

VII. Applications

In this section we add some remarks and discuss some applications of the foregoing theory. For reasons of space we will refer to the literature whenever proofs are concerned which are minor variants of proofs which appear in the literature.

(A) First we note that formula (2.8), which is a consequence of corollary 2 to theorems 1,2 and of Lemma 10, somewhat loosely speaking says that if $v \in (H^2_{per}(Q_L))^n$ is an equilibrium solution of (1.1) which is unstable against small perturbations $\varphi \in (H^2(Q_L))^n$, then it is that $\varphi_0 \equiv 0$ is an equilibrium solution of (2.7), unstable against small perturbations $\varphi \in (H^2(\mathbf{R}^m))^n$. In fact, if we disregard the exceptional case where $\sigma^2_{per} \subseteq \{\lambda/\mathrm{re}(\lambda) \leq 0\}$ and $\sigma^2_{per} \cap \{\lambda/\mathrm{re}(\lambda) = 0\} \neq \emptyset$, instability with respect to perturbations $\varphi \in (H^2_{per}(Q_L))^n$ is equivalent to say that σ^2_{per} contains an eigenvalue λ with $\mathrm{re}(\lambda) > 0$. By Lemma 10 then $\lambda \in \sigma_{per}$. But since σ_{per} is just $\sigma(A_0(0)+B(0))$ in the terminology of sect. IV, we infer from corollary 2 to Theorems 1,2 that $\lambda \in \sigma_{\mathcal{L}^2}$, whence $\lambda \in \sigma^2_{\mathcal{L}^2}$ follows from Lemma 10. By the principle of linearized instability, discussed in sections II, VI, it follows that v is unstable against small perturbations $\varphi \in (H^2(\mathbf{R}^m)))^n$. Thus the passage from

L-periodic perturbations to smooth $\mathcal{L}^2$-perturbations can only decrease stability or eventually leave the stability status unchanged.

(B) In our first example we consider a space periodic equilibrium solution of the scalar equation

$$u_t = \triangle u + f(u), \tag{7.1}$$

where $f(u)$ is a polynomial in u with constant coefficients. Our main result about such solutions is provided by

Theorem 3 *Let $v \in H^2_{per}(Q_L)$ be an equilibrium solution of (7.1) which is not equal to a constant. Then both σ^2_{per} and $\sigma^2_{\mathcal{L}^2}$ contain points $\lambda > 0$, ie. v is unstable against perturbations $\varphi \in H^2_{per}(Q_L)$ and $\psi \in H^2(\mathbf{R}^m)$.*

PROOF: By definition, v is a solution of

$$(a) \qquad \triangle v + f(v) = 0.$$

By well known regularity results, based on bootstrap-arguments and the polynomiality of $f(u)$ one infers $v \in \bigcap_k H^k_{per}(Q_L)$; by embedding theorems ([1], pg. 97) which are valid in the periodic case we have that $v \in C^\infty_{per}(\mathbf{R}^m)$. We now assume that v is not a constant. Hence $\partial_j v \not\equiv 0$ for some j, e.g. $j = 1$. We can differentiate (a) with respect to x_1 so as to get

$$(b) \qquad \triangle(\partial_1 v) + (d_u f)(v)(\partial_1 v) = 0.$$

That is, $\partial_1 v$ is a nontrivial eigenfunction to the eigenvalue $\lambda = 0$ of the linearization $A = \triangle + (d_u f)(v)$, which acts on $H^2_{per}(Q_L)$ and has $H^4_{per}(Q_L)$ as domain. Next let A' be $\triangle + (d_u f)(v)$, but now on $\mathcal{L}^2(Q_L)$ with $H^2_{per}(Q_L)$ as domain. In the notation of Lemma 10, $\sigma(A) = \sigma^2_{per}$ and $\sigma(A') = \sigma_{per}$, whence $0 \in \sigma(A')$ by Lemma 10. To $-A'$ we now apply the same reasoning used in the proof of Theorem 1 in [22]. That is $-A'$ is easily seen to be selfadjoint and positivity improving in the sense of [16], pg. 201 (see also appendix I in [22]). Moreover, since $\partial_1 v$ is continuous, L-periodic and $\not\equiv 0$, each of the sets $E_- = \{x/x \in Q_L, \quad (\partial_1 v)(x) < 0\}$ and $E_+ = \{x/x \in Q_L, \quad (\partial_1 v)(x) > 0\}$ has positive Lebesgue measure, ie.

$$(c) \qquad \mu(E_-) > 0, \qquad \mu(E_+) > 0.$$

Now $\sigma(-A')$ consists of isolated eigenvalues of finite multiplicity, accumulating only at $+\infty$; therefore the leftmost point of $\sigma(-A')$ is necessarily an eigenvalue. It then follows from (c) above and from Theorem XIII, 44 in [16] that $\lambda = 0$ is not the leftmost point in $\sigma(-A')$. That is, there must be some $\lambda > 0$ in $\sigma(A')$ and hence $\sigma(A)$, by Lemma 10, implying that v is unstable against perturbations $\varphi \in H^2_{per}(Q_L)$. On the other hand, let T, T_0 be the operators introduced subsequently to Lemma 10, ie. denoting $\triangle + (d_u f)(v)$ on $H^2(\mathbf{R}^m)$ and $\mathcal{L}^2(\mathbf{R}^m)$ respectively, with domains $H^4(\mathbf{R}^m)$ and $H^2(\mathbf{R}^m)$ respectively. Since $\sigma(A') = \sigma_{per}$, $\sigma(T_0) = \sigma_{\mathcal{L}^2}$ we have $\sigma_{per} \subseteq \sigma_{\mathcal{L}^2}$ by corollary 2 to Theorems 1,2 and hence $\lambda \in \sigma(T) = \sigma^2_{\mathcal{L}^2}$ by (b) of Lemma 10, implying that v is unstable against perturbations $\varphi \in H^2(\mathbf{R}^m)$, what proves the theorem. ■

Theorem 3 depends on the parabolic maximum principle which is implicit in the notation of "positivity improving" and in Theorem XIII 44, [16]. Its extension to systems is therefore limited. Such an extension is provided by

Theorem 4 *Let $f = (f_1, \dots, f_n)$ be a gradient, ie. $f_j = \frac{\partial F}{\partial u_j}$ for some polynomial $F(u)$. Let $v \in H^2_{per}(Q_L))^n$ be an equilibrium solution of*

$$v_t = D\triangle v + f(v) \tag{7.2}$$

such that $\frac{\partial f_j}{\partial u_k}(v) \geq 0$ for $j \neq k$. Then both σ^2_{per}, $\sigma^2_{\mathcal{L}^2}$ contain points $\lambda > 0$, ie. v is unstable against perturbations $\varphi \in (H^2_{per}(Q_L))^n$ and $\psi \in (H^2(\mathbf{R}^m))^n$.

For reasons of space we omit the proof which is essentially the same as the proof of theorem 3 in [22]. The notion of "positivity improving" is now replaced by a variant, provided by definitions 1,2 in [22], and the role of Theorem XIII,44 is taken over by Lemmas 5,6,7 in [22], whose proofs carry over to the present situation practically without changes. The parabolic maximum principle is guaranteed by the well known assumption $\frac{\partial f_j}{\partial u_k}(v) \geq 0$, $j \neq k$; see Protter-Weinberger [14] or appendix I in [22]. The assumption that f is a gradient is needed in order to secure that the linearization $D\triangle + (d_u f)(v)$ is selfadjoint if considered on $(\mathcal{L}^2(Q_L))^n$ (resp. $(\mathcal{L}^2(\mathbf{R}^m))^n$), with domain $(H^2_{per}(Q_L))^n$ (resp. $(H^2(\mathbf{R}^m))^n$), an assumption crucial to all considerations in [16],[22]. The selfadjointness on the other hand guarantees the validity of the principle of linearized instability; in fact the validity of this principle in this case is assured by theorem 5 in [22] which applies to situations more general than those described by theorems 3,4. Effective constructions of smooth, 2π-periodic equilibrium solutions of (7.1) which are not constant are performed by Bandle-Tesei [3] for large classes of nonlinearities f which include a variety of polynomials; there it is also recognized (Theorem 2.5) that such a solution is unstable against 2π-periodic perturbations.

(C) Theorem 3 shows that in the scalar case nonconstant equilibrium solutions are unstable against periodic perturbations and against smooth $\mathcal{L}^2$-perturbations. If we admit also constant solutions then stability is possible. A result, proved in Eastham [6] then takes care of the situation. In our context it says that the rightmost points of σ_{per} and $\sigma_{\mathcal{L}^2}$ coincide. This implies that with the eventual exeption where the rightmost point is $= 0$, periodic and $\mathcal{L}^2$-stability (resp. instability) coincide. Thus in order to obtain results which distinguish between the two notions of stability one is forced to look at systems. This will be done in the sequel, whereby we heavily rely on the results in [21]. We consider a parabolic system of type (1.1) with two unknown functions (ie. $n = 2$) and of space dimension $m = 2$. We assume that the system has the special form

$$u_t = D\triangle u + (1+\delta)(Bu + B_2u^2 + \ldots + B_pu^p) \tag{7.3}$$

where $D = (\delta_{jk}\tau_k)$, $\tau_k > 0$ and $j,k \leq 2$; δ is a small bifurcation parameter. The B_j, $j \geq 2$ are multilinear functionals from $(\mathbf{R}^2)^j$ into $\mathbf{R}^2$ which in the usual way give rise to monomials B_ju^j, $u \in \mathbf{R}^2$. Finally, B is a real 2×2 matrix, subject to some conditions. The period L is henceforth $L = 2\pi$ or an integer multiple thereof, ie. $nL = 2n\pi$; for simplicity we retain the notation Q_L, Q_{nL} resp.. For the first of our results below we need the Sobolev spaces $H^{p,e}_{per}(Q_L) \subseteq H^p_{per}(Q_L)$ of even elements of $f \in H^p_{per}(Q_L)$, ie. satisfying $f(x_1,x_2) = f(-x_1,-x_2)$. We also need the matrices

$$M_n(k) = -k^2n^{-2}D + B, \qquad M(k) = M_1(k), \tag{7.4}$$

where $k = (k_1,k_2) \in \mathbf{Z}^2$ and $k^2 = k_1^2 + k_2^2$; the k's are referred to as wave vectors. We now state our basic assumptions on which our results are based:

(a) there is a $k_0 \in \mathbf{Z}$, $k_0 \neq 0$ such that $\lambda = 0$ is a simple eigenvalue of $M(k_0)$, and if $k \in \mathbf{Z}$ is such that $k^2 \neq k_0^2$ then $0 \notin \sigma(M(k))$,

(b) if $\lambda \in \bigcup \sigma(M(k))$, $k \in \mathbf{Z}$ and $\lambda \neq 0$ then $\mathrm{re}(\lambda) < 0$,

(c) if $\eta, \zeta \in \mathbf{R}^2$ satisfy $\eta \neq 0$, $\zeta \neq 0$ and $\eta^tM(k) = 0$, $M(k)\zeta = 0$ then $(\eta,\zeta)^{-1}(\eta, B\zeta) > 0$,

(d) $B_2 = 0$ and $(\eta, B_3\zeta^3) < 0$.

Here, η^t means the transpose, and (,) is the scalar product in $\mathbf{R}^2$. Conditions (a)-(d) are of the type encountered in bifurcation theory; how to satisfy them will be discussed in the appendix. Our results are

Theorem 5 *Assume (a)-(d). There exists a branch $\delta(r) \in \mathbf{R}$, $u(r) \in (H^{2,e}_{per}(Q_L))^2$ of equilibrium solutions of (7.3) (ie. $\delta = \delta(r), u = u(r)$) such that $\delta(r), u(r)$ are real holomorphic on $|r| < \varepsilon^*$ (some ε^*) and such that:*
(1) $u(0) = 0$, $\delta(0) = 0$, (2) $u(r)$ is not constant for $r \neq 0$, (3) $u(r)$ is asymptotically stable against small perturbations $\varphi \in (H^{2,e}_{per}(Q_L))^2$, (4) $u(r)$ is Ljapounov unstable in $(H^2(\mathbf{R}^2))^2$.

Theorem 6 *Assume (a)-(c). For sufficiently small $\varepsilon > 0$, $u_0 \equiv 0$ is an equilibrium solution of*

$$u_t = D\triangle u + (B - \varepsilon I)u + B_2u^2 + \ldots \tag{$*$}$$

which is asymptotically stable in $(H^2_{per}(Q_L))^2$ and Ljapounov unstable in $(H^2(\mathbf{R}^2))^2$.

Prior to pass to the proof of Theorems 5,6 we briefly discuss their content. In accordance with our preivious sections, asymptotic stability means all eigenvalues in the left half plane, bounded away from the imaginary axes, while Ljapounov instability means that the spectrum of the relevant linearization contains some $\lambda \in \mathbf{C}$ with $\mathrm{re}(\lambda) > 0$. The larger part of Theorem 5 is already proved in [21], theorems 1,2 where a family $\delta_j(r), u_j(r)$ of branches of equilibrium solutions of (7.3) is constructed, which under assumptions (a)-(d) satisfy (1)-(3) of Theorem 5 (an omission is on pg. 497 in [21] in the bottom line where a minus sign should preceed λ_2^{ij}). Thus what remains to be shown is clause (4) of Theorem 5. Theorem 5 is second best to what one would like to have: a nonconstant periodic equilibrium solution to (7.3) which is asymptotically stable against all small periodic perturbations but Ljapounov unstable with respect to smooth $\mathcal{L}^2$-perturbations. However such an example is likely to present technical difficulties. In fact if a nonconstant equilibrium solution u of (7.3) is at disposal which eventually satisfies $\partial_j u \not\equiv 0$, $j = 1, 2$ then the periodic spectrum of the linearization would contain the at least twofold degenerate eigenvalue $\lambda = 0$. A stability discussion would then have to take the center manifold into account, which is associated with the eigenvalue $\lambda = 0$. This seems to be delicate work which has still to be done.

The situation is different in case of Theorem 6. Here the result is in sharp contrast to the scalar case, where periodic and $\mathcal{L}^2$-stability imply each other by virtue of Easthams result. What is still open is whether the nonlinearity in (7.3) can be chosen to be a gradient; in our example this is not the case since B is not symmetrical by virtue of assumptions (a),(b).

Prior to proceed to the proofs of Theorems 5,6 we need some remarks. We set $G(u) = Bu + B_2u^2 + \ldots$; for the derivative we then have

$$(dG)(u)h = Bh + B_2uh + \ldots. \tag{7.5}$$

The solution branch $\delta(r), u(r)$ of Theorem 5 then gives rise to the expression

$$T(r) = D\triangle + (1 + \delta(r))(dG(u)(u(r)) \tag{7.6}$$

which is the linearization of the righthandside of (7.3) at $u = u(r)$ with $\delta = \delta(r)$ kept fixed. Actually we should distinguish between (5.6) and the operator it defines on different spaces. For simplicity we write $T(r)$ for any of the operators which arises if we take $(\mathcal{L}^2(Q_{nL}))^2$ or $\mathcal{L}^2(\mathbf{R}^m))^2$ as basic space with $(H^2_{per}(Q_{nL}))^2$ or $(H^2(\mathbf{R}^m))^2$ resp. as domain; with $\sigma(X/T(r))$ we denote its associated spectrum, where X is the basic space. If however $(\mathcal{L}^2(Q_{nL}))^2$ is the underlying space and $(H^2_\theta(Q_{nL}))^2$ the domain of $T(r)$ then we denote its spectrum by $\sigma_\theta((\mathcal{L}^2(Q_{nL}))^2/T(r))$.

We also need a lemma from [21] (Lemma 14), namely

Lemma* *Under assumptions (a),(c) there is an $\varepsilon_0 > 0$ and an $n_0 > 0$ with the property: for $n \geq n_0$ there is a $k \in \mathbf{Z}^2$ such that $M_n(k)$ has a simple eigenvalue $\lambda^* \geq \varepsilon_0$.*

This lemma is proved in [21] under the additional assumption $(\eta, D\zeta) \neq 0$, with

η, ζ as in (c). However this assumption is superfluous since it is already guaranteed by (a),(c) (by (A2) in [21]): since $(\eta, M(k_0)\zeta) = 0$ by (c) we have $-k_0^2(\eta, D\zeta) + (\eta, B\zeta) = 0$ whence $(\eta, D\zeta) \neq 0$ by (c) and $k_0^2 \neq 0$.

PROOF OF THEOREM 5: First recall that $T(0) = D\triangle + B$. By straightforward functional analytic arguments, based eg. on Fourier series expansions, one shows that

$$(a) \qquad \sigma((\mathcal{L}^2(Q_{nL}))^2/T(0)) = \bigcup \sigma(M_n(k)), \qquad k \in \mathbf{Z}^2$$

for any integer $n > 0$. Next let ε_0,n_0 be as in Lemma* and pick $n \geq n_0$ arbitrarily but fixed; let $\lambda^* \geq \varepsilon_0$ and $k \in \mathbf{Z}^2$ be associated with n according to Lemma*, ie. such that $\lambda^* \in \sigma(M_n(k))$. By these choices and (a) we obtain

$$(b) \qquad \lambda^* \in \sigma((\mathcal{L}^2(Q_{nL}))^2/T(0)) \quad \text{and} \quad \lambda^* \geq \varepsilon_0.$$

Next recall that by Theorem 5 we have $u(r) \in (H^2_{per}(Q_L))^2$ and hence $u(r) \in (H^2_{per}(Q_{nL}))^2$. Thus we may consider $dG(u(r))$ as a bounded linear operator on $(\mathcal{L}^2(Q_{nL}))^2$. In addition, the mappings $r \longrightarrow \delta(r) \in \mathbf{R}$ and $r \longrightarrow u(r) \in (H^2_{per}(Q_{nL}))^2$ are holomorphic by Theorem 5 and satisfy $\delta(0) = 0$, $u(0) = 0$. By the form of $dG(u(r))$ as given by (5.5) (with $u = u(r)$) this implies

$$(c) \qquad \lim_{r \to 0} \|B - (1 + \delta(r))dG(u(r))\|_{n,\infty} = 0$$

where $\| \quad \|_{n,\infty}$ denotes the operator norm on the Banach space of bounded linear operators on $(\mathcal{L}^2(Q_{nL}))^2$. By classical results from the perturbation theory of real, isolated, simple eigenvalues of closed operators with compact resolvents ([9], [17]) one infers from (c) that the following holds:

(d) there is an $\varepsilon' > 0$ with the property: if $0 \leq |r| \leq \varepsilon'$ then there exists a real λ_r in $\sigma(\mathcal{L}^2(Q_{nL}))^2/T(r))$ such that $|\lambda^* - \lambda_r| \leq \varepsilon_0/2$.

From $\lambda^* \geq \varepsilon_0$ we infer that $\lambda_r \geq \varepsilon_0/2$ for $|r| \leq \varepsilon'$. Since $T(r)$ is now considered as an unbounded linear operator having $(\mathcal{L}^2(Q_{nL}))^2$ as basic space and $(H^2_{per}(Q_{nL}))^2$ as its domain, we can apply corollary 2 to Theorems 1,2 to this situations and infer as in previous cases:

$$(e) \qquad \sigma((\mathcal{L}^2(Q_{nL}))^2/T(r)) \subseteq \sigma(\mathcal{L}^2(\mathbf{R}^2))^2/T(r)).$$

It then follows from (d),(e) that $\sigma((\mathcal{L}^2(\mathbf{R}^2))^2/T(r))$ contains an eigenvalue $\lambda_r \geq \varepsilon_9/2$. According to our exposition in section VI, (Lemma 10), this implies the Ljapounov instability of $u(r)$ against small perturbations $\varphi \in (H^2(\mathbf{R}^2))^2$, provided that $|r| \leq \varepsilon'$. By replacing the original ε^* in Theorem 5, provided by Theorems 1,2 in [21], by the eventually smaller ε', clause (4) of the theorem is also satisfied. ■

PROOF OF THEOREM 6: We assume that the matrices D, B satisfy assumptions (a)-(c). Furthermore we rely on the proof of Theorem 5. According to the assumptions (a)-(c) and clause (a) in the proof of Theorem 5 we have that

$$(i) \qquad \lambda \in \sigma((\mathcal{L}^2(Q_L))^2/T(0)) \rightarrow \mathrm{re}(\lambda) \leq 0.$$

On the other hand it follows from (b) in the proof of Theorem 5 that there is an integer $n > 0$ and a λ^* such that

$$(ii) \qquad \lambda^* \in \sigma((\mathcal{L}^2(Q_{nL}))^2/T(0)) \quad \text{and} \quad \lambda^* > 0.$$

We now apply corollary 2 to Theorems 1,2 to the present situation by considering $T(0)$ first as an operator on $(\mathcal{L}^2(Q_{nL}))^2$ with domain $(H^2_{per}(Q_{nL}))^2$ and second as an operator on $(\mathcal{L}^2(\mathbf{R}^2))^2$ with $(H^2(\mathbf{R}^2))^2$ as domain. By arguing via corollary 2 as in the previous cases we infer

$$(iii) \qquad \sigma((\mathcal{L}^2(Q_{nL}))^2/T(0)) \subseteq \sigma((\mathcal{L}^2(\mathbf{R}^2))^2/T(0))$$

Now pick $\varepsilon > 0$ so small that $\lambda^* - \varepsilon > 0$ and set $T_\varepsilon = T(0) - \varepsilon I$ (with I=Identity). The spectrum of T_ε is then obtained from the spectrum of $T(0)$ through translation by ε to the left, regardless on which of the spaces $(\mathcal{L}^2(Q_L))^2$ resp $(\mathcal{L}^2(\mathbf{R}^2))^2$ we consider $T(0)$. It then follows from (i), Lemma 10 and $\varepsilon > 0$ that $u_0 \equiv 0$ is an asymptotically stable equilibrium solution of

$$(iv) \qquad u_t = D\triangle u + (B - \varepsilon I)u + B_2 u^2 + \dots$$

if one considers (iv) on $(H^2_{per}(Q_L))^2$ as basic space with $\mathrm{dom}(D\triangle) = (H^4_{per}(Q_L))^2$. On the other hand it follows from (ii),(iii) section VI and $\lambda^* - \varepsilon > 0$ that $u_0 \equiv 0$ is a Ljapunov unstable equilibrium solution of (iv), if one considers (iv) on the space $(H^2(\mathbf{R}^2))^2$ with $\mathrm{dom}(D\triangle) = (H^4(\mathbf{R}^2))^2$. This is precisely the claim of Theorem 6. ■

Corollary *There is a $\theta \in [0, 2\pi]^2$ and a $\lambda^* > 0$ such that $\lambda^* \in \sigma_\theta((\mathcal{L}^2(Q_L))^2/T(0))$, while re$(\lambda) \leq 0$ for all $\lambda \in \sigma((\mathcal{L}^2(Q_L))^2/T(0))$.*

PROOF: The second part of the statement is just clause (i) in the proof of Theorem 6. As to the first part we invoke the basic formula in corollary 2 to Theorems 1,2 according to which

$$(*) \qquad \sigma((\mathcal{L}^2(\mathbf{R}^2))^2/T(0)) = \bigcup \sigma_\theta((\mathcal{L}^2(Q_L))^2/T(0)), \qquad \theta \in [0, 2\pi]^2.$$

By combining $(*)$ with (ii), (iii) in the proof of Theorem 5, the existence of θ follows. ■

Remark: As noted earlier, the corollary is false in case of single scalar equation by virtue of Easthams result.

VIII. Appendix

It remains to show that the matrices D, B, B_2, B_3 can indeed be chosen so that assumptions (a)-(d) are satisfied, a problem not considered in[21]. This amounts to find D, B which satisfy (a)-(c); one sets $B_2 = 0$ and it is then easy to find B_3 so that (d) is satisfied. To start with, we introduce some notation. As before, $D = (\delta_{jk}\tau_k), j, k \leq 2$ and $\tau_k \geq 2$. B is a real 2×2 matrix whose first row is (a, b), whose seconde row is (c, d). We now pick a wave vector $k_0 \in \mathbf{Z}$ with $k_0^2 > 0$; throughout what follows k_0 is fixed and supposed to play the role of k_0 in (a)-(c). With $k_- \in \mathbf{Z}$ we denote a fixed wave vector such that

$$(8.1) \qquad k \in \mathbf{Z} \quad \text{and} \quad k^2 < k_0^2 \quad \text{iff} \quad k^2 \leq k_-^2.$$

Finally let for any $k \in Z$ $\triangle(k)$ be the determinant of $M(k)$, ie. $\triangle(k) = \det(M(k))$. We now impose three conditions on B, D, namely:

(A.1) $a < 0$ and $a + d < 0$

(A.2) $\triangle(k_0) = 0$

(A.3) $k_0^2 + k_-^2 < \frac{a}{\tau_1} + \frac{d}{\tau_2} < 2k_0^2$.

Lemma 11 *If the matrices D, B satisfy (A.1)-(A.3) then (a)-(c) holds.*

It is advantageous to reduce the Lemma to two propositions.

Proposition 1 *Let D, B satisfy* (A.1)-(A.3). *Then conditions* (a), (b) *hold.*

PROOF: One has to discuss the eigenvalues of $M(k)$ for arbitrary $k \in \mathbf{Z}$, that is the roots of $\det(M(k) - \lambda I) = 0$. Since $\triangle(k_0) = 0$ we have that $\lambda = 0$ is an eigenvalue of $M(k_0)$. The second eigenvalue of $M(k_0)$ is obtained by computation and is given by

$$(1) \qquad \lambda = (a + d) - k_0^2(\tau_1 + \tau_2)$$

According to (A.1) this implies $\lambda < 0$. Next note that by virtue of $\triangle(k_0) = 0$ we have that

$$(2) \qquad \det(B) = k_0^2(a\tau_2 + d\tau_1) - k_0^4\tau_1\tau_2.$$

Using (2) we get for $\triangle(k)$ the expression

$$(3) \qquad \triangle(k) = (k^2 - k_0^2)\tau_1\tau_2 \left\{ -\left(\frac{a}{\tau_1} + \frac{d}{\tau_2}\right) + k^2k_0^2 \right\}.$$

We now compare (3) with assumption (A.3). A distinction of cases according to whether $k^2 < k_0^2$ or $k_0^2 < k^2$ then easily yields:

(4) $$\text{if} \quad k^2 \neq k_0^2 \quad \text{then} \quad \triangle(k) > 0.$$

For the roots of $\det(M(k) - \lambda I) = 0$ we find

(5) $$2\lambda = -\chi \pm (\chi^2 - 4\triangle(k))^{\frac{1}{2}}$$

where $\chi = |a+d-k^2(\tau_1+\tau_2)|$.By virtue of (4) this implies $\mathrm{re}(\lambda) < 0$, what proves the proposition. ∎

Prior to procced to the next proposition it is advantageous to introduce$\alpha, \beta, \gamma, \delta$ according to

(8.2) $$a = \alpha\tau_1, \qquad b = \beta\tau_2, \qquad c = \gamma\tau_1, \qquad d = \delta\tau_2.$$

In addition we introduce a fixed parameter $\mu > 0$ and set

(8.3) $$\gamma = \mu(\alpha - k_0^2), \qquad \delta = \mu\beta + k_0^2.$$

It is then clear that by expressing $a, \ldots, d$ via (8.2), (8.3), condition (A.2) is automatically satisfied. The other conditions, now expressed in terms of α, β now become

(A'.1) $\alpha < 0$ and $\alpha\tau_1 + \tau_2(\mu\beta + k_0^2) < 0$,

(A'.2) $k_-^2 < \alpha + \mu\beta < k_0^2$.

In order to satisfy (A'.1),(A'.2) one first fixes $\tau_1 > 0$, $\alpha < 0$, then we pick $\beta > 0$ so that (A'.2) is satisfied and finally one choses $\tau_2 > 0$ so small, that the second part of (A'.1) holds. The two vectors η, ζ in condition (c) are determined up to a scalar multiple. As representants we choose:

(8.4) $$\zeta^t = (-\beta\tau_2, \tau_1(\alpha - k_0^2)), \qquad \eta^t = (-\mu, 1)$$

where the superscript t means the transpose. Computation based on (8.2),(8.4) shows that $\eta^t M(k_0) = 0$, $M(k_0)\zeta = 0$.

Proposition 2 $(\eta, \zeta) < 0$ *and* $(\eta, B\zeta) < 0$.

PROOF: By computation we find that

$$(\eta, \zeta) = \alpha\tau_1 + \mu\beta\tau_2 - \tau_1 k_0^2 < 0$$

by virtue of (A'.1). On the other hand we have that $M(k_0)\zeta = (-k_0^2 D + B)\zeta = 0$ whence $(\eta, B\zeta) = k_0^2(\eta, D\zeta)$. Now

$$(\eta, D\zeta) = \tau_1\tau_2(\mu\beta + \alpha - k_0^2) < 0$$

by virtue of (A'.3) whence $(\eta, B\zeta) < 0$. This proves the claim of the proposition. ∎

Proof of Lemma 11: Immediate via propositions 1,2 and(8.2), (8.3), (A'.1), (A'.2).

■

Remark: By arguments similiar to the above one can show that the matrices D, B are stable against small perturbations in the following sense: if B' is close to B (in some suitable metric) then there exists a real γ close to $\lambda = 1$ such that $D, \gamma B'$ still satisfy (A.1)-(A.3) and hence (a)-(c).

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