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Periodic solutions of asymptotically linear dynamical systems*

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0 Introduction

Let $V = V(x,t)$, $x \in \mathbf{R}^n$ and $t \in \mathbf{R}$, be a C^2 real function T_0-periodic in the t variable.

We shall study the following equation:

$$\ddot{x} + V'_x(x,t) = 0, \quad x = x(t) \qquad kT_0\text{-periodic curve } (k \in \mathbf{N}) \text{ in } \mathbf{R}^n \tag{0.1}$$

where $V'_x(x,t)$ denotes the gradient of V with respect to x.

We assume that the potential function V is asymptotically quadratic, i.e.

$$V(x,t) = (1/2)\,(A_\infty x|x) + U(x,t) \tag{0.2}$$

where $(\,|\,)$ denotes the standard inner product in $\mathbf{R}^n$, $A_\infty = A_\infty(t)$ is a symmetric, real, T_0-periodic $n \times n$ matrix and $U(x,t)$ is a function which is bounded, having bounded gradient U'_x and whose Hessian matrix (with respect to x) $U''_{xx}(x,t)$ tends to zero (uniformly in t) as $|x|$ goes to infinity.

We shall also assume that

$$V'_x(0,t) = 0 \qquad \text{and} \qquad V(0,t) = 0 \tag{0.3}$$

then 0 is a solution of (0.1).

Assumptions (0.2), (0.3) allow to consider the linearized equations at ∞ and at zero which are respectively

$$\ddot{x} + A_\infty(t)x = 0 \qquad \text{and} \qquad \ddot{x} + A_0(t)x = 0 \tag{0.4}$$

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where $A_\infty(t)$ is the matrix introduced in (0.2) and $A_0(t)$ denotes the Hessian matrix $V''_{xx}(0,t)$ of V at $x = 0$.

We denote by $L^T_\infty = L_\infty$ (respectively $L^T_0 = L_0$) the self- adjoint realization in L^2 with $T = kT_0$-periodicity conditions of the operator $x \to -\ddot{x} - A_\infty(t)x$ (respectively of the operator $x \to -\ddot{x} - A_0(t)x$).

Problem (0.1), in the framework of hamiltonian systems, has been studied under nonresonance conditions at ∞ (see [1, 6]), i.e. assuming that L^T_∞ is invertible for all $T = kT_0,\ k \in \mathbf{N}$.

The nonresonance condition at infinity permits to get suitable a priori bounds on the solutions of (0.1) and consequently the action functional related to (0.1) (see (3.1) in section 3) satisfies the compactness Palais-Smale condition.

Some results are available also in the strong resonance case, i.e. when the function U in (0.2) goes to zero at infinity (see [3], [5]).

The aim of this paper is to prove existence results of (0.1) without the non-resonance assumption at ∞ and without the strong resonance assumption.

In order to state the results we need to recall the definition of twist number (see section 2).

We set

$$\tau_0 = \lim_{k\to\infty} j(A_0, kT_0)/kT_0\ , \qquad \tau_\infty = \lim_{k\to\infty} j(A_\infty, kT_0)/kT_0 \tag{0.5}$$

where $j(A_0, kT_0)$ (respectively $j(A_\infty, kT_0)$) is the number of the negative eigenvalues, counted with their multiplicity, of L^T_0, $T = kT_0$ (respectively L^T_∞, $T = kT_0$). The limits τ_0 and τ_∞ in (0.5) exist (see e.g. [2]) and they are called twist number at 0 and at ∞ respectively.

Now we can state the theorem we shall prove in this paper

Theorem 0.1 *Assume that V satisfies assumptions (0.2), (0.3). Assume moreover that there is not resonance at the origin (i.e. the linearized operator at 0 L^T_0 is invertible for all $T = kT_0$, $k \in \mathbf{N}$) and that $\tau_0 \neq \tau_\infty$ (see (0.5)). Then equation (0.1) has a non zero solution for all $k \in \mathbf{N}$ s. t.*

$$kT_0 > (2n+1)/|\tau_0 - \tau_\infty|\ .$$

1 The Morse inequalities

In this section we shortly review some basic facts on Morse theory. In particular we recall some recent results obtained for functionals with degenerate critical points (see [4]).

Let E be a real Hilbert space and J a C^2 functional on E. We denote by K the set of the critical points of J

$$K = \{x \in E \ \mid\ J'(x) = 0\}\ .$$

Definition 1.1 *Let $x \in K$ and let $J''(x)$ be the second Fréchet differential of J at x.*

The Morse index $m(x, J)$ of x (for J) is the cardinal number (possibly infinite) defined by

$$m(x,J) = \max\{\dim(S) \mid S \text{ is a linear subspace of } E \text{ s.t. } \langle J''(x)v, v\rangle < 0 \text{ for any } v \in S,\ v \neq 0\}. \tag{1.1}$$

Moreover the large Morse index $m^*(x, J)$ of x is defined by

$$m^*(x,J) = m(x,J) + \dim \operatorname{Ker} J''(x).$$

Usually we shall write $m(x)$, $m^*(x)$ instead of $m(x, J)$, $m^*(x, J)$.

The critical point x is called non degenerate if $\operatorname{Ker} J''(x) = \{0\}$.

Now if $a < b$ (b possibly infinite) we set

$$J^b = \{x \in E \mid J(x) < b\}, \qquad J_a^b = \{x \in E \mid a < J(x) < b\}.$$

If $a < b$ are real numbers we set

$$K_a^b = \{x \in K \mid a \leq J(x) \leq b\}. \tag{1.2}$$

Moreover we set

$$P_t(J^b, J^a) = \sum_{q \geq 0} \dim H_q(J^b, J^a, \mathbf{K})\, t^q \tag{1.3}$$

where $H_q(J^b, J^a, \mathbf{K})$ denotes the q-th singular relative homology of J^b with respect to J^a with coefficients in some field $\mathbf{K}$. $P_t(J^b, J^a)$ is a formal series whose coefficients are cardinal numbers called "Betti numbers".

We recall the Morse relations

Theorem 1.1 *Let a, b (b possibly infinite) be regular values for J (i.e. if $x \in K$ then $J(x) \neq a, b$). Assume moreover the following:*

- *J satisfies the Palais-Smale condition in (a, b) (i.e. any sequence $\{x_n\} \subseteq J_a^b$, s.t. $J(x_n)$ is bounded, contains a convergent subsequence)* (1.4)
- *Any $x \in K_a^b$ is nondegenerate and has finite Morse index $m(x)$* (1.5)

Then

$$\sum_{x \in K_a^b} t^{m(x)} = P_t(J^b, J^a) + (1+t)\, Q(t) \tag{1.6}$$

where $Q(t)$ is a formal series whose coefficients are cardinal numbers (possibly infinite).

We point out that since J satisfies the (P.S.) condition K_a^b is finite, if b is a real number, and it is at most countable, if $b = +\infty$.

If assumption (1.5) is not satisfied a relation between the set of the critical points K_a^b and the Poincarè polynomial $P_t(J^b, J^a)$ still holds. In fact, by using a generalized Morse index, the following theorem can be proved (see [4]).

Theorem 1.2 *Let a, b (b possibly infinite) be regular values for J. Assume that J satisfies the* (P.S) *condition (1.4). Assume moreover that for any $x \in K_a^b$, x degenerate, 0 is an isolated eigenvalue of $J''(x)$ having finite multiplicity. Then a formal series with positive coefficients exists*

$$i(K_a^b) = \sum_q a_q t^q$$

satisfying the following properties:

$$(a_q \neq 0) \implies (\textit{there exists } x \in K_a^b \textit{ s.t. } m(x) \leq q \leq m^*(x)) \tag{1.7}$$

$$(\textit{there exists } x \in K_a^b \textit{ non degenerate and } m(x) < \infty) \implies (a_{m(x)} \neq 0) \tag{1.8}$$

$$i(K_a^b) = P_t(J^b, J^a) + (1 + t)\, Q(t) \tag{1.9}$$

where $Q(t)$ is a formal series whose coefficients are cardinal numbers (possibly infinite).

We recall that if any $x \in K_a^b$ is non degenerate then $i(K_a^b)$ reduces to the Morse polynomial

$$i(K_a^b) = \sum_{x \in K_a^b} t^{m(x)} \, .$$

The following corollary holds:

Corollary 1.1 *Let a, b and J as in theorem 1.2. Let $y \in K_a^b$ be a non degenerate critical point with $m(y) < \infty$. Assume that*

$$P_t(J^b, J^a) = \sum_q \beta_q t^q \qquad \textit{with } \beta_{m(y)} = 0\,. \tag{1.10}$$

Then J has a critical point $x \neq y$ such that

$$m(x) \leq m(y) + 1 \qquad \textit{and} \qquad m(y) - 1 \leq m^*(x)\,. \tag{1.11}$$

Proof By (1.8) we have

$$i(K_a^b) = \sum_q a_q t^q \qquad \text{with } a_{m(y)} \neq 0\,. \tag{1.12}$$

By (1.9), (1.10), (1.12) we deduce that $(1+t)Q(t)$ "contains" the monomial $t^{m(y)}$, i.e.

$$(1+t)\,Q(t) \;=\; \sum_q b_q t^q + \sum_q b_q t^{q+1}$$

with

$$b_{m(y)} \neq 0 \qquad \text{or} \qquad b_{m(y)-1} \neq 0\,. \tag{1.13}$$

Clearly by (1.9), (1.12) and (1.13) we have

$$\begin{aligned} (\,b_{m(y)} \neq 0\,) \;&\Longrightarrow\; (\,b_{m(y)}t^{m(y)+1} \neq 0\,) \;\Longrightarrow\; (\,a_{m(y)+1} \neq 0\,) \\ (\,b_{m(y)-1} \neq 0\,) \;&\Longrightarrow\; (\,a_{m(y)-1} \neq 0\,)\,. \end{aligned}$$

Then by (1.7) we deduce that there exists $x \in K_a^b$ s.t.

$$m(x) \,\leq\, m(y)+1 \,\leq\, m^*(x) \qquad \text{or} \qquad m(x) \,\leq\, m(y)-1 \,\leq\, m^*(x)\,. \tag{1.14}$$

Since y is nondegenerate (1.14) implies that $y \neq x$. Moreover from (1.14) we deduce that

$$m(x) \,\leq\, m(y)+1 \qquad \text{and} \qquad m(y)-1 \,\leq\, m^*(x)\,.$$

□

2 Twist number

In this section we recall some basic facts on the twist number. For a more extensive treatment we refer to [2], [7, 8], [6].

Let $A = A(t)$ be a family of real symmetric $n \times n$ matrices depending continuously on t and T_0-periodic. Consider the second order, linear differential operator

$$x \,\longrightarrow\, -\ddot{x} - A(t)x \tag{2.1}$$

and denote by L^T, $T = kT_0$ ($k \in \mathbf{N}$), its self-adjoint realization in the L^2 space with T-periodicity conditions.

L^T has discrete spectrum with only a finite number of negative eigenvalues. We set

$$j(A,T) \;=\; \text{number of negative eigenvalues of } L^T \text{ counted with their multiplicity.} \tag{2.2}$$

We call $j(A,T)$ the CZ (Conley-Zehnder) index in $[0,T]$ relative to the equation

$$\ddot{x} + A(t)x \,=\, 0\,. \tag{2.3}$$

It is possible to prove that the number

$$\tau \,=\, \tau(A) \,=\, \lim_{k\to\infty} j(A,kT_0)/kT_0 \tag{2.4}$$

is well defined and it is called twist number of the operator (2.1).

The proof of (2.4) in this context can be found in [2]. A formula like (2.4) has been previously proved by Ekeland in the context of convex Hamiltonian systems [7, 8].

Now if we set

$$j^*(A,T) = j(A,T) + \dim \operatorname{Ker} L^T \tag{2.5}$$

it is possible to prove that (see e.g. [2])

$$\tau(A)T - n \leq j(A,T) \leq j^*(A,T) \leq \tau(A)T + n\,. \tag{2.6}$$

In order to get another characterization of the twist number we need to introduce some definitions.

Let $W(t)$ be the Wronskian matrix relative to the equation (2.3), i.e. the matrix which sends the initial data $(x(0), \dot{x}(0))$ to $(x(t), \dot{x}(t))$.

The complex eigenvalues $\lambda_1(t), \ldots, \lambda_n(t)$ of $W(t)$ are continuous functions of t. Then, if we set $\lambda_j(t) = \rho_j(t) \exp i\vartheta_j(t)$, $\rho_j(0) = 1$, $\vartheta_j(0) = 0$, the numbers $\vartheta_j(t)$ are uniquely determined.

The map $W(T_0)$ from C^{2n} to C^{2n} is called the Poincaré map or the monodromy map. The eigenvalues $\lambda_j(T_0)$ of $W(T_0)$ are usually called Floquet multipliers of (2.3).

We shall consider the Floquet multipliers on the unit circle S^1

$$\exp i\omega_1 T_0, \ldots, \exp i\omega_p T_0\,; \qquad \omega_1, \ldots, \omega_p \in \mathbf{R}\,.$$

The numbers $i\omega_j T_0 = i\vartheta_j(T_0)$ $(j = 1, \ldots, p)$ are called Floquet exponents of (2.3) and ω_j are the fundamental frequencies of (2.3).

Proposition 2.1 *Let $\omega_1, \ldots, \omega_p$ be the fundamental frequencies of (2.3). Then L^T is invertible for all $T = kT_0$ $(k \in \mathbf{N})$, if and only if for all $j = 1, \ldots, p$ the numbers $\omega_j T_0/2\pi$ are irrational (non resonance condition).*

Proof Assume that there exists a positive integer k such that L^T, with $T = kT_0$, is not invertible. This amounts to say that there exists a nontrivial kT_0-periodic solution of (2.3), then there exists $q \in \{1, \ldots, p\}$ s.t. $\exp i\omega_q kT_0 = 1$ and this means that $\omega_q T_0/2\pi$ is rational.

□

Remark 2.1 *It can be proved that $\tau(A) = \omega_1 + \ldots + \omega_p$ $(\omega_1, \ldots, \omega_p$ being the fundamental frequencies of (2.3)).*

3 Proof of Theorem 0.1

Problem (0.1) can be reduced to the study of the critical points of the C^2 functional

$$f(x) = \int_0^T (1/2\,|\dot{x}(t)|^2 - V(x(t),t))\,dt\,, \qquad T = kT_0,\ x \in H^1_T \tag{3.1}$$

where H^1_T is the Sobolev space of the absolutely continuous T- periodic curves in $\mathbf{R}^n$ with square integrable derivative.

We denote by $|\ |_1$ the standard norm in H^1_T. We denote by L_∞ the linearized operator at infinity, i.e. the operator

$$x \in H^1_T \ \rightarrow \ -\ddot{x} - A_\infty(t)x \tag{3.2}$$

where the matrix $A_\infty(t)$ has been introduced in (0.2).

The positive (respectively negative) span of L_∞ will be denoted by H^+ (respectively H^-). We shall set $H^0 = \operatorname{Ker} L_\infty$. Then

$$H^1_T \ = \ H^+ \oplus H^- \oplus H^0\,. \tag{3.3}$$

We denote by P^+ the projection operator on H^+ and by P the projection operator on $H^- \oplus H^0$, moreover we set

$$x \ = \ x^+ + x^- + x^0\,, \qquad \text{with } x^+ \in H^+,\ x^- \in H^-,\ x^0 \in H^0\,.$$

The standard Sobolev norm $|\ |_1$ in H^1_T is equivalent to the norm

$$|x|^* \ = \ (<L_\infty x^+, x^+>)^{1/2} + (-<L_\infty x^-, x^->)^{1/2} + |x^0|_1\,. \tag{3.4}$$

Clearly the functional (3.1) can be written

$$f(x) \ = \ (1/2)<L_\infty x, x> - \int_0^T U(x,t)\,dt \qquad U \in H^1_T \tag{3.5}$$

where $U = U(x,t)$ has been introduced in (0.2).

Since U is bounded it is not difficult to realize that (3.5) does not satisfy in general the Palais-Smale condition when $\operatorname{Ker} L_\infty \neq \{0\}$.

To overcome this difficulty we shall add to (3.4) a penalizing term which is "sensitive" for large values of $|Px|_1$.

More precisely for $R > 0$ we set

$$\phi(t) \ = \ \begin{cases} t^4 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} \qquad g_R(x) \ = \ |Px|_1^2 - R$$

and

$$F_R^+(x) \ = \ f(x) + \phi(g_R(x))\,, \qquad F_R^-(x) \ = \ f(x) - \phi(g_R(x)) \qquad x \in H^1_T\,.$$

Lemma 3.1 *If the gradient U'_x of U is bounded, then for any $R > 0$ the functionals F_R^+, F_R^- satisfy the Palais-Smale condition, namely any sequence $\{x_n\}$ in H^1_T s.t. $F_R^+(x_n)$ (respectively $F_R^-(x_n)$) is bounded and $dF_R^+(x_n) \to 0$ (respectively $dF_R^-(x_n) \to 0$) contains a convergent subsequence.*

Proof Let $\{x_n\}$ be a sequence in H^1_T such that

$$F^+_R(x_n) = (1/2)\langle L_\infty x_n, x_n\rangle - \int_0^T U(x_n,t)\,dt + \phi(g_R(x_n)) \text{ is bounded} \tag{3.6}$$

and

$$dF^+_R(x_n) = L_\infty x_n - U'_x(x_n,t) + 2\phi'(g_R(x_n))\,Px_n = v_n \tag{3.7}$$

where v_n goes to zero in the dual of H^1_T.

Set

$$x_n = x^+_n + x^-_n + x^0_n \qquad x^+_n \in H^+,\ x^-_n \in H^-,\ x^0_n \in H^0\,. \tag{3.8}$$

Testing (3.7) with x^+_n we get

$$\langle L_\infty x_n, x^+_n\rangle - \langle U'_x(x_n,t), x^+_n\rangle = \langle v_n, x_n\rangle\,. \tag{3.9}$$

Since U'_x is bounded and v_n goes to zero, from (3.9) we deduce that $|x_n|^*$ (see (3.4)) is bounded.

Testing now (3.7) with $Px_n = x^-_n + x^0_n$ we get

$$-(|x^-_n|^*)^2 - \langle U'_x(x_n,t), Px_n\rangle + 2\phi'(g_R(x_n))\,|Px_n|_1^2 = \langle v_n, Px_n\rangle\,. \tag{3.10}$$

arguing by contradiction assume that for a suitable subsequence

$$|Px_n|_1^2 \to \infty\,. \tag{3.11}$$

As a consequence

$$g_R(x_n) = |Px_n|_1^2 - R \qquad \text{for large } n.$$

Then from (3.10) we have

$$-(|x^-_n|^*)^2 - \langle U'_x(x_n,t), Px_n\rangle + 8(|Px_n|_1^2 - R)^3|Px_n|_1^2 = \langle v_n, Px_n\rangle$$

which contradicts (3.11).

Finally we conclude that $|x_n|_1 = (|x^+_n|_1^2 + |Px_n|_1^2)^{1/2}$ is bounded.

Now standard arguments show that $\{x_n\}$ contains a strongly convergent subsequence.

A similar proof shows that F^-_R satisfies the Palais-Smale condition.

□

Lemma 3.2 *Let U'_x be bounded. Then there exists $M > 0$ such that for all $R > 0$ and for any critical point y of F^+_R (or F^-_R) we have*

$$|P^+y|_1 \leq M\,.$$

Proof Let y be a critical point of F_R^+, then

$$dF_R^+(y) = L_\infty y - U_x'(y,t) + 2\phi'(g_R(y))\, Py = 0\,.$$

Testing with $y^+ = P^+y$ we get

$$(|y^+|^*)^2 - \langle U_x'(y,t), y^+\rangle = 0\,. \tag{3.12}$$

Since U_x' is bounded, (3.12) implies the conclusion.

Analogous proof holds for F_R^-.

□

It will be convenient to set

$$\begin{aligned} m(\infty) &= j(A_\infty, T)\,, \quad & m^*(\infty) &= j^*(A_\infty, T) \\ m(0) &= j(A_0, T)\,, & m^*(0) &= j^*(A_0, T) \end{aligned} \tag{3.13}$$

where j and j^* have been defined in (2.2), (2.5).

Clearly

$$m(\infty) = \dim H^- \qquad \text{and} \qquad m^*(\infty) = \dim(H^- \oplus H^0)\,. \tag{3.14}$$

A critical point y of F_R^- (respectively F_R^+) with $|Py|_1$ sufficiently large has a Morse index which is "large" (respectively "small") when compared with $m(\infty)$. More precisely the following lemma holds

Lemma 3.3 *Let V satisfy assumption (0.2). Then there exists $Q > 0$ (Q independent of R) s.t. if y is a critical point of F_R^- (respectively F_R^+) with $|Py|_1 \geq Q$, then we have*

$$m(\infty) \leq m(y) \tag{3.15}$$

$$(\textit{respectively } m^*(y) \leq m^*(\infty)) \tag{3.16}$$

where $m(y) = m(y, F_R^-)$ and $m^*(y) = m^*(y, F_R^+)$ (see Def. 1.1).

Proof For simplicity we set

$$F_R^- = F_R$$

and denote by $F_R''(y)$ (respectively $f''(y)$) the second Fréchet differential of F_R (respectively f) at y.

Clearly for all $v \in H_T^1$ we have

$$F_R''(y)[v,v] = f''(y)[v,v] - 2\phi'(g_R(y))(Pv|v) - 4\phi''(g_R(y))(Py|v)^2 \leq f''(y)[v,v]\,. \tag{3.17}$$

We shall assume $H^- \neq \{0\}$, otherwise (3.15) is trivial.

We show that, if y is critical point of F_R and $|Py|_1$ is large enough, then

$$\forall\, v \in H^-,\ v \neq 0 \ :\ f''(y)[v,v] < 0\,. \tag{3.18}$$

(3.18) and (3.17) easily will imply (3.15).

The norm $|\ |^*$ defined in (3.4) is equivalent to the norm $|\ |_1$ in H^1_T, then there exists a constant $c > 0$ s.t.

$$|v|^2_\infty \leq c(|v|^*)^2 \qquad \text{for all } v \in H^1_T$$

where $|v|_\infty$ denotes the L^∞ norm.

By the above inequality we easily get, $\forall\ v \in H^-$

$$\begin{aligned} f''(y)[v,v] &= <L_\infty v, v> - \int_0^T (U''_{xx}(y,t)v|v)\,dt \leq \qquad (3.19) \\ &\leq -(|v|^*)^2 + \left(\int_0^T |U''_{xx}(y,t)|\,dt\right)|v|^2_\infty \leq (|v|^*)^2\left(c\int_0^T |U''_{xx}(y,t)|\,dt - 1\right). \end{aligned}$$

Since y is a critical point of F^+_R, by lemma 3.2 we have

$$y = y^+ + Py\,, \qquad |y^+|_1 \leq M \qquad (3.20)$$

where M is independent of R.

Moreover assumption (0.2) implies that

$$|U''_{xx}(x,t)| \rightarrow 0 \qquad \text{for } |x| \rightarrow +\infty\,. \qquad (3.21)$$

By (3.20), (3.21) and using lemma 3.2 in [3], we deduce that there exists $Q > 0$ such that

$$(\,|Py|_1 > Q\,) \implies \left(\int_0^T |U''_{xx}(y,t)|\,dt < 1/c\right). \qquad (3.22)$$

Then, if $|Py| > Q$, (3.18) easily follows from (3.22) and (3.19) and the proof of (3.15) is complete.

Let us now prove (3.16). To this end we set for simplicity

$$F^+_R = F_R$$

and evaluate for $v \in H^1_T$

$$F''_R(y)[v,v] = f''(y)[v,v] + 2\phi'(g_R(y))(Pv|v) + 4\phi''(g_R(y))(Py|v)^2 \geq f''(y)[v,v]. \qquad (3.23)$$

We show that, if y is critical point of F_R and $|Py|_1$ is large enough, then

$$\forall\ v \in H^+,\ v \neq 0\ :\ f''(y)[v,v] > 0\,. \qquad (3.24)$$

Clearly (3.24) and (3.23) easily will imply (3.16).

As in the proof of (3.19) we get

$$\begin{aligned} \forall\ v \in H^+ \quad f''(y)[v,v] &= <L_\infty v, v> - \int_0^T (U''_{xx}(y,t)v|v)\,dt \geq \\ &\geq (|v|^*)^2\left(1 - c\int_0^T |U''_{xx}(y,t)|\,dt\right). \qquad (3.25) \end{aligned}$$

Arguing as in the proof of the first part we deduce that (3.25) implies (3.24)

$\square$

We shall set

$$F_+ = F_Q^+, \qquad F_- = F_Q^-$$

Q being the positive number introduced in Lemma 3.3.

We shall evaluate the relative homology of H_T^1 with respect to suitable sublevels of F_-. The following result holds

Proposition 3.1 *Assume that U and U_x' are bounded. Then there exists $c < 0$ sufficiently small in order that*

$$P(H_T^1, F_-^c) = t^{m^*(\infty)}$$

where $F_-^c = \{ x \in H_T^1 \mid F_-(x) < c \}$, $P(H_T^1, F_-^c)$ denotes the Poincaré polynomial of H_T^1 relatively to F_-^c and $m^*(\infty)$ has been defined in (3.13).

Proof The proof is divided in various steps. First we introduce the "cylinder"

$$C = \{ (u,w) \ : \ u \in H^+, \ w \in H^- \oplus H^0, \ |w|_1 \leq 1 \}$$

and its boundary

$$\partial C = \{ (u,w) \in C \ : \ |w|_1 = 1 \} .$$

Step 1. There exists $K > \sqrt{Q}$ such that for all $(u,w) \in \partial C$ the real map $\sigma(s) = F_-(u + sw)$, $s \in [K, +\infty[$, is strictly decreasing.

In fact, for $(u,w) \in \partial C$ and $s > \sqrt{Q}$, we have

$$\begin{aligned} \sigma'(s) \ &= \ <dF_-(u+sw), w> \ = \ <L_\infty(u+sw), w> - \int_0^T (U_x'(u+sw)|w)\, dt - \\ & 8(s^2 - Q)^3 s |w|_1^2 \ \leq \ |U_x'|_\infty \, |w|_{L^1} - 8s(s^2 - Q)^3 . \end{aligned} \tag{3.26}$$

Then if $s > K$, with K sufficiently large, we have

$$\sigma'(s) < 0 .$$

Step 2. Let

$$c < -|U|_\infty T - 2K^8 . \tag{3.27}$$

Then for all $x = (u,w) \in \partial C$

$$\text{there exists only one } s = s(x) > K \text{ s.t. } F_-(u + sw) = c . \tag{3.28}$$

In fact for all $(u,w) \in \partial C$ we have

$$\sigma(K) = F_-(u + Kw) \geq -|U|_\infty T - (K^2 - Q)^4 > c .$$

Then, since $\sigma(s)$ is strictly decreasing in $[K, +\infty[$ (see Step 1) and $\sigma(s)$ diverges to $-\infty$ as t goes to $+\infty$, the conclusion easily follows.

Step 3. Consider the map ψ defined on the cylinder C by

$$\forall\ (u,w) \in C \qquad \psi(u,w) \ = \begin{cases} u & \text{if } w=0 \\ u+s(x)w & \text{if } w \neq 0 \end{cases} \tag{3.29}$$

where $x = (u, w/|w|_1)$ and $s(x)$ has been defined in (3.28).

It can be seen that ψ is an homeomorphism of C onto

$$H^1_T / F^c_- \ = \{\, x \ \mid \ F_-(x) \geq c \,\}\,.$$

Step 4. Let us finally evaluate the Poincaré polynomial $P(H^1_T, F^c_-)$. We denote by B the unit ball in $H^- \oplus H^0$ and by ∂B its boundary. Then we have

$P(H^1_T, F^c_-) =$ (by excision) $= P(H^1_T/F^c_-, \partial F^c_-) =$ (by Step 3) $= P(C, \partial C) =$
$= P(B \times H^+, \partial B \times H^+) = P(B, \partial B) =$ (since $\dim(H^- \oplus H^0) = m^*(\infty)$) $=$
$= t^{m^*(\infty)}$.

□

Finally we are ready to prove Theorem 0.1

Proof of Theorem 0.1. In order to prove Theorem 0.1 we distinguish two cases: $\tau_0 > \tau_\infty$ and $\tau_\infty > \tau_0$.

Assume first that

$$\tau_0 \ > \ \tau_\infty\,. \tag{3.30}$$

In this case we shall consider the penalized functional F_+.

The assumption

$$T \ > \ (2n+1)/(\tau_0 - \tau_\infty)$$

and the inequalities (2.6), used with $A = A_\infty$ and $A = A_0$, imply that

$$m^*(\infty) \ < \ m(0) - 1\,. \tag{3.31}$$

Since F_+ is bounded from below, the sublevel

$$F^a_+ \ = \{\, x \ \mid \ F_+(x) \leq a \,\}$$

is empty if $a \in \mathbf{R}$ is sufficiently small. Then

$$P(H^1_T, F^a_+) \ = \ P(H^1_T, \emptyset) \ = \ 1\,. \tag{3.32}$$

By (3.31) $m(0) > 0$ then, by (3.32) $t^{m(0)}$ is not "contained" in $P(H^1_T, F^a_+)$. Then, using corollary 1.1, there exists a critical point $x \neq 0$ of F_+ with

$$m^*(x) \ \geq \ m(0) - 1\,. \tag{3.33}$$

Clearly x will be a critical point of the action functional f if we show that

$$|Px|_1 \ < \ Q\,. \tag{3.34}$$

In fact in this case the penalizing term has no influence and we will have

$$f'(x) = F'_+(x) = 0 .$$

Arguing by contradiction assume that (3.34) does not hold, then by lemma 3.3 (see (3.16)) we have

$$m^*(x) \leq m^*(\infty) \tag{3.35}$$

(3.33) and (3.35) imply that

$$m(0) - 1 \leq m^*(\infty)$$

which contradicts (3.31).

Finally consider the case

$$\tau_0 < \tau_\infty . \tag{3.36}$$

In this case we consider the penalized functional F_-.

Since

$$T > (2n+1)/(\tau_\infty - \tau_0)$$

(2.6) will imply that

$$m(0) + 1 < m(\infty) . \tag{3.37}$$

Now by proposition 3.1 there exists $c < 0$ such that

$$P(H^1_T, F^c_-) = t^{m^*(\infty)} . \tag{3.38}$$

By (3.38) and (3.37) we see that $t^{m(0)}$ is not "contained" in $P(H^1_T, F^c_-)$. Then, using corollary 1.1, there exists a critical point x of F_- such that

$$x \neq 0 \qquad \text{and} \qquad m(x) \leq m(0) + 1 . \tag{3.39}$$

As before we argue by contradiction and assume that (3.34) does not hold. Then by lemma 3.3 (see (3.15)) we have

$$m(\infty) \leq m(x) . \tag{3.40}$$

Clearly (3.40) and (3.39) contradict (3.37).

□

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