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Global inversion of functions: an introduction *

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Dedicated to Roberto Conti on the occasion of his 70th birthday

Abstract

This is an exposition of some basic ideas in the realm of Global Inverse Function theorems. We address ourselves mainly to readers who are interested in the applications to Differential Equations. But we do not deal with those applications and we give a 'self-contained' elementary exposition.

The first part is devoted to the celebrated Hadamard-Caccioppoli theorem on proper local homeomorphisms treated in the framework of the Hausdorff spaces. In the proof, the concept of 'ω-limit set' is used in a crucial way and this is perhaps the novelty of our approach.

In the second part we deal with open sets in Banach spaces. The concept of 'attraction basin' here is the main tool of our exposition which also shows a few recent results, here extended from finite dimensional to general Banach spaces, together with the classical theorem of Hadamard-Levy which assumes that the operator norm of the inverse of the derivative does not grow too fast (roughly at most linearly).

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Introduction

A fundamental problem in Analysis is the existence and/or uniqueness of the solutions to the equation $y = f(x)$ in the unknown x. The function $f : X \to Y$ relates two spaces X, Y with some structure, otherwise we are impotent. From the other side, the concrete case where X, Y are subsets of the n-space $\mathbb{R}^n$ is often too restrictive, and actually many applications arise in more general spaces. We especially think about injectivity and surjectivity problems in Differential Equations which are not discussed in this paper but constitute one of the reasons of our discussion.

The books Prodi and Ambrosetti [31], and Chow and Hale [9], give the proof of global inversion theorems in general spaces and show applications to differential equations. Let us also refer to Invernizzi and Zanolin [21], Brown and Lin [6], and Radulescu and Radulescu [33] among the papers which could be mentioned for results in differential equations obtained by means of the inversion of functions in infinite dimensional Banach spaces. Finite dimensional problems are also important. The research field of the Jacobian conjectures deals with deep questions of invertibility linked to global stability problems, see Olech [27], Meisters [23], Meisters and Olech [25], [26], and the references contained therein. The inversion of functions, of course, also plays a role in the applied sciences, e.g. Economics and Network Theory.

More references are listed at the end of the paper with no claim to completeness. The present paper is not a survey on the rich literature on these topics.

Section 1 below is devoted to the following theorem which we call after Hadamard and Caccioppoli since Hadamard was probably the first to have the idea in finite dimension, and Caccioppoli was perhaps the most important author in the process of clarification and generalization to abstract spaces (but other mathematicians also gave a contribution).

Theorem 0.1 (Hadamard-Caccioppoli) . *Let $f : X \to Y$ be a local homeomorphism with X, Y path connected Hausdorff spaces and Y simply connected. Then f is a homeomorphism onto Y if and only if it is a proper function, namely if and only if the inverse image $f^{\leftarrow}(K)$ of any compact set $K \subset Y$ is compact.*

The proof below uses, in a crucial way, the concept of ω-limit set. This is perhaps the main novelty of our approach.

The statements of the Theorem in the books of Prodi and Ambrosetti [31], and Chow and Hale [9] (whose treatment of this topic is based on [31]), seem different from Theorem 1 at a first glance since they mention possible singular points of f; however those statements actually follow at once from the one above. Incidentally, those books state the theorem in metrizable spaces. We believe that the more general framework of Hausdorff spaces does not cost more than usual presentations in metrizable spaces even if these are, of course, the relevant case for applications. And generality usually favours understanding the essence of a subject. The framework of Theorem 1 is somehow essential, in particular it is false in non-Hausdorff topological spaces as a simple counterexample will show.

Finally we show an application of the theorem to Algebra, due to Gordon. Namely we show, following [14], that there cannot be a product in $\mathbb{R}^n$ for $n \geq 3$ (see Proposition 1.3 below for a precise formulation). This is related to the fact that $\mathbb{R}^n \setminus \{0\}$ is simply-connected if and only if $n \geq 3$. We quote this application to convince the reader of the depth of the Hadamard-Caccioppoli theorem in a concise way.

In Section 2 we deal with local homeomorphisms $f : D \to Y$ from an open connected set of a Banach space X to a Banach space Y. In order to briefly mention the ideas discussed there, let us here refer to the particular case of a local diffeomorphism f. Then the celebrated Ważewski equation with parameter $v \in Y$,

$$\dot{x} = f'(x)^{-1}\, v \tag{0.1}$$

is often used in the literature to deal with invertibility problems. Ważewski introduced (0.1) in [40], for $X = Y = \mathbb{R}^n$, to give an estimate for a ball, around a given point $x_0 \in D$, where the inverse function can be defined. Instead of (0.1) we consider

$$\dot{x} = F(x)\,, \qquad F : D \to X\,,\ x \mapsto -f'(x)^{-1}\left(f(x) - f(x_0)\right)\,, \tag{0.2}$$

whose trajectories are also trajectories of the family of equations (0.1) (as $v \in Y$) but with different parametrization (incidentally, remark that the family (0.1) has many more trajectories).

The point x_0 is an asymptotically stable equilibrium for (0.2) and its attraction basin $\mathcal{A}$ will be proved to coincide with the maximal open subset of D, containing x_0, such that $f|\mathcal{A}$ is injective and, at the same time, the image $f(\mathcal{A})$ is star-shaped with respect to $y_0 := f(x_0)$. Using these ideas we show some criteria for the injectivity of f. Moreover, we shall see that the solutions to the equation (0.2) are all defined on the whole $\mathbb{R}$ if and only if f is a global homeomorphism onto Y. In particular, this fact leads to the following:

Theorem 0.2 (Hadamard-Levy) *Let $f : X \to Y$ be a local diffeomorphism with X, Y Banach spaces. Then f is a diffeomorphism onto Y if there exists a continuous (weakly) increasing map $\beta : \mathbb{R}_+ \to \mathbb{R}_+ \setminus \{0\}$ such that*

$$\int_0^{+\infty} \frac{1}{\beta(s)} ds = +\infty\,, \qquad \|f'(x)^{-1}\| \leq \beta(\|x\|). \tag{0.3}$$

In particular this holds if, for some $a, b \in \mathbb{R}_+$, we have

$$\|f'(x)^{-1}\| \leq a + b\|x\|. \tag{0.4}$$

This theorem was discovered by Hadamard in $\mathbb{R}^n$. Then it was generalized by Levy to infinite dimension under condition (0.4) with $b = 0$. Meyer dealt with the full condition (0.4), and finally Plastock gave a proof for the general statement. In the literature it is often named after Hadamard only.

Finally, we deal with the injectivity of f (together with the star-shape of the image) by means of global Lyapunov functions. We extend to general Banach spaces some results previously obtained in [17] by two of the authors for $\mathbb{R}^n$.

Our approach to the invertibility of functions, by means of attraction basins for (0.2), is one of the ingredients used in [26] by Meisters and Olech to prove one of the results in that paper, namely the global asymptotic stability for certain polynomial vector fields. We hope that it can lead to further consequences, in particular for the Differential Equations.

1 The Hadamard-Caccioppoli Theorem

In this Section X, Y, Z will always be topological Hausdorff spaces.

Local homeomorphism. As is well known the function $f : X \to Y$ is called a local homeomorphism at $x_0 \in X$ if there exist open neighbourhoods U, V of x_0 and $y_0 := f(x_0)$ respectively, such that $f(U) = V$ and the restriction $f|U : U \to V$ is a homeomorphism. Then $g := (f|U)^{-1} : V \to U$ is called a local inverse of f at y_0. Moreover we say that $f : X \to Y$ is a local homeomorphism if it is a local homeomorphism at any $x_0 \in X$. Such a mapping is clearly continuous and open, namely inverse-images and images of open sets are open sets.

Lifting. Let $f : X \to Y$ be a local homeomorphism and let $p : Z \to Y$ be a continuous function. A continuous function $\tilde{p} : Z \to X$ is called a lifting of p by f whenever $f \circ \tilde{p} = p$, that is, if the following diagram commutes:

$$\begin{array}{ccc} & & X \\ & {}^{\tilde{p}}\nearrow & \downarrow f \\ Z & \xrightarrow{p} & Y \end{array}$$

Lemma 1.1 *(Uniqueness). Let $f : X \to Y$ be a local homeomorphism between Hausdorff spaces and let $p : Z \to Y$ be continuous with Z connected. If $\tilde{p}_1, \tilde{p}_2 : Z \to X$ are both liftings of p then either $\tilde{p}_1 = \tilde{p}_2$ or $\tilde{p}_1(z) \neq \tilde{p}_2(z)$ for every $z \in Z$.*

Proof. Let $C := \{z \in Z : \tilde{p}_1(z) = \tilde{p}_2(z)\}$. Let us see that C is open in Z. If $C = \emptyset$ then it is open; otherwise take $z_0 \in C$ and let $x_0 := \tilde{p}_1(z_0) = \tilde{p}_2(z_0)$. Moreover let U, V and $g : V \to U$ be as in the definition of local homeomorphism above. The set $W := \tilde{p}_1^{\leftarrow}(U) \cap \tilde{p}_2^{\leftarrow}(U)$ is an open neighbourhood of z_0 and we have $\tilde{p}_1|W = \tilde{p}_2|W = g \circ p|W$. Thus $W \subseteq C$ and C is open.

Now, $Z \setminus C$ is open by an easy standard argument (which uses that X is Hausdorff), so we are done since Z is connected.

□

Path-lifting property. We say that the local homeomorphism $f : X \to Y$ lifts the paths if, for every continuous function $\alpha : [0,1] \to Y$, with $\alpha(0) \in f(X)$ (called a path in Y with origin in $f(X)$), and for every $x_0 \in f^{\leftarrow}(0)$, there exists a lifting $\tilde{\alpha} : [0,1] \to X$ of α with $\tilde{\alpha}(0) = x_0$. By Lemma 1.1, if f lifts the paths then it does it with uniqueness, that is the $\tilde{\alpha}$ above is unique.

Homotopy-lifting property. A continuous map $H : Z \times [0,1] \to Y$ is called a homotopy with base $H_0 : Z \to Y$, $z \mapsto H(z,0)$. We say that $f : X \to Y$ lifts the homotopies if, for any such H, and any continuous map $\tilde{H}_0 : Z \to X$ such that $f \circ \tilde{H}_0 = H_0$ ($\tilde{H}_0$ is a lifting of the base of the homotopy), there exists a continuous lifting $\tilde{H}$ with base $\tilde{H}_0$, that is $f \circ \tilde{H} = H$ and $\tilde{H}(z,0) = \tilde{H}_0(z)$ for all $z \in Z$.

The path-lifting property is clearly a particular case of the homotopy-lifting property, with Z a one-point space. It is then remarkable the following

Lemma 1.2 *(Path-lifting $\Longrightarrow$ Homotopy-lifting). If the local homeomorphism between Hausdorff spaces $f : X \to Y$ lifts the paths, then it lifts the homotopies.*

Proof. With the notations as in the above definitions, let $t \mapsto \tilde{H}(z,t)$ be the unique lifting of the path $t \mapsto H(z,t)$, with origin $\tilde{H}_0(z)$, for any $z \in Z$. Clearly $f \circ \tilde{H} = H$, and $\tilde{H}(z,0) = \tilde{H}_0(z)$. So starting from H and $\tilde{H}_0$ as above, we have defined $\tilde{H}$, all we are left to prove is its continuity on $Z \times [0,1]$. Take $z_0 \in Z$, and let D be the subset of $[0,1]$ consisting of all $t \in [0,1]$ such that $\tilde{H}$ is not continuous at (z_0, t). We argue by contradiction: assuming D non empty, D has an infimum $a \geq 0$; since $t \mapsto \tilde{H}(z_0,t)$ is continuous, given any neighborhood U of $\tilde{H}(z_0,t_0)$ in X there exists an interval J_1, an open neighborhood of a in $[0,1]$, such that $\tilde{H}(z_0,t) \in U$ for every $t \in J_1$. By restricting U if necessary we can assume U open, and that f induces a homeomorphism $f|U : U \to V$ onto a neighborhood V of $H(z_0,a)$. By continuity of H there exists a neighborhood W_1 of z_0 in Z, and another interval J_2, open neighborhood of a in $[0,1]$, such that $H(W_1 \times J_2) \subseteq V$. Let $J = J_1 \cap J_2$, and pick $b \in J$, with $b < a$ if $a > 0$; if $a = 0$ let $b = 0$; in both cases $z \mapsto \tilde{H}(z,b)$ is continuous at z_0 (as a function from Z to X), and since $\tilde{H}(z_0,b) \in U$, with U open, there exists a neighborhood W_2 of z_0 in Z such that $\tilde{H}(W_2 \times \{b\}) \subseteq U$; put $W = W_1 \cap W_2$. We claim that

$$\tilde{H}|W \times J = (f|U)^{-1} \circ H|W \times J;$$

in fact these functions coincide on $W \times \{b\}$; but then, for every $z \in W$ the functions defined on J by $t \mapsto \tilde{H}(z,t)$, $t \mapsto (f|U)^{-1} \circ H(z,t)$ are liftings of $t \mapsto H(z,t)$ which coincide on $b \in J$, an hence coincide on all of J. The equality just proved shows that $\tilde{H}$ is continuous at (z_0,t), for every $t \in J$, $\quad t \geq a$, contradicting the minimality of a.

□

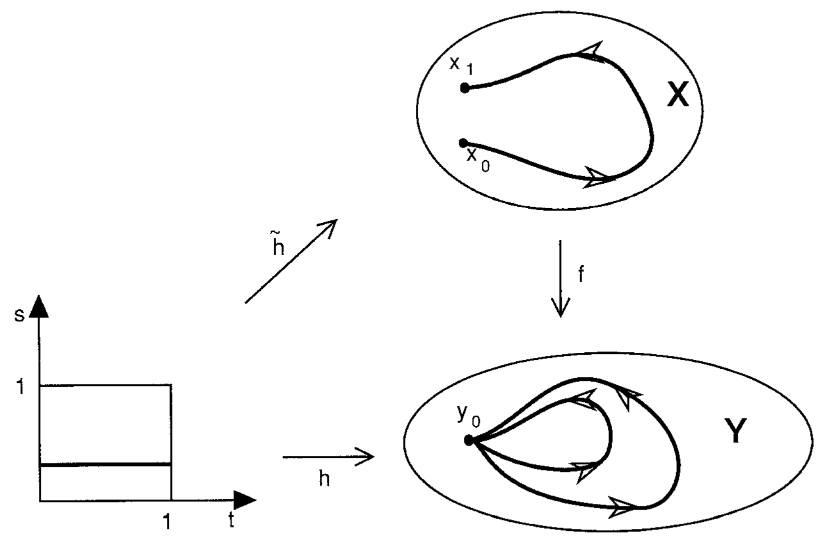
Lemma 1.3 *(Simply connected codomain). Let $f : X \to Y$ be a local homeomorphism between Hausdorff spaces which lifts the paths. If X, Y are path connected and Y is simply connected, then f is a homeomorphism.*

Proof. First of all let us see the surjectivity. Let $y_0 \in f(X)$, $x_0 \in f^{\leftarrow}(y_0)$, and let $\alpha : [0,1] \to Y$ be a path with $\alpha(0) = y_0$ and $\alpha(1) = y$. There exists a (unique) lifting $\tilde{\alpha}$ of α with $\tilde{\alpha}(0) = x_0$. The formula $f \circ \tilde{\alpha} = \alpha$ gives $f(\tilde{\alpha}(1)) = y$.

Now, let us see the injectivity of f. Let $x_0, x_1 \in X$ satisfy $f(x_0) = f(x_1) =: y_0$. Since X is path connected we can consider a path $\sigma : [0,1] \to X$ joining x_0, x_1, that is with $\sigma(0) = x_0$ and $\sigma(1) = x_1$. The formula $\alpha := f \circ \sigma$ defines a circuit in Y (i.e. a closed path) with $\alpha(0) = \alpha(1) = y_0$. Since Y is simply connected there exists a homotopy with fixed end-points h between α and the constant path $[0,1] \to Y, t \mapsto y_0$, namely a continuous function $h : [0,1]^2 \to Y$ such that $h(t,0) = \alpha(t)$, $h(t,1) = y_0$ for all $t \in [0,1]$, and $h(0,s) = y_0 = h(1,s)$, for all $s \in [0,1]$ (see the figure below).

Since f lifts paths, then, by Lemma 1.2, there exists a unique $\tilde{h} : [0,1]^2 \to X$ which lifts h and which satisfies $\tilde{h}(t,0) = \sigma(t)$, for all $t \in [0,1]$.

In the rest of the proof we use the following important fact: a constant path is lifted to a constant path (which works being continuous and which is the unique lifting by Lemma 1.1). Thus $\tilde{h}(0,s) = \sigma(0) = x_0$, $\tilde{h}(1,s) = \sigma(1) = x_1$, for all $s \in [0,1]$; and since $t \mapsto \tilde{h}(t,1)$ is also constant, we have $x_0 = \tilde{h}(0,1) = \tilde{h}(1,1) = x_1$. □



Maximal path-lifting. Let $f : X \to Y$ be a local homeomorphism, let $\alpha : [0,1] \to Y$ be a path with $\alpha(0) \in f(X)$, and let $x_0 \in f^{\leftarrow}(\alpha(0))$. We define the maximal lifting $\phi : J \to X$ of α with $\phi(0) = x_0$ in the following way. There certainly exists a continuous map $\phi_I : I \to X$, with $I = [0, b[\subset [0,1]$, such that $\phi_I(0) = x_0$ and $f \circ \phi_I = \alpha | I$. By the uniqueness Lemma 1.1, the formula $\phi | I = \phi_I$ defines the mapping $\phi : J \to X$ on the union J of all the intervals I.

ω-limit set. Let $\phi : [0, b[\to X,\ 0 < b \leq +\infty$ be a continuous function. Then the following formula, where 'cl' denotes the closure in X, defines the ω-limit set of ϕ:

$$\omega_\phi := \bigcap_{t \in [0,b[} \operatorname{cl} \phi([t, b[)\,.$$

Equivalently, $x \in \omega_\phi$ if and only if x is a cluster point of a sequence $(\phi(t_n))$, for some sequence $t_n \in [0, b[$ which converges to b; in the particular case of X metrizable, $x \in \omega_\phi$ if and only if there exists a sequence (t_n) with $t_n \in [0, b[$ such that $t_n \to b$ and $\phi(t_n) \to x$ as $n \to \infty$.

If ϕ were a solution of an autonomous differential equation $\dot{x} = F(x)$, then the terminology 'ω-limit set' would be usual. This concept has paramount importance since one of the main goal of Dynamics is precisely to say what is the destiny of the motions (incidentally, recall that ω is the last letter of the Greek alphabet).

Lemma 1.4 *(ω-limit set of a maximal path lifting). Let $f : X \to Y$ be a local homeomorphism between Hausdorff spaces, and let $\phi : J \to X$ be the maximal lifting of $\alpha : [0,1] \to Y$ with $\phi(0) = x_0 \in f^{\leftarrow}(\alpha(0))$. If $J \neq [0,1]$ then it is open to the right, i.e. $J = [0, b[$ with $b \in]0,1]$, and the ω-limit set of ϕ is empty: $\omega_\phi = \emptyset$.*

Proof. We argue by contradiction by assuming that $J = [0, a]$ with $0 < a < 1$. We consider a local inverse of f at $f(\phi(a))$ and we easily extend ϕ to a lifting defined on a larger domain, this contradicts the maximality of ϕ. So $\phi : [0, b[\to X$ for a suitable $b \in]0, 1]$.

Now, let us contradict $\omega_\phi = \emptyset$ and let $x_0 \in \omega_\phi$. Then $f(x_0) = \alpha(b)$ since by continuity $f(\operatorname{cl} \phi([t, b[)) \subseteq \operatorname{cl} f(\phi([t, b[)$ and

$$\bigcap_{t \in [0,b[} \operatorname{cl} f(\phi([t, b[) = \bigcap_{t \in [0,b[} \alpha([t, b]) = \{\alpha(b)\}$$

(in metric spaces we could just argue with sequences).

Consider open neighbourhoods U, V, of x_0 and $f(x_0)$ respectively, such that $f|U : U \to V$ be a homeomorphism, and let g be the inverse function. We can consider $a \in [0, b[$ such that $\alpha([a, b]) \subset V$, and such that $\phi(a) \in U$. Moreover, we can define $\psi : [0, b] \to X$ lifting of $\alpha|[0, b]$ by $\psi|[0, a] = \phi|[0, a]$ and by $\psi|]a, b] = g \circ \alpha|]a, b]$. This contradicts the maximality of ϕ.

□

Now we are ready to prove Theorem 0.1 of the Introduction.

Proof of the Hadamard-Caccioppoli Theorem. Let f be proper (in the other sense the theorem is trivial). We are going to prove that f lifts the paths. This gives the theorem by means of Lemma 1.3.

We argue by contradiction by assuming the existence of a path $\alpha : [0,1] \to Y$ and a point $x_0 \in f^{\leftarrow}(\alpha(0))$ such that the maximal lifting ϕ of α, with $\phi(0) = x_0$, is defined on $[0, b[$, with $b \leq 1$ (but not on $[0,1]$). Then Lemma 1.4 says that $\omega_\phi = \emptyset$.

But $\phi([0,b[\,) \subset f^{\leftarrow}(\alpha([0,1]))$ and this last set is compact since f is proper. Since every finite family of closed sets $\{\operatorname{cl}\phi([t_i, b[\,)\}_i$ has nonempty intersection, then

$$\omega_\phi := \bigcap_{t \in [0,b[} \operatorname{cl}\phi([t,b[\,) \neq \emptyset \,,$$

a contradiction.

□

Closed local homeomorphisms. The hypothesis of properness of f can be replaced by closedness of f: that is, a local homeomorphism between Hausdorff spaces which maps closed subsets of X into closed subsets of Y has the path lifting property. To see this, argue as above: to prove that ω_ϕ is non-empty, take a sequence $t_n \in [0, b[$ converging to b and such that $\alpha(t_n)$ consists of distinct points, and is never equal to $\alpha(b)$ (such a sequence certainly exists, unless α is constant on some left neighborhood of b). If the sequence $(\phi(t_n))$ has no cluster point, then its range $R = \{\phi(t_n) : n \in \mathbb{N}\}$ is a closed set in X; but then $\{\alpha(t_n) : n \in \mathbb{N}\} = f(R)$ is closed in Y; this is plainly absurd, since $\alpha(b) \notin f(R)$, but $(\alpha(t_n))$ converges to $\alpha(b)$. There are relations between properness and closedness, see Proposition 1.1 below.

A counterexample. We are going to show that the preceding theorem is not true if we drop the Hausdorff property. Let $S = \mathbb{R} \cup \{c\}$ with $c \notin \mathbb{R}$ with the following topology: the open sets in $\mathbb{R}$, $\{c\} \cup A$, with A open neighbourhood of 0 in $\mathbb{R}$, and $\{c\} \cup A \setminus \{0\}$. The topological space S can be said 'the line with two origins', it is path connected but the Hausdorff property does not hold true. We easily check that the function $f : S \to \mathbb{R}$ whose restriction to $\mathbb{R}$ is the identity, and with $f(c) = 0$, is a proper local homeomorphism but it is not injective.

Incidentally, also simple connectedness is essential, at least for locally well behaved spaces.

Proper maps. Now, let us state two Propositions, whose proofs are easy, to remind what proper functions are in the context of metrizable spaces and for maps $\mathbb{R}^n \to \mathbb{R}^m$.

Proposition 1.1 *(Proper maps in metrizable spaces). Let $f : X \to Y$ be a continuous function between the metrizable spaces X, Y. Then f is proper if and only if every sequence (x_n) in X admits a converging subsequence whenever $(f(x_n))$ converges. Moreover, if such a function f is proper then it is closed. Finally, a* *closed local homeomorphism between metrizable spaces without isolated points is a proper map.*

Proposition 1.2 *(Proper maps between Euclidean spaces). A continuous function $f:\mathbb{R}^n \to \mathbb{R}^m$ is proper if and only if it is coercive, namely*

$$|f(x)| \to \infty\,, \qquad \text{as} \qquad |x| \to \infty\,.$$

Finally, let us see Gordon's application of the Hadamard-Caccioppoli Theorem to Algebra. We give some more details than the original paper [14].

Proposition 1.3 *(Nonexistence of a product in n-space for $n \geq 3$). The n-space $\mathbb{R}^n$ with $n \geq 3$ cannot be endowed of a product operation $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $(x,y) \mapsto xy$ which has the following properties for any $x,y,z \in \mathbb{R}^n$ and any $a \in \mathbb{R}$*

- *(i)* $x\,(a\,y) = (a\,x)\,y = a\,x\,y\,,$
- *(ii)* $x\,(y+z) = x\,y + x\,z\,,$
- *(iii)* $x\,y = 0 \implies$ *either* $x = 0$ *or* $y = 0\,,$
- *(iv)* $x\,y = y\,x\,.$

In other words: $\mathbb{R}^n$, with $n \geq 3$, does not have a commutative algebra structure without zero divisors. Remark that the associative property $x\,(y\,z) = (x\,y)\,z$ is not required.

Proof. We again argue by contradiction, and we denote by $F:(x,y)\mapsto xy$ the product. Consider the function $f: X \to Y$, $x \mapsto x^2 = F(x,x)$, with $X = Y = \mathbb{R}^n \setminus \{0\}$. First, note that f is a C^∞ function on X: if $x = \sum_{k=1}^n x_k e_k$, where $e_1,\ldots,e_n$ is the standard base of $\mathbb{R}^n$, then $f(x) = \sum_{k,l=1}^n x_k x_l F(e_k,e_l)$, a quadratic polynomial function, hence C^∞. Next, denoting by m, M the minimum, respectively the maximum, value of $|f(x)|$ when x ranges over the unit sphere of $\mathbb{R}^n$, we have

$$0 < m|x|^2 \leq |f(x)| \leq M|x|^2, \qquad \text{for every} \quad x \in X = \mathbb{R}^n \setminus \{0\}$$

this follows from $|f(x)| = |f(|x|(x/|x|))| = |x|^2|f(x/|x|)|$, valid for every $x \in X$ (note that, by (i), $f(tx) = t^2 f(x)$ for every non-zero real number t), and readily implies that f is a proper map. The differential of f is given by $df(x)v = 2xv$, for every $x \in X$ and $v \in \mathbb{R}^n$. In fact, by (i) and (ii),

$$f(x+tv) - f(x) = xx + txv + tvx + t^2vv - xx = txv + tvx + t^2 f(v)\,;$$

by (iv) we then have $f(x+tv)-f(x)=2txv+t^2f(v)$, so that

$$\lim_{t\to 0}\frac{f(x+tv)-f(x)}{t}=2xv+\lim_{t\to 0}(tf(v))=2x\,v.$$

By (iii), $xv=0, x\neq 0$ imply $v=0$. Thus $df(x)$ is nonsingular, for every $x\in X$. Now all the hypotheses of the Hadamard-Caccioppoli theorem are satisfied (in particular Y is simply connected), and so f is a homeomorphism, in particular it is injective; but clearly $f(x)=f(-x)$, a contradiction.

□

Remark to the proof. $Y=\mathbb{R}^n\setminus\{0\}$ is simply connected if and only if $n\geq 3$, and actually commutative division algebra structures exist on $\mathbb{R}^n$ if $n\leq 2$; the quaternions prove that commutativity is essential for the above result (what fails is that $df(x)$, now given by $df(x)v=xv+vx$, is singular for some $x\in X$).

2 Star-shaped images

In this Section X,Y will always be Banach spaces, D an open connected set, with $\emptyset\neq D\subseteq X$, and $f:D\to Y$ a local homeomorphism.

The auxiliary flow. Let $x_0\in D$, and $y_0=f(x_0)$. We are going to define a flow $\Phi: D_\Phi\to D$ which will be our tool in investigating the invertibility of f around x_0. The basic properties of Φ, so that it is called a flow in D, are the following:

- (i) D_Φ is an open subset of $D\times\mathbb{R}$, and $\Phi\colon D_\Phi\to D$ is continuous;
- (ii) for all $x\in D$, the set $\{t\in\mathbb{R} : (x,t)\in D_\Phi\}$ is an interval containing 0;
- (iii) $\Phi(x,0)=x$ for all $x\in D$;
- (iv) $(x,t_1),(x,t_1+t_2)\in D_\Phi \Rightarrow \big(\Phi(x,t_1),t_2\big)\in D_\Phi$ and $\Phi\big(\Phi(x,t_1),t_2\big)=\Phi(x,t_1+t_2)$ for all $x\in D$, $t_1,t_2\in\mathbb{R}$.

If $\{x\}\times[0,+\infty[\subset D_\Phi$ we say that the trajectory through x is *global in the future*. Moreover, whenever $D_\Phi=D\times\mathbb{R}$ we say that Φ is a (global) *dynamical system* in D.

To define Φ we start from the following dynamical system in Y:

$$\Psi\colon Y\times\mathbb{R}\to Y,\qquad \Psi(y,t):=y_0+e^{-t}(y-y_0),\tag{2.1}$$

whose trajectories are the half-lines hinged at y_0, but with an exponential parameter instead of a linear one, so that $\Psi(y,0)=y$, $\Psi(y,t)\to y_0$ as $t\to+\infty$. It is indeed a dynamical system, because $\Psi(\Psi(y,t_1),t_2)=\Psi(y,t_1+t_2)$.

Lemma 2.1 *(The auxiliary flow). Let X, Y be Banach spaces, let $D \subseteq X$ be open and connected, let $x_0 \in D$, and let $f : D \to Y$ be a local homeomorphism. Then there exists a flow $\Phi : D_\Phi \to D$ which satisfies the following formula*

$$f\big(\Phi(x,t)\big) = \Psi\big(f(x),t\big) \quad \text{for all } (x,t) \in D_\Phi\,, \tag{2.2}$$

and two such flows coincide in the intersection of their domains (so Φ will be maximal in the sequel). In the particular case where f is a local diffeomorphism (namely it is also C^1 together with all its local inverses), the mapping Φ is C^1 and it is the flow of the following differential equation

$$\dot{x} = F(x)\,, \qquad F : D \to X\,,\ x \mapsto -f'(x)^{-1}\left(f(x) - f(x_0)\right)\,. \tag{2.3}$$

In other words we could say that Φ is the maximal lifting of $\Psi \circ (f \times id)$ (where id the identity in $\mathbb{R}$) such that $\Phi(x,0) = x$ for all $x \in D$.

Proof. Fix $x \in D$ and consider the continuous function $\mathbb{R} \to Y\,,\ t \mapsto \Psi(f(x),t)$. By similar arguments as in Section 1 we prove the existence of a unique maximal lifting $]a(x), b(x)[\to D$, $t \mapsto \Phi(x,t)$, with $\Phi(x,0) = x$, $-\infty \le a(x) < 0 < b(x) \le +\infty$. Let $D_\Phi := \bigcup_{x\in D}\{x\}\times]a(x), b(x)[$. All the properties above are easy to check except (i) which requires some arguments.

We consider the subset $D \times [0,+\infty[$ only; the set $D\times]-\infty, 0]$ is handled similarly. Let $x_0 \in D$ be given. First consider the supremum $\tau(x_0)$ of all real numbers $t \ge 0$ such that $\{x_0\} \times [0,t[$ is contained in the interior of D_Φ (if no such $t > 0$ exists, then $\tau(x_0) = 0$). Next, define E to be the set of all real $t \in [0, \tau(x_0)]$ such that Φ is not continuous at (x_0, t); arguing as in Lemma 1.2 one easily sees that E is empty. And still arguing as in Lemma 1.2, with $\tau(x_0)$ in place of a, it is also easy to see that $\tau(x_0) = b(x_0)$, hence that D_Φ is open.

□

The attraction basin. Let $f : D \to Y$, x_0, Φ be as in Lemma 2.1 (in the general case), and let $y_0 = f(x_0)$. Let U be an open neighbourhood of x_0 where f is injective and let $g := (f|U)^{-1}$. For any small $r > 0$, the ball $B(y_0; r)$ (with center at y_0 and radius r) is contained in $f(U)$, and for such r let $U_r := g^{-1}(B(y_0; r))$. Then U_r is a neighbourhood of x_0, and for all $x \in U_r$ the trajectories $t \mapsto \Phi(x,t)$ of Φ are defined globally in the future, belong to U_r for all $t \ge 0$ and converge to x_0 as $t \to +\infty$. Then x_0 is an *attractor*, namely it attracts a whole neighbourhood (any U_r will do), and it is *stable*, that is, any of its neighbourhoods contains a positively invariant neighbourhood with global existence in the future, indeed again we can consider U_r, with small enough r (we remind that positive invariance means that $\Phi(x,t) \in U_r$ for any $x \in U_r$ and $t > 0$, such that $(x,t) \in D_\Phi$). So we just proved that x_0 is asymptotically stable, i.e. a stable attractor.

The maximal neighbourhood $\mathcal{A}$ of x_0 such that, for all $x \in \mathcal{A}$, the trajectories $t \mapsto \Phi(x,t)$ of Φ are defined globally in the future, and converge to x_0 as $t \to +\infty$, is called the basin of attraction of x_0.

Proposition 2.1 *(Injectivity in the attraction basin). Under the hypotheses of the first part of Lemma 2.1 the attraction basin $\mathcal{A}$ of x_0 for Φ is open. Moreover:*

- *(i) the restriction of f to $\mathcal{A}$ is injective,*
- *(ii) $f(\mathcal{A})$ is star-shaped with respect to $y_0 := f(x_0)$, and*
- *(iii) $\mathcal{A}$ is the maximal connected subset of D which contains x_0 and has the properties (i) and (ii).*

Proof. $\mathcal{A}$ is open because D_Φ is open in $X \times \mathbb{R}$ and Φ is continuous.

To prove that f is injective on $\mathcal{A}$, let $x_1, x_2 \in \mathcal{A}$ be such that $f(x_1) = f(x_2)$. Then for all $t \geq 0$

$$f(\Phi(x_1, t)) = y_0 + e^{-t}(f(x_1) - y_0) = y_0 + e^{-t}(f(x_2) - y_0) = f(\Phi(x_2, t)).$$

Since, for large t, both $\Phi(x_1, t)$ and $\Phi(x_2, t)$ enter a neighbourhood of x_0 where f is injective, we have that $\Phi(x_1, t) = \Phi(x_2, t)$ for large t. Thus for large t we have $x_1 = \Phi(\Phi(x_1, t), -t) = \Phi(\Phi(x_2, t), -t) = x_2$. The image $f(\mathcal{A})$ is star-shaped with respect to y_0 because

$$f(\mathcal{A}) = \{y_0\} \cup \{y_0 + e^{-t}(f(x) - y_0) : (x, t) \in D_\Phi\}.$$

The maximality is also easily verified.

□

Proposition 2.2 *(Bijectivity $\iff$ $D_\Phi = D \times \mathbb{R}$). Let X and Y be Banach spaces, let $D \subseteq X$ be open and connected, let $x_0 \in D$, let $f : D \to Y$ be a local homeomorphism, and let Φ be the auxiliary flow as above. Then f is a global homeomorphism onto Y if and only if the flow Φ is a global dynamical system.*

Proof. Suppose first that f is a global homeomorphism onto Y. Then the inverse mapping f^{-1} is defined and continuous on Y and the expression $\Phi(x,t) = f^{-1}(y_0 + e^{-t}(f(x) - y_0))$ is defined and continuous for all $(x, t) \in D \times \mathbb{R}$.
Conversely, suppose that $D_\Phi = D \times \mathbb{R}$. Let $y \in \mathbb{R}^n$ and $\varepsilon > 0$ such that $y_0 + \varepsilon(y - y_0) \in f(\mathcal{A})$, and let $g := (f|\mathcal{A})^{-1}$. Then

$$f(D) \supset f(\mathcal{A}) \ni f\Big(\Phi\big(g(y_0 + \varepsilon(y - y_0)), \ln \varepsilon\big)\Big) = y_0 + e^{-\ln \varepsilon}\big(y_0 + \varepsilon(y - y_0) - y_0\big) = y,$$

and $f|\mathcal{A}$ is proved to be onto Y. To verify that f is also one-to-one on all of D, i.e., that $\mathcal{A} = D$, it suffices to prove that $\mathcal{A}$ is a closed subset of D, because we already know that it is open and nonempty. Let then $x_n \in \mathcal{A}$ be a sequence converging to $x \in D$. Since $f(\mathcal{A}) = Y$, there exists $\bar{x} \in \mathcal{A}$ such that $f(\bar{x}) = f(x)$. Recalling that $(f|\mathcal{A})^{-1} : Y \to \mathcal{A}$ is continuous, from $f(x_n) \to f(\bar{x})$ we get that $x_n \to \bar{x}$, whence $x = \bar{x} \in \mathcal{A}$.

□

Now, let us prove Theorem 0.2 in the Introduction.

Proof of the Hadamard-Levy Theorem. By the preceding Proposition 2.3 we can just show that the solutions to the equation (2.3) are defined on the whole $\mathbb{R}$. First remark that by (2.1), and (2.2),

$$\|f(\Phi(\bar{x},t)) - y_0\| = e^{-t}\|f(\bar{x}) - y_0\|$$

so this is bounded whenever t ranges on a bounded interval. Then, along a trajectory $\gamma :]a,b[\to D$, $\gamma(t) = \Phi(\bar{x},t)$, defined in a bounded interval of time $]a,b[$, we have the following estimate for the vector field in (2.3):

$$\|F(\gamma(t))\| \le \|f'(\gamma(t))^{-1}\| \, \|f(\gamma(t)) - y_0\| \le c\,\beta(\|\gamma(t)\|)\,,$$

for a suitable $c > 0$ (the function β was introduced in (0.3)).

From now on the arguments are standard, however we prefer to complete the proof to be self-contained. Let $r(t) := \|\gamma(t)\|$. Then for $a \le t_1 \le t_2 \le b$ we have

$$\|r(t_2) - r(t_1)\| \le \|\gamma(t_2) - \gamma(t_1)\| \le c \int_{t_1}^{t_2} \beta\left(\|\gamma(t)\|\right) dt\,. \tag{2.4}$$

The function $x \mapsto \|x\|$ is Lipschitz continuous and the function γ is C^1 (remind that f is a local diffeomorphism in the present theorem), so that $t \mapsto r(t)$ is locally absolutely continuous and it has derivative almost everywhere. By the previous estimate, dividing by $t_2 - t_1$ and going to the limit we have $\|r'(t)\| \le c\,\beta(r(t))$ almost everywhere. Now, for $t, t_0 \in]a,b[$

$$\left|\int_{r(t_0)}^{r(t)} \frac{1}{\beta(s)}\,ds\right| = \left|\int_{t_0}^{t} \frac{r'(s)}{\beta(r(s))}\,ds\right| \le \left|\int_{t_0}^{t} \left|\frac{r'(s)}{\beta(r(s))}\right| ds\right| \le c\,|t - t_0| \le c\,|b - a|\,.$$

Then $r(t)$ for $t \in]a,b[$ is bounded from above by any $r_0 > 0$ large enough to give $\int_{r(t_0)}^{r_0} \frac{1}{\beta(s)}\,ds \ge c\,|b-a|$ (remind the first formula in (0.3)). Using again the inequality (2.4) and this time the monotonicity of β we see that $\|\gamma'(t)\| \le c\,\beta(r_0)$. Then γ is Lipschitz continuous on $]a,b[$ and it can be extended by continuity to a and b.

□

In the sequel we shall need the following Lemma:

Lemma 2.2 *(On $\partial\mathcal{A}$ the trajectories have finite life). Let us assume the hypotheses of the first part of Lemma 2.1. Then the attraction basin $\mathcal{A}$ is invariant, namely $x \in \mathcal{A} \implies \Phi(x,t) \in \mathcal{A}$ for all t such that $(x,t) \in D_\Phi$, and also $\partial\mathcal{A}$ (the boundary of $\mathcal{A}$ in D) is invariant. Moreover, there is not global existence in the future for $t \mapsto \Phi(x,t)$ if $x \in \partial\mathcal{A}$.*

Proof. First of all let us see that

$$f(\partial\mathcal{A}) \subseteq \partial f(\mathcal{A})\,. \tag{2.5}$$

The set $\mathcal{A}$ is open in X and f is a one-to-one local homeomorphism on $\mathcal{A}$, so that $f(\mathcal{A})$ turns out to be open, too, and $f|\mathcal{A}\colon\mathcal{A}\to f(\mathcal{A})$ is a homeomorphism. $f(\partial\mathcal{A})$ is contained in the closure of $f(\mathcal{A})$ because f is continuous. Let $\bar{x}$ be a point in the closure of $\mathcal{A}$ such that $f(\bar{x})\in f(\mathcal{A})$, i.e., $f(\bar{x})=f(x)$ for some $x\in\mathcal{A}$. Let x_n, $n\geq 1$, be a sequence of points of $\mathcal{A}$ converging to $\bar{x}$. By continuity of f we have $f(x_n)\to f(\bar{x})=f(x)$, and by continuity of $(f|\mathcal{A})^{-1}$ we have $x_n = (f|\mathcal{A})^{-1}(f(x_n)) \to (f|\mathcal{A})^{-1}(f(x))=x$, so that $\bar{x}=x\in\mathcal{A}$. From (2.5) and the fact that $f(\mathcal{A})$ is a neighbourhood of $y_0=f(x_0)$, there exists $\varepsilon>0$ such that

$$x\in\partial\mathcal{A} \qquad\Longrightarrow\qquad \|f(x)-y_0\|\geq\varepsilon\,. \tag{2.6}$$

It is obvious from its definition that $\mathcal{A}$ is invariant for the flow $(x,t)\mapsto\Phi(x,t)$. The same holds for $\partial\mathcal{A}$: In fact, let $x\in\partial\mathcal{A}$, $x_n\in\mathcal{A}$, $x_n\to x$, $(x,t)\in D_\Phi$. Then $(x_n,t)\in D_\Phi$ for all large n, because D_Φ is open, and, by continuity $\mathcal{A}\ni\Phi(x_n,t)\to\Phi(x,t)$. The point $\Phi(x,t)$ belongs to the closure of $\mathcal{A}$, but not to $\mathcal{A}$, because otherwise $x=\Phi(\Phi(x,t),-t)$ itself would be in $\mathcal{A}$.

Finally, from (2.6) we get:

$$x\in\partial\mathcal{A} \implies \varepsilon\leq\|f(\Phi(x,t))-y_0\| = e^{-t}\|f(x)-y_0\| \implies t\leq\ln\frac{\|f(x)-y_0\|}{\varepsilon}\,.$$

□

Bounded sets in D. In the sequel we say that a set $B\subseteq D$ is bounded in D if (i) it is bounded as a subset of X, and (ii) its closure in X is contained in D.

Trapped trajectories. We need to guarantee that the trajectories of Φ which are trapped into a closed and bounded subset of D are defined globally in the future (condition (c) in Lemma 2.3 below). This is familiar and always true for solutions to differential equations which are 'trapped' into compact sets in finite dimension. The following Lemma 2.3 shows few technical conditions each of which implies this property. In the statement we denote by $[f(x_0);f(x)]\subset Y$ the line segment from $f(x_0)$ to $f(x)$.

Lemma 2.3 *(Trapped trajectories never die). Let X and Y be Banach spaces, let $D\subseteq X$ be open and connected, let $x_0\in D$, let $f:D\to Y$ be a local homeomorphism, and let Φ be the auxiliary flow as above. Consider the following conditions:*

- *(a-1) the restriction $f|B$ is proper for any set B closed and bounded in D;*
- *(a-2) f is a local C^1 diffeomorphism and for each bounded and closed set $B\subset D$ we have*

$$\sup_{x\in B}\|f'(x)^{-1}\|<+\infty\,. \tag{2.7}$$

- *(b) for any B, closed and bounded subset of D, and any $x \in B$, the connected components of $f^{\leftarrow}([f(x_0); f(x)]) \cap B$ are compact;*
- *(c) for any B, closed and bounded subset of D, and any $x \in B$, if $\Phi(x,t) \in B$ for all $t > 0$ such that $(x,t) \in D_\Phi$, then the trajectory through x is global in the future (in other words: trajectories which are eventually in bounded closed sets never die).*

Then either one of (a-1) and (a-2) imply (b), which implies (c). All conditions are trivially satisfied if X is finite dimensional.

Proof. The proof is trivial except for (a-2) $\Rightarrow$ (b). Let L be a component of $f^{\leftarrow}([f(x_0); f(x)]) \cap B$. Pick $x_1 \in L$, and let $v = f(x) - f(x_0)$. If $v = 0$, then $[f(x_0); f(x)]$ consists of the single point $f(x_0)$ and L is then also a singleton, since f is a local homeomorphism. Assume then $v \neq 0$. Since $f(L)$ is connected, the set $\{t \in \mathbb{R} : f(x_1) + tv \in f(L)\}$ is a bounded interval I of $\mathbb{R}$ containing 0. Let $\alpha : J \to L$ be the maximal lifting of the path $\ell(t) = f(x_1) + tv$ $(t \in I)$ with origin $\alpha(0) = x_1$. Since f is a local diffeomorphism, such an α is differentiable, and differentiating $f(\alpha(t)) = f(x_1) + tv$ we get $f'(\alpha(t))(\alpha'(t)) = v$, whence $\alpha'(t) = f'(\alpha(t))^{-1}v$. Since $\sup_{x \in L} \|f'(x)\|$ is finite, α' is bounded on its maximal interval J of existence; thus the ω-limit set of α is nonempty, and it is contained in the closed set L. It follows that $J = I$, and by the same token, that $\inf I \in I$, and $\sup I \in I$, that is, I is compact. It is now obvious that f induces a homeomorphism of L onto $f(L)$, which has $\alpha \circ \ell^{-1}$ as inverse. Thus L is compact, since $f(L)$ is homeomorphic to I *via* ℓ. □

A class of functions satisfying (a-1). The condition (a-1) is fulfilled if $f = p + c$ with p proper and c compact, i.e., mapping closed bounded sets to compact sets. Indeed, remind Proposition 1.1, and consider a sequence (x_n) in the closed bounded set B, with $(f(x_n))$ convergent. Since c is compact, it maps a subsequence (x_{n_k}) to a convergent sequence $(c(x_{n_k}))$, thus $p(x_{n_k}) = f(x_{n_k}) - c(x_{n_k})$ converges and finally (x_{n_k}) has a convergent subsequence since p is proper.

Coercive auxiliary functions. The nonnegative continuous function $k : D \to \mathbb{R}$ is called coercive whenever for any $a > 0$ the inverse image $k^{\leftarrow}([0,a])$ is bounded in D.

Global Lyapunov functions. In our framework the function $k : D \to \mathbb{R}_+$ is called a global Lyapunov function for the flow Φ above, if it is continuous, nonnegative, coercive, and weakly decreasing along the trajectories, namely $t \mapsto k(\Phi(x,t))$ weakly decreases for all $x \in D$.

Proposition 2.3 *(Injectivity and star-shaped image by Lyapunov functions). Let X, Y be Banach spaces, let $D \subseteq X$ be open and connected, let $x_0 \in D$, and let $f : D \to Y$ be a local homeomorphism. Then f is injective, and the image $f(D)$ is* *star-shaped with respect to* $f(x_0)$*, if there exists a global Lyapunov function for* Φ*, and* f *satisfies any of the conditions (a-1), (a-2), (b), (c) in Lemma 2.3.*

Proof. We are going to prove that $D = \mathcal{A}$. So we are done by Proposition 2.1. It is enough to show that the boundary $\partial\mathcal{A}$ (of $\mathcal{A}$ in D) is empty. We argue by contradiction and assume that $x \in \partial\mathcal{A}$ By Lemma 2.2 the maximal positive trajectory through x, $\gamma : [0, b[\to D$, $t \mapsto \Phi(x,t)$, lies in $\partial\mathcal{A}$, and has a finite life: $\gamma([0, b[) \subseteq \partial\mathcal{A}$, and $b < +\infty$.

The Lyapunov function $k : D \to \mathbb{R}_+$ is coercive and, in particular, $B := k^{\leftarrow}([0, b])$ is bounded in D. Moreover, $k \circ \gamma$ is decreasing and so $\gamma([0, b[) \subseteq B$. Now Lemma 2.3 says that condition (c) above holds true, namely $b = +\infty$, a contradiction.

□

The preceding result, as well as the following one, extend some results in [17] (by two of the authors) where the finite dimensional case is treated. That paper also shows that the converse of Proposition 2.3 holds true in $\mathbb{R}^n$ (and proves other related facts). In the following statement we consider an Hilbert space X with scalar product '$\cdot$', and $B(x_0; r)$ will denote the open ball $\|x - x_0\| < r$. We could formulate an analogous fact in general Banach spaces but it would be more complicated to be stated (but not to be proved).

Proposition 2.4 *(A criterion of injectivity on a ball). Let* X *be a Hilbert space,* $x_0 \in X$*,* Y *be a Banach space,* $f\colon B(x_0; r_0) \to Y$ *be a local* C^1 *diffeomorphism satisfying any of the conditions of Lemma 2.3. Then the following two conditions are equivalent:*

- *(a)* f *is injective and* $f(B(x_0; r))$ *is star-shaped with respect to* $f(x_0)$ *for all positive* $r \le r_0$*;*
- *(b) the following inequality holds for all* $x \in B(x_0; r_0)$

$$(x - x_0) \cdot f'(x)^{-1}\big(f(x) - f(x_0)\big) \ge 0\,. \tag{2.8}$$

Proof. The left-hand side of (2.8) is the derivative with respect to t at $t = 0$ of the scalar function

$$t \mapsto \frac{1}{2}\|\Phi(x,t) - x_0\|^2\,.$$

Asking it to be nonnegative is the same as asking the scalar function $x \mapsto (1/2)\|x - x_0\|^2$ to be weakly decreasing along the flow Φ, which in turn is the same as requiring the same from each of the functions $x \mapsto 1/(r^2 - \|x - x_0\|^2)$ on $B(x_0; r)$, $0 < r \le r_0$. These last functions have the advantage of being coercive on $B(x_0; r)$. Hence condition (b) is satisfied, Proposition 2.6 can be applied to get condition (a).

Conversely, if condition (a) holds, then the sets $B(x_0; r)$ are positively invariant for Φ and the (square) norm of $\Phi(x,t)$ must be a weakly decreasing function of t, whence inequality (2.8).

□

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