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Optimal laminate design subject to single membrane loads

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Abstract In this paper we investigate optimization of laminates for maximal membrane stiffness under single in-plane loads. The design parameters are the relative ply thicknesses and fiber orientations of an arbitrary number of plies. The design is allowed to vary in a pointwise fashion throughout the structure.

From prior work on lamination parameters (Hammer *et al.* 1997), it is known that the optimal design is given by either some sort of two ply lay-up in special strain situations or otherwise by just a single rotated ply. This is exploited in the present analysis to derive analytically the unique parameters of the optimal design (cross-ply, angle-ply or single ply) as expressions of the membrane forces. Both high and low shear stiffness material are treated. Furthermore the analysis covers all possible local strain or membrane force situations.

Finally, it is shown how these expressions for the optimal configuration of the laminate also appear as bounds on the principal membrane forces in order to obtain alignment between the numerically largest principal membrane force and principal strain.

1 Introduction

A proper and thorough design process of fibrous laminates is of prior importance in order to fully exploit the directional properties of the material.

The present paper treats the objective of how to obtain maximal membrane stiffness of a structure. The optimization covers laminates of any imaginable type with an arbitrary number of plies. Despite the innumerable design possibilities, the optimal laminate still turns out to be given by the simple types of either a single ply, a cross-ply or an angle-ply. Two different types of optimal designs occur depending on the type of ply material, whether the material has what is called high or low shear stiffness. This characteristic is not normally recognized or considered as significant. Both cases are thoroughly treated. The optimal design parameters, i.e. the two ply rotations and thicknesses are derived as explicit functions of the membrane forces. From this a given structure can readily in an easy and fast way be designed to yield optimal stiffness properties.

When designing a laminate the parameterization of the problem plays an important role. In general, one can choose either to work directly with the dimensioning parameters, i.e. the number of plies, their thicknesses and orientations, or one can keep the formulation in the integrated properties of the laminate, the so-called lamination parameters. In general, if the design variables are those of ply thicknesses and fiber orientations, one gains the insight of always knowing exactly how the laminate looks. However, this approach ultimately involves an unfortunate mixture of integer variables (number of plies) and real variables (thicknesses and orientations). Numerous authors have treated the subject of laminate optimization in this way with different objective functions and by different means. See, e.g. Chapter 11 in the book by Haftka *et al.* (1990), or the paper by Fang and Springer (1993) for references to the literature.

The lamination parameters introduced by Miki (1982) are the effective integrated properties of the laminate, and are given as moments relative to the plate mid-plane of the trigonometric functions entering in the rotation formulae for the stiffness matrices. Only four lamination parameters describe the membrane state for a general laminate irrespective of the number of plies. An advantage is also that the strain energy and thus the objective function is linear in terms of these parameters, thus further facilitating the optimization. This parameterization has been used for membrane stiffness optimization by Fukunaga and Vanderplaats (1991), Fukunaga and Sekine (1992, 1993), and Hammer *et al.* (1997).

Fukunaga and Vanderplaats (1991), and Fukunaga and Sekine (1992) performed the analysis on symmetric eight ply laminates for which the authors could solve the identification problem of finding a laminate configuration with given lamination parameters. Fukunaga and Sekine (1993) maximized the average membrane stiffness of a single element laminate and combined the result to a strength optimization. The general identification problem for the four membrane lamination parameters was solved by Lipton (1994), who showed how to realize any set of lamination parameters by a laminate of at most three plies. This result was applied by Hammer *et al.* (1997) to show that the stiffest laminate is characterized by just a single ply solution, or a two ply thick cross-ply or angle-ply. Both cases of a ply material with high or low shear stiffness were treated.

The present paper can be seen as a continuation hereto, taking the optimization results of a general laminate from the paper by Hammer *et al.* (1997) and combining these with optimization results from the design problem of orienting just a single ply of orthotropic material. When seen in the light of the laminate optimization it also becomes possible to explain some of the problems encountered in the design process for the single ply. More precisely, these are the problems of nonexistence in the special strain situations of the two principal strains being numerically equal.

The single ply design giving the extreme value of the local energy density was found by Seregin and Troitskii (1982), Fedorov and Cherkaev (1983), and Pedersen (1989, 1990).

The analysis was also performed in the stress space by Cheng and Pedersen (1997). Some of the results from these analyses reappear in the general case of designing a laminate.

The rather special material with so-called high shear stiffness is only treated in a few papers (Pedersen 1990; Cheng and Pedersen 1997; Hammer *et al.* 1997), although the special material characteristics cause the optimal design to differ significantly. In the compliance optimization problem at hand, the use of such a material actually results in an infinity of optimal designs all yielding extreme stiffness. This is thus a good example clearly demonstrating that the optimal compliance design is not necessarily always unique.

The outline of the paper is as follows. The constitutive relations for a single ply of orthotropic material and for a laminate are stated in Section 2. Section 3 summarizes the results obtained in the literature on which the analysis in this present paper builds. Then the optimization problem of maximizing the compliance of a structure is presented. The formulation is kept in terms of membrane forces and complimentary energy. The optimal laminate configuration is then derived first in Section 4.1 for a ply material with low shear stiffness and then afterwards in Section 4.2 for the more special case of a high shear stiffness material. A numerical example is given in Section 5 demonstrating the optimization procedure as well as illustrating the change in design caused by altering the shear stiffness of the ply material. Some of the expressions for the optimal laminate design also appear as bounds on the relation between the principal membrane forces. Where these bounds come from and how they relate to the optimal designs is discussed in Section 6. Finally, a conclusion and summary are given.

2 Constitutive relations

In the following, the constitutive relations for a single ply of material and for a laminate of several plies are stated. The presentation is given for plies of orthotropic material in any lay-up. The resulting laminate thus in general has overall anisotropic properties. The relations can readily be simplified to, say, the case of a symmetric or orthotropic laminate.

The elasticity tensor $\mathbf{C}_{ijk\ell}$ of the ply material is for convenience written in matrix form as

$$[\mathbf{C}]_x = \begin{bmatrix} \mathrm{C}_{1111} & \mathrm{C}_{1122} & \sqrt{2}\mathrm{C}_{1112} \\ \mathrm{C}_{1122} & \mathrm{C}_{2222} & \sqrt{2}\mathrm{C}_{2212} \\ \sqrt{2}\mathrm{C}_{1112} & \sqrt{2}\mathrm{C}_{2212} & 2\mathrm{C}_{1212} \end{bmatrix}_x . \tag{1}$$

The relation between strains and stresses is thus

$$\{\sigma\} = [\mathbf{C}]_x\{\varepsilon\}\,, \tag{2}$$

where the $\sqrt{2}$-notation is used [i.e. $\{\sigma\} = \{\sigma_{11}, \sigma_{22}, \sqrt{2}\sigma_{12}\}^T$, etc. (see e.g. Pedersen 1995)]. The index indicates that the constitutive parameters are given in the coordinate system x. If the material is orthotropic in the x-system, $\mathrm{C}_{1112} = \mathrm{C}_{2212} = 0$. In another coordinate system X rotated at the angle γ, $[\mathbf{C}]_X$ is most easily expressed using the material parameters C_{1-5}, introduced by Tsai and Hahn (1980). The constitutive matrix $[\mathbf{C}]_X$ is written in terms of five symmetric matrices containing the material parameters as

$$[\mathbf{C}]_X = [\mathbf{\Gamma}_0] + [\mathbf{\Gamma}_1]\cos 2\gamma +$$
$$[\mathbf{\Gamma}_2]\cos 4\gamma + [\mathbf{\Gamma}_3]\sin 2\gamma + [\mathbf{\Gamma}_4]\sin 4\gamma\,, \tag{3}$$

$$[\mathbf{\Gamma}_0] = \begin{bmatrix} \mathrm{C}_1 & \mathrm{C}_4 & 0 \\ & \mathrm{C}_1 & 0 \\ \text{symm.} & & 2\mathrm{C}_5 \end{bmatrix},$$

$$[\mathbf{\Gamma}_1] = \begin{bmatrix} \mathrm{C}_2 & 0 & 0 \\ & -\mathrm{C}_2 & 0 \\ \text{symm.} & & 0 \end{bmatrix},$$

$$[\mathbf{\Gamma}_2] = \begin{bmatrix} \mathrm{C}_3 & -\mathrm{C}_3 & 0 \\ & \mathrm{C}_3 & 0 \\ \text{symm.} & & -2\mathrm{C}_3 \end{bmatrix},$$

$$[\mathbf{\Gamma}_3] = \begin{bmatrix} 0 & 0 & -\frac{1}{\sqrt{2}}\mathrm{C}_2 \\ & 0 & -\frac{1}{\sqrt{2}}\mathrm{C}_2 \\ \text{symm.} & & 0 \end{bmatrix},$$

$$[\mathbf{\Gamma}_4] = \begin{bmatrix} 0 & 0 & -\sqrt{2}\mathrm{C}_3 \\ & 0 & \sqrt{2}\mathrm{C}_3 \\ \text{symm.} & & 0 \end{bmatrix},$$

where the material parameters C_{1-5} are defined as

$$C_1 = \frac{1}{2}(C_{1111} + C_{2222})_x - C_3\,,$$

$$C_2 = \frac{1}{2}(C_{1111} - C_{2222})_x\,,$$

$$C_3 = \frac{1}{8}(C_{1111} + C_{2222} - 2C_{1122} - 4C_{1212})_x\,,$$

$$C_4 = (C_{1122})_x + C_3\,,$$

$$C_5 = (C_{1212})_x + C_3 = \frac{1}{2}(C_1 - C_4)\,. \tag{4}$$

If the material is isotropic in the x-system $C_2 = C_3 = 0$. The parameter C_3 characterizes whether the ply material has so-called low ($C_3 > 0$) or low ($C_3 < 0$) shear stiffness. By far most materials possess low shear stiffness and so this situation is often the only one treated in the literature. However, $C_3 < 0$ is also physically possible and this case is therefore also thoroughly analysed in the following. Figure 1 shows the different coordinate systems of the ply material.

The material is now used to build a laminate consisting of a number of plies stacked on top of each other. The laminate is of the total thickness h and each ply is characterized by its relative thickness t_i and orientation γ_i. The stacking sequence is of no importance here as only the membrane case is considered. All the plies consist of the same material.

In the classical plate theory the relation between the membrane forces $\{\mathbf{N}\}$ and the strains $\{\varepsilon\}$ is

$$\{\mathbf{N}\} = [\mathbf{A}]\{\varepsilon\}\,. \tag{5}$$

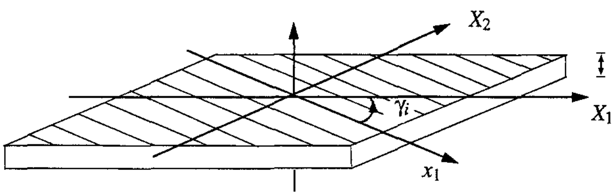

Fig. 1. Sketch of the i-th ply of material with the global coordinate system X, a material system x, and orientation γ_i of the ply material shown. Note the sign of γ_i

Again the $\sqrt{2}$-notation is used (i.e. $\{\mathbf{N}\} = \{N_{11}, N_{22}, \sqrt{2}N_{12}\}^T$). The membrane stiffness matrix for the whole laminate can be expressed in a similar way as the constitutive matrix in (3) as

$$[\mathbf{A}]_X = h\sum_i t_i[\mathbf{C}]_X^i = h\sum_i t_i\{[\mathbf{\Gamma}_0] + [\mathbf{\Gamma}_1]\cos(2\gamma_i) +$$

$$[\mathbf{\Gamma}_2]\cos(4\gamma_i) + [\mathbf{\Gamma}_3]\sin(2\gamma_i) + [\mathbf{\Gamma}_4]\sin(4\gamma_i)\}\ . \tag{6}$$

The specific energy (per area of the laminate) is then given either in terms of the strains or membrane forces as

$$u = \frac{1}{2}\{\varepsilon\}^T[\mathbf{A}]\{\varepsilon\}\,,\quad u^c = \frac{1}{2}\{\mathbf{N}\}^T[\mathbf{A}]^{-1}\{\mathbf{N}\}\,. \tag{7}$$

The optimization in Section 4 is formulated in terms of the membrane forces and complementary energy. This choice is made to avoid a problem of nonuniqueness to the local optimization problem otherwise arising from a formulation in the strains and strain energies (Hammer *et al.* 1997). The price paid for this is then a far more complicated objective function as all linearity in the design variables is lost.

Finally, one can in the membrane situation choose to consider the laminate as one single ply of anisotropic material with the laminate constitutive matrix $[\mathbf{C}]_X^{\text{lam}}$. This is then related to the membrane stiffness matrix $[\mathbf{A}]$ as

$$[\mathbf{C}]_X^{\text{lam}} = \frac{1}{h}[\mathbf{A}]_X\,. \tag{8}$$

From the constitutive matrix a set of material parameters $(C_{1-7})^{\text{lam}}$ can be derived (seven as the laminate material in general is anisotropic). The relations between the laminate parameters and the ply material parameters are derived in the paper by Hammer *et al.* (1997). Of special interest here is $(C_3)^{\text{lam}}$ giving information about the laminate shear stiffness. For an orthotropic ply material $(C_3)^{\text{lam}}$ is given by

$$(C_3)^{\text{lam}} = C_3\sum_i t_i\cos(4\gamma_i)\,. \tag{9}$$

One can thus build a laminate with high shear stiffness properties $[(C_3)^{\text{lam}} < 0]$ by a proper stacking of plies having low shear stiffness ($C_3 > 0$) and vice versa. For instance, an angle-ply $[\gamma/-\gamma]$ where $\gamma > 22.5^\circ$ yields $(C_3)^{\text{lam}} < 0$ if $C_3 > 0$, and the other way around gives $(C_3)^{\text{lam}} > 0$ if $C_3 < 0$.

3 Background

The optimization results to be derived in Section 4 are based on earlier results in compliance optimization on laminates and orthotropic materials. The relatively new results from laminate design using lamination parameters are exploited together with the results on orthotropic material directions yielding extreme local energy densities.

Hammer *et al.* (1997) formulated the laminate theory and the optimization problem of designing a general laminate in terms of the lamination parameters. The main conclusion in this context is that the optimal laminate design under a single membrane load is shown to be either a single rotated ply, a cross-ply in some special strain situations or an angle-ply. However, as Hammer *et al.* (1977) performed the analysis in the strain-space, a problem of nonuniqueness of the local optimization problem arises. The full characterization of the optimal laminate is therefore not given. Still, the analysis is complete in the sense that an optimal lay-up can be found numerically (as shown in the paper) without difficulties.

The results of Hammer *et al.* (1977) are repeated below. The material direction is given in every design point relative to the coordinate system of the principal strains (see Fig. 2). These are ordered so $|\varepsilon_I| \geq |\varepsilon_{II}|$. Similarly, the orthotropic ply material is per definition oriented so $|C_{1111}| \geq |C_{2222}|$. In the case of a ply material having low shear stiffness, $C_3 > 0$ [cf. (4)], the optimal design is given by

$$\begin{aligned}&\underline{\text{a single ply}}:\quad \gamma = 0^\circ\,,\quad \text{if}\quad |\varepsilon_I| \neq |\varepsilon_{II}|\,,\\ &\underline{\text{a cross-ply}}:\quad [0^\circ_t/90^\circ_{1-t}]\,,\quad \text{if}\quad |\varepsilon_I| = |\varepsilon_{II}|\,.\end{aligned} \tag{10}$$

The relative thickness t remains to be determined through the equilibrium conditions on the structure. In all cases the optimal *laminate* has overall orthotropic properties.

The solution is more complex if the material has high shear stiffness, $C_3 < 0$. For this case the optimal lay-up is

$$\begin{array}{lll}\underline{\text{a cross-ply}}: [0^\circ_t/90^\circ_{1-t}] & & \text{if } \varepsilon_I = \varepsilon_{II}\,,\\ \underline{\text{a single ply}}: \gamma = 0^\circ\,, & \text{if } |K| > 1 & \left.\vphantom{\begin{array}{l}a\\a\\a\end{array}}\right\}\ \text{if } \varepsilon_I \neq \varepsilon_{II}\,,\\ \underline{\text{an angle-ply}}: [\gamma_t/-\gamma_{1-t}] & & \\ \quad\cos 2\gamma = -K\,, & \text{if } |K| < 1 & \\ \quad K = \dfrac{C_2}{4C_3}\dfrac{\varepsilon_I+\varepsilon_{II}}{\varepsilon_I-\varepsilon_{II}} & & \end{array} \tag{11}$$

Again the relative thickness t remains to be determined. For this type of material the optimal *laminate* is not necessarily orthotropic when given by the angle-ply solution.

In the following derivations we use mainly the fact that the numerically largest principal strain $|\varepsilon_I| \geq |\varepsilon_{II}|$ is always aligned with the numerically largest principal membrane force $|N_I| \geq |N_{II}|$ in the optimal configuration (see e.g. Pedersen and Bendsøe 1995). This is irrespective of the sign of the C_3-parameter. This result is found by showing that $(A_{1111})_\varepsilon \geq (A_{2222})_\varepsilon$ in the coordinate system of the principal strains, also when the optimal orientation is $\cos 2\gamma = -\dfrac{C_2}{4C_3}\dfrac{\varepsilon_I+\varepsilon_{II}}{\varepsilon_I-\varepsilon_{II}}$ as in (11).

This is exploited in the following where the problem is formulated in terms of the membrane forces. As the coordinate

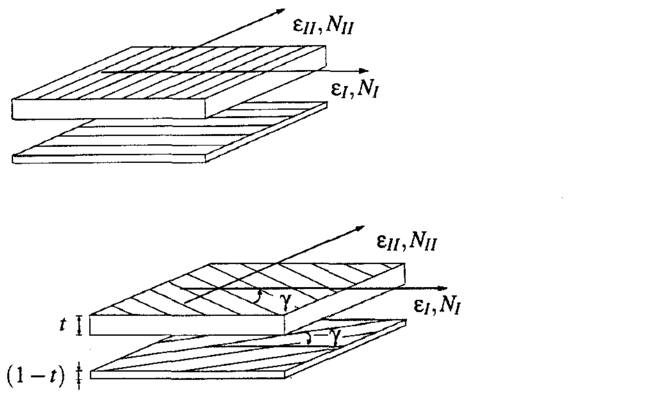

Fig. 2. The coordinate systems of the principal strains (ε_I, ε_{II}), and principal membrane forces (N_I, N_{II}), relative to the cross-ply laminate $[0^\circ_t/90^\circ_{1-t}]$ to the left and the angle-ply $[\gamma_t/-\gamma_{1-t}]$ to the right

systems of the principal strains and membrane forces are co-aligned, the optimal laminate for $C_3 > 0$ is characterized by a cross-ply of the type $[0^\circ_t/90^\circ_{1-t}]$. This is the only solution if if $C_3 > 0$. As the laminate is orthotropic the main direction of orthotropy characterized by $A_{1111} \geq A_{2222}$ will also be aligned with N_I and ε_I (Pedersen and Bendsøe 1995). To ensure this, the feasible relative ply thicknesses are restricted to $t \in [\frac{1}{2};1]$.

If $C_3 < 0$, the the optimal laminate can also be characterized by an angle-ply $[\gamma_t/-\gamma_{1-t}]$, still in the coordinate system of the principal membrane forces. Here the thickness can take all values in the range $t \in [0;1]$.

What remains is thus to determine the relative ply thickness t in the cross-ply solution $[0^\circ_t/90^\circ_{1-t}]$ (for all C_3), and the orientation γ and thickness t of the angle-ply $[\gamma_t/-\gamma_{1-t}]$ (for $C_3 < 0$) yielding the maximum stiffness.

4 Compliance optimization

In this section, the compliance optimization problem expressed in terms of the membrane forces is solved for each of the two possibilities of ply material, $C_3 \lesseqgtr 0$. First the general problem is stated and it is shown how the global optimization problem is made local in character. Thereafter the two different cases of C_3 negative or positive are solved and discussed in turn.

The objective of the optimization is to minimize the compliance of the laminate. The design variables are simply the relative thickness t and the angle of rotation γ in the angle-ply. These design parameters are allowed to vary independently from point to point throughout the plate. The minimization problem can thus be formulated as

$$\min_{t,\gamma} W(v^*)\,, \tag{12}$$

where $W(v^*)$ is the compliance given by the displacement field at equilibrium v^*; W is then the work done by the external forces. The problem (12) written in terms of membrane forces and complementary energy yields

$$\min_{t,\gamma} W(v^*) = -2 \min_{t,\gamma} \min_{\{\mathbf{N}\}\in\mathbf{Q}} U^C(\{\mathbf{N}\})\,. \tag{13}$$

The inner problem is the principle of minimum complementary energy and the minimization is performed over the space $\mathbf{Q}$ of all statically admissible membrane forces $\{\mathbf{N}\}$; U^C is the total complementary energy

$$U^C(\{\mathbf{N}\}) = \int_\Omega \frac{1}{2}\{\mathbf{N}\}^T[\mathbf{A}]^{-1}\{\mathbf{N}\}\,\mathrm{d}\Omega\,, \tag{14}$$

integrating over the structural area Ω. The two min-problems are independent and can be interchanged. Thereby the inner problem yields a minimization of the complementary energy over the design variables for a fixed field of membrane forces. From the formulation in lamination parameters (Hammer *et al.* 1997) it is furthermore shown that there is existence of a solution.

4.1 *Low shear stiffness material,* $C_3 > 0$

In the case of plies of orthotropic material with low shear stiffness (i.e. $C_3 \geq 0$), the *laminate* direction of orthotropy (choosing $A_{1111} \geq A_{2222}$) is aligned with the direction of the principal laminate membrane forces $|N_I| \geq |N_{II}|$, and principal strains $|\varepsilon_I| \geq |\varepsilon_{II}|$ in the optimal configuration, cf. Section 3. Furthermore, the optimal laminate is a cross-ply $[0^\circ_t/90^\circ_{1-t}]$, where the thickness is restricted to the interval $[\frac{1}{2};1]$ in order to assure $A_{1111} \geq A_{2222}$. Thereby the membrane stiffness matrix in the orthotropic direction using (6) becomes

$$[\mathbf{A}] = h \begin{bmatrix} C_1+(2t-1)C_2+C_3 & C_4-C_3 & 0 \\ & C_1-(2t-1)C_2+C_3 & 0 \\ \text{symm.} & & 2(C_5-C_3) \end{bmatrix} \tag{15}$$

The specific stress energy for the laminate is then given as

$$u^c = \frac{1}{2}\{\mathbf{N}\}^T[\mathbf{A}]^{-1}\{\mathbf{N}\} =$$

$$\frac{1}{2}\frac{N_I^2}{A_{1111}A_{2222}-A_{1122}^2}\left[A_{1111}\left(\frac{N_{II}}{N_I}\right)^2 - \right.$$

$$\left. 2A_{1122}\frac{N_{II}}{N_I} + A_{2222}\right] = \frac{1}{2}\frac{R-S-T}{U}\,, \tag{16}$$

where $R = (C_1+C_3)(N_I^2+N_{II}^2)$, $S = 2(C_4-C_3)N_I N_{II}$, $T = (2t-1)C_2(N_I^2-N_{II}^2)$, and $U = (C_1+C_3)^2-(C_4-C_3)^2-(2t-1)^2C_2^2$.

By differentiating the expression for u^c with respect to t and setting it equal to zero, the optimal thickness is found to be (afer some cumbersome calculations)

$$2t-1 =$$

$$\begin{cases} \dfrac{N_I+N_{II}}{N_I-N_{II}}\dfrac{C_1-C_4+2C_3}{C_2} & \text{for } \dfrac{N_{II}}{N_I} \leq -\dfrac{C_1-C_4-C_2+2C_3}{C_1-C_4+C_2+2C_3} \\ \dfrac{N_I-N_{II}}{N_I+N_{II}}\dfrac{C_1+C_4}{C_2} & \text{for } \dfrac{N_{II}}{N_I} \geq \dfrac{C_1+C_4-C_2}{C_1+C_4+C_2} \\ 1 & \text{otherwise} \end{cases} \,. \tag{17}$$

These expressions for the optimal thickness are shown in Fig. 3 based on the material data for graphite/epoxy and glass/epoxy [cf. the paper by Tsai and Hahn (1980), see Table 2]. The materials are highly orthotropic and possess low shear stiffnesses of $C_3 = 19.7$ GPa and $C_3 = 3.33$ GPa, respectively.

In the intervals where the thickness t is given by the first expression in (17), the strain situation calculated from $\{\varepsilon\} = [\mathbf{A}]^{-1}\{\mathbf{N}\}$ will be that of pure shear $\varepsilon_I = -\varepsilon_{II}$, independently of the membrane force situation. Similarly, the second expression in (17) leads to pure strain dilation $\varepsilon_I = \varepsilon_{II}$ for all $\frac{N_{II}}{N_I} \geq \frac{C_1+C_4-C_2}{C_1+C_4+C_2}$. Thus an optimal design in general can be dominated by large regions of the extreme strain situations of $|\varepsilon_I| = |\varepsilon_{II}|$.

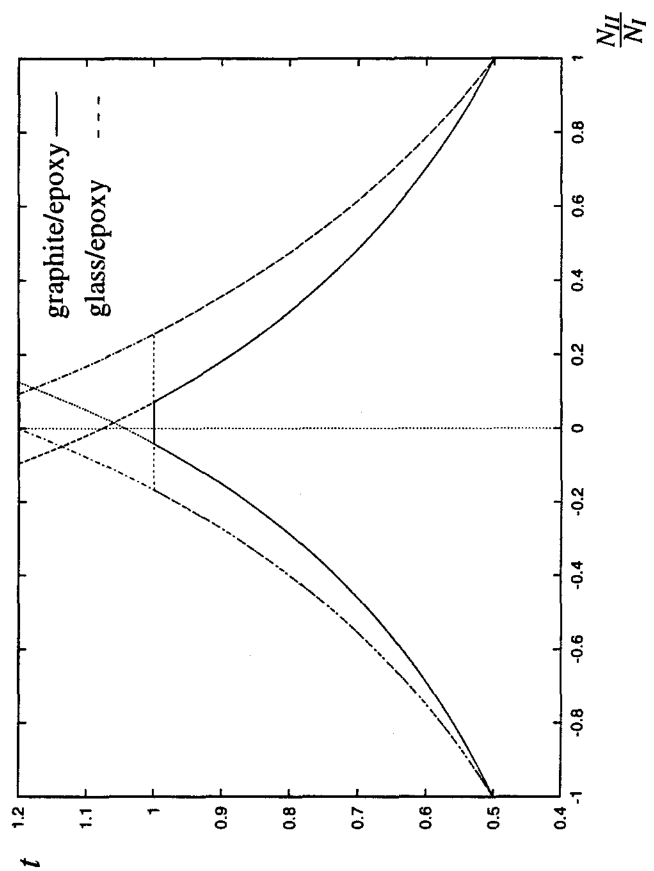

Fig. 3. The optimal relative ply thickness t of the cross-ply $[0^\circ_t/90^\circ_{1-t}]$ as a function of membrane stresses $\frac{N_{II}}{N_I}$. The solution is not symmetric

4.2 High shear stiffness material

Now, if the material has high shear stiffness $C_3 < 0$, the optimal laminate is either of the form $[0^\circ_t/90^\circ_{1-t}]$ or $[\gamma_t/-\gamma_{1-t}]$.

In the case of a cross-ply design the derivations of the optimal relative thickness are identical to the ones in Section 4.1 for the low shear stiffness material. The relative thickness t is thus again given by (17).

If the laminate is an angle-ply $[\gamma_t/-\gamma_{1-t}]$, the laminate is only orthotropic for special values of t and γ. The membrane stiffness matrix in this case is

$$[\mathbf{A}] = h[\mathbf{Z}]\,, \tag{18}$$

where $[\mathbf{Z}]$ is given in Table 1.

As above this is used in the specific complementary energy u^c which can be written in the following form:

$$u^c = \frac{1}{2}\{\mathbf{N}\}^T[\mathbf{A}]^{-1}\{\mathbf{N}\} = \cdots = \frac{f_1(\gamma) + (2t-1)^2 f_2(\gamma)}{f_3(\gamma) + (2t-1)^2 f_4(\gamma)}\,, \tag{19}$$

where $f_i(\gamma)$ are functions of the angle γ alone. First the gradient of the above expression with respect to t is put equal to zero. Thus

$$\frac{\partial u^c}{\partial t} = 8(2t-1)\frac{f_2(\gamma)f_3(\gamma) - f_1(\gamma)f_4(\gamma)}{\left[f_3(\gamma) + (2t-1)^2 f_4(\gamma)\right]^2} = 0\,. \tag{20}$$

This yields the three feasible solutions of $[\gamma/-\gamma]$, $[0^\circ]$ or $[\gamma_t/-\gamma_{1-t}]$, where $\cos 2\gamma = \frac{N_I+N_{II}}{N_I-N_{II}}\frac{C_2(C_1-C_4-2C_3)}{C_2^2-4C_3(C_1+C_4)}$ and t can take all values in $[0;1]$. In the first two cases we have an angle-ply with equal ply thickness or a single ply and the *laminate* itself is orthotropic.

The next step is to find the derivatives of u^c with respect to the orientation γ. The expression is lengthy and is thus not given here, but it can be shown that $\frac{\partial u^c}{\partial \gamma} = 0$ for all thicknesses t in the case of $\cos 2\gamma = \frac{N_I+N_{II}}{N_I-N_{II}}\frac{C_2(C_1-C_4-2C_3)}{C_2^2-4C_3(C_1+C_4)}$. Similarly for the $[\gamma_t/-\gamma_{1-t}]$ design (yielding $\frac{\partial u^c}{\partial t} = 0$), the ply angle yielding extreme complementary energy density is given by

$$\cos 2\gamma = \begin{cases} \dfrac{N_I+N_{II}}{N_I-N_{II}}\dfrac{(C_1-C_4+2C_3)}{C_2^2-4C_3(C_1+C_4)}\,, \\ \qquad \text{for } \dfrac{N_{II}}{N_I} \leq -\dfrac{C_2(C_1-2C_3-C_4)+4C_3(C_1+C_4)-C_2^2}{C_2(C_1-2C_3-C_4)-4C_3(C_1+C_4)+C_2^2} \\ \dfrac{N_I-N_{II}}{N_I+N_{II}}\dfrac{C_1+C_4}{C_2}\,, \\ \qquad \text{for } \dfrac{N_{II}}{N_I} \geq \dfrac{C_1+C_4-C_2}{C_1+C_4+C_2} \\ 1 \qquad \text{otherwise} \end{cases} \tag{21}$$

By comparison of the energies for the different design possibilities, it is found that the optimal design all in all is given by an angle-ply for lower values of $\frac{N_{II}}{N_I}$, a cross-ply for the highest values and a single ply in between

$$[\gamma_t/-\gamma_{1-t}] : \cos 2\gamma = \frac{N_I + N_{II}}{N_I - N_{II}}\frac{C_2(C_1 - C_4 - 2C_3)}{C_2^2 - 4C_3(C_1 + C_4)}$$

Table 1. Scaled stiffness matrix [**Z**]

$$h\begin{bmatrix} C_1+C_2\cos 2\gamma+C_3\cos 4\gamma & C_4-C_3\cos 4\gamma & -\sqrt{2}(2t-1)(\frac{1}{2}C_2\sin 2\gamma+C_3\sin 4\gamma) \\ & C_1-C_2\cos 2\gamma+C_3\cos 4\gamma & -\sqrt{2}(2t-1)(\frac{1}{2}C_2\sin 2\gamma-C_3\sin 4\gamma) \\ \text{symm.} & & 2C_5-2C_3\cos 4\gamma \end{bmatrix}.$$

Table 2. The material data used in the numerical examples

	E_L (GPa)	E_T (GPa)	ν_{LT}	G_{LT} (GPa)	C_3 (GPa)
graphite/epoxy from Tsai and Hahn (1980)	181	10.3	0.28	7.17	19.7
graphite/epoxy with adjusted G_{LT} giving high shear stiffness	181	10.3	0.28	47.6	-0.5
glass/epoxy from Tsai and Hahn (1980)	38.6	8.27	0.26	4.14	3.33
glass/epoxy with adjusted G_{LT} giving high shear stiffness	38.6	8.27	0.26	11.8	-0.5

with any t,

$$\text{for } \frac{N_{II}}{N_I} \le -\frac{C_2(C_1-2C_3-C_4)+4C_3(C_1+C_4)-C_2^2}{C_2(C_1-2C_3-C_4)-4C_3(C_1+C_4)+C_2^2},$$

$$[0^\circ_t/90^\circ_{1-t}] : 2t-1 = \frac{N_I-N_{II}}{N_I+N_{II}}\frac{C_1+C_4}{C_2},$$

$$\text{for } \frac{N_{II}}{N_I} \ge \frac{C_1+C_4-C_2}{C_1+C_4+C_2},$$

$$[0^\circ] \quad \text{otherwise}. \tag{22}$$

Figure 4 shows the optimal solution as a function of the membrane forces. The material data are still those of graphite/epoxy and glass/epoxy with the exception of the shear moduli G_{LT} which are altered to 47.6 GPa and 11.8 GPa, respectively. Hereby the modified materials get high shear stiffness with $C_3 = -0.5$ GPa, see Table 2.

That the thickness t can be chosen arbitrarily in the angle-ply solution of (22) can be explained by noticing that all members in the membrane stiffness matrix [**A**] except A_{1112} and A_{2212} are unaffected by the thickness t. This is still in the coordinate system of the principal strains and membrane forces. Thus a given strain situation will result in the same membrane forces and thus in the same energy density for any t. This is then an example of having an infinity of designs all yielding minimum compliance although having quite different appearances.

The expression for the optimal orientation in (22) is seen to be the same as in the optimization problem of rotating a single ply of orthotropic material with $C_3 < 0$ (Cheng and Pedersen 1997). The solutions only coincide in this region. Apart from that, the designs and the nature of the problem cannot be compared as the design freedom is much bigger when starting out from a laminate with an arbitrary number of plies.

As already mentioned, the cross-ply solution in (22) results in the strain situation of $\varepsilon_I = \varepsilon_{II}$. Thus instead of finding the optimal laminate type by comparing the energies for the angle-ply and the cross-ply designs, one could reach the same conclusion by exploiting the result from the optimization in the lamination parameters in the strains (Hammer *et al.* 1997), that the cross-ply is only optimal in the situation of $\varepsilon_I = \varepsilon_{II}$. As to the angle-ply solution, the relation $\varepsilon_I = -\varepsilon_{II}$ is only obtained if $\gamma = \pm 45^\circ$ (independently of t).

5 Numerical examples

In this section (17) and (21) giving the optimal cross-ply and angle-ply configurations are used in a numerical example to find the optimal laminate design of a structure subject to a single load case. The same structure is optimized for a ply material with either high or low shear stiffness.

The structure is analysed by means of the finite element method giving information about the strains and membrane forces throughout the structure for a given laminate design. The material parameters and the laminate configuration is kept constant within each finite element but the design is allowed to vary independently from element to element. An initial design is arbitrarily chosen. Based on the calculated membrane forces the laminate design is then changed into the optimal design solution. That is a cross-ply with relative ply thickness given by (17) and oriented along the direction of N_I if $C_3 > 0$, or if $C_3 < 0$ an angle-ply or a cross-ply as stated in (22). Then a new finite element analysis is performed, the design changed and so on until convergence is obtained.

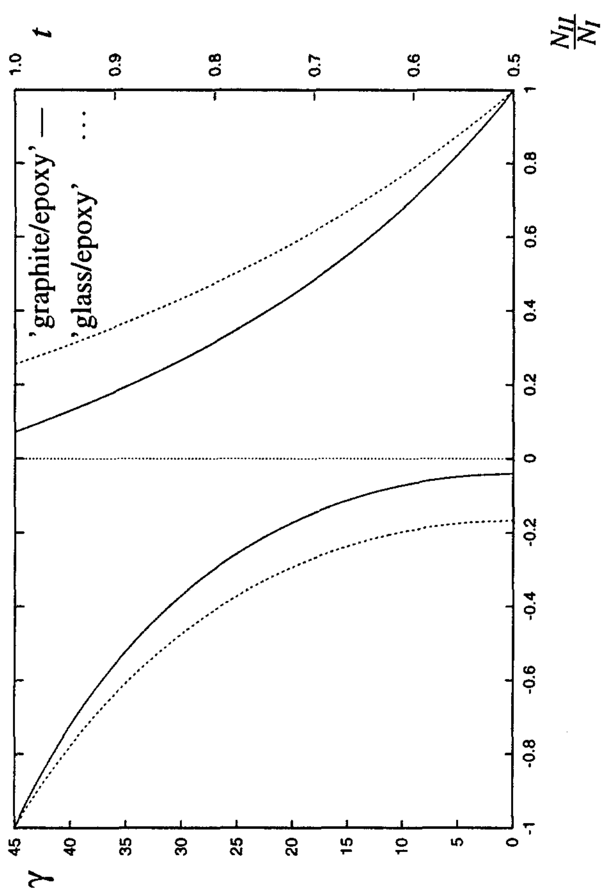

Fig. 4. The optimal angle-ply rotation of $[\gamma_t / -\gamma_{1-t}]$ and cross-ply thickness of $[0^\circ_t/90^\circ_{1-t}]$ as a function of membrane stresses $\frac{N_{II}}{N_I}$ for two materials with $C_3 < 0$

The structure to be designed is sketched in Fig. 5 and consists of a short cantilever fixed at one end and with a point load at the other.

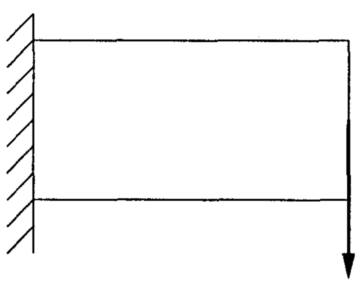

Fig. 5. A cantilever with a point load

The optimal design found using the procedure described above is shown in Fig. 6. The ply material is here graphite/epoxy and has thus low shear stiffness $C_3 > 0$, see Table 2. The small lines show the direction of the fibers whereas the shading shows the relative thickness of each of the two plies. Black means the thickness is zero and white corresponds to the maximum relative thickness of one. As can be seen, the optimal solution consists of a single ply in the outermost regions and a cross-ply in the interior. The strain situation in each finite element in the optimal design of Fig. 6 is shown in Fig. 7. Here black corresponds to pure dilation $\varepsilon_I = \varepsilon_{II}$, and white to pure shear $\varepsilon_I = -\varepsilon_{II}$. It is evident that these special cases which one might consider as rare exceptions, actually end up being dominant after the optimization.

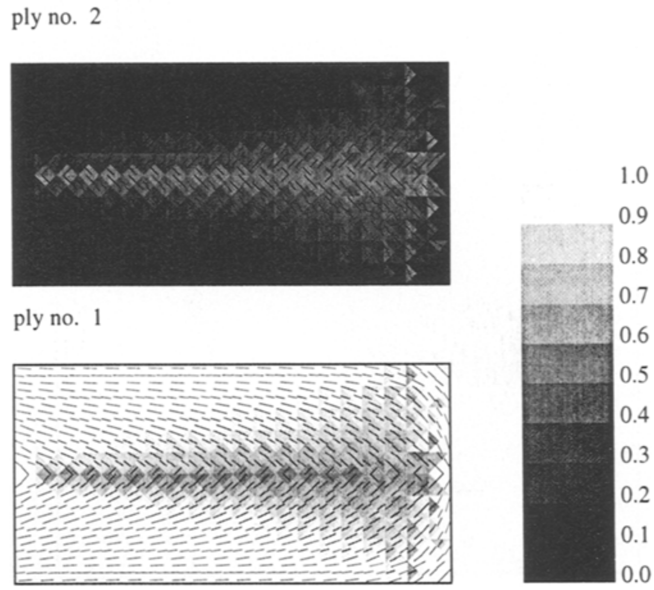

Fig. 6. The optimal cross-ply for a ply material with low shear stiffness $C_3 > 0$. The fiber directions are shown by the lines. Black shading is zero thickness and white shading corresponds to the maximum relative thickness of one

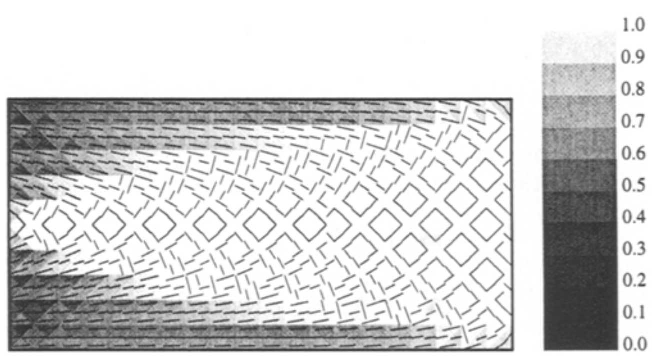

Fig. 7. $(|\varepsilon_I| - |\varepsilon_{II}|)/2|\varepsilon_I|$ for the optimal design for $C_3 > 0$. Black corresponds to pure dilation $\varepsilon_I = \varepsilon_{II}$, and white to pure shear $\varepsilon_I = -\varepsilon_{II}$

Using the full parameterization of lamination parameters in the optimization process as described by Hammer *et al.* (1997) leads to the same optimal design (as it should), but this method is less efficient in the sense that there one must solve a linear optimization problem with nonlinear side constraints in each iteration and therefore it takes more computational time. The iteration history for both techniques is shown in Fig. 8, redesigning according to (17) or using the lamination parameters both with and without a penalty function that makes the problem strictly concave. See the paper by Hammer *et al.* (1997) for the computational details.

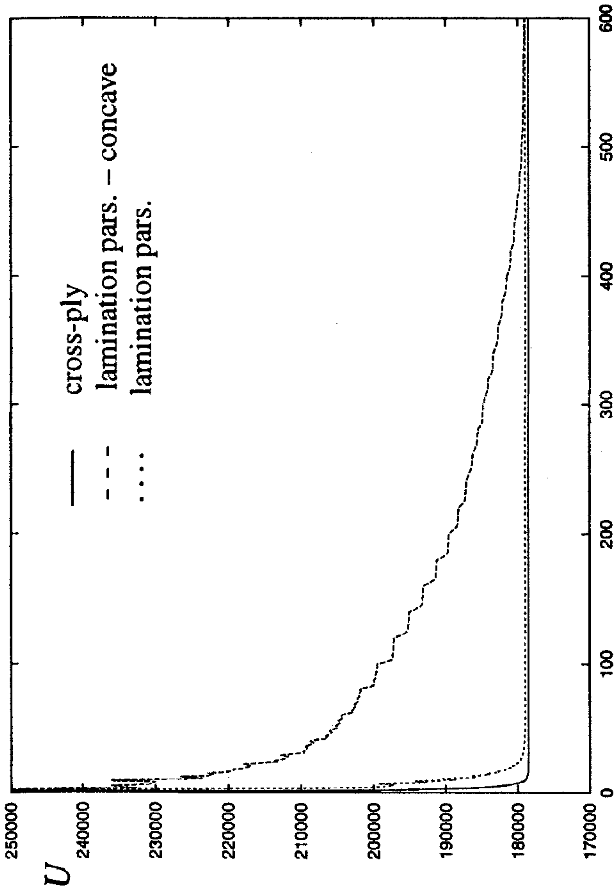

Fig. 8. The iteration history for the problem in Fig. 6. The total energy is plotted as a function of the iteration number. The design is found either by employing the lamination parameters or by redesigning a cross-ply

Using a ply material with high shear stiffness, $C_3 < 0$, yields the optimal designs in Fig. 9. In this example, it is shown numerically that the membrane forces are everywhere and at all times during the optimization history so that the optimal design is either a single ply or an angle-ply and never a cross-ply. Here the outer regions consist of a single ply solution whereas the inner part is an angle-ply. The resulting designs for two different choices of relative ply thickness t in the angle-ply solution are shown in Fig. 9. To the left $t = \frac{1}{2}$ and to the right $t = 1$ (thus a single ply design). The two designs yield the same compliance (down to the 8-th digit). The designs are quite different from that of Fig. 6 (where $C_3 > 0$) and it is clear how the highly improved shear stiffness of the material now plays an important role in the design.

No matter what sort of ply material, convergence is obtained in very few iterations of the order of 10-20. Moreover, the optimization process is very computationally efficient as it is both simple and fast.

6 Bounds on $\frac{N_{II}}{N_I}$

In the case of a material with low shear stiffness $C_3 > 0$, the optimal relations between the membrane forces and lay-up parameters also appear in another context. Since the overall

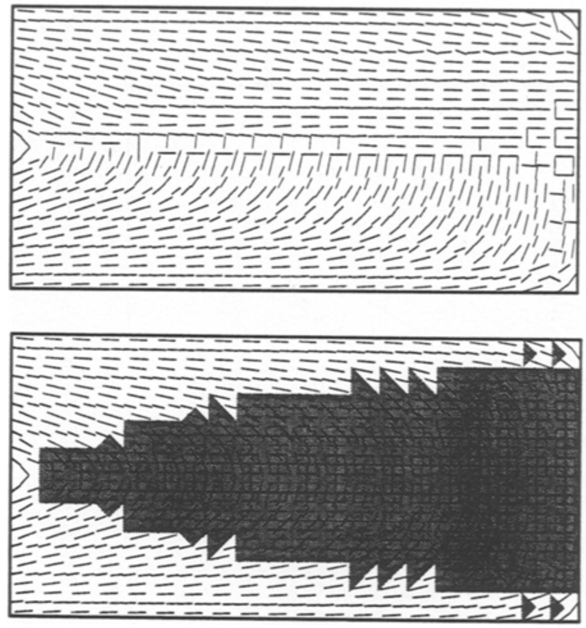

Fig. 9. The optimal angle-ply design for a ply material with high shear stiffness, $C_3 < 0$. The fiber directions are shown by the lines. White colouring corresponds to the maximum relative thickness of one and the grey to equal thickness of the two plies

laminate is orthotropic here, the numerically largest principal strain ε_I, membrane force N_I and laminate direction of orthotropy must be coaligned in the optimal design. However, the requirement that $|\varepsilon_I| \geq |\varepsilon_{II}|$ given that $|N_1| \geq |N_{II}|$ and $A_{1111} \geq A_{2222}$, imposes the following bounds on the membrane forces (Pedersen and Bendsøe 1995):

$$-\frac{A_{2222} - A_{1122}}{A_{1111} - A_{1122}} \leq \frac{N_{II}}{N_I} \leq \frac{A_{2222} + A_{1122}}{A_{1111} + A_{1122}} . \tag{23}$$

The constraining values can be very severe depending on the anisotropy of the material.

For the cross-ply $[0^\circ_t/90^\circ_{1-t}]$ the limits become

$$-\frac{C_1 - C_4 + 2C_3 - (2t-1)C_2}{C_1 - C_4 + 2C_3 + (2t-1)C_2} \leq \frac{N_{II}}{N_I} \leq$$

$$\frac{C_1 + C_4 - (2t-1)C_2}{C_1 + C_4 + (2t-1)C_2} . \tag{24}$$

If the inequalities above are rearranged so the thickness term $(2t - 1)$ is isolated, the expressions in (17) for the optimal thickness distribution reappear. In other words, the optimal cross-ply is always so that the relation $\frac{N_{II}}{N_I}$ goes to the limits.

7 Conclusions

The optimal laminate configuration for a structure subject to single membrane load cases is found. Both ply materials with low and high shear stiffnesses are treated. In the first case, the optimal design is either a single ply of material or a cross-ply. In the latter case, a single ply, an angle-ply, or a cross-ply is optimal. The parameters in the form of the material orientation and the relative ply thicknesses are derived and epxressed as functions of the membrane forces in the design area.

The optimization is performed in the stress space (optimization on the complementary stress energy), whereby the design solution is determined directly from the necessary conditions. This is thus advantageous compared to the equivalent formulation in the displacements where the objective function indeed turns out to be linear in the design parameters but where the solution to the inner local problems becomes nonunique, thus necessitating the use of the equilibrium equations.

Finally, it is shown how the optimal relations for the laminate appears as material bounds on the membrane forces in order to assure the required alignment of the orthotropy direction, the numerically largest principal force and strain.

Numerical examples are given on optimal designs also demonstrating how the structure seems to be dominated by pure shear or dilation strains in the optimal configuration.

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