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An evolutionary method for optimal design of plates with discrete variable thicknesses subject to constant weight

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Abstract This paper presents a simple evolutionary method for optimization of plates subject to constant weight, where design variable thicknesses are discrete. Sensitivity numbers for sizing elements are derived using optimality criteria methods. An optimal design with minimum displacement or minimum strain energy is obtained by gradually shifting material from elements to the others according to their sensitivity numbers. A simple smoothing technique is additionally employed to suppress formation of checkerboard patterns. It is shown that the proposed method can directly deal with discrete design variables. Examples are provided to show the capacity of the proposed evolutionary method for structural optimization with discrete design variables.

1 Introduction

In many engineering applications of structural optimization, the design variables must be selected from a given set of discrete values. For example, structural members may have to be selected from standard sections or thicknesses commercially available from manufacturers. Traditional optimization techniques such as mathematical programming and optimality criteria methods are valid for problems where sizing design variables are continuous in order to calculate derivatives of objective and constraint functions with respect to design variables. These methods usually require sophisticated mathematical treatment when dealing with discrete design variables. Often the problem is solved for a continuous optimal solution assuming all designs variables are continuous. Then, one of the methods, such as rounding-off, branch and bound methods, simulated annealing, genetic algorithm, Lagrangian relaxation methods, is used to get a discrete solution (Huang and Arora 1995; Olsen and Vanderplaats 1989; Ringertz 1988; Sandgren 1990; Schmit and Fleury 1980). Usually the values given for each discrete design variable are required be close to each other for the validity of converting the continuous optimal solution to the discrete one (Ringertz 1988). The branch and bound method, basically known as an enumeration method, is more widely used for discrete optimization problems. After obtaining a continuous optimal solution for the problem, each variable is assigned a discrete value in sequence and the problem is solved again in the remaining variables. It can be seen that the number of times the problem needs to be re-solved increases enormously with the number of variables (Huang and Arora 1995).

Recently, a simple method, called Evolutionary Structural Optimization (ESO), which is based on the concept of gradually removing redundant material to achieve an optimal design, was proposed by Xie and Steven (1993). The ESO method was developed by Xie and Steven (1993, 1994) first for shape and layout problems with stress consideration and then for frequency optimization (Xie and Steven 1996). The ESO method for shape and topology problems with displacement constraints has been presented recently by Chu *et al.* (1995, 1996). It is found that the ESO method for shape and layout problems is simple and can be easily implemented into any general purpose finite element analysis (FEA) program. Many structural shape and topology optimization solutions obtained by other mathematically more complicated methods have been reproduced by the ESO method (Xie and Steven 1993, 1994, 1996; Chu *et al.* 1995, 1996, 1997, 1998; Chu and Xie 1997; Chu 1997). It appears that the ESO method can be also applied to sizing optimization problems. An earlier development in this direction was presented by Manickarajah *et al.* (1995) to increase the buckling load capacity of structures while keeping their volumes constant. More recently, an evolutionary procedure for discrete variable problems of weight minimization subject to displacement constraints has been presented recently by Chu (1997) and Chu, Xie and Steven (1998).

This paper presents an ESO method for problems of minimizing a specified displacement or the strain energy (equivalently maximizing stiffness) of structures subject to constant weight, where sizing design variables are discrete. Sensitivity numbers for sizing elements are formulated using optimality criteria methods. An optimal design of the structure will be obtained by repeating the cycle of finite element analysis and material shifting until no further improvement in the objective can be achieved. An additional smoothing technique is used to suppress formation of checkerboard patterns in the resulting designs. It will be seen that the proposed ESO method for sizing problems is simple and capable of dealing with discrete design variables.

2 Problem formulation

Consider the problem of minimizing a displacement component at a point of a structure for a given weight. It is realized that when material is removed from an element by reducing its sizes, the displacement will generally increase. In contrast, when material is added to an element by increasing its thickness, the element becomes stiffer and consequently, it increases the overall stiffness of the structure. As a result, the displacement is generally reduced in the absolute value. So the displacement can be significantly reduced when the material is shifted from the locations where it has a small effect on the increase in the displacement to the locations where it has a large effect on the reduction of the displacement. A substantial reduction in the structure's strain energy can also be obtained when a similar material shifting technique is used, which results in a stiffer structure.

The problem can be stated as follows.

By shifting materials between elements,

minimize *the specified response of a structure*
subject to *a constraint on the weight.*

Shifting material from one element to the other can be done by reducing the thickness of one element and increasing the thickness of another one. In general, when elements vary differently, it may not be possible to shift the same amount of material from one element to another. However, it is possible to shift the same amount of material from a number of elements to other elements. In some cases, additional scaling design variables are needed to keep the weight unchanged. At any step, every element can be either reduced or increased. Therefore, to solve this problem we need to evaluate the effects on the specified response (displacement or strain energy) by removing and adding material due to decrease and increase in the element thickness, respectively.

3 Sensitivity analysis

3.1 Sensitivity numbers for displacement minimization

Suppose the i-th element is to be sized to the next lower or upper available dimension. This results in the change in the element weight by the value $\Delta w_i = w_i^{\text{new}} - w_i$ and the change in its stiffness matrix by $[\Delta \mathbf{K}^i] = [\mathbf{K}^i]^{\text{new}} - [\mathbf{K}^i]$. The change in displacements can be determined by considering equilibrium conditions before and after sizing the i-th element. This gives

$$[\mathbf{K}]\{\mathbf{u}\} = \{\mathbf{P}\}, \tag{1}$$

and

$$[\mathbf{K} + \Delta\mathbf{K}]\{\mathbf{u} + \mathbf{\Delta u}\} = \{\mathbf{P}\}, \tag{2}$$

where $[\mathbf{K}]$ is the global stiffness matrix, $\{\mathbf{u}\}$ is the global nodal displacement vector, and $\{\mathbf{P}\}$ is the nodal load vector. No change in the nodal load vector is assumed. By subtracting (1) from (2) and ignoring the higher-order term, the change in the displacement vector can be found as

$$\{\mathbf{\Delta u}\} = -[\mathbf{K}]^{-1}[\mathbf{\Delta K}]\{\mathbf{u}\}. \tag{3}$$

To find the change in the specified j-th displacement component u_j, a virtual unit load vector $\{\mathbf{F}^j\}$, in which only corresponding j-th component is equal to unity and all the others are equal to zero, is introduced. Multiplying (3) by $\{\mathbf{F}^j\}^T$, the change in the specified displacement due to sizing the i-th element is determined by

$$\begin{aligned}\Delta u_j &= \{\mathbf{F}^j\}^T\{\mathbf{\Delta} u\} = -\{\mathbf{F}^j\}^T[\mathbf{K}]^{-1}[\mathbf{\Delta K}]\{\mathbf{u}\} = \\ &-\{\mathbf{u}^j\}^T[\mathbf{\Delta K}]\{\mathbf{u}\} = -\{\mathbf{u}^{ij}\}^T[\mathbf{\Delta K}^i]\{\mathbf{u}^i\} = \\ &\{\mathbf{u}^{ij}\}^T[\mathbf{K}^i]\{\mathbf{u}^i\} - \{\mathbf{u}^{ij}\}^T[\mathbf{K}^i]^{\text{new}}\{\mathbf{u}^i\},\end{aligned} \tag{4}$$

where $[\mathbf{\Delta K}] = [\mathbf{\Delta K}^i] = [\mathbf{K}^i]^{\text{new}} - [\mathbf{K}^i]$ is employed, $\{\mathbf{u}^j\}$ is the solution of (1) for the virtual unit load $\{\mathbf{F}^j\}$, $\{\mathbf{u}^i\}$ and $\{\mathbf{u}^{ij}]$ are the displacements of the i-th element due to the real load $\{\mathbf{P}\}$ and the virtual unit load $\{\mathbf{F}^j\}$, respectively. It should be noted that Δu_j can be positive or negative, which implies that u_j may increase or decrease.

The value

$$\alpha_{ij} = \{\mathbf{u}^{ij}\}^T[\mathbf{K}^i]\{\mathbf{u}^i\} \quad (i = 1, n) \tag{5}$$

is known as the virtual energy of the i-th element. Furthermore, by taking the sum of α_{ij} in (5) over all elements, we have the well-known relationship

$$u_j = \sum_{i=1}^{n} \alpha_{ij}, \tag{6}$$

where α_{ij} is also referred to as the element contribution. Assuming small changes in the displacements due to sizing an element, the change in the element virtual energy can be approximately determined as

$$\begin{aligned}\Delta\alpha_{ij} &= \alpha_{ij}^{\text{new}} - \alpha_{ij} = \\ &\{\mathbf{u}^{ij}\}^T[\mathbf{K}^i]^{\text{new}}\{\mathbf{u}^i\} - \{\mathbf{u}^{ij}\}^T[\mathbf{K}^i]\{\mathbf{u}^i\} \quad (i = 1, n).\end{aligned} \tag{7}$$

Therefore, (4) and (7) give

$$\Delta u_j = -\{\mathbf{u}^{ij}\}^T[\mathbf{\Delta K}^i]\{\mathbf{u}^i\} = -\Delta\alpha_{ij} \quad (i = 1, n), \tag{8}$$

which means that, in absolute values, the change in the specified displacement is equal to the change in the virtual energy within an element due to changing its sizes.

By using (8), the change in the specified displacement is estimated by the change in the element virtual energy due to changing the element sizes. It is obvious that reducing the element, whose $\Delta\alpha_{ij}$ is close to zero or $|\Delta\alpha_{ij}|$ is the lowest, will result in the minimum change in the displacement. When all elements can be reduced by equal weight portions, reduction of the element with the lowest $|\Delta\alpha_{ij}|$ is always the best choice, because the weight of structure is reduced by the same amount with the least change in the specified performance. However, when elements are reduced differently, the efficiency of reduction of an element depends also on how much the change Δw_i in its own weight w_i. Comparing two elements whose reductions result in the same $|\Delta\alpha_{ij}|$, it is obvious that reduction of the element, which gives a larger reduction in weight, will result in a lighter structure with an equal response. This means that reduction of the element with lower ratio $|\Delta\alpha_{ij}|/|\Delta w_i|$ is more efficient. The question of whether the ratio $|\Delta\alpha_{ij}|/|\Delta w_i|$ can be used as sensitivity numbers for material reduction and addition to elements for minimum displacement design can be addressed using a Lagrange multiplier approach.

For the problem of minimizing the absolute value of the displacement $|u_j|$ subject to a weight constraint

$$W = \sum_{i=1}^{n} w_i = W^* , \tag{9}$$

the Lagrangian is defined as

$$L = |u_j| - \lambda \left(\sum_{i=1}^{n} w_i - W^* \right) , \tag{10}$$

where λ is a Lagrange multiplier and W^* is the prescribed weight for the structure. Taking w_i as the design variables, the optimality condition for the problem is

$$\frac{\partial L}{\partial w_i} = \left| \frac{\partial u_j}{\partial w_i} \right| - \lambda = 0 \quad (i = 1, n) , \tag{11}$$

which can be approximated by

$$\left| \frac{\Delta u_j}{\Delta w_i} \right| - \lambda = 0 \quad (i = 1, n) . \tag{12}$$

Recalling relationship (8), which implies that the change in the specified displacement, in the absolute value, is equal to the change in the element virtual energy due to sizing the element, the above optimality condition becomes

$$\left| \frac{\Delta\alpha_{ij}}{\Delta w_i} \right| - \lambda = 0 \quad (i = 1, n) , \tag{13}$$

or

$$\gamma_i = \left| \frac{\Delta\alpha_{ij}}{\Delta w_i} \right| = \lambda = \text{const.} \quad (i = 1, n) . \tag{14}$$

The optimality criterion (14) states that *at the optimum the absolute ratio of the change in the element virtual energy and the change in the element weight is equal for all elements.* The value γ_i in (14) can be viewed as a measure of effectiveness of the material within the portion to be removed from or added to an element. Thus, the sensitivity number for the sizing element in the displacement minimization problem is defined as

$$\gamma_i = \frac{|\Delta\alpha_{ij}|}{|\Delta w_i|} = \frac{|\{\mathbf{u}^{ij}\}^T [\mathbf{\Delta K}^i] \{\mathbf{u}^i\}|}{|\Delta w_i|} \quad (i = 1, n) . \tag{15}$$

It is obvious that the most effective way of removing material is to remove the material from the element with the lowest value of γ_i because it will have the smallest effect on increase in the displacement. On the contrary, the most effective way of adding material is to add the material to the element with the highest value of γ_i because this amount of material will have the largest effect on decrease in the displacement. This sizing strategy can result in more efficient design in the sense that more uniform values of γ_i can be obtained. In some cases, it is possible to reach a uniform state of γ_i by which the optimality criterion (14) is satisfied for all elements.

Based on the sensitivity numbers defined by (15), different optimization procedures can be employed to obtain the solution. One way is to use only the material removal (element size reduction) technique to solve this problem. Using (15), the sensitivity numbers for element reduction are calculated for all elements. Material is removed from elements with the lowest sensitivity numbers by changing their thicknesses to the next lower values. As a result, the weight of the structure decreases. In this case scaling the design variables is necessary to satisfy the weight constraint (9). The disadvantage of this procedure is that by using the design variable scaling, the resulting thicknesses of elements will not have the exact values as given in the sets of discrete values. It is preferable to use a procedure which will keep the thicknesses within the given sets.

It is possible to keep the weight of the structure unchanged, without involvement of design variable scaling, by shifting material between elements, i.e. the same amount of material, which has been removed from some elements, is added to others. To do this, the best elements to remove material from and the best elements to add material to need to be identified. Because the increase and decrease in thicknesses can have different effects, two sensitivity numbers need to be calculated for each element using (15).

Assume that a plate structure to be optimized is modelled by elements with thicknesses chosen from the given set $t_i = \{t_1, t_2, \ldots, t_{s-1}, t_s, t_{s+1}, \ldots, t_r\}$. For convenience, thicknesses in the set are often put in ascending order. This way, if the current thickness of an element is t_s, the next lower and higher thicknesses are t_{s-1} and t_{s+1}, respectively. Materials can be removed from or added to an element by selecting the next lower or higher thickness in the given set. Due to reduction and increase in the i-th element thickness, the changes in the element weight are

$$(\Delta w_i)^- = w_i(t_{s-1}) - w_i(t_s) \quad (i = 1, n) , \tag{16}$$

and

$$(\Delta w_i)^+ = w_i(t_{s+1}) - w_i(t_s) \quad (i = 1, n) . \tag{17}$$

Correspondingly, the changes in the stiffness matrix of the element are given as

$$[\mathbf{\Delta K}^i]^- = [\mathbf{K}^i(t_{s-1})] - [\mathbf{K}^i(t_s)] \quad (i = 1, n) , \tag{18}$$

and

$$[\mathbf{\Delta K}^i]^+ = [\mathbf{K}^i(t_{s+1})] - [\mathbf{K}^i(t_s)] \quad (i = 1, n) , \tag{19}$$

where the superscript "−" denotes thickness reduction and "+" denotes thickness increase. The change in the element virtual energy can be calculated using (8), which gives

$$(\Delta\alpha_{ij})^- = \{\mathbf{u}^{ij}\}^T[\mathbf{\Delta K}^i]^-\{\mathbf{u}^i\} \quad (i = 1, n), \tag{20}$$

$$(\Delta\alpha_{ij})^+ = \{\mathbf{u}^{ij}\}^T[\mathbf{\Delta K}^i]^+\{\mathbf{u}^i\} \quad (i = 1, n). \tag{21}$$

Therefore, for each element two following sensitivity numbers are calculated:

$$\gamma_i^- = \left|\frac{(\Delta\alpha_{ij})^-}{(\Delta w_i)^-}\right| = \frac{|\{\mathbf{u}^{ij}\}^T[\mathbf{\Delta K}^i]^-\{\mathbf{u}^i\}|}{|(\Delta w_i)^-|} \quad (i = 1, n), \tag{22}$$

$$\gamma_i^+ = \left|\frac{(\Delta\alpha_{ij})^+}{(\Delta w_i)^+}\right| = \frac{|\{\mathbf{u}^{ij}\}^T[\mathbf{\Delta K}^i]^+\{\mathbf{u}^i\}|}{|(\Delta w_i)^+|} \quad (i = 1, n). \tag{23}$$

It is obvious that removing material from elements will generally increase the displacement and adding material will reduce the displacement in absolute values. Reducing the thickness of the element with the smallest γ_i^- will result in the minimum increase in the objective displacement. Conversely, increasing the thickness of the element with largest γ_i^+ will make the maximum decrease in the objective displacement. If the maximum value of γ_i^+ is much larger than the minimum value of γ_i^-, the specified displacement will be significantly reduced when the material is shifted from the element with minimum γ_i^- to the element with maximum γ_i^+.

3.2 Sensitivity numbers for strain energy minimization

The strain energy of a structure is defined as

$$S = \frac{1}{2}\{\mathbf{P}\}^T\{\mathbf{u}\}, \tag{24}$$

where $\{\mathbf{u}\}$ is the displacement due to the real load $\{\mathbf{P}\}$. The element strain energy is

$$s_i = \frac{1}{2}\{\mathbf{u}^i\}^T[\mathbf{K}^i]\{\mathbf{u}^i\} \quad (i = 1, n), \tag{25}$$

where $\{\mathbf{u}^i\}$ is the displacement of the i-th element, which is obtained from $\{\mathbf{u}\}$. Similar to the change in the displacement given by (8), the change in the strain energy of the structure is equal, with an opposite sign, to the change in the element strain energy due to the change in the element size, i.e.

$$\Delta S = -\frac{1}{2}\{\mathbf{u}^i\}^T[\mathbf{\Delta K}^i]\{\mathbf{u}^i\} = -\Delta s_i \quad (i = 1, n). \tag{26}$$

For a weight constraint given by (9), the Lagrangian for the problem of minimizing the strain energy is defined as

$$L = S + \lambda\left(\sum_{i=1}^{n} w_i - W^*\right), \tag{27}$$

where λ is a Lagrange multiplier. In this case, by using (26), the optimality conditions become

$$-\frac{\Delta s_i}{\Delta w_i} + \lambda = 0 \quad (i = 1, n), \tag{28}$$

or

$$\gamma_i = \frac{\Delta s_i}{\Delta w_i} = \lambda = \text{const.} \quad (i = 1, n). \tag{29}$$

The sensitivity number for sizing elements to minimize the strain energy is

$$\gamma_i = \frac{\Delta s_i}{\Delta w_i} = \frac{\frac{1}{2}\{\mathbf{u}^i\}^T[\mathbf{\Delta K}^i]\{\mathbf{u}^i\}}{\Delta w_i} \quad (i = 1, n). \tag{30}$$

To shift material within the structure for minimizing the strain energy, the following two sensitivity numbers for each elements are calculated:

$$\gamma_i^- = \frac{(\Delta s_i)^-}{(\Delta w_i)^-} = \frac{\frac{1}{2}\{\mathbf{u}^i\}^T[\mathbf{\Delta K}^i]^-\{\mathbf{u}^i\}}{(\Delta w_i)^-} \quad (i = 1, n), \tag{31}$$

$$\gamma_i^+ = \frac{(\Delta s_i)^+}{(\Delta w_i)^+} = \frac{\frac{1}{2}\{\mathbf{u}^i\}^T[\mathbf{\Delta K}^i]^+\{\mathbf{u}^i\}}{(\Delta w_i)^+} \quad (i = 1, n), \tag{32}$$

where the superscript "−" denotes thickness reduction and "+" denotes thickness increase. The changes in the element weight and the element stiffness matrix are calculated by (16)-(19). Similarly, the strain energy can be significantly reduced if material is shifted from the elements with lowest γ_i^- to the elements with largest γ_i^+ defined by (31) and (32), respectively.

4 Optimization procedure

To obtain an accurate solution, an iterative procedure, where only a small amount of material is shifted in each iteration, should be employed. The evolutionary optimization procedure for the displacement minimization problem is as follows.

Step 1. Model the structure by finite elements with intermediate thicknesses.

Step 2. Analyse the structure for the given real load and the virtual unit load corresponding to the objective displacement.

Step 3. If convergence in the objective displacement has been reached, go to Step 7. Otherwise, go to Step 4.

Step 4. Calculate the sensitivity numbers γ_i^- and γ_i^+ for each element.

Step 5. If the sensitivity numbers are uniform, go to Step 7. Otherwise, go to Step 6.

Step 6. Shift a specified amount of material in the structure by selecting the next lower thicknesses for a number of elements which have the lowest γ_i^- and the next larger thicknesses for other elements with highest γ_i^+. Go back go to Step 2.

Step 7. Stop.

Exactly the same evolutionary optimization procedure is used for strain energy minimization, except that the structure is analysed for the real load only. For the proposed evolutionary procedure two parameters, the tolerance and the amount of material to be shifted at each iteration, need to be specified. Tolerance is used to check the convergence of the objective displacement. Convergence is reached when the relative change in the objective between two successive iterations is less than the given tolerance δ, i.e.

$$\left|\frac{u_j^{\text{new}} - u_j^{\text{old}}}{u_j^{\text{new}}}\right| \leq \delta \quad \text{or} \quad \left|\frac{S^{\text{new}} - S^{\text{old}}}{S^{\text{new}}}\right| \leq \delta\,. \tag{33}$$

The optimization process will also be terminated when the sensitivity numbers are uniform, by which the optimality conditions are satisfied and the optimal solution is reached.

The amount of material to be shifted can be prescribed by *the material shifting ratio* (MSR), which is defined as the ratio of the portion of the weight (material) to be shifted at each iteration to the total weight (material) of the structure. This parameter controls the change of the design in each step. Depending on the step size, it controls the number of elements subjected to thickness decrease and increase. In the most general case where elements can change differently, the number of elements subjected to thickness reduction and the number of elements subjected to thickness increase are different and vary from iteration to iteration. When all elements can change by equal weight portions, as in the case where all elements are identical and equal step sizes are used, the number of elements subjected to decrease equal to the number of elements subjected to increase and is constant. In this case, an equivalent parameter called *the element shifting ratio* (ESR), defined by the ratio of the number of elements to be reduced (or increased) at each iteration to the total number of elements, can be used. Values less than 1% or 2% can be adopted for MSR although the influence of the MSR on the final design needs further investigation.

During the optimization process, elements having their thicknesses changed to the maximum or minimum values are not allowed to be further increased or decreased. It is noted that the thicknesses initially chosen for all elements must be other than maximum or minimum thicknesses, otherwise shifting the material within the structure cannot be carried out. The initial thicknesses for all elements are determined from the given weight for the structure.

It is worth noting that the order of the sensitivity numbers of an element is more important than their absolute values. For the simplest case where there is only one point load Q acting on a structure and the objective is to minimize the displacement at the same location and in the direction of the load, we have $\{\mathbf{u}^j\} = (1/Q)\{\mathbf{u}\}$ and there is no need to analyse the structure for the virtual unit load. The sensitivity numbers for this problem are exactly of the same order as in the problem of minimizing strain energy. This means solutions of these two problems are identical if the same mesh and the same ESR or MSR are used.

A computer program has been written to calculate the sensitivity numbers and shift a specified amount of material over elements. A batch file is set up to link this program to a finite element program and a loop is created to carry out the iterative process of optimization. The thicknesses t_s ($s = 1, r$) are input in either ascending or descending order as property number increases. Initially, all elements are assigned an intermediate thickness which is determined from the weight (or volume) given for the structure. Removing or adding material from or to an element is done by substituting the current property by the preceding or exceeding property. The properties with maximum and minimum thickness are identified so that elements, which have the thicknesses changed to the maximum or minimum thicknesses, will not be further increased or decreased. The input data file for the optimization subprogram includes the value for tolerance, the material (or element) shifting ratio, the parameter indicates the order of thicknesses (ascending or descending) input in the plate property file, the parameter for keeping symmetry, the properties with the minimum and maximum thicknesses. It also includes the maximum number of iterations which is allowed for a particular problem.

5 Smoothing technique for suppression of checkerboard patterns

It is observed that checkerboard patterns often appear in solutions derived by the proposed method for two-dimensional continuum structures (see Figs. 2a and 3a). This is quite typical in solutions for optimization problems using finite elements. With checkerboard patterns the shapes obtained may become practically unacceptable. The origin of checkerboard patterns is still not fully understood but it is likely to be related to the finite element approximation as a numerical phenomenon (Bendsøe *et al.* 1993). Díaz and Sigmund (1995) pointed out that material in checkerboard arrangements in 4-node element meshes appears to be locally stiffer than any real material built. It is desirable to suppress of the formation of checkerboard patterns. One way is to use higher-order elements. Rodrigues and Fernandes (1993) showed that the checkerboard patterns which appear by using 4-node elements, can be avoided by using 9-node elements. Another way is to use a special algorithm to control formation of checker board patterns as proposed by Bendsøe *et al.* (1993). A simpler algorithm for density redistribution was suggested by Youn and Park (1995) to suppress formation of checkerboard patterns.

In our method for minimizing structural response (displacement or strain energy), the sensitivity numbers defined by (15) or (30) can be expressed by the following formula:

$$\gamma_i = \frac{e_i}{\rho_i} \quad (i = 1, n)\,, \tag{34}$$

where ρ_i is mass density and

$$e_i = \left|\frac{\Delta\alpha_{ij}}{\Delta v_i}\right| \quad \text{or} \quad e_i = \left|\frac{\Delta s_i}{\Delta v_i}\right| \quad (i = 1, n) \tag{35}$$

is the virtual energy density or the strain energy density in the part of the element to be removed, and Δv_i is the volume of this part. It is obvious that in continuum structures, the energy density is a continuous function. However, the energy density e_i determined by (35) is only an average value and therefore becomes a stepwise function, experiencing discontinuity at the nodes. To provide continuity of the energy density at the nodes, a smoothing technique is needed. The simplest procedure for smoothing energy density and sensitivity numbers is as follows.

1. Calculate the virtual energy density at each node by averaging the virtual energy densities of all elements connecting to the node.

2. Calculate the smoothed virtual energy density for each element by averaging the obtained energies densities at its nodes.

3. The smoothed sensitivity numbers for elements are then calculated from the smoothed energy densities according to (34).

It will be seen through examples that the optimization procedure based on the smoothed sensitivity numbers is able to suppress formation of checkerboard patterns.

6 Examples

We shall illustrate the capability of the proposed method for minimizing the response of structures subject to a constant weight where the design variables are discrete. All examples are solved using a 586 Pentium/150 MHz personal computer. The number of design variables ranges from four to eight hundred with sets of three to ten discrete values for element thicknesses.

6.1 Minimum displacement design for a simply supported plate

Consider an example of minimum displacement design for a plate in bending. A simply supported square plate with sides of 8 m is carrying at the centre a point load $P = 100$ kN normal to its plane as shown in Fig. 1. Young's modulus $E =$ 30 GPa and Poisson's ratio $\nu = 0.2$ are assumed. The plate is designed for different sets of discrete thicknesses ranging from the minimum 0.2 m to the maximum 0.4 m. Because of symmetry, only a quarter of the plate is analysed using 400 quadrilateral plate elements. Initially, all elements are assigned the thickness 0.3 m, which defines the weight of the plate. The initial out-of-plane displacement at the centre is 4.22 mm. To show the effect of the proposed smoothing technique, the problem is solved first using the original and then using smoothed sensitivity numbers. Adopting MSR = 1% and tolerance $\delta = 0.01$, solutions are obtained for the following two sets of thicknesses.

1. *Design for three thicknesses*: $\Delta t = 0.1$ m, ESR = 3% (i.e. 12 elements having thicknesses reduced and 12 elements having thicknesses increased in each iteration), the optimal designs are shown in Fig. 2.

2. *Design for five thicknesses*: $\Delta t = 0.05$ m, ESR = 6% (i.e. 24 elements having thicknesses reduced and 24 elements having thicknesses increased), the optimal designs are shown in Fig. 3.

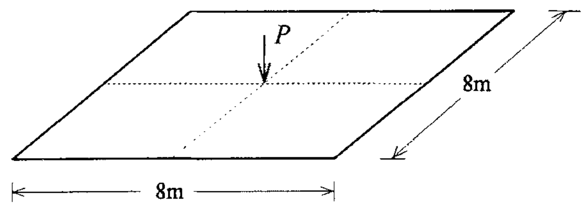

Fig. 1. Initial uniform thickness design for a plate in bending

It can be seen that the solutions obtained by using original sensitivity numbers contain checkerboard patterns as shown in Figs. 2a and 3a. However, the use of the proposed smoothing technique effectively avoids formation of checkerboard patterns in the solutions as observed in Figs. 2b and 3b. To keep symmetry of resulting designs for this example, *a design element linking* is used. This type of element linking is based on the fact that elements in the symmetrical positions have equal sensitivity numbers. The linked elements (elements have equal sensitivity numbers) are sized equally.

It is seen that the optimal designs for two cases (Figs. 2b and 3b) have similar thickness distributions. The areas with the minimum thickness in all cases are located along the elastic hinge lines, which have been reported by Suzuki *et al.* (1992), and Tenek and Hagiwara (1994). The structural responses of the optimal designs for both cases are significantly improved compared to the initial uniform design while the weight remains the same. The objective displacement (out-of-plane displacement at the centre) is reduced from 4.22 mm to 2.46 mm by 41.7% with respect to the initial value for the case of three thicknesses and to 2.47 mm by 41.5% for the case of five thicknesses.

Optimization histories for two cases using the smoothing technique are very similar as given in Fig. 4. It is seen that the displacement is sharply reduced in first several iterations. The change in the displacement is more steady in the following iterations and becomes smaller for the later iterations. The same computational time of 11 minutes is needed for each solution. This illustrates the effectiveness of the proposed sensitivity numbers for shifting material and efficiency of the proposed ESO method.

It is observed from this example (Fig. 4), that the value MSR = 1% gives very similar change in the objective displacement despite big differences in the step size (0.1 m and 0.05 m) and in ESR (3% and 6%). This illustrates that the parameter MSR plays a more important role in controlling the level of change in the objective displacement during the optimization process using the proposed ESO method.

6.2 Minimum strain energy design of a cantilever plate

This serves as an example of minimization of the strain energy of a plate subject to constant volume, which is equivalent to maximizing the overall stiffness of the plate. A rectangular plate with dimensions $L_x = 4$ m and $L_y = 2$ m is clamped at one long edge. Three point loads $P = 20$ kN, normal to the plate, act at the free corners and in the middle of the free edge (see Fig. 5). Young's modulus $E = 30$ GPa and Poisson's ratio $\nu = 0.2$ are assumed. The plate is being designed for different sets of thicknesses ranging from the minimum of 0.1

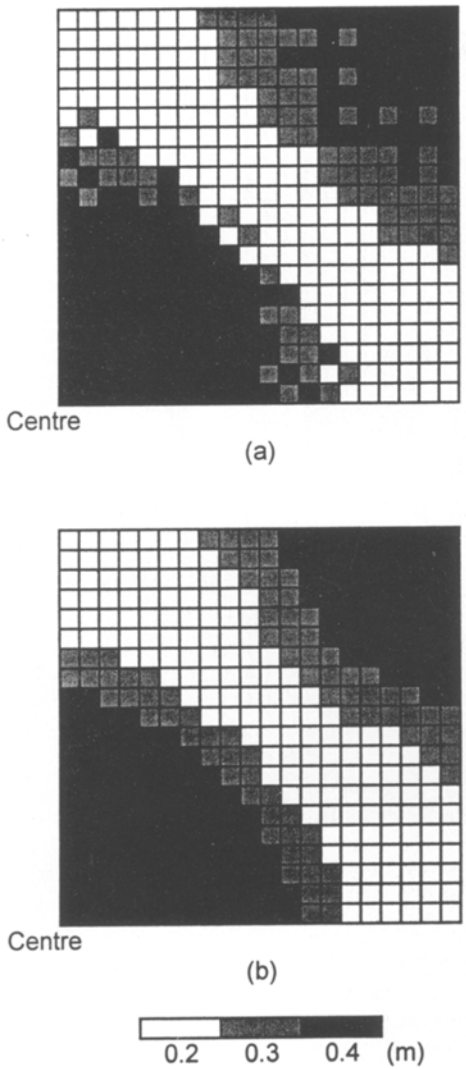

Fig. 2. Optimal designs for a quarter of the simply supported plate, three thicknesses, MSR = 1% (a) without smoothing, (b) with smoothing

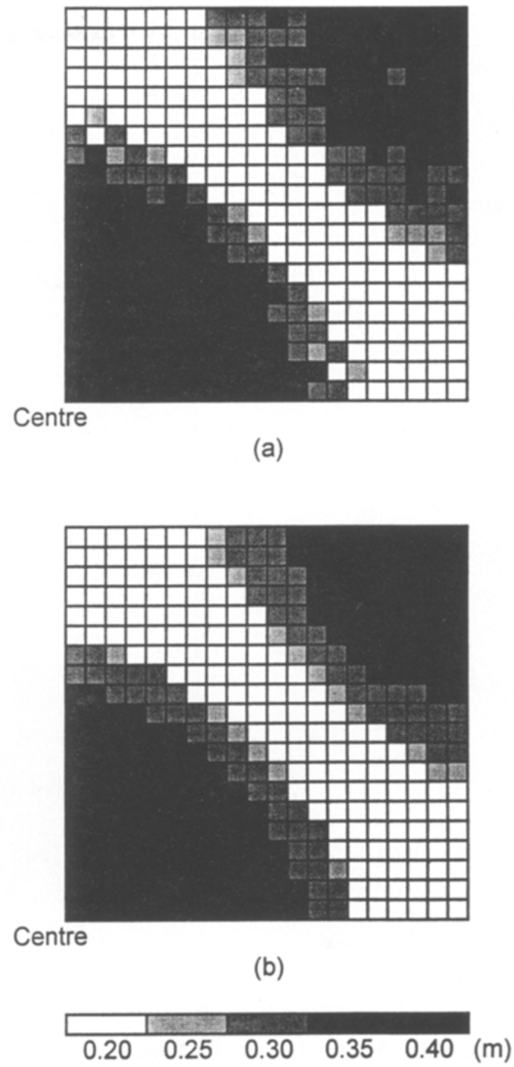

Fig. 3. Optimal designs for a quarter of the simply supported plate, five thicknesses, MSR = 1% (a) without smoothing, (b) with smoothing

m to the maximum of 0.2 m. Because of the symmetry, only a half of the plate is analysed using 400 quadrilateral plate elements. Initially, all elements are assigned a thickness of 0.15 m, which is defined from the weight of the plate. The initial strain energy is 74.0 Nm. Adopting MSR = 0.5% and tolerance $\delta = 0.01$, solutions are obtained using the smoothing technique in the following cases.

1. *Design for three thicknesses*: $\Delta t = 0.05$ m, ESR = 1.5% (i.e. 6 elements having thicknesses reduced and 6 elements having thicknesses increased), the optimal design is shown in Fig. 6. The strain energy is reduced from 74.0 to 49.4 Nm by about 33.2% with respect to the initial value. The computation time is only 13 minutes.
2. *Design for five thicknesses*: $\Delta t = 0.025$ m, ESR = 3% (i.e. 12 elements having thicknesses reduced and 12 elements having thicknesses increased), the optimal design is shown in Fig. 7. The strain energy is reduced to 49.0 Nm by about 33.8% with respect to the initial value. For this case 14 minutes computation time is needed.

The optimal designs in both cases are similar. No checkerboard patterns have been observed during the solution process. The structural responses of the optimal designs are much more improved in the sense that the strain energy is significantly reduced while the weight remains unchanged. The optimization history is given in Fig. 8. The same trend of reduction in the objective, as shown in the previous example, is also observed in this example. Once again, using the same value MSR = 0.5% produces almost the same change in the strain energy in both sets of thicknesses.

6.3 Minimum strain energy design for a plate under torsional loading

A plate of dimensions 0.20 m × 0.10 m is clamped along one short edge and is under the influence of a torsional loading, which is created by applying two loads of the same magnitude $P = 1$ N but opposite direction at the corner nodes of the free boundary (see Fig. 9). Young's modulus is $E = 90$ GPa and Poisson's ratio is $\nu = 0.3$. A similar example was considered

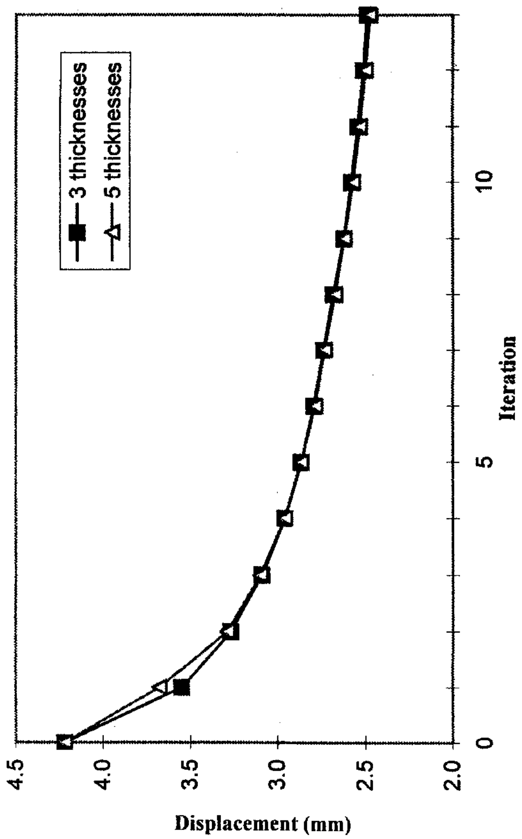

Fig. 4. History of a minimum displacement for the simply supported plate, MSR = 1%, tolerance = 0.01

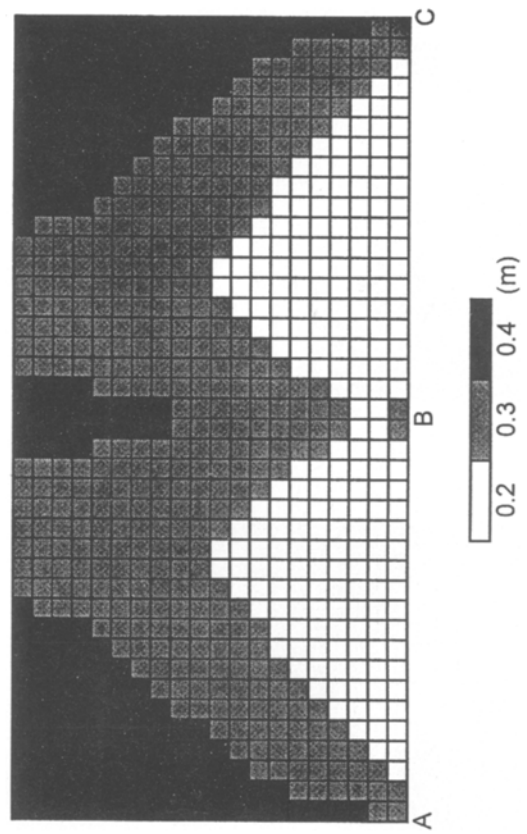

Fig. 6. Minimum strain energy design for the cantilever plate, three thicknesses, MSR = 0.5%, tolerance = 0.01

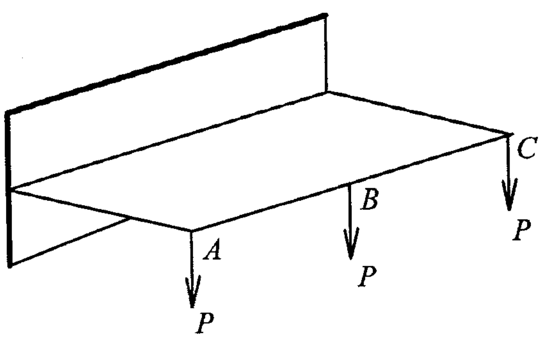

Fig. 5. Initial uniform design for a cantilever plate

by Tenek and Hagiwara (1994). The plate is designed for the set of 10 thicknesses ranging from the minimum of 0.1 mm to the maximum of 1.0 mm with the step size $\Delta t = 0.1$ mm. The whole plate is modelled by 800 quadrilateral plate elements. All elements are initially given the thickness of 0.7 mm and the initial strain energy is of $2.22 \times 10^{7-3}$ Nm.

Using MSR = 0.5%, the optimal design obtained after 50 iterations is shown in Fig. 10. It takes 2 hours 29 minutes to derive to this solution. It is seen that the thicknesses are mainly bounded at the minimum and maximum values. From the result, there is a tendency for thickening along the edges and across the plate, which suggests for such structures that edge beams and stiffeners are useful. The strain energy is reduced to 1.18×10^{-3} Nm, by 46.8% with respect to the initial value. The structure in much more efficient in comparison with the initial uniform design for the same weight. The strain energy of the plate is steadily reduced as observed in Fig. 11.

7 Conclusions

It has been seen that the proposed ESO method can directly deal with discrete sizing problems. Optimality criteria are formulated for the problems. The sensitivity numbers for shifting materials over the structure are proposed, which allow us to reach the optimal solution by an iterative process. The thicknesses of elements are gradually changed to neighbouring values from the given set of discrete values according to their sensitivity numbers. The simple smoothing technique is able to suppress formation of checkerboard patterns creating more practically acceptable solutions. Compared with other existing solution methods, where repeated analyses are needed in order to find a continuous solution and the problem is re-solved to obtain a discrete solution, the proposed evolutionary optimization method is simple and efficient. Only one finite element analysis is required for each design step. The structural response (displacement or strain energy) of the optimal designs obtained by the proposed method for structures is significantly improved in comparison with the initial uniform design for the same weight.

It is found that in the proposed method for optimization of structures with a given weight, the material shifting ratio

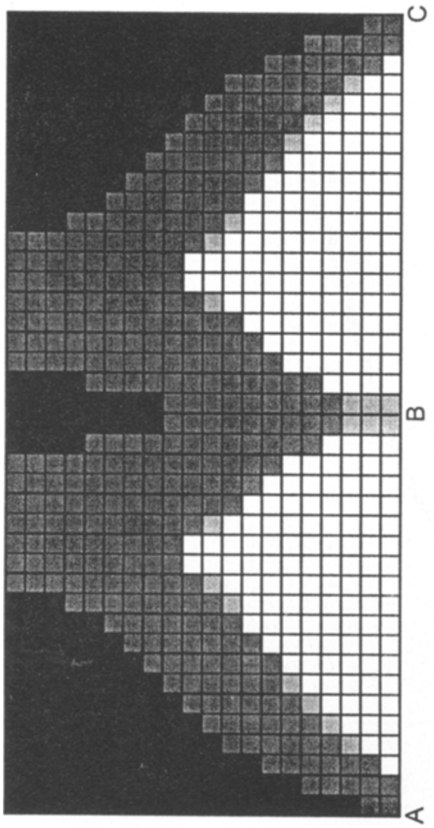

Fig. 7. Minimum strain energy designs for the cantilever plate, five thicknesses, MSR = 0.5%, tolerance = 0.01

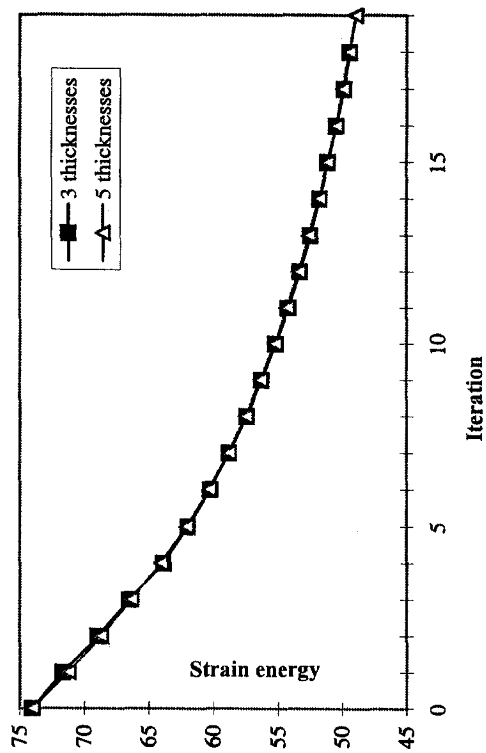

Fig. 8. History of minimum strain energy design for the cantilever plate, MSR = 0.5%, tolerance = 0.01

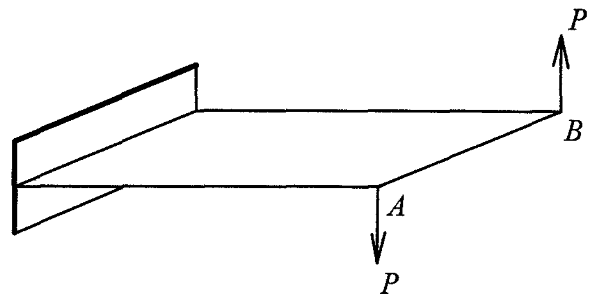

Fig. 9. Initial uniform design for the plate under torsional loading

is more important than the element shifting ratio. The value of the material shifting ratio is more closely related to the level of change between successive designs. It is obvious that the smaller value for the material shifting ratio is used, the more accurate the solution will be, but at higher computational costs. Although the values of 0.5% and 1% for the material shifting ratio are used, which give good results, the influences of this parameter on the final designs needs further investigation.

It should be noted that the sensitivity numbers are formulated from results of the general static problems, so the proposed method can be applied to other types of structures such as trusses or frames, where bar elements can be selected from commercially available sections. An extension of the proposed method for truss and frame structures will be reported in the near future.

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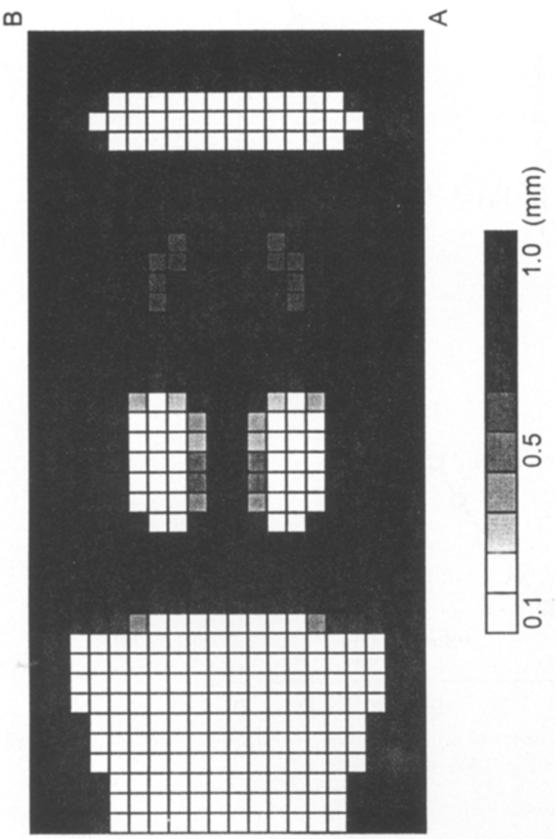

Fig. 10. Minimum strain energy design for the plate under torsional loading after 50 iterations, MSR = 0.5%

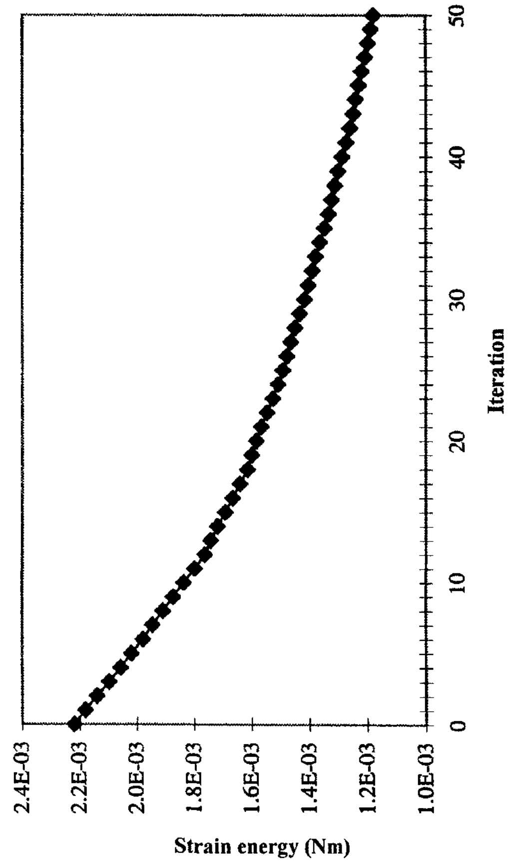

Fig. 11. Optimization history for the plate under torsional loading, ten thicknesses,MSR = 0.5%

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