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On the Bergman space norm of the Cesàro operator

By

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1. Introduction. Let $\mathbb{D}$ denote the unit disc in the complex plane $\mathbb{C}$, and $dm = (1/\pi)\,dx\,dy$ the normalized Lebesgue measure on $\mathbb{D}$. For $1 \leqq p < \infty$ the Bergman space A^p is the closed subspace of all analytic functions in $L^p(\mathbb{D}, dm)$. For f analytic on $\mathbb{D}$ the A^p norm is

$$\|f\|_p^p = \int_{\mathbb{D}} |f(z)|^p \, dm(z), \tag{1.1}$$

while for $p = 2$ we can use the expression

$$\|f\|_2^2 = \sum_{n \geqq 0} |a_n|^2/(n+1), \tag{1.2}$$

where $f(z) = \sum_{n \geqq 0} a_n z^n$. For such f analytic on $\mathbb{D}$ the Cesàro transformation C is defined by

$$C(f)(z) = \frac{1}{z}\int_0^z f(\zeta)\frac{1}{1-\zeta}\,d\zeta = \sum_{n \geqq 0}\left(\frac{1}{n+1}\sum_{k=0}^{n} a_k\right) z^n. \tag{1.3}$$

The averaging operator C and its continuous analogues have been studied on various spaces including sequence spaces and the Hardy spaces [1, 2, 3, 7, 9, 11]. In the case of Hardy spaces, C has been related to a semigroup of composition operators [2, 3, 11], thereby giving a method of studying C by studying the semigroup. The observation providing this link is that on the space of all analytic functions on $\mathbb{D}$, $(-C)^{-1}(g)(z) = -z(1-z)g'(z) - (1-z)g(z)$, and the restriction of this differential operator on Hardy spaces is found to be the infinitesimal generator of a specific strongly continuous composition semigroup.

Any semigroup of composition operators is also strongly continuous on the Bergman spaces A^p, in fact on their weighted versions A^p_α with weights $w(r) = (1-r^2)^\alpha$. This was the main result of [12], along with the identification of the corresponding infinitesimal generators. In addition as an application we found in [12] that the operator

$$\mathscr{A}(f)(z) = \sum_{n \geqq 0}\left(\sum_{k \geqq n}\frac{a_k}{k+1}\right)z^k, \quad \text{where } f(z) = \sum_{k \geqq 0} a_k z^k,$$

is bounded on A^p_α if and only if $\alpha + 2 < p$ [12, Theorem 3]. In particular with $\alpha = 0$ $\mathscr{A}$ is bounded on A^p if and only if $p > 2$. It is clear that the operators C and $\mathscr{A}$ are induced by infinite matrices that are transposes of each other, in fact on the Hardy space H^2 with the usual inner product, C and $\mathscr{A}$ are Hilbert space adjoints of each other. This is no longer true on the Bergman space A^2 ($\mathscr{A}$ is not bounded on A^p $p \leqq 2$), and in fact even for $p > 2$ using the usual pairing for the duality of Bergman spaces we can see by an easy calculation that C and $\mathscr{A}$ do not form a dual pair. Thus the partial results obtained for $\mathscr{A}$ on Bergman spaces in [12] cannot be used to study C.

The main question here is to find the norm of the Cesàro operator on Bergman spaces. We use a specific semigroup of composition operators, induced by functions $\{\phi_t\}$, much like we did for Hardy spaces in [11]. We recall that for the case of Hardy spaces, we obtained growth estimates for the semigroup by transfering to Hardy spaces on a half-plane and doing the calculations there. In our present case however we exploit an identity satisfied by the functions $\{\phi_t\}$ (see 2.4 below) and a change of variables in the integrals, to obtain the analogous growth estimates for the semigroup. We should stress here that this identity could not have been usef for the case of Hardy spaces because the Hardy space norm is obtained by integrating the boundary function. In contrast the boundary values of Bergman space functions are defined only as distributions and because of this we cannot replicate the calculations in [11] to obtain the growth of the semigroup on Bergman spaces.

2. The semigroup and its properties. For $t \geqq 0$ let $\phi_t\colon \mathbb{D} \to \mathbb{D}$ be given by

$$\phi_t(z) = \frac{e^{-t}z}{(e^{-t}-1)z+1}, \tag{2.1}$$

and let the operators S_t be defined by

$$S_t(f)(z) = (\phi_t(z)/z)\, f(\phi_t(z)). \tag{2.2}$$

We next show that $\{S_t\}$ is a strongly continuous semigroup of bounded operators on A^p. We remark that this does not follow directly from results in [12] because in [12] we had considered only unweighted composition semigroups. The following reasoning however gives the desired conclusion.

Let $1 \leqq p < \infty$. It is well known that composition by any analytic self map of $\mathbb{D}$ is a bounded operator on A^p. Also if $f \in A^p$ then $\| z f(z)\|_p \leqq \| f\|_p$, and if $f \in A^p$ with $f(0) = 0$ then $\| f(z)/z\|_p \leqq K(p)\| f\|_p$ where $K(p)$ is a constant depending on p but not on f [14, p. 75]. Combining these we see that each S_t is a bounded operator on A^p.

Further for s, $t \geqq 0$ we have $\phi_t \circ \phi_s = \phi_{t+s}$ and we easily see that $S_t S_s = S_{t+s}$ and $S_0 = I$, the identity operator. Thus $\{S_t\}$ is a semigroup of bounded operators on A^p.

For the strong continuity let f be any function in A^p, and set $g(z) = z f(z)$. The function $g \circ \phi_t - g$ vanishes at 0, and we have

$$\| S_t(f) - f\|_p = \| (g(\phi_t(z)) - g(z))/z\|_p \leqq K(p)\| g \circ \phi_t - g\|_p.$$

Using Theorem 1 of [12] we know that composition semigroups are strongly continuous on A^p. Thus, since $g \in A^p$ we have $\lim\limits_{t\to 0} \| g \circ \phi_t - g\|_p = 0$ and we conclude $\lim\limits_{t\to 0} \| S_t(f) - f\|_p = 0$ implying that $\{S_t\}$ is strongly continuous on A^p.

To identify the infinitesimal generator of $\{S_t\}$ let $z \in \mathbb{D}$, then

$$\lim_{t\to 0} \frac{\partial}{\partial t}(S_t(f)(z)) = -z(1-z)f'(z) - (1-z)f(z).$$

Using this observation and applying the reasoning of [12] we find that the infinitesimal generator Δ_p of $\{S_t\}$ on A^p is given by

$$\Delta_p(f)(z) = -z(1-z)f'(z) - (1-z)f(z) = -(1-z)(zf(z))', \tag{2.3}$$

with domain $\mathrm{Dom}(\Delta_p) = \{f \in A^p \colon (1-z)(zf(z))' \in A^p\}$.

Next we study $\{S_t\}$ and Δ_p. Recall that $(1-z)^\lambda \in A^p$ if and only if $\mathrm{Re}(\lambda) > -2/p$. We use $\sigma(\,)$ and $\sigma_\pi(\,)$ to denote the spectrum and the point spectrum of an operator.

Lemma 1. *If* $1 \leqq p < \infty$ *then* $\{z \colon \mathrm{Re}(z) \leqq -2/p\} \subset \sigma(\Delta_p)$.

Proof. First we find the point spectrum. If $\Delta_p(f) = \lambda f$ then an easy calculation shows that $\lambda = -k-1$ for some $k \in \{0, 1, 2, \ldots\}$ and if $\lambda = -k-1$ the corresponding eigenfunction has the form $f_k(z) = cz^k(1-z)^{-(k+1)}$ with c a nonzero constant. This function is not in A^p for $p \geqq 2$ so for such p the point spectrum is empty. If $1 \leqq p < 2$ then $f_k \in A^p$ for $k = 0$ only so $\sigma_\pi(\Delta_p) = \{-1\}$ in this case.

Let $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \leqq -2/p$ (and if $1 \leqq p < 2$, $\lambda \neq -1$). Choose an integer n such that $\mathrm{Re}(\lambda + n + 1) > -2/p$ and let

$$P_{n,\lambda}(z) = (\lambda+1) \sum_{k=1}^{n} ((-1)^k/k)\binom{n}{k} z^k.$$

Then the function $(1-z)^{\lambda+n+1}\exp(P_{n,\lambda}(z))$ is in A^p. If $\lambda - \Delta_p$ is invertible then the equation $(\lambda - \Delta_p)(y)(z) = (\lambda+1)(1-z)^{\lambda+n+1}\exp(P_{n,\lambda}(z))$ has a solution $y(z)$ analytic on $\mathbb{D}$, and $y(z) \in \mathrm{Dom}(\Delta_p) \subset A^p$. From (2.3) we see that this equation is

$$(\lambda + 1 - z)y(z) + z(1-z)y'(z) = (\lambda+1)(1-z)^{\lambda+n+1}\exp(P_{n,\lambda}(z)),$$

and a routine calculation shows that $y(z) = (1-z)^\lambda \exp(P_{n,\lambda}(z))$ is the only analytic solution. But $y(z)$ is not in A^p because $\mathrm{Re}(\lambda) \leqq -2/p$. Thus $\{z \colon \mathrm{Re}(z) \leqq -2/p\} \subset \sigma(\Delta_p)$.

Lemma 2. *Suppose* $p \geqq 4$. *Then* $\|S_t\|_p \leqq e^{(-2/p)t}$ *for each* $t \geqq 0$.

Proof. An easy calculation shows that the functions ϕ_t defined in (2.1) satisfy

$$(\phi_t(z)/z)^2 = e^{-t}\phi_t'(z), \quad z \in \mathbb{D},\ t \geqq 0. \tag{2.4}$$

For $f \in A^p$, using (2.4) we have

$$\begin{aligned}\|S_t(f)\|_p^p &= \int_{\mathbb{D}} |\phi_t(z)/z|^p\, |f(\phi_t(z))|^p\, dm(z)\\ &= e^{-2t} \int_{\mathbb{D}} |\phi_t(z)/z|^{p-4}\, |f(\phi_t(z))|^p\, |\phi_t'(z)|^2\, dm(z)\\ &\leqq e^{-2t} \int_{\mathbb{D}} |f(\phi_t(z))|^p\, |\phi_t'(z)|^2\, dm(z),\end{aligned}$$

where in the last step we have used $|\phi_t(z)/z| \leqq 1$, a consequence of Schwarz's lemma. Since ϕ_t is $1-1$, by a change of variable in the last integral we further obtain

$$= e^{-2t} \int_{\phi_t(\mathbb{D})} |f(w)|^p \, dm(w)$$
$$\leqq e^{-2t} \int_{\mathbb{D}} |f(w)|^p \, dm(w) = e^{-2t} \| f \|_p^p,$$

and the conclusion follows.

For $1 \leqq p < 4$ we have to use a different version of the semigroup. For these values of p let

$$T_t(f)(z) = (\phi_t(z)/z)^{4/p} f(\phi_t(z)), \quad t \geqq 0. \tag{2.5}$$

Since $|\phi_t(z)/z| \leqq 1$ the operators T_t are bounded on A^p, and a calculation similar to the one in Lemma 2 shows that $\| T_t \|_p \leqq e^{(-2/p)t}$. The strong continuity of $\{T_t\}$ can be shown by following the proof of [11, Theorem 1], we omit the details. The infinitesimal generator in this case is found to be

$$E_p(f)(z) = -z(1-z) f'(z) - (4/p)(1-z) f(z). \tag{2.6}$$

3. The result. Let C_p denote C acting on A^p. We prove the following.

Theorem 1.

(i) *Suppose* $4 \leqq p < \infty$. *Then* $\| C_p \| = p/2$ *and* $\sigma(C_p) = \{z : |z - (p/4)| \leqq p/4\}$.
(ii) *Suppose* $1 \leqq p < 4$. *Then* $p/2 \leqq \| C_p \| \leqq 2$ *and* $\{z : |z - (p/4)| \leqq p/4\} \subset \sigma(C_p)$.

Proof. (i) From Lemma 2 and [4, Theorem VIII.1.11] we see that the spectrum $\sigma(\Delta_p)$ is contained in $\{z : \operatorname{Re}(z) \leqq -2/p\}$. From Lemma 1 then we have $\sigma(\Delta_p) = \{z : \operatorname{Re}(z) \leqq -2/p\}$. It follows in particular that 0 is in the resolvent set of Δ_p. Let $R(\lambda, \Delta_p) = (\lambda - \Delta_p)^{-1}$ denote the resolvent operator for λ in the resolvent set. We see that $R(0, \Delta_p) = C_p$. Further for the spectrum of C_p we have $\sigma(C_p) = \{-1/z : z \in \sigma(\Delta_p)\} = \{z : |z - (p/4)| \leqq p/4\}$. This shows also that $\| C_p \| \geqq p/2$. For the opposite inequality [4, Corollary VIII.1.14] applies and gives $\| R(0, \Delta_p) \| \leqq p/2$. Thus $\| C_p \| = p/2$.

(ii) Using the estimate $\| T_t \|_p \leqq e^{(-2/p)t}$ we see that the spectrum of E_p is contained in $\{z : \operatorname{Re}(z) \leqq -2/p\}$. We calculate the resolvent of E_p at $\lambda = 0$ and we find that for $f(z) = \sum_{n \geqq 0} a_n z^n \in A^p$,

$$R(0, E_p)(f)(z) = \sum_{n \geqq 0} \left(\frac{1}{n + (4/p)} \sum_{k=0}^{n} a_k \right) z^n. \tag{3.1}$$

Using the estimate $\| T_t \|_p \leqq e^{(-2/p)t}$ and [4, Corollary VIII.1.14] we see that $\| R(0, E_p) \| \leqq p/2$. Now let L_p be the operator defined for $f(z) \in A^p$ by

$$L_p(f)(z) = f(z) + ((4/p) - 1) \frac{1}{z} \int_0^z f(\zeta) \, d\zeta. \tag{3.2}$$

We see that L_p acts on $f(z) = \sum_{n \geqq 0} a_n z^n \in A^p$ as a multiplier of Taylor coefficients: $\{a_n\} \to \{\lambda_n a_n\}$ where $\lambda_n = (n + (4/p))/(n+1)$. Also since the operator of integration has norm equal to 1 on A^p, from (3.2) we find $\| L_p \| \leqq 1 + |(4/p) - 1| = 4/p$. Observe now that $C_p = L_p \circ R(0, E_p)$ so for the norm we have $\| C_p \| \leqq \| L_p \| \, \| R(0, E_p) \| = 2$. In particular C_p is bounded on A^p for $1 \leqq p < 4$ and this implies that the point 0 is in the resolvent set of Δ_p. From Lemma 1 we then have that $\sigma(C_p)$ contains the set $\{-1/z : \operatorname{Re}(z) \leqq -2/p\} = \{z : |z - (p/4)| \leqq p/4\}$. Thus $\| C_p \| \geqq p/2$ and this completes the proof.

Remark. There is no hope to prove $\| C_p \| = p/2$ for the full range $1 \leqq p < \infty$, since on the Hilbert space A^2 we have $\| C_2 \| > 1$. Indeed for the constant function $f = 1$, $\| 1 \|_2 = 1$, $C(1)(z) = 1 + (1/2) z + (1/3) z^2 + \cdots$, and $\| C_2 \| \geqq \| C(1) \|_2 = \sum_{n \geqq 1} 1/n^3 > 1$. In fact setting $q = 2 \| C(1) \|_2 > 2$ and letting $p \in (2, q)$ we have $\| C(1) \|_p > \| C(1) \|_2 = q/2 > p/2$ so $\| C_p \| \geqq \| C(1) \|_p > p/2$ for $2 < p < q$. The same phenomenon of bad behaviour of the norm for small values of p appeared on Hardy spaces [11], where the bad range is $1 \leqq p < 2$.

4. Spectra of composition operators. As a byproduct of the above we obtain information for the spectrum of composition operators

$$C_\phi(f) = f \circ \phi, \quad f \in A^p,$$

induced by the maps $\phi(z) = r z/((r-1) z + 1)$, $0 < r < 1$. The spectra of composition operators on Hardy spaces were studied by H. Kamowitz [6] and by many others. Functions ϕ of the above form are a special case of those considered in [6, Theorem 3.8]. For our ϕ the results of [6] give the spectrum of C_ϕ on H^p to be $\sigma(C_\phi : H^p) = \{z : |z| \leqq r^{1/p}\} \cup \{1\}$. Using the estimates of Lemma 1 and Lemma 2 we are going to find that a similar result holds for the spectrum of C_ϕ on A^p. Setting $r = e^{-t}$, $0 < t < \infty$, we have the composition operators $C_t(f) = f \circ \phi_t$ where ϕ_t are given by (2.1). From (2.2) we see that $M_z \circ S_t = C_t \circ M_z$, $t > 0$, where M_z is the operator of multiplication by z. We can relate the spectra of S_t and C_t by the following lemma.

Lemma 3. *Suppose* $1 \leqq p < \infty$. *Then for each* $t > 0$, $\sigma(C_t : A^p) = \sigma(S_t : A^p) \cup \{1\}$.

Proof. First we determine the point spectra. Suppose $\lambda \in \mathbb{C}$ with $C_t(f) = \lambda f$ i.e. $f \circ \phi_t = \lambda f$. This is a Schroeder's equation and has analytic solutions only for $\lambda \in \{(\phi_t'(0))^n : n = 0, 1, 2, \ldots\}$. For each such λ the solution is unique [10, p. 93]. Since $\phi_t'(0) = e^{-t}$ and since the function $h_n(z) = z^n/(1-z)^n$ satisfies Schroeder's equation for $\lambda = e^{-nt}$, $n = 0, 1, 2, \ldots$, we see that $\sigma_\pi(C_t : A^p) = \{e^{-nt} : z^n/(1-z)^n \in A^p\} = \{1\}$ for $p \geqq 2$ and $\sigma_\pi(C_t : A^p) = \{1, e^{-t}\}$ for $1 \leqq p < 2$. Similarly we find $\sigma_\pi(S_t : A^p) = \{e^{-t}\}$ for $1 \leqq p < 2$ and $\sigma_\pi(S_t : A^p)$ is empty for $p \geqq 2$.

Next let $\lambda \in \mathbb{C}$ such that $\lambda - C_t$ is invertible. Then $\lambda - S_t$ is injective and we will show that it is also surjective and thus invertible. Indeed let $g \in A^p$ then the function $y(z) = (\lambda - C_t)^{-1}(z g(z))$ is in A^p and vanishes at 0 (because $\lambda \neq 1$) so from [14, p. 75] we have $y(z)/z \in A^p$ and it is easy to see that $(\lambda - S_t)(y(z)/z) = g(z)$ so $\lambda - S_t$ is surjective. Conversely suppose $\lambda - S_t$ is invertible and $\lambda \neq 1$, then $\lambda - C_t$ is injective. If $g \in A^p$ then $(g(z) - g(0))/z \in A^p$. Set $y(z) = (\lambda - S_t)^{-1}((g(z) - g(0))/z)$ and $y_1(z) = z y(z) + (\lambda - 1)^{-1} g(0) \in A^p$. It is easy to see that $(\lambda - C_t)(y_1) = g$ so $\lambda - C_t$ is also surjective and the proof is complete.

We now use Lemma 1 which states that $\{z : \operatorname{Re}(z) \leqq -2/p\} \subset \sigma(\Delta_p)$. From the general theory of strongly continuous semigroups [8, Theorem 2.3] we have $\{e^{t\lambda} : \lambda \in \sigma(\Delta_p)\} \subset \sigma(S_t : A^p)$ hence $\{z : |z| \leqq e^{(-2/p)t}\} \subset \sigma(S_t : A^p)$ for $1 \leqq p < \infty$. For $p \geqq 4$ using Lemma 2 we then obtain $\sigma(S_t : A^p) = \{z : |z| \leqq e^{(-2/p)t}\}$. Using these and Lemma 3 and recalling that $e^{-t} = r$ we obtain:

Proposition 1. *Let $\phi(z) = rz/((r-1)z+1)$, $0 < r < 1$, and C_ϕ the operator of composition by ϕ on A^p. The following holds.*

(i) *If $p \geqq 4$, then $\sigma(C_\phi) = \{z : |z| \leqq r^{2/p}\} \cup \{1\}$.*
(ii) *If $1 \leqq p < 4$, then $\{z : |z| \leqq r^{2/p}\} \cup \{1\} \subset \sigma(C_\phi)$.*

5. A general result on the Cesàro operator. In this section we state a general necessary condition for C to be bounded on certain Banach spaces of analytic functions. Let X be a Banach space consisting of analytic functions on $\mathbb{D}$ such that:

(P 1) X contains the constant functions.
(P 2) The multiplication operator $M_z(f)(z) = zf(z)$ is bounded on X.
(P 3) Point evaluations are continuous linear functionals of X.

Examples of spaces satisfying these conditions include:

(1) The Hardy spaces H^p $(1 \leqq p < \infty)$.
(2) The Bergman spaces A^p and their weighted versions A^p_α, $1 \leqq p < \infty$, $\alpha > -1$, with norm $\| f \|^p_{p,\alpha} = \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha \, dm(z)$.
(3) The family of weighted Dirichlet spaces $\mathscr{D}_\alpha$, $\alpha > -1$, with norm $\| f \|^2_\alpha = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^\alpha \, dm(z)$.

The spaces in the last example are Hilbert spaces, $\mathscr{D}_0$ is the classical Dirichlet space.

In order for the Cesàro operator to be bounded on such spaces X, it is necessary that X contains relatively fast growing functions. More precisely:

Theorem 2. *Let X be a Banach space of analytic functions on $\mathbb{D}$ satisfying the properties* (P 1), (P 2) *and* (P 3). *Assume C is bounded on X, then*

(i) *$(1-z)^{-s} \in X$ for all sufficiently small positive s.*
(ii) *$\| C \|_X \geqq c/s_X$ where $c = 1/\| M_z \|_X$ and $s_X = \sup\{s : (1-z)^{-s} \in X\}$.*

Proof. Since point evaluations are bounded functionals we have $g_n(z) \to g(z)$ for each $z \in \mathbb{D}$ whenever g_n, $g \in X$ and $\| g_n - g \|_X \to 0$. Assume C is bounded and let $B = M_z \circ C$. Applying B to the constant function 1 we find $B(1)(z) = \log(1/(1-z))$ and by iterating,

$$B^n(1)(z) = \frac{1}{n!} \log^n \left(\frac{1}{1-z}\right), \quad n = 1, 2, 3, \ldots$$

Let s be positive such that $s \| B \|_X < 1$, then we obtain

$$\left\| \frac{1}{n!} s^n \log^n \left(\frac{1}{1-z}\right) \right\|_X \leqq (s \| B \|_X)^n \, \| 1 \|_X,$$

and it follows that the series

$$\sum_{n \geqq 0} \frac{1}{n!} (s \log (1/(1 - z)))^n$$

converges in X. Since point evaluations are continuous on X, we see that the sum of the series coincides with the pointwise sum which is the function $(1 - z)^{-s}$. It follows that $(1 - z)^{-s} \in X$ for each s with $0 < s < 1/\| B \|_X$, providing the first part. Further $s_X = \sup \{s : (1 - z)^{-s} \in X\} \geqq 1/\| B \|_X$ and since $\| C \|_X \geqq \| B \|_X / \| M_z \|_X$ the second part follows.

Theorem 2 can be applied to give lower estimates on the norm of C on various spaces.

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