Structural Optimization 13, 271–274 © Springer-Verlag 1997

Partial relaxation of the orthogonality requirement for classical Michell trusses

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Abstract It is often stated, even in standard references, that in classical Michell trusses (i.e. least-weight trusses for one load condition with a stress or compliance constraint) a pair of intersecting compression and tensile bars must always be orthogonal. The aim of this brief note is to show that there are important exceptions to this rule and that the modification of this restriction enables us to obtain new classes of solutions.

1 Introduction

The theory of least-weight trusses for one load condition and a stress constraint was established in a milestone contribution by Michell (1904). His pioneering work remained largely unnoticed until the late fifties, after which Hemp (e.g. 1973) developed a systematic and rigorous theory of Michell trusses, which was partially based on the theory of slip-lines (Hencky-Prandtl nets) for plane perfectly plastic solids.

Hencky-Prandtl nets consist, in general, of a system of *orthogonal* intersecting curves. In his outstanding book on optimal structures, for example, Hemp (1973) states about Michell trusses: "If a pair of tension and compression members meet at a point, they must be orthogonal ... No other members can be coplanar with them".

It will be clear from this note that, in the context of the problems treated by Hemp (1973), the above restriction is completely justifiable. However, it will be shown that there are classes of correct optimal solutions for Michell's problem, which could not be obtained within this orthogonality restriction.

Whilst the above comments refer to one load condition with a stress or compliance constraint, it was shown in the early nineties (e.g. Rozvany 1992; Rozvany *et al.* 1993; Rozvany and Birker 1995) that for several load conditions or for displacement constraints, least-weight trusses in general consist of *nonorthogonal* networks of members. Moreover, whilst the optimal microstructure for perforated plates in plane stress with a compliance constraint (e.g. Lurie *et al.* 1982; Kohn and Strang 1986) consists of an *orthogonal* system or rank-1 and rank-2 ribs, Lurie (1994, 1995) showed recently that for so-called non-selfadjoint problems (e.g. those with displacement constraints) the optimal microstructure is in general *nonorthogonal.* Explicit optimal solutions for non-selfadjoint plate problems were presented recently by Károlyi and Rozvany (1997).

The subsequent discussion, however, is restricted to the orthogonality condition for the *"classical" Michell problem.*

The conclusions of this note were mentioned in an invited principal lecture by the author at a recent AIAA/ASME/ASCE/AHS/ASC meeting (Rozvany 1997).

2 A brief summary of Michell's optimality criteria and of optimal regions in Michell trusses

In his classical paper, Michell (1904) considered the optimization of truss layouts subject to the stress constraints

$$-\sigma_o^- \le \sigma \le \sigma_o^+ , \tag{1}$$

where σ is the stress in a truss member and (σ_o^-, σ_o^+) are the permissible stresses in compression and tension. Michell's optimality condition states that for a virtual strain field $\bar{\varepsilon}$ on the structural domain ($\mathcal{D} \subset I\!R^2$ or $I\!R^3$) we must have

$$\bar{\varepsilon} = k \operatorname{sgn} f \quad (\text{for } f \neq 0), \tag{2}$$

$$|\,\bar{\varepsilon}\,| \le k \quad (\text{for } f = 0), \tag{3}$$

where k is a positive constant, f is the force in a bar and sgn f is the usual sign function (sgn $f = 1$ for $f > 0$ and sgn $f = -1$ for $f < 0$). This means that on line segments of the structural domain along which there is no truss element we must still satisfy the inequality condition (3).

It was shown recently (Rozvany 1996) that Michell's (1904) optimality criteria in (2) and (3) are generally valid only for equal permissible stresses in tension and compression ($\sigma_o^- = \sigma_o^+$). We shall therefore consider this latter case herein.

For *plane* trusses, the optimality criteria in (2) and (3) permit the following five regions at any point with a member in at least one direction (Prager and Rozvany 1977):

$$\begin{array}{llll}
R^+: \ \bar{\varepsilon}_I = k, & |\,\bar{\varepsilon}_I\,| < k\,, & f_I > 0, & f_{II} = 0, \\
R^-: \ \bar{\varepsilon}_{II} = -k, & |\,\bar{\varepsilon}_I\,| < k\,, & f_I < 0, & f_{II} = 0, \\
S^+: \ \bar{\varepsilon}_{II} = \bar{\varepsilon}_{II} = k, & & f_I > 0, & f_{II} > 0, \\
S^-: \ \bar{\varepsilon}_{II} = \bar{\varepsilon}_{II} = -k, & & f_I < 0, & f_{II} < 0, \\
T\ : \ \bar{\varepsilon}_I = k, & \bar{\varepsilon}_{II} = -k, & f_I > 0, & f_{II} < 0,
\end{array} \tag{4}$$

where $\bar{\varepsilon}_I$ and $\bar{\varepsilon}_{II}$ are the principal adjoint strains, and f_I and f_{II} the corresponding member forces.

As can be seen from (4), in R regions members run in only one direction and have forces of the same sign; in S regions members with forces of a given sign may run in any direction, and in T regions a set of members in compression and a set of members in tension intersect each other at right angles.

The layout of various types of optimal regions, together with the symbols commonly used for them, are shown in Fig. 1, in which continuous and broken thin lines, respectively, denote tension and compression bars.

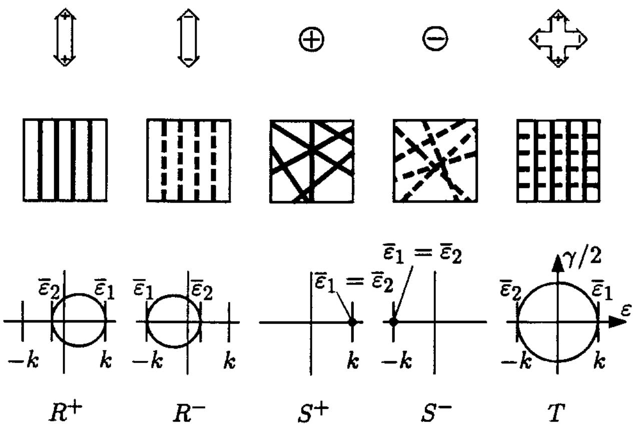

Fig. 1. Types of optimal regions in Michell trusses

3 Nonorthongonal member junctions in solutions with T regions

Hemp's (e.g. 1973) brilliant contributions to this field included not only the correct optimality criteria for different permissible stresses in tension or compression ($\sigma_0^+ \neq \sigma_0^-$), but also a most systematic treatment of possible geometrical properties of T regions.

In the context of optimal layouts consisting of T regions, the orthogonality condition quoted in the Introduction is usually satisfied. The only exceptions are isolated points where "fans" with forces of different signs meet. This is the case at point A of Michell's (1904) classical solution (Fig. 2), but also at point B of its extension (Fig. 3) by Hemp (1973). At these points, an infinite number of members meet and only the outside members satisfy the above orthogonality condition. However, these two isolated points seem to be the only exceptions to the orthogonality rule in all the solutions discussed by Hemp, who treated mostly layouts consisting of T regions.

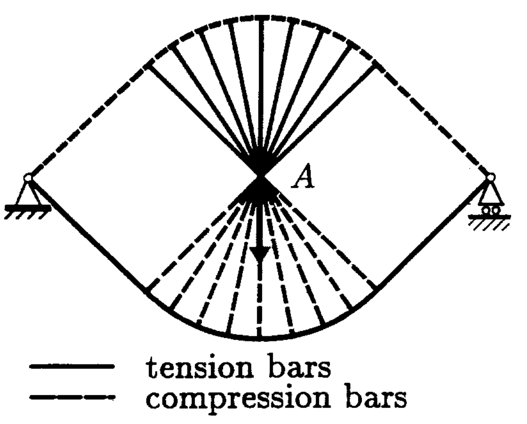

Fig. 2. Michell's solution for a simply supported truss with a central point load

4 Nonorthogonal tension and compression members along the boundary of an R^+ and an R^- region

Whilst for T regions the violation of the orthogonality rule is most exceptional, it was found in the study of R regions

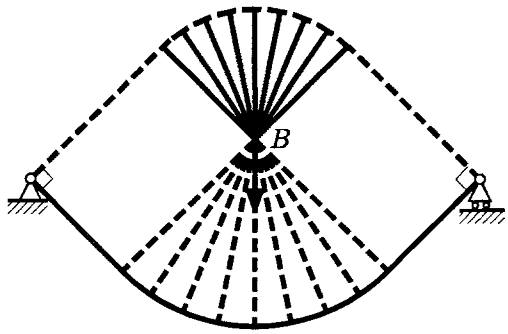

Fig. 3. Hemp's generalization of Michell's solution to a load at a higher level

(e.g. Rozvany and Gollub 1990) that along the *boundary of an R^+ and an R^- region* the members in these two regions are *in general not orthogonal to each other*. The above authors developed a systematic method for deriving the optimal Michell layout for any convex polygonal boundary formed by supporting lines and their solutions confirm the above conclusion. Typical optimal truss layouts for a quadrilateral boundary are given in Fig. 4. In Fig. 4a, one tension member and one compression member meet at the region boundary and in Fig. 4b, two compression members and one tension member.

In the detailed example that follows, an optimal truss layout that violates the orthogonality condition is derived.

4.1 Two pin supports with a point load

The problem under consideration, together with the optimal truss layout is shown in Fig. 5a. The kinematic boundary conditions for the supports are (Fig. 5b)

$$\bar{u} = \bar{v} = 0 \,, \tag{5}$$

where $\bar{u}$ and $\bar{v}$ are the adjoint displacements in the x and y directions. Due to skew-symmetry, we have $\bar{u} = 0$ along $y = 0$ (Fig. 5b) and along the top and bottom bars, respectively, we must have $\bar{\varepsilon} = k$ and $\bar{\varepsilon} = -k$ (see the double arrows denoting R^+ and R^- regions in Fig. 5b). Using the normalized value of $k = 1$, the Mohr circle for the adjoint strains in the top region is shown in Fig. 5c. In the latter, we have (for $\alpha > 45°$)

$$\varepsilon_I = R(1 - \cos 2\alpha) = 1 \,, \tag{6}$$

$$R = 1/(1 - \cos 2\alpha) \,, \tag{7}$$

where R is the radius of the Mohr circle. It follows then from Fig. 5c that

$$\varepsilon_{II} = -R(1 + \cos 2\alpha) = -\frac{1 + \cos 2\alpha}{1 - \cos 2\alpha} = -\cot^2 \alpha \,. \tag{8}$$

Then the inequality optimality condition in (3) requires

$$|\, \varepsilon_{II} \,| = \cot^2 \alpha < k = 1 \Rightarrow \alpha \geq 45° \,. \tag{9}$$

This means that the limiting case for this optimal layout is the one shown in Fig. 5e. For $\alpha \geq 45°$, the solution satisfies all optimality, boundary and continuity conditions for the adjoint strain fields.

For $\alpha = 90°$, the obvious optimal solution is shown in Fig. 5d, where for $\sigma_0^+ = \sigma_0^-$ the load can arbitrarily be shared by the top and bottom bars. The solution in Fig. 5e is well-known, together with the solutions for $\alpha < 45°$ in Fig. 5f (e.g. Hemp 1973). The optimal solution derived in this note, therefore, is also confirmed by its limiting cases, which are either obviously optimal (Fig. 5d) or known from earlier studies by others.

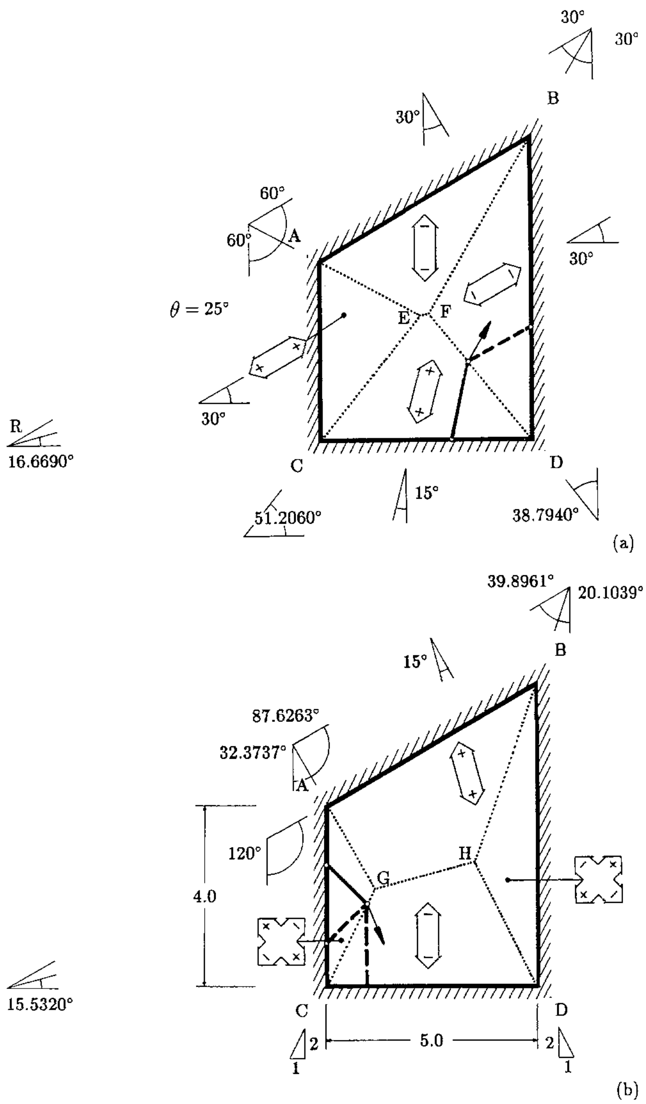

Fig. 4. Michell trusses for polygonal supports with nonorthogonal member junctions along R^+/R^- type region boundaries

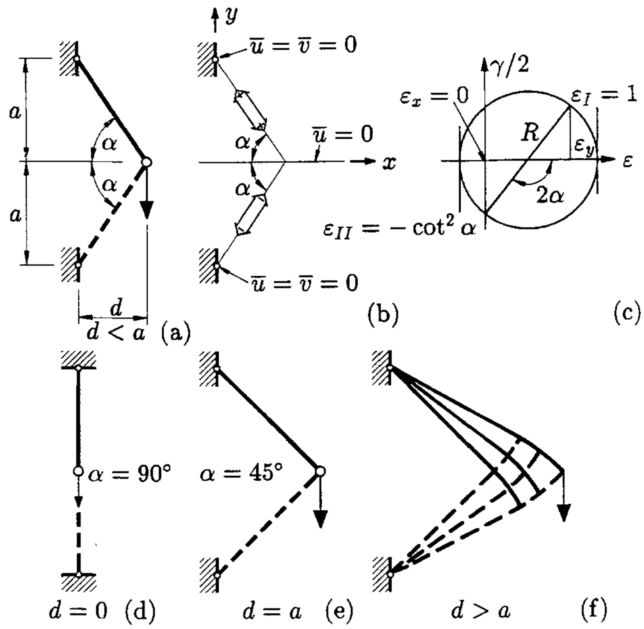

Fig. 5. Detailed example of a Michell truss with nonorthogonal compression and tension members

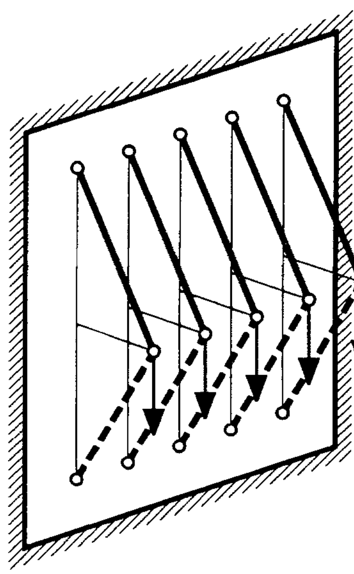

Fig. 6. Michell space-truss with nonorthogonal tension and compression members

5 Extension to space trusses

Hemp's (1973) orthogonality requirement was formulated in the context of 3D or space trusses. The example in Fig. 5 can also be extended to a 3D truss system, if we have a supporting plane and a series of point loads along a line (Fig. 6). In this example, the adjoint strain in the third direction (normal to the xy plane) is $\overline{w} = 0$.

6 Concluding remarks

It is clear from the above discussion that the orthogonality requirement for a pair of interseting tension and compression bars in a Michell truss is in general violated along R^+/R^- type region boundaries. This means, however, that the considered members in a given direction do not pass through their point of intersection but end at that point. The orthogonality condition, therefore, would become of general validity, if we replaced the word "meet" with "cross", resulting in the statement: "If a pair of tension and compression members *cross each other*, they must be orthogonal... No other members can be coplanar with them". The above formulation excludes region boundaries where members "meet" but do not "cross each other".

The second part of the modified statement becomes also generally valid, because we may only have tension and compression members in more than two directions at region boundaries (see Fig. 4b).

Although the above modification of the orthogonality requirement may appear trivial, it opens up new avenues of optimal Michell layouts (see Figs. 4 and 5).

Acknowledgements

The author is indebted to the NATO Scientific and Environmental Affairs Division for financial support; to Elke Becker (drafting), and to Susann Rozvany (text processing and editing).

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Received April 12, 1997