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Approximation and derivatives of probabilities of survival in structural analysis and design

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Abstract Yield stresses, allowable stresses, moment capacities (plastic moments), external loadings, manufacturing errors, etc., are not fixed quantities in practice, but must be modelled as random variables with a certain joint probability distribution. In reliability-oriented structural optimization the violation of the random behavioural constraints are evaluated by means of the corresponding probability p_s of survival. Hence, the approximative computation of p_s and its sensitivities is of utmost importance. After the consideration of lower bounds of p_s based on a selection of certain redundants in the vector of internal forces/bending moments, and the consideration of upper bounds of p_s based on an optimizational representation of the yield or safety constraints by a pair of dual linear programs, a conical representation of p_s is introduced based on a cone Y_o of admissible pairs of external loads/strength increaments. Approximations of p_s can be constructed then by replacing the (finitely generated) cone Y_o by more simple ones, e.g. spherical or ellipsoidal cones. For the direct numerical computation of sensitivities of p_s and its bounds or approximations by using e.g. sampling methods or asymptotic expansion techniques based on Laplace integral representation of multiple integrals, exact differentiation formulae - of arbitrary order - for p_s and its bounds or approximations with respect to deterministic input or design variables are obtained by applying the transformation method/stochastic completion techniques; the derivatives of p_s are represented again by certain expectations or multiple integrals.

1 Limit (collapse) load analysis of structures as a linear programming problem

Assuming that the material behaves as an elastic-perfectly plastic material (Hodge 1959; Neal 1965) a conservative estimate of the collapse load factor λ_T is based (Haftka *et al.* 1990; Kirsch 1993; Tin-Loi 1995) on the following linear programm:

$$\text{maximize } \lambda \tag{1a}$$

$$\text{s.t.}\quad \mathbf{F}^{\mathrm{L}} \le \mathbf{F} \le \mathbf{F}^{\mathrm{U}}, \tag{1b}$$

$$\mathbf{C}\mathbf{F} = \lambda \mathbf{R}_o . \tag{1c}$$

Here, (1c) is the equilibrium equation of a statically indeterminate loaded structure involving an $m \times n$ matrix $\mathbf{C} = (c_{ij})$, $m < n$, of given coefficients c_{ij}, $1 \le i \le m$, $1 \le j \le n$, depending on the undeformed geometry of the structure having n_0 members (elements); we suppose that rank $\mathrm{C} = m$. Furthermore, $\mathbf{R}_o$ is an external load m-vector, and $\mathbf{F}$ denotes the n-vector of internal forces and bending-moments in the relevant points (sections, nodes or elements) of lower and upper bounds $\mathbf{F}^{\mathrm{L}}$, $\mathbf{F}^{\mathrm{U}}$.

For a *plane or spatial truss* (Lawo *et al.* 1980; Spillers 1972) we have that $n = n_o$, the matrix $\mathbf{C}$ contains the direction cosines of the members, and $\mathbf{F}$ involves only the normal (axial) forces moreover,

$$\mathbf{F}_j^{\mathrm{L}} := \sigma_{yj}^{\mathrm{L}} A_j\,, \quad \mathbf{F}_j^{\mathrm{U}} := \sigma_{yj}^{\mathrm{U}} A_j\,, \quad j = 1, \ldots, n (= n_o)\,, \tag{2}$$

where A_j is the (given) cross-sectional area, and σ_{yj}^{L}, σ_{yj}^{U}, respectively, denotes the yield stress in compression (negative values) and tension (positive values) of the j-th member of the truss. In case of a *plane frame*, $\mathbf{F}$ is composed of subvectors (Spillers 1972),

$$\mathbf{F}^{(k)} = \begin{pmatrix} F^{(k)}{}_1 \\ F^{(k)}{}_2 \\ F^{(k)}{}_3 \end{pmatrix} = \begin{pmatrix} t_k \\ m_k^+ \\ m_k^- \end{pmatrix}, \tag{3a}$$

where $F^{(k)}{}_1 = t_k$ denotes the normal (axial) force, and $F^{(k)}{}_2 = m_k^+$, $F^{(k)}{}_3 = m_k^-$ are the bending-moments at the positive, negative end of the k-th member. In this case $\mathbf{F}^{\mathrm{L}}$, $\mathbf{F}^{\mathrm{U}}$ contain - for each member k - the subvectors

$$\mathbf{F}^{(k)\mathrm{L}} = \begin{pmatrix} \sigma_{yk}^{\mathrm{L}} A_k \\ -M_{kp\ell} \\ -M_{kp\ell} \end{pmatrix}, \quad \mathbf{F}^{(k)\mathrm{U}} = \begin{pmatrix} \sigma_{yk}^{\mathrm{U}} A_k \\ M_{kp\ell} \\ M_{kp\ell} \end{pmatrix}, \tag{3b}$$

respectively, where $M_{kp\ell}$, $k = 1, \ldots, n_o$, denotes the plastic moments (moment capacities) (Hodge 1959; Neal 1965) given by

$$M_{kp\ell} = \sigma_{yk}^{\mathrm{U}} W_{kp\ell}\,, \tag{3c}$$

and $W_{kp\ell} = W_{kp\ell}(A_k)$ is the plastic section modulus of the cross-section of the k-th member (beam) with respect to the local z-axis.

For a *spatial frame* (Lawo 1980; Spillers 1972), corresponding to the k-th member (beam), $\mathbf{F}$ contains the subvector

$$\mathbf{F}^{(k)} := (t_k, m_{kT}, m_{k\bar{y}}^+ m_{k\bar{z}}^+, m_{k\bar{y}}^-, m_{k\bar{z}}^-)'\,, \tag{4a}$$

where t_k is the normal (axial) force, m_{kT} the twisting moment, and $m_{k\overline{y}}^+$, $m_{k\overline{z}}^+$, $m_{k\overline{y}}^-$, $m_{k\overline{z}}^-$ denote four bending moments with respect to the local $\overline{y}$-, z- axis at the positive , negative end of the beam, respectively. Finally, the bounds $\mathbf{F}^{\mathrm{L}}$, $\mathbf{F}^{\mathrm{U}}$ for $\mathbf{F}$ are given by

$$\mathbf{F}^{(k)\mathrm{L}} = \left(\sigma_{yk}^{\mathrm{L}} A_k, -M_{kp\ell}^{\overline{p}}, -M_{kp\ell}^{\overline{y}}, -M_{kp\ell}^{\overline{z}}, -M_{kp\ell}^{\overline{y}}, -M_{kp\ell}^{z}\right)' \tag{4b}$$

$$\mathbf{F}^{(k)\mathrm{U}} = \left(\sigma_{yk}^{\mathrm{U}} A_k, M_{kp\ell}^{\overline{p}}, M_{kp\ell}^{\overline{y}}, M_{kp\ell}^{\overline{z}}, M_{kp\ell}^{\overline{y}}, M_{kp\ell}^{\overline{z}}\right)'\,, \tag{4c}$$

where, (cf. Hodge 1959; Neal 1965),

$$M^{\overline{p}}_{kp\ell} := \tau_{yk} W^{\overline{p}}_{kp\ell}, M^{\overline{y}}_{kp\ell} := \sigma^{\mathrm{U}}_{yk} W^{\overline{y}}_{kp\ell}, M^{\overline{z}}_{kp\ell} := \sigma^{\mathrm{U}}_{yk} W^{\overline{z}}_{kp\ell}, \tag{4d}$$

are the plastic moments of the cross-section of the k-th element with respect to the local twisting axis, the local $\overline{y}$-, $\overline{z}$-axis, respectively. In (4d), $W^{\overline{p}}_{kp\ell} = W^{\overline{p}}_{kp\ell}(\mathbf{X})$ and $W^{\overline{y}}_{kp\ell} = W^{\overline{y}}_{kp\ell}(\mathbf{X})$, $W^{\overline{z}}_{kp\ell} = W^{\overline{z}}_{kp\ell}(\mathbf{X})$, respectively, denote the polar, axial modulus of the cross-sectional area of the k-th beam and τ_{yk} denotes the yield stress with respect to torsion; we suppose that $\tau_{yk} = \sigma^{\mathrm{U}}_{yk}$.

Remark 1.1. Possible *plastic hinges* (Hodge 1959; Lawo 1987; Neal 1965) are taken into account by inserting appropriate eccentricities $e_{\mathrm{kl}} > 0$, $e_{\mathrm{kr}} > 0$, $k = 1, \ldots, n_o$, with e_{kl}, $e_{\mathrm{kr}} << L_k$, where L_k is the length of the k-th beam.

Remark 1.2. Working with more general *yield polygons* (Arnbjerg-Nielsen 1991; Tin-Loi 1990; Zimmermann 1991), the stress condition (1b) is replaced by the more general system of inequalities

$$\mathbf{H}(\mathbf{F}^{\mathrm{U}}_d)^{-1}\mathbf{F} \le \mathbf{h}. \tag{5a}$$

Here, $(\mathbf{H}, \mathbf{h})$ is a given $\nu \times (n+1)$ matrix, and $\mathbf{F}^{\mathrm{U}}_d := (F^{\mathrm{U}}_j \delta_{ij})$ denotes the $n \times n$ diagonal matrix of *principal axial and bending plastic capacities*

$$F^{\mathrm{U}}_j := \sigma^{\mathrm{U}}_{yk_j} A_{kj}, \quad F^{\mathrm{U}}_j := \sigma^{\mathrm{U}}_{yk_j} W^{\kappa j}_{k_j p\ell}, \tag{5b}$$

where k_j, κ_j are indices as arising in (3b)-(4d). The more general case (5a) can be treated by similar methods as the case (1b) which is considered here.

2 Plastic and elastic design of structures

In the plastic design of trusses and frames (Kirsch 1993) having n_o members, the n-vectors $\mathbf{F}^{\mathrm{L}}$, $\mathbf{F}^{\mathrm{U}}$ of lower and upper bounds

$$\mathbf{F}^{\mathrm{L}} = \mathbf{F}^{\mathrm{L}}(\boldsymbol{\sigma}^{\mathrm{L}}_y, \boldsymbol{\sigma}^{\mathrm{U}}_y, \mathbf{X}), \quad \mathbf{F}^{\mathrm{U}} = \mathbf{F}^{\mathrm{U}}(\boldsymbol{\sigma}^{\mathrm{L}}_y, \boldsymbol{\sigma}^{\mathrm{U}}_y, \mathbf{X}), \tag{6}$$

for the n-vector $\mathbf{F}$ of internal member forces and bending-moments F_j, $j = 1, \ldots, n$, are determined (Haftka *et al.* 1990; Kirsch 1993) by the yield stresses, i.e. compressive limiting stresses (negative values) $\boldsymbol{\sigma}^{\mathrm{L}}_y = (\sigma^{\mathrm{L}}_{y1}, \ldots, \sigma^{\mathrm{L}}_{yn_o})'$, the tensile yield stresses $\boldsymbol{\sigma}^{\mathrm{U}}_y = (\sigma^{\mathrm{U}}_{y1}, \ldots, \sigma^{\mathrm{U}}_{yn_o})'$, and the r-vector

$$\mathbf{X} = (X_1, X_2, \ldots, X_r)' \tag{7}$$

of design variables of the structure. In case of trusses we have that, cf. (2),

$$\mathbf{F}^{\mathrm{L}} = \boldsymbol{\sigma}^{\mathrm{L}}_{yd}\mathbf{A}(\mathbf{X}) = \mathbf{A}(\mathbf{X})_d \boldsymbol{\sigma}^{\mathrm{L}}_y,$$

$$\mathbf{F}^{\mathrm{U}} = \boldsymbol{\sigma}^{\mathrm{U}}_{yd}\mathbf{A}(\mathbf{X}) = \mathbf{A}(\mathbf{X})_d \boldsymbol{\sigma}^{\mathrm{U}}_y, \tag{8}$$

where $n = n_o$, and $\boldsymbol{\sigma}^{\mathrm{L}}_{yd}$, $\boldsymbol{\sigma}^{\mathrm{U}}_{yd}$ denote the $n \times n$ diagonal matrices having the diagonal elements σ^{L}_{yj}, σ^{U}_{yj}, respectively, $j = 1, \ldots, n$; moreover,

$$\mathbf{A}(\mathbf{X}) = [A_1(\mathbf{X}), \ldots, A_n(\mathbf{X})]' \tag{9}$$

is the n-vector of cross-sectional areas $A_j = A_j(\mathbf{X})$, $j = 1, \ldots, n$, depending on the r-vector $\mathbf{X}$ of design variables X_k, $k = 1, \ldots, r$, and $\mathbf{A}(\mathbf{X})_d$ denotes the $n \times n$ diagonal matrix having the diagonal elements $A_j = A_j(\mathbf{X})$, $1 \le j \le n$.

Corresponding to (1c), here the equilibrium equation reads

$$\mathbf{CF} = \mathbf{R}_u, \tag{10}$$

where R_u describes (Kirsch 1993) the ultimate load [representing constant external loads or self-weight expressed in linear terms of $\mathbf{A}(\mathbf{X})$].

The *plastic design* of structures can be represented then (Arnbjerg-Nielsen 1991; Augusti *et al* 1984; Kirsch 1993; Rozvany *et al.* 1995) by the optimization problem

$$\min G(\mathbf{X}), \tag{11a}$$

$$\text{s.t. } \mathbf{F}^{\mathrm{L}}(\boldsymbol{\sigma}^{\mathrm{L}}_y, \boldsymbol{\sigma}^{\mathrm{U}}_y, \mathbf{X}) \le \mathbf{F} \le \mathbf{F}^{\mathrm{U}}(\boldsymbol{\sigma}^{\mathrm{L}}_y, \boldsymbol{\sigma}^{\mathrm{U}}_y, \mathbf{X}), \tag{11b}$$

$$\mathbf{CF} = \mathbf{R}_u, \tag{11c}$$

where $G = G(\mathbf{X})$ is a certain objective function, e.g. the volume or weight of the structure.

Remark 2.1. As mentioned in *Remark 1.2*, working with more general yield polygons, (11b) is replaced by the condition

$$\mathbf{H}[\mathbf{F}^{\mathrm{U}}(\boldsymbol{\sigma}^{\mathrm{U}}_y, \mathbf{X})_d]^{-1}\mathbf{F} \le \mathbf{h}. \tag{11d}$$

For the *elastic design* we must replace the yield stresses $\boldsymbol{\sigma}^{\mathrm{L}}_y$, $\boldsymbol{\sigma}^{\mathrm{U}}_y$ by the allowable stresses $\boldsymbol{\sigma}^{\mathrm{L}}_a$, $\boldsymbol{\sigma}^{\mathrm{U}}_a$ and instead of ultimate loads we consider service loads $\mathbf{R}_s$. Hence, instead of (11) we have the related program

$$\min G(\mathbf{X}), \tag{12a}$$

$$\text{s.t.} \quad \mathbf{F}^{\mathrm{L}}(\boldsymbol{\sigma}^{\mathrm{L}}_a, \boldsymbol{\sigma}^{\mathrm{U}}_a, \mathbf{X}) \le \mathbf{F} \le \mathbf{F}^{\mathrm{U}}(\boldsymbol{\sigma}^{\mathrm{L}}_a, \boldsymbol{\sigma}^{\mathrm{U}}_a, \mathbf{X}), \tag{12b}$$

$$\mathbf{CF} = \mathbf{R}_s, \tag{12c}$$

$$\mathbf{X}^{\mathrm{L}} \le \mathbf{X} \le \mathbf{X}^{\mathrm{U}}, \tag{12d}$$

where $\mathbf{X}^{\mathrm{L}}$, $\mathbf{X}^{\mathrm{U}}$ still denote lower and upper bounds for $\mathbf{X}$.

3 Analysis and design of structures in the case of random data

In practice, yield stresses, allowable stresses, the loads applied to the structure, other material properties and the manufacturing errors are not given fixed quantities, but must be treated as random variables on a certain probability space $(\Omega, \mathcal{A}, p)$. Hence, (1), (11), (12) are stochastic programs which have the same basic structures represented by a random objective function

$$Z(\omega) := G(\omega, \mathbf{X}), \quad \omega \in \Omega, \tag{13a}$$

and by stochastic constraints of the type

$$\mathbf{CF} = \mathbf{R}(\omega), \tag{13b}$$

$$\mathbf{F}^{\mathrm{L}}(\omega) \le \mathbf{F} \le \mathbf{F}^{\mathrm{U}}(\omega), \tag{13c}$$

where $\mathbf{R} = \mathbf{R}(\omega)$, $\omega \in \Omega$, is a random load m-vector given by

$$\mathbf{R}(\omega) = \lambda \mathbf{R}_o(\omega), \quad \mathbf{R}(\omega) = \mathbf{R}_u(\omega), \quad \mathbf{R}(\omega) = \mathbf{R}_s(\omega), \tag{14}$$

respectively and for the n-vector $\mathbf{F} = (F_j)$ of internal member forces and bending-moments we have the n-vectors of random bounds

$$\mathbf{F}^{\mathrm{L}}(\omega) = \mathbf{F}^{\mathrm{L}}(\omega, \mathbf{X}), \quad \mathbf{F}^{\mathrm{U}}(\omega) = (\omega, \mathbf{X}), \omega \in \Omega, \tag{15}$$

depending on an r-vector $\mathbf{X}$ of design variables X_k, $k = 1, \ldots, r$.

Obviously, each realization of the random element $\omega \in \Omega$ yields new loading conditions, represented by the vector

$\mathbf{R} = \mathbf{R}(\omega)$, and therefore new arrangements $\mathbf{F} = \mathbf{F}(\omega)$ of internal forces and bending-moments. Hence, the survival of the structure can be evaluated by the probability of survival

$$p_s := P\left[\text{there is } \mathbf{F} = \mathbf{F}(\omega) \text{ such that } \mathbf{CF}(\omega) = \mathbf{R}(\omega) \text{ and } \mathbf{F}^{\mathrm{L}}(\omega) \le \mathbf{F}(\omega) \le \mathbf{F}^{\mathrm{U}}(\omega)\right], \tag{16}$$

(see Augusti *et al.* 1984; Nafday and Corotis 1987), assuming that

$$S[\mathbf{F}^{\mathrm{L}}(\cdot), \mathbf{F}^{\mathrm{U}}(\cdot), \mathbf{R}(\cdot)] := \{\omega \in \Omega : \text{there is a vector} \mathbf{F} = \mathbf{F}(\omega) \text{fulfilling (13b) and (13c)}\} \tag{17}$$

is a measurable set. Denoting by $[\mathbf{F}^{\mathrm{L}}, \mathbf{F}^{\mathrm{U}}]$ the n-dimensional interval

$$[\mathbf{F}^{\mathrm{L}}, \mathbf{F}^{\mathrm{U}}] := \{\mathbf{F} \in I\!R^n : \mathbf{F}^{\mathrm{L}} \le \mathbf{F} \le \mathbf{F}^{\mathrm{U}}\}, \tag{18}$$

we find that

$$S[\mathbf{F}^{\mathrm{L}}(\cdot), \mathbf{F}^{\mathrm{U}}(\cdot), \mathbf{R}(\cdot)] := \{\omega \in \Omega : \mathbf{R}(\omega) \in \mathbf{C}[\mathbf{F}^{\mathrm{L}}(\omega), \mathbf{F}^{\mathrm{U}}(\omega)]\} \tag{19}$$

with $\mathbf{C}[\mathbf{F}^{\mathrm{L}}, \mathbf{F}^{\mathrm{U}}] = \{\mathbf{CF} : \mathbf{F}^{\mathrm{L}} \le \mathbf{F} \le \mathbf{F}^{\mathrm{U}}\}$, and therefore

$$p_s = P\left\{S[\mathbf{F}^{\mathrm{L}}(\cdot), \mathbf{F}^{\mathrm{U}}(\cdot), \mathbf{R}(\cdot)]\right\} = P\left\{\mathbf{R}(\omega) \in \mathbf{C}[\mathbf{F}^{\mathrm{L}}(\omega), \mathbf{F}^{\mathrm{U}}(\omega)]\right\} = \int P\left\{\mathbf{R}(\omega) \in \mathbf{C}[\mathbf{F}^{\mathrm{L}}, \mathbf{F}^{\mathrm{U}}] | \mathbf{F}^{\mathrm{L}}(\omega) = \mathbf{F}^{\mathrm{L}}, \mathbf{F}^{\mathrm{U}}(\omega) = \mathbf{F}^{\mathrm{U}}\right\} \pi(\mathrm{d}\mathbf{F}^{\mathrm{L}}, \mathrm{d}\mathbf{F}^{\mathrm{U}}), \tag{20}$$

where π denotes the distribution of the bounds $[\mathbf{F}^{\mathrm{L}}(\omega), \mathbf{F}^{\mathrm{U}}(\omega)]$. Since the bounds $\mathbf{F}^{\mathrm{L}}$, $\mathbf{F}^{\mathrm{U}}$ in (13c) depend also on the vector $\mathbf{X}$ of design variables X_k, $k = 1, \ldots, r$, cf. (15), we have $p_s = P(\mathbf{X})$ with the probability function

$$P(\mathbf{X}) = P\left\{\mathbf{R}(\omega) \in \mathbf{C}[\mathbf{F}^{\mathrm{L}}(\omega, \mathbf{X}), \mathbf{F}^{\mathrm{U}}(\omega, \mathbf{X})]\right\}; \tag{21}$$

furthermore, if the external load $\mathbf{R} = \mathbf{R}(\omega)$ is given by

$$\mathbf{R}(\omega) = \mathbf{R}(\omega, \boldsymbol{\lambda}) := \sum_{i=1}^{m_R} \lambda_i \mathbf{R}^{(i)}(\omega), \quad \lambda := (\boldsymbol{\lambda}_1, \ldots, \lambda_{m_R})', \tag{22}$$

with random m-vector $\mathbf{R}^{(i)} = \mathbf{R}^{(i)}(\omega)$, $i = 1, \ldots, m_R$, and deterministic coefficients $\lambda_i, i = 1, \ldots, m_R$, then $p_S = P(\lambda, \mathbf{X})$ with

$$P(\lambda, \mathbf{X}) := P\left\{\sum_{i=1}^{m_R} \lambda_i \mathbf{R}^{(i)}(\omega) \in \mathbf{C}[\mathbf{F}^{\mathrm{L}}(\omega, \mathbf{X}), \mathbf{F}^{\mathrm{U}}(\omega, \mathbf{X})]\right\}. \tag{23}$$

Especially, in the case of trusses and with $\mathbf{R}(\omega) := \lambda \mathbf{R}_o(\omega)$, for the consideration of p_s we have the following probability functions:

$$P_A(\lambda, \mathbf{X}) := P\left\{\lambda \mathbf{R}_o(\omega) \in \mathbf{C}[\mathbf{A}(\mathbf{X})_d \boldsymbol{\sigma}_y^{\mathrm{L}}(\omega), \mathbf{A}(\mathbf{X})_d \boldsymbol{\sigma}_y^{\mathrm{U}}(\omega)]\right\}, \ \lambda \in I\!R, \tag{24a}$$

$$P_u(\mathbf{X}) := P\left\{\mathbf{R}_u(\omega) \in \mathbf{C}[\mathbf{A}(\mathbf{X})_d \boldsymbol{\sigma}_y^{\mathrm{L}}(\omega), \mathbf{A}(\mathbf{X})_d \boldsymbol{\sigma}_y^{\mathrm{U}}(\omega)]\right\}, \tag{24b}$$

$$P_s(\mathbf{X}) := P\left\{\mathbf{R}_s(\omega) \in \mathbf{C}[\mathbf{A}(\mathbf{X})_d \boldsymbol{\sigma}_a^{\mathrm{L}}(\omega), \mathbf{A}(\mathbf{X})_d \boldsymbol{\sigma}_a^{\mathrm{U}}(\omega)]\right\}. \tag{24c}$$

Since $p_s = P(\mathbf{X})$ are very complicated expressions in general, in the following we are looking for approximations of $p_s = P(\mathbf{X})$ by simpler probability functions. For simplification, $P(\mathbf{X})$ is used also to denote $P(\lambda, \mathbf{X})$.

4 Lower and upper bounds for $p_s = P(\mathbf{X})$

According to (19) we have that

$$S\left[\mathbf{F}^{\mathrm{L}}(.), \mathbf{F}^{\mathrm{U}}(.), \mathbf{R}(.)\right] \subset \bigcap_{i=1}^{m} S_i\left[\mathbf{F}^{\mathrm{L}}(.), \mathbf{F}^{\mathrm{U}}(.)\right], \tag{25a}$$

where

$$S_i\left[\mathbf{F}^{\mathrm{L}}(.), \mathbf{F}^{\mathrm{U}}(.), \mathbf{R}(.)\right] := \left\{\omega \in \Omega : \mathbf{R}_i(\omega) \in \mathbf{C}_i[\mathbf{F}^{\mathrm{L}}(\omega), \mathbf{F}^{\mathrm{U}}(\omega)]\right\}, \ i = 1, \ldots, m, \tag{25b}$$

and $\mathbf{C}_i$ denotes the i-th row of $\mathbf{C}$. Hence, we have the inequality (Cornell 1967; Ditlevsen 1979)

$$P(\mathbf{X}) \le \min_{1 \le i \le m} P_i(\mathbf{X}), \tag{26a}$$

where

$$P_i(\mathbf{X}) := P\left\{R_i(\omega) \in \mathbf{C}_i[\mathbf{F}^{\mathrm{L}}(\omega, \mathbf{X}), \mathbf{F}^{\mathrm{U}}(\omega, \mathbf{X})]\right\}. \tag{26b}$$

We find that

$$\mathbf{C}_i\left[\mathbf{F}^{\mathrm{L}}(\omega, \mathbf{X}), \mathbf{F}^{\mathrm{U}}(\omega, \mathbf{X})\right] = \left[\gamma_i^{\mathrm{L}}(\omega, \mathbf{X}), \gamma_i^{\mathrm{U}}(\omega, \mathbf{X})\right] \tag{27a}$$

is an interval in $I\!R^1$ having the bounds

$$\gamma_i^{\mathrm{L}}(\omega, \mathbf{X}) = \min_{i \le \iota \le J} \mathbf{C}_i \mathbf{G}^{\iota}(\omega, \mathbf{X}), \quad \gamma_i^{\mathrm{U}}(\omega, \mathbf{X}) = \max_{1 \le \iota \le J} \mathbf{C}_i \mathbf{G}^{\iota}(\omega, \mathbf{X}), \tag{27b}$$

where $\mathbf{G}^{\iota} = \mathbf{G}^{\iota}(\omega, \mathbf{X}), \iota = 1, \ldots, J$, are the extereme points of the interval $[\mathbf{F}^{\mathrm{L}}(\omega, \mathbf{X}), \mathbf{F}^{\mathrm{U}}(\omega, \mathbf{X})]$. Since the components $\mathbf{G}_j^{\iota}(\omega, \mathbf{X}), j = 1, 2, \ldots, n$, of $\mathbf{G}^{\iota}(\omega, \mathbf{X}), \iota = 1, \ldots, J$, are certain elements of $[\mathbf{F}^{\mathrm{L}}(\omega, \mathbf{X}), \mathbf{F}^{\mathrm{U}}(\omega, \mathbf{X})]$, the measurability of the bounds $\mathbf{F}^{\mathrm{L}}$, $\mathbf{F}^{\mathrm{U}}$ with respect to $\omega \in (\Omega, A, P)$ yields the measurability of γ_i^{L} and $\gamma_i^{\mathrm{U}}, i = 1, \ldots, m$, with respect to ω. Hence, $S_i[\mathbf{F}^{\mathrm{L}}(.), \mathbf{F}^{\mathrm{U}}(.), \mathbf{R}(.)], i = 1, \ldots, m$, are measurable sets, and $P_i(\mathbf{X})$ is well-defined. Using (25), more exact upper bounds follow from the work of Cornell (1967), Ditlevsen (1979), Galambos (1977) and Kounias (1968).

4.1 Lower bounds by selection of redundants

For the construction of lower bounds for $P(\mathbf{X})$, the vector $F = F(\omega)$ is partitioned

$$\mathbf{F}(\omega) = \begin{pmatrix} \mathbf{F}_I \\ \mathbf{N} \end{pmatrix} \tag{28}$$

into a certain $(n-m)$-vector $\mathbf{N} = (F_{j_\ell})$, $1 \le \ell \le n-m$ of *redundants* (Kirsch 1993) F_{j_ℓ}, $\ell = 1, \ldots, n-m$, and m-vector $\mathbf{F}_I$ of statically determined member forces/bending-moments. Hence, with a corresponding partition of the $m \times n$ matrix $\mathbf{C}$ into $m \times m$, $m \times (n-m)$ submatrices $\mathbf{C}_I$, $\mathbf{C}_{II}$, respectively, where

$$\operatorname{rank} \mathbf{C}_I = \operatorname{rank} \mathbf{C} = m, \tag{29}$$

the equilibrium equation (13b) yields for $\mathbf{F}(\omega)$ the representation

$$\mathbf{F}(\omega) = \begin{pmatrix}\mathbf{F}_I\\ \mathbf{N}\end{pmatrix} = \begin{pmatrix}\mathbf{C}_I^{-1}\mathbf{R}(\omega)\\ 0\end{pmatrix} + \begin{pmatrix}-\mathbf{C}_I^{-1}\mathbf{C}_{II}\\ \mathbf{I}\end{pmatrix}\mathbf{N}. \tag{30}$$

Consequently, selecting for each $\omega \in \Omega$ a vector of redundants $\mathbf{N} = \mathbf{N}(\omega) = [F_{j_\ell}(\omega)], 1 \le \ell \le n-m$, such that $\mathbf{N}(.)$ is a measurable function on (Ω, A, P), we have

$$S\left[\mathbf{F}^{\mathrm{L}}(.,X), \mathbf{F}^{\mathrm{U}}(.,X), \mathbf{R}(.)\right] \supset \tilde{S}[\mathbf{X}, \mathbf{N}(.), \mathbf{R}(.)], \tag{31}$$

where $\tilde{S}[\mathbf{X}, \mathbf{N}(.), \mathbf{R}(.)]$ is the measurable set given by

$$\tilde{S}[\mathbf{X}, N(.), \mathbf{R}(.)] := \left\{\omega \in \Omega : \mathbf{F}^{\mathrm{L}}(\omega, \mathbf{X}) \le \begin{pmatrix}\mathbf{C}_I^{-1}[\mathbf{R}(\omega) - \mathbf{C}_{II}\mathbf{N}(\omega)]\\ \mathbf{N}(\omega)\end{pmatrix} \le \mathbf{F}^{\mathrm{U}}(\omega, \mathbf{X})\right\}. \tag{32}$$

Thus, for ($P(\mathbf{X})$ corresponding to (Augusti *et al* 1984) we find the lower bound

$$P(\mathbf{X}) \ge \tilde{P}[(\mathbf{X}, \mathbf{N}(.)], \tag{33}$$

where

$$\tilde{P}[\mathbf{X}, \mathbf{N}(.)] := P\begin{pmatrix}\mathbf{F}_I^{\mathrm{L}}(\omega, \mathbf{X}) & \le \mathbf{C}_I^{-1}[\mathbf{R}(\omega) - \mathbf{C}_{II}\mathbf{N}(\omega)] \le \mathbf{F}_I^{\mathrm{U}}(\omega, \mathbf{X})\\ \mathbf{F}_{II}^{\mathrm{L}}(\omega, \mathbf{X}) \le \mathbf{N}(\omega) & \le \mathbf{F}_{II}^{\mathrm{U}}(\omega, \mathbf{X})\end{pmatrix} \tag{34}$$

and $\mathbf{F}_I^{\mathrm{L}}, \mathbf{F}_{II}^{\mathrm{L}}$ and $\mathbf{F}_I^{\mathrm{U}}, \mathbf{F}_{II}^{\mathrm{U}}$ denotes the partition of $\mathbf{F}^{\mathrm{L}}$, $\mathbf{F}^{\mathrm{U}}$ respectively, corresponding to (28). Note that inequality (33) holds for any choice $\mathbf{N} = (F_{j_\ell}), 1 \le \ell \le n-m$, of an $(n-m)$-subvector of redundants such that (29) holds and any representation of $\mathbf{N}$ as a random vector $\mathbf{N} = \mathbf{N}(\omega)$ on (Ω, A, P); especially, $\mathbf{N}$ can be selected as a *deterministic* vector of redundants:

$$\mathbf{N}(\omega) = \mathbf{z} \quad \text{a.s.} \quad (\text{almost sure}), \tag{35}$$

where $\mathbf{z} \in \mathbb{R}^{n-m}$ is a deterministic vector; in this case we set $\tilde{P}[\mathbf{X}, \mathbf{N}(.)] = \tilde{P}(\mathbf{X}, \mathbf{z})$.

4.1.1 Special cases

(a) In case of trusses, cf. (2), (8), we have that

$$\mathbf{F}_I^{\mathrm{L}}(\omega, \mathbf{X}) = \mathbf{A}_I(\mathbf{X})_d \boldsymbol{\sigma}_I^{\mathrm{L}}(\omega), \mathbf{F}_I^{\mathrm{U}}(\omega, \mathbf{X}) = \mathbf{A}_I(\mathbf{X})_d \boldsymbol{\sigma}_I^{\mathrm{U}}(\omega)$$
$$\mathbf{F}_{II}^{\mathrm{L}}(\omega, \mathbf{X}) = \mathbf{A}_{II}(\mathbf{X})_d \boldsymbol{\sigma}_{II}^{\mathrm{L}}(\omega), \mathbf{F}_{II}^{\mathrm{U}}(\omega, \mathbf{X}) = \mathbf{A}_{II}(\omega)_d \boldsymbol{\sigma}_{II}^{\mathrm{U}}(\omega) \tag{36}$$

where $\mathbf{A}_I, \mathbf{A}_{II}, \boldsymbol{\sigma}_I^{\mathrm{L}}, \boldsymbol{\sigma}_{II}^{\mathrm{L}}, \boldsymbol{\sigma}_I^{\mathrm{U}}, \boldsymbol{\sigma}_{II}^{\mathrm{U}}$ are the partitions of $\mathbf{A}$, $\boldsymbol{\sigma}^{\mathrm{L}}$, $\boldsymbol{\sigma}^{\mathrm{U}}$, corresponding to the partition $\mathbf{F}_I$, F_{II} of $\mathbf{F}$, and $\mathbf{A}_I(\mathbf{X})_d$ denotes the diagonal matrix has the components of $\mathbf{A}_I(\mathbf{X})$ as its diagonal elements. Thus, we have

$$\tilde{P}(\mathbf{X}, \mathbf{z}) = P\begin{pmatrix}\mathbf{A}_I(\mathbf{X})_d\boldsymbol{\sigma}_I^{\mathrm{L}}(\omega) \le \quad \mathbf{C}_I^{-1}[\mathbf{R}(\omega) - \mathbf{C}_{II}\mathbf{z}]\\ \le \mathbf{A}_I(\mathbf{X})_d\boldsymbol{\sigma}_I^{\mathrm{U}}(\omega)\\ \mathbf{A}_{II}(\mathbf{X})_d\boldsymbol{\sigma}_{II}^{\mathrm{L}}(\omega) \le \mathbf{z}\\ \le \mathbf{A}_{II}(\mathbf{X})_d\boldsymbol{\sigma}_{II}^{\mathrm{U}}(\omega)\end{pmatrix} = P\begin{pmatrix}\boldsymbol{\sigma}_I^{\mathrm{L}}(\omega) \le \mathbf{A}_I(\mathbf{X})_d^{-1}\mathbf{C}_I^{-1}[\mathbf{R}(\omega) - \mathbf{C}_{II}\mathbf{z}]\\ \le \boldsymbol{\sigma}_I^{\mathrm{U}}(\omega)\\ \mathbf{A}_{II}(\mathbf{X})_d\boldsymbol{\sigma}_{II}^{\mathrm{L}}(\omega) \le \mathbf{z}\\ \le \mathbf{A}_{II}(\mathbf{X})_d\boldsymbol{\sigma}_{II}^{\mathrm{U}}(\omega)\end{pmatrix}. \tag{37}$$

(b) Suppose that the partition of $\mathbf{F}^{\mathrm{L}}$, $\mathbf{F}^{\mathrm{U}}$ into $\mathbf{F}_I^{\mathrm{L}}$, $\mathbf{F}_I^{\mathrm{U}}$ and $\mathbf{F}_{II}^{\mathrm{L}}$, $\mathbf{F}_{II}^{\mathrm{U}}$ can be chosen such that $[\mathbf{F}_I^{\mathrm{L}}(\omega, \mathbf{X}), \mathbf{F}_I^{\mathrm{U}}(\omega, \mathbf{X})]$, $[\mathbf{F}_{II}^{\mathrm{L}}(\omega), \mathbf{F}_{II}^{\mathrm{U}}(\omega)]$ are stochastically independent. If (35) holds, then

$$\tilde{P}(\mathbf{X}, \mathbf{z}) := P\left(\mathbf{F}_I^{\mathrm{L}}(\omega, \mathbf{X}) \le \mathbf{C}_I^{-1}[\mathbf{R}(\omega) - \mathbf{C}_{II}z] \le \mathbf{F}_I^{\mathrm{U}}(\omega, \mathbf{X})\right) \times P\left(\mathbf{F}_{II}^{\mathrm{L}}(\omega, \mathbf{X}) \le \mathbf{z} \le \mathbf{F}_{II}^{\mathrm{U}}(\omega, \mathbf{X})\right). \tag{38}$$

5 Failure modes, limit state functions and upper bounds for $P(\mathbf{X})$

According to (16) and (21) we have that

$$P(X) = P\ (\text{there is } \mathbf{F} = \mathbf{F}(\omega) \text{ such that } \mathbf{CF}(\omega) = \mathbf{R}(\omega) \text{ and } F_j(\omega) - F_j^{\mathrm{U}}(\omega, \mathbf{X}) \le 0, \quad j = 1, \ldots, n, \quad F_j^{\mathrm{L}}(\omega, \mathbf{X}) - F_j(\omega) \le 0, \quad j = 1, \ldots, n), \tag{39}$$

where we suppose that all bounds F_j, $\mathbf{F}_j^{\mathrm{U}}$ are finite, i.e.

$$-\infty < F_j^{\mathrm{L}}(\omega, \mathbf{X}) \le F_j^{\mathrm{U}}(\omega, \mathbf{X}) < +\infty \text{ a.s.}, 1 \le j \le n, \forall \mathbf{X}. \tag{40}$$

Defining

$$t[\omega, \mathbf{F}(\omega), \mathbf{X}] := \max_{1\le j\le n}\left\{F_j(\omega) - F_j^{\mathrm{U}}(\omega, \mathbf{X}), F_j^{\mathrm{L}}(\omega, \mathbf{X}) - F_j(\omega)\right\}, \tag{41}$$

we obtain

$$P(\mathbf{X}) = P\ (\text{there is } \mathbf{F} = \mathbf{F}(\omega) \text{ such that } \mathbf{CF}(\omega) = \mathbf{R}(\omega) \text{ and } t[\omega, \mathbf{F}(\omega), \mathbf{X}] \le 0) = P\left(\inf\{t(\omega, \mathbf{F}(\omega), \mathbf{X}) : \mathbf{CF}(\omega) = \mathbf{R}(\omega)\} \le 0\right) = P[t^*(\omega, \mathbf{X}) \le 0], \tag{42}$$

where

$$t^*(\omega, \mathbf{X}) := \inf\{t[\omega, \mathbf{F}(\omega), \mathbf{X}] : \mathbf{CF}(\omega) = \mathbf{R}(\omega)\} \tag{43}$$

is the minimal value of the program

$$\min t[\omega, \mathbf{F}(\omega), \mathbf{X}]), \tag{44}$$
$$\text{s.t. } \mathbf{CF}(\omega) = \mathbf{R}(\omega)$$

being equivalent to the *linear programm*

$$\min t \tag{45a}$$
$$\text{s.t. } F_j - F_j^{\mathrm{U}}(\omega, \mathbf{X}) - t \le 0, \quad j = 1, \ldots, n, \tag{45b}$$
$$F_j^{\mathrm{L}}(\omega, \mathbf{X}) - F_j - t \le 0, \quad j = 1, \ldots, n, \tag{45c}$$
$$\mathbf{CF}(\omega) = \mathbf{R}(\omega), \tag{45d}$$

with the variables $F_1, F_2, \ldots, F_n, t$.

Because of condition (40), for each $(\omega, \mathbf{X})$ we have

$$t[\omega, \mathbf{F}(\omega), \mathbf{X}] \ge \max_{1\le j\le n} \frac{1}{2}\left[F_j^{\mathrm{L}}(\omega, \mathbf{X}) - F_j^{\mathrm{U}}(\omega, \mathbf{X})\right] > -\infty, \tag{46}$$

for arbitrary $F(\omega)$; hence, the objective function of the linear program (45) is bounded from below for each $(\omega, \mathbf{X})$. Since the LP (45) always has a feasible solution, for each $(\omega, \mathbf{X})$ an optimal solution $\binom{\mathbf{F}^*}{t^*}$ of (45) is guaranteed, and we have that

$$t^* = t[\omega, \mathbf{F}^*(\omega), \mathbf{X}] = t^*(\omega, \mathbf{X}) . \quad (47)$$

Consequently, by means of duality theory the optimal value $t^*(\omega, \mathbf{X})$ of the equivalent programs (44) and (45) can be represented also by the optimal value of the dual program of (45) given by

$$\max \mathbf{R}(\omega)'\mathbf{u} - \mathbf{F}^{\mathrm{U}}(\omega, \mathbf{X})'\tilde{\mathbf{u}}^+ + \mathbf{F}^{\mathrm{L}}(\omega, \mathbf{X})'\tilde{\mathbf{u}}^- , \quad (48a)$$

$$\text{s.t. } \mathbf{C}'\mathbf{u} - \tilde{\mathbf{u}}^+ + \tilde{\mathbf{v}}^- = 0 , \quad (48b)$$

$$\mathbf{1}'\tilde{\mathbf{u}}^+ + \mathbf{1}'\tilde{\mathbf{u}}^- = 1 , \quad (48c)$$

$$\tilde{\mathbf{u}}^+ \geq 0 , \quad \tilde{\mathbf{v}}^- \geq 0 , \quad (48d)$$

where $\mathbf{u} \in \mathbb{R}^n$ is not restricted.

5.1 Remark

Obviously, (48b) is the member-node (or joint) displacement equation. According to its mechanical meaning, we call (45) and (48), respectively, the *static kinematic* linear programm (LP) cf. Augusti *et al.* 1984; Ditlevsen 1984; Nafday and Corotis 1987; Simoes 1990; Zimmermann *et al.* 1991).

Having (48), $t^*(\omega, \mathbf{X})$ reads

$$t^*(\omega, \mathbf{X}) =$$

$$\max \left\{ \begin{pmatrix} \mathbf{R}(\omega) \\ -\mathbf{F}^{\mathrm{U}}(\omega, \mathbf{X}) \\ \mathbf{F}^{\mathrm{L}}(\omega, \mathbf{X}) \end{pmatrix}' \delta : \begin{pmatrix} \mathbf{u} \\ \tilde{\mathbf{u}}^+ \\ \tilde{\mathbf{v}}^- \end{pmatrix} =: \delta \in \Delta_o \right\} , \quad (49)$$

where Δ_o denotes the convex polyhedron in $\mathbb{R}^{m+2n}$ represented by the constraints (48b-(48d) of the LP (48). Taking any subset $\Delta_1 \subset \Delta_o$ of Δ_o, and defining then $t_1^*(\omega, \mathbf{X})$ by

$$t_1^*(\omega, \mathbf{X}) = \sup \left\{ \begin{pmatrix} \mathbf{R}(\omega) \\ -\mathbf{F}^{\mathrm{U}}(\omega, \mathbf{X}) \\ \mathbf{F}^{\mathrm{L}}(\omega, \mathbf{X}) \end{pmatrix}' \delta : \delta \in \Delta_1 \right\} , \quad (50)$$

corresponding to Augusti *et al.* (1984) we have

$$t^*(\omega, \mathbf{X}) \geq t_1^*(\omega, \mathbf{X}) , \quad (51)$$

which yields for $P(\mathbf{X})$ the following upper bound:

$$P(\mathbf{X}) = P[t^*(\omega, \mathbf{X}) \leq 0] \leq P[t_1^*(\omega, \mathbf{X}) \leq 0] . \quad (52)$$

Moreover, if

$$\delta^{(\ell)} = \begin{pmatrix} \mathbf{u}^{(\ell)} \\ \tilde{\mathbf{u}}^{+(\ell)} \\ \tilde{\mathbf{u}}^{-(\ell)} \end{pmatrix} , \quad \ell = 1, \ldots, \ell_o , \quad (52)$$

denote the extreme points of the convex polyhedron Δ_o, then

$$t^*(\omega, \mathbf{X}) = \max_{1 \leq \ell \leq \ell_o} \mathbf{R}(\omega)'\mathbf{u}^{(\ell)} - \mathbf{F}^{\mathrm{U}}(\omega, \mathbf{X})'\tilde{\mathbf{u}}^{+(\ell)} +$$

$$\mathbf{F}^{\mathrm{L}}(\omega, \mathbf{X})'\tilde{\mathbf{u}}^{-(\ell)} , \quad (53)$$

which shows that $t^*(., \mathbf{X})$ is measurable. Hence, $S\left[\mathbf{F}^{\mathrm{L}}(.), \mathbf{F}(.), \mathbf{R}(.)\right] = \{\omega \in \Omega : t^*(\omega, \mathbf{X}) \leq 0\}$ is meaurable, cf. (21), (42), and we have as in the book by Augusti *et al.* (1984)

$$P(\mathbf{X}) = P\left[\mathbf{R}(\omega)'\mathbf{u}^{(\ell)} - \mathbf{F}^{\mathrm{U}}(\omega, \mathbf{X})'\tilde{\mathbf{u}}^{+(\ell)} + \right.$$

$$\left. \mathbf{F}^{\mathrm{L}}(\omega, \mathbf{X})'\tilde{\mathbf{u}}^{-(\ell)} \leq 0, 1 \leq \ell \leq \ell_o \right] . \quad (54)$$

According to (13b) and (13c), the survival or failure of the underlying structure can be described by the inequality

$$t^*(\omega, \mathbf{X}) \leq 0 , t^*(\omega, \mathbf{X}) > 0 , \text{respectively.} \quad (55)$$

Thus, the structure fails if and only if

$$\mathbf{R}(\omega)'\mathbf{u}^{(\ell)} + \mathbf{F}^{\mathrm{U}}(\omega, \mathbf{X})'\tilde{\mathbf{u}}^{+(\ell)} - \mathbf{F}^{\mathrm{L}}(\omega, \mathbf{X})'\tilde{\mathbf{u}}^{-(\ell)} > 0$$

$$\text{for at least one } 1 \leq \ell \leq \ell_o ; \quad (56)$$

obviously, (56) represents the different *failure modes* of the structure.

Having a certain number $\ell_1 \leq \ell_o$ of basic solutions $\sigma^{(\ell_\tau)}$, $\tau = 1, \ldots, \ell_1$, of the LP (48), and defining

$$\tilde{t}_1^*(\omega, \mathbf{X}) := \max_{1 \leq \tau \leq \ell_1} \mathbf{R}(\omega)'\mathbf{u}^{(\ell_\tau)} - \mathbf{F}^{\mathrm{U}}(\omega, \mathbf{X})'\tilde{\mathbf{u}}^{+(\ell_\tau)} +$$

$$\mathbf{F}^{\mathrm{L}}(\omega, \mathbf{X})'\tilde{\mathbf{u}}^{-(\ell_\tau)} , \quad (57)$$

corresponding to (51), here we have, cf. (Augusti *et al.* 1984),

$$t^*(\omega, \mathbf{X}) \geq \tilde{t}_1^*(\omega, \mathbf{X}) \quad (58)$$

and therefore

$$P(\mathbf{X}) = P\left[t^*(\omega, \mathbf{X}) \leq 0\right] \leq P\left[\tilde{t}_1^*(\omega, \mathbf{X}) \leq 0\right] . \quad (59)$$

6 The probability of failure p_f

According to (16), (21), (42) and (55), for the probability of failure $p_f := 1 - p_s = 1 - P(X)$ we obtain

$$p_f = P\left[t^*(\omega, \mathbf{X}) > 0\right] =$$

$$P\Bigg[\mathbf{R}(\omega)'\mathbf{u} - \mathbf{F}^{\mathrm{U}}(\omega, \mathbf{X})'\tilde{\mathbf{u}}^+ + \mathbf{F}^{\mathrm{L}}(\omega, \mathbf{X})'\tilde{\mathbf{u}}^- > 0$$

$$\text{for at least one } \begin{pmatrix} \mathbf{w} \\ \tilde{\mathbf{u}}^+ \\ \tilde{\mathbf{u}}^- \end{pmatrix} \in \Delta_o \Bigg] =$$

$$P\left[\mathbf{R}(\omega)'\mathbf{u}^{(\ell)} - \mathbf{F}^{\mathrm{U}}(\omega, \mathbf{X})'\tilde{\mathbf{u}}^{+(\ell)} + \mathbf{F}^{\mathrm{L}}(\omega, \mathbf{X})'\tilde{\mathbf{u}}^{-(\ell)} > 0\right.$$

$$\left.\text{for at least one } 1 \leq \ell \leq \ell_o\right] =$$

$$P\left[\bigcup_{\ell=1}^{\ell_o} \mathbf{F}_\ell(\mathbf{X})\right] , \quad (60)$$

where $\mathbf{F}_\ell(\mathbf{X})$ denotes the ℓ-th failure domain

$$\mathbf{F}_\ell(\mathbf{X}) := \left\{\omega \in \Omega : \mathbf{R}(\omega)'\mathbf{u}^{(\ell)} - \mathbf{F}^{\mathrm{U}}(\omega, \mathbf{X})'\tilde{\mathbf{u}}^{+(\ell)} + \right.$$

$$\left.\mathbf{F}^{\mathrm{L}}(\omega, \mathbf{X})'\tilde{\mathbf{u}}^{-(\ell)} > 0\right\} =$$

$$\{\omega \in \Omega : M_\ell(\omega, \mathbf{X}) < 0\} . \quad (61)$$

with the corresponding *limit state functions* (Augusti *et al.* 1984; Nafday 1987; Rackwitz and Cuntze 1987)

$$M_\ell(\omega, \mathbf{X}) := \mathbf{F}^{\mathrm{U}}(\omega, \mathbf{X})'\tilde{\mathbf{u}}^{+(\ell)} - \mathbf{F}^{\mathrm{L}}(\omega, \mathbf{X})'\tilde{\mathbf{u}}^{-(\ell)} - \mathbf{R}(\omega)'u^{(\ell)} ,$$

$$\ell = 1, \ldots, \ell_o \quad (62)$$

especially, for trusses, cf. (8), we find

$$M_\ell(\omega, \mathbf{X}) := \boldsymbol{\sigma}^{\mathrm{U}}(\omega)'\mathbf{A}(\mathbf{X})_d\tilde{\mathbf{u}}^{+(\ell)} - \boldsymbol{\sigma}^{\mathrm{L}}(\omega)'\mathbf{A}(\mathbf{X})_d\tilde{\mathbf{u}}^{-(\ell)} -$$

$$\mathbf{R}(\omega)'\mathbf{u}^{(\ell)} . \quad (63)$$

Using known inequalities for probabilities (Cornell 1967; Ditlevsen 1979; Galambos 1977; Kounias 1968), for p_f we find the bounds

$$\max_{1 \leq \ell \leq \ell_o} p_{f,\ell} \leq p_f \leq \sum_{\ell=1}^{\ell_o} p_{f,\ell} , \quad (64)$$

where $p_{f,\ell}$ is given by

$$p_{f,\ell} := P\left[\mathbf{F}_\ell(\mathbf{X})\right] = P\left[M_\ell(\omega,\mathbf{X}) < 0\right] =$$
$$P\left[\mathbf{F}^U(\omega,\mathbf{X})'\tilde{\mathbf{u}}^{+(\ell)} - \mathbf{F}^L(\omega,\mathbf{X})'\tilde{\mathbf{u}}^{-(\ell)} < \mathbf{R}(\omega)'\mathbf{u}^{(\ell)}\right] =$$
$$1 - P\left[\mathbf{R}(\omega)'u^{(\ell)} \le \mathbf{F}^U(\omega,\mathbf{X})'\tilde{\mathbf{u}}^{+(\ell)} - \mathbf{F}^L(\omega,\mathbf{X})'\tilde{\mathbf{u}}^{-(\ell)}\right] \quad (65)$$

and sharper bounds can be obtained by using more general inequalities for probabilities.

7 Conical representation of p_s

According to (20) we have that

$$p_s = P\left\{\mathbf{R}(\omega) \in \mathbf{C}\left[\mathbf{F}^L(\omega), \mathbf{F}^U(\omega)\right]\right\},$$

where $\left[\mathbf{F}^L, \mathbf{F}^U\right]$ is given by (18). Representing therefore the vector $\mathbf{F}$ of internal member forces/bending-moments by

$$\mathbf{F} = \mathbf{F}^L + \Delta\mathbf{F}^L = \mathbf{F}^U - \Delta\mathbf{F}^U,$$

with n-vectors $\Delta\mathbf{F}^U$, $\Delta\mathbf{F}^L \ge 0$, the condition $\mathbf{R} \in \mathbf{C}\left[\mathbf{F}^L, \mathbf{F}^U\right]$ can be represented by

$$\mathbf{R} - \mathbf{C}\mathbf{F}^U = -\mathbf{C}\Delta\mathbf{F}^U$$
$$-(\mathbf{F}^U - \mathbf{F}^L) = -\Delta\mathbf{F}^U - \Delta\mathbf{F}^L, \quad \Delta\mathbf{F}^U \ge 0, \Delta\mathbf{F}^L \ge 0. \quad (66)$$

Thus, we consider the cone $Y_o \subset I\!R^{m+n}$ defined by

$$Y_o := \left\{\begin{pmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{I} & \mathbf{I}\end{pmatrix}\begin{pmatrix}\Delta\mathbf{F}^U \\ \Delta\mathbf{F}^L\end{pmatrix} : \Delta\mathbf{F}^U \ge 0, \Delta\mathbf{F}^L \ge 0\right\} =$$
$$\left\{\sum_{k=1}^{2n} \alpha_k \mathbf{y}_k : \alpha_k \ge 0, k = 1,\ldots,2n\right\}, \quad (67)$$

where the *cone-generators* $\mathbf{y}^{(k)}$, $k = 1,\ldots,2n$, are given by

$$\mathbf{y}^{(k)} := \begin{pmatrix}\mathbf{c}_k \\ \mathbf{e}_k\end{pmatrix}, 1 \le k \le n, \mathbf{y}^{(k)} := \begin{pmatrix}\mathbf{0} \\ \mathbf{e}_k\end{pmatrix},$$
$$n < k \le 2n, \quad (68)$$

and $\mathbf{c}_k$, $\mathbf{e}_k$ denotes the k-th column of $\mathbf{C}$, of the $n \times n$ identity matrix $\mathbf{I}$, respectively. Having Y_o, the set $S\left[\mathbf{F}^L(.), \mathbf{F}^U(.), \mathbf{R}(.)\right]$ defined in (17) can be described by

$$S\left[\mathbf{F}^L(.), \mathbf{F}^U(.), \mathbf{R}(.)\right] =$$
$$\left\{\omega \in \Omega : \begin{pmatrix}\mathbf{R}(\omega) - \mathbf{C}\mathbf{F}^U(\omega,\mathbf{X}) \\ -\mathbf{F}^U(\omega,\mathbf{X}) + \mathbf{F}^L(\omega,\mathbf{X})\end{pmatrix} \in (-1)Y_o\right\}, \quad (69)$$

which shows again the measurability of $S\left[\mathbf{F}^L(.), \mathbf{F}^U(.), \mathbf{R}(.)\right]$.

Moreover, the probability function $P = P(\lambda, \mathbf{X})$ defined by (23) can be represented by

$$P(\lambda,\mathbf{X}) = P\left[\begin{pmatrix}\mathbf{C}\mathbf{F}^U(\omega,\mathbf{X}) & - & \lambda\mathbf{R}_o(\omega) \\ \mathbf{F}^U(\omega,\mathbf{X}) & - & \mathbf{F}^L(\omega,\mathbf{X})\end{pmatrix} \in Y_o\right]. \quad (70)$$

Remark 7.1

The cone Y_o can be interpreted as the cone containing all parts $(\mathbf{C}\Delta\mathbf{F}^U, \Delta\mathbf{F}^L + \Delta\mathbf{F}^U)$ of *admissble* external load/strength increments. A big advantage of this conical representation of p_s is that the cone Y_o is given *explicitly* by the cone-generators $\mathbf{y}^{(k)}$, $k = 1,\ldots,2n$ whereas in the representation of p_s by means of the limit state functions $M_\ell = M_\ell(\omega,\mathbf{X})$, $\ell = 1\ldots,\ell_o$, the extreme points $\delta^{(\ell)}$, $\ell = 1,\ldots,\ell_o$, of the convex polyhedron Δ_o are not known explicitly, but must be computed in general by a very expensive enumeration procedure.

According to the reperesentation (67) of Y_o there are a finite number of boundary hyperplanes in $I\!R^{m+n}$, represented by vectors $\eta^{(\ell)} = \binom{\mathbf{w}^{(\ell)}}{\mathbf{v}^{(\ell)}}$, $\ell = 1,\ldots,\ell'_o$, such that

$$Y_o = \left\{\mathbf{y} = \begin{pmatrix}\mathbf{w} \\ \mathbf{v}\end{pmatrix} \in I\!R^{m+n} : \mathbf{y}'\eta^{(\ell)} = \mathbf{w}'\mathbf{w}^{(\ell)} + \mathbf{v}'\mathbf{v}^{(\ell)} \ge\right.$$
$$\left.0, 1 \le \ell \le \ell'_o\right\}. \quad (71)$$

Hence, because of (69) and (71), the survival of the structure can be represented also by the inequalities

$$\left[\mathbf{R}(\omega) - \mathbf{C}\mathbf{F}^U(\omega,\mathbf{X})\right]'\mathbf{w}^{(\ell)} + \left[-\mathbf{F}^U(\omega,\mathbf{X}) +\right.$$
$$\left.\mathbf{F}^L(\omega,\mathbf{X})\right]'\mathbf{v}^{(\ell)} \le 0 \quad 1 \le \ell \le \ell_o',$$

which yields

$$\mathbf{R}(\omega)'\mathbf{w}^{(\ell)} - \mathbf{F}^U(\omega,\mathbf{X})'\left[\mathbf{C}'\mathbf{w}^{(\ell)} + \mathbf{v}^{(\ell)}\right] +$$
$$\mathbf{F}^L(\omega,\mathbf{X})'\mathbf{v}^{(\ell)} \le 0, 1 \le \ell \le'_o, \quad (72)$$

and therefore

$$p_s = P\left[\mathbf{R}(\omega)'\mathbf{w}^{(\ell)} - \mathbf{F}^U(\omega,\mathbf{X})'[\mathbf{C}'\mathbf{w}^{(\ell)} + \mathbf{v}^{(\ell)}] +\right.$$
$$\left.\mathbf{F}^L(\omega,\mathbf{X})'\mathbf{v}^{(\ell)} \le 0, 1 \le \ell \le \ell'_o\right]. \quad (73)$$

Obviously, the conditions for structural safety given by (55) and (72) coincide.

For an arbitrary subset $Y_o^{(\ell)}$, $\ell = 1, 2$, such that

$$Y_o^{(1)} \subset Y_o \subset Y_o^{(2)}, \quad (74)$$

we have

$$p_s^{(1)} \le p_s \le p_s^{(2)}, \quad (75)$$

where the bounds $p_s^{(\ell)}$, $\ell = 1, 2$, are defined by

$$p_s^{(\ell)} := P\left[\begin{pmatrix}\mathbf{C}\mathbf{F}^U(\omega,\mathbf{X}) & - & \mathbf{R}(\omega) \\ \mathbf{F}^U(\omega,\mathbf{X}) & - & \mathbf{F}^L(\omega,\mathbf{X})\end{pmatrix} \in Y_o^{(\ell)}\right],$$
$$\ell = 1, 2. \quad (76)$$

7.1 Construction of approximating cones

Suppose next to that we have a cone axis (centre or middle line)

$$g = \{\lambda\bar{\mathbf{y}} : \lambda \ge 0\} \quad (77)$$

generated by a vector $\bar{\mathbf{y}} \in Y_o$, $\bar{y} \ne 0$, which is defined later.

Let E_o denote the hyperplane

$$(\mathbf{y} - \overline{\mathbf{y}})'\overline{\mathbf{y}} = 0 \Leftrightarrow \mathbf{y}'\overline{\mathbf{y}} = \|\overline{\mathbf{y}}\|^2, \quad (78)$$

through $\overline{\mathbf{y}}$ and orthogonal to axis g. Consider then the points $\tilde{\mathbf{y}}^{(\ell)} \in Y_o$, $\ell = 1,\ldots,2n$, lying on E_o and on the straight lines through $\mathbf{0}$ and $\mathbf{y}^{(\ell)}$, $\ell = 1,\ldots,2n$, hence,

$$\tilde{\mathbf{y}}^{(\ell)} := \frac{\|\overline{\mathbf{y}}\|^2}{y^{(\ell)'}\overline{\mathbf{y}}} y^{(\ell)}, \quad \ell = 1,\ldots,2n, \quad (79)$$

see Fig. 1. Obviously, we have

$$\mathbf{y}^{(\ell)'}\overline{\mathbf{y}} > 0, \quad \ell = 1,\ldots,2n, \quad (80)$$

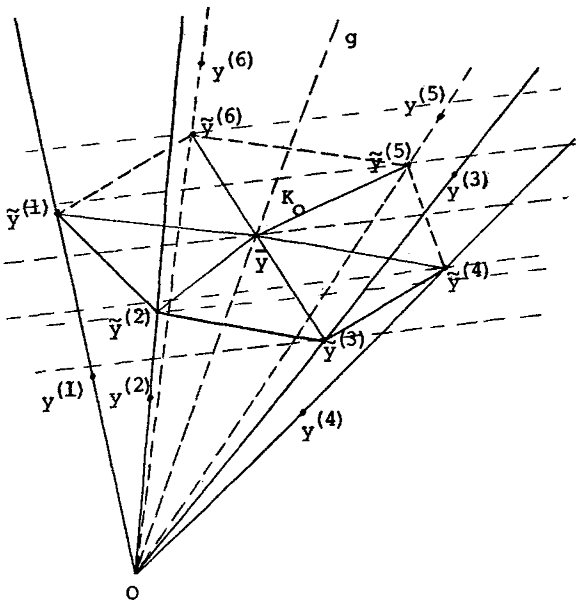

Fig. 1. Cone Y_o with cone-generators $y^{(\ell)}$, $\ell = 1, \ldots, 6$, cone-axis g, hyperplane E_o and convex polyhedron $K_o = \operatorname{conv}\{\tilde{\mathbf{y}}^{(1)}, \ldots, \tilde{\mathbf{y}}^{(6)}\}$

as a condition for $\overline{\mathbf{y}}$. Since the equation $\sum_{k=1}^{2n} \alpha_k \mathbf{y}^{(k)} = 0$, $\alpha_k \geq 0$, $k = 1, \ldots, 2n$, cf. (67) and (68), has no nonzero solution α, according to Gordan's transposition theorem, system (80) has solutions $\overline{\mathbf{y}} \neq 0$. Let

$$K_o := \operatorname{conv}\{\tilde{\mathbf{y}}^{(1)}, \tilde{\mathbf{y}}^{(2)}, \ldots, \tilde{\mathbf{y}}^{(2n)}\}, \tag{81}$$

denote the convex polyherdron on E_o generated by the points $\tilde{\mathbf{y}}^{(\ell)}$, $\ell = 1, \ldots, 2n$. By (80), (81) we have

$$\mathbf{y}'\overline{\mathbf{y}} > 0, \forall \mathbf{y} \in K_o. \tag{82}$$

According to (67), (79)-(81), the cone Y_o can be represented by

$$Y_o = \bigcup_{\lambda \geq 0} \lambda K_o = \{\lambda \mathbf{y} : \lambda \geq 0, \mathbf{y} \in K_o\}. \tag{83}$$

Based on the above representation of Y_o, approximations $\tilde{Y}_o$ of Y_o of the type (74) can be obtained by replacing the generating polyhedron K_o in Y_o by suitable approximations, e.g. ellipsoids, spheres, denoted by $\tilde{K}_o$.

(a) *Ellipsoidal approximations.* According to (82), an outer ellipsoidal approximation $K_o^{(\text{ell},2)} \supset K_o$ can be defined, for $j = 2$, by

$$K_o^{(\text{ell},j)} := \{\mathbf{y} \in E_o : \mathbf{y}'\overline{\mathbf{y}} > 0,$$

$$(\mathbf{y} - \overline{\mathbf{y}})'\mathbf{\Gamma}^{(j)}\mathbf{\Gamma}^{(j)'}(\mathbf{y} - \overline{\mathbf{y}}) \leq 1\Big\} \tag{84}$$

where the $(n+m) \times (n+m)$ matrix $\mathbf{\Gamma}^{(2)} = \mathbf{\Gamma}$ is chosen such that

$$(\tilde{\mathbf{y}}^{(\ell)} - \overline{\mathbf{y}})'\mathbf{\Gamma}\mathbf{\Gamma}'(\tilde{\mathbf{y}}^{(\ell)} - \overline{\mathbf{y}}) \leq 1, \quad \ell = 1, \ldots, 2n$$

$$\mathbf{y}'\overline{\mathbf{y}} = 0 \Longrightarrow (\mathbf{\Gamma}y)'\overline{\mathbf{y}} = 0,$$

$$\sum_{\ell=1}^{2n} \left[1 - (\tilde{\mathbf{y}}^{(\ell)} - \overline{\mathbf{y}})'\mathbf{\Gamma}\mathbf{\Gamma}'(\tilde{\mathbf{y}}^{(\ell)} - \overline{\mathbf{y}})\right] \longrightarrow \min. \tag{85}$$

Conversely, an "inner" ellipsoidal approximation $K_o^{(\text{ell},1)}$ of K_o can be determined by choosing a matrix $\Gamma^{(1)} = \Gamma$ such that

$$(\tilde{\mathbf{y}}^{(\ell)} - \overline{\mathbf{y}})'\mathbf{\Gamma}\mathbf{\Gamma}'(\tilde{\mathbf{y}}^{(\ell)} - \overline{\mathbf{y}}) \geq 1, \quad \ell = 1, \ldots, 2n$$

$$\mathbf{y}'\overline{\mathbf{y}} = 0 \Longrightarrow (\mathbf{\Gamma}\mathbf{y})'\overline{\mathbf{y}} = 0,$$

$$\sum_{\ell=1}^{2n} (\tilde{\mathbf{y}}^{(\ell)} - \overline{\mathbf{y}})'\mathbf{\Gamma}\mathbf{\Gamma}'(\tilde{\mathbf{y}}^{(\ell)} - \overline{\mathbf{y}}) - 1 \longrightarrow \min. \tag{86}$$

Note that $K_o^{(\text{ell},1)} \setminus K_o \neq \emptyset$ may happen.

(b) *Spherical approximations:* An outer spherical approximation $K_o^{(\text{sph},2)} \supset K_o$ is defined by

$$K_o^{(\text{sph},j)} := \left\{\mathbf{y} \in E_o : \mathbf{y}'\overline{\mathbf{y}} > 0, \|\mathbf{y} - \overline{\mathbf{y}}\| < \rho_j\right\}, \tag{87}$$

$j = 2$, where the radius ρ_2 is given by

$$\rho_2 := \max_{1 \leq \ell \leq 2n} \|\tilde{\mathbf{y}}^{(\ell)} - \overline{\mathbf{y}}\| \tag{88}$$

and $\|.\|$ denotes the Euclidean norm. Likewise, an "inner" spherical approximation $K_o^{(\text{sph},1)}$ of K_o can be determined by choosing the radius

$$\rho_1 := \min_{1 \leq \ell \leq 2n} \|\tilde{\mathbf{y}}^{(\ell)} - \overline{\mathbf{y}}\|, \tag{89}$$

where, as for $K_o^{(\text{ell},1)}$, the relation $K_o^{(\text{sph},1)} \setminus K_o \neq \emptyset$ is not excluded in general. Of course, an inner spherical approximation $K_o^{(\text{sph},1)} \subset K_o$ is obtained by selecting the radius

$$\rho_1' := \min\{\|\mathbf{y} - \overline{\mathbf{y}}\| : \mathbf{y} \in \partial K_o\}, \tag{90}$$

where ∂K_o denotes the boundary of K_o.

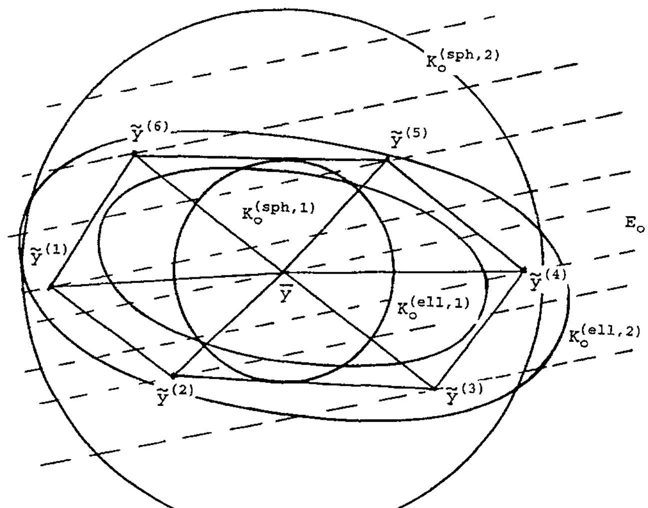

Fig. 2. Convex polyhedron K_o in hyperplane E_o and (second-order) approximations $\tilde{K}_o = K_o^{(\text{ell},j)}$, $\tilde{K}_o = K_o^{(\text{sph},j)}$, $j = 1, 2$

Having an approximation $\tilde{K}_o$ of K_o as described by (84), (87), the cone Y_o can be approximated now by the cone

$$\tilde{Y}_o := \left\{\lambda \mathbf{y} : \lambda \geq 0, \quad \mathbf{y} \in \tilde{K}_o\right\}. \tag{91}$$

For any point $\mathbf{y} \in \mathbb{R}^{m+n}$, $\mathbf{y} \neq 0$, the intersection $\mathbf{y}_o$ of the hyperplane E_o and the straight line through $\mathbf{0}$ and $\mathbf{y}$ is given by

$$\mathbf{y}_o := \frac{||\overline{\mathbf{y}}||^2}{\mathbf{y}'\overline{\mathbf{y}}}\mathbf{y}\,. \tag{92}$$

Hence, according to the definition (91) of $\tilde{Y}_o$, in the case of (84) and (87) we have that $\mathbf{y} \in \tilde{Y}_o$, $\mathbf{y} \neq 0$, if and only if the following simple relations for $\mathbf{y}$ hold:

$$\mathbf{y}'\overline{\mathbf{y}} > 0\,, \tag{93a}$$

$$||\mathbf{\Gamma}'(||\overline{\mathbf{y}}||^2\mathbf{y} - \mathbf{y}'\mathbf{y}\overline{\mathbf{y}})|| \leq \mathbf{y}'\overline{\mathbf{y}}\,, \tag{93b}$$

$$||\mathbf{y}|| \leq \frac{(\rho^2 + ||\overline{\mathbf{y}}||^2)^{1/2}}{||\overline{\mathbf{y}}||^2}\mathbf{y}'\overline{\mathbf{y}}\,,\ \text{respectively,} \tag{93c}$$

where $\mathbf{\Gamma} = \mathbf{\Gamma}^{(1)}, \mathbf{\Gamma}^{(2)}$ and $\rho = \rho_1, \rho_2$. Obviously, (93a)-(93c) are convex conditions for y.

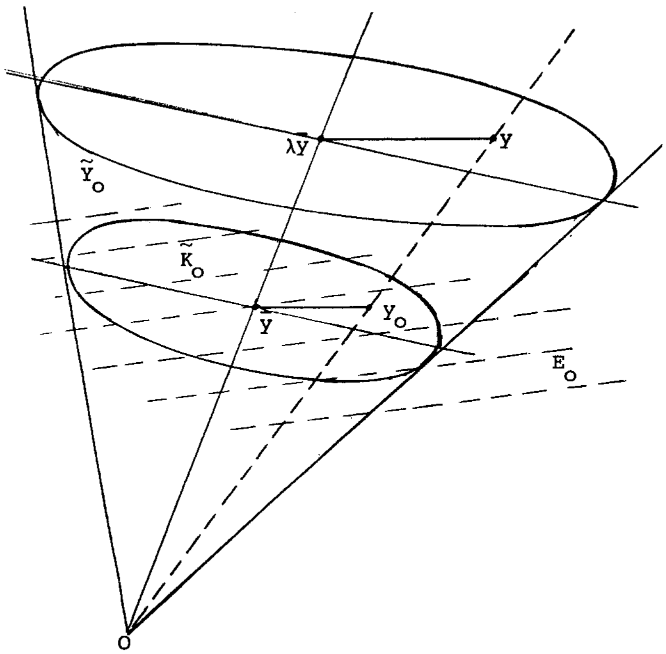

Fig. 3. Approximating cone $\tilde{Y}_o$ generated by $\tilde{K}_o$

Having an approximation $\tilde{Y}_o$ of Y_o, the probability function $P = P(\lambda, \mathbf{X})$ given by (70) can be appoximated then, cf. (76), by

$$\tilde{P}(\lambda, \mathbf{X}) := P\left[\begin{pmatrix} \mathbf{C}\mathbf{F}^{\mathrm{U}}(\omega, \mathbf{X}) & - & \lambda R_o(\omega) \\ \mathbf{F}^{\mathrm{U}}(\omega, \mathbf{X}) & - & \mathbf{F}^{\mathrm{L}}(\omega, \mathbf{X}) \end{pmatrix} \in \tilde{Y}_o\right]. \tag{94}$$

Hence, in the above case from (93) we obtain

$$\tilde{P}(\lambda, \mathbf{X}) = P\left\{||\mathbf{\Gamma}'\left[||\overline{\mathbf{y}}||^2\mathbf{y}(\omega) - \overline{\mathbf{y}}'y(\omega)\overline{\mathbf{y}}\right]|| \leq \mathbf{y}(\omega)'\overline{\mathbf{y}}\right\}, \tag{95}$$

$$\tilde{P}(\lambda, \mathbf{X}) = P\left[||\mathbf{y}(\omega)|| \leq \frac{(\rho^2 + ||\overline{\mathbf{y}}||^2)^{1/2}}{||\overline{\mathbf{y}}||^2}\mathbf{y}(\omega)'\overline{\mathbf{y}}\right], \tag{96}$$

where

$$\mathbf{y}(\omega) := \begin{pmatrix} \mathbf{C}\mathbf{F}^{\mathrm{U}}(\omega, \mathbf{X}) & - & \lambda \mathbf{R}_o(\omega) \\ \mathbf{F}^{\mathrm{U}}(\omega, \mathbf{X}) & - & \mathbf{F}^{\mathrm{L}}(\omega, \mathbf{X}) \end{pmatrix}. \tag{97}$$

Finally, we must determine the axis $g = \{\lambda\overline{\mathbf{y}} : \lambda \geq 0\}$ by selecting a generating vector $\overline{\mathbf{y}} \in Y_o$, $\overline{\mathbf{y}} \neq \mathbf{0}$. Having $\overline{\mathbf{y}} = \sum_{k=1}^{2n} \alpha_k \mathbf{y}^{(k)}$ with $\alpha_k \geq 0$, $k = 1, \ldots, 2n$, condition (80) can be fulfilled by choosing the coefficients $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{2n})'$ such that

$$\mathbf{Y}_o^M \boldsymbol{\alpha} > 0\,, \tag{98}$$

where the $2n \times 2n$ matrix $\mathbf{Y}_o^M$ is given by

$$\mathbf{Y}_o^M := (\mathbf{y}^{(\ell)'}\mathbf{y}^{(k)})_{\ell,k=1,\ldots,2n} =$$

$$\left(\begin{array}{cccc|c} ||\mathbf{c}_1||^2+1 & \mathbf{c}_1'\mathbf{c}_2 & \ldots & \mathbf{c}_1'\mathbf{c}_n & \\ \mathbf{c}_2'\mathbf{c}_1 & ||\mathbf{c}_2||^2+1 & \ldots & \mathbf{c}_2'\mathbf{c}_n & \\ \vdots & \vdots & & \vdots & \mathbf{I}_n \\ \mathbf{c}_n'\mathbf{c}_1 & \mathbf{c}_n'\mathbf{c}_2 & \ldots & ||\mathbf{c}_n||^2+1 & \\ \hdashline & & \mathbf{I}_n & & \mathbf{I}_n \end{array}\right) \tag{99}$$

and $\mathbf{I}_n$ denotes the $n \times n$ identity matrix. Thus, we may select then $\boldsymbol{\alpha}$ such that $\min_{1\leq\ell\leq 2n} \mathbf{y}^{(\ell)'}\overline{\mathbf{y}}$ is maximized, which yields the linear program

$$\max t \tag{100a}$$

s.t.

$$\left(\begin{array}{c|c} -\mathbf{Y}_o^M & 1 \\ \hdashline \mathbf{I}_{2n} & 0 \end{array}\right) \cdot \begin{pmatrix} \boldsymbol{\alpha} \\ t \end{pmatrix} \leq \begin{pmatrix} 0 \\ \mathbf{a}_o \end{pmatrix}, \tag{100b}$$

$$\boldsymbol{\alpha} \geq 0\,, \quad t \geq 0\,, \tag{100c}$$

where $\mathbf{a}_o = (a_{o1}, a_{o2}, \ldots, a_{o2n})'$ is a given $2n$-vector having positive components, and the constraints $\boldsymbol{\alpha} \leq \mathbf{a}_o$ is imposed because the direction of $\overline{\mathbf{y}}$ is needed only.

On the other hand, for any vector $\overline{\mathbf{y}}, \overline{\mathbf{y}} \neq \mathbf{0}$, we consider the hyperplane E_o defined by (78) and the points $\tilde{\mathbf{y}}^{(\ell)}, \ell = 1, \ldots, 2n$, on E_o defined by (79).

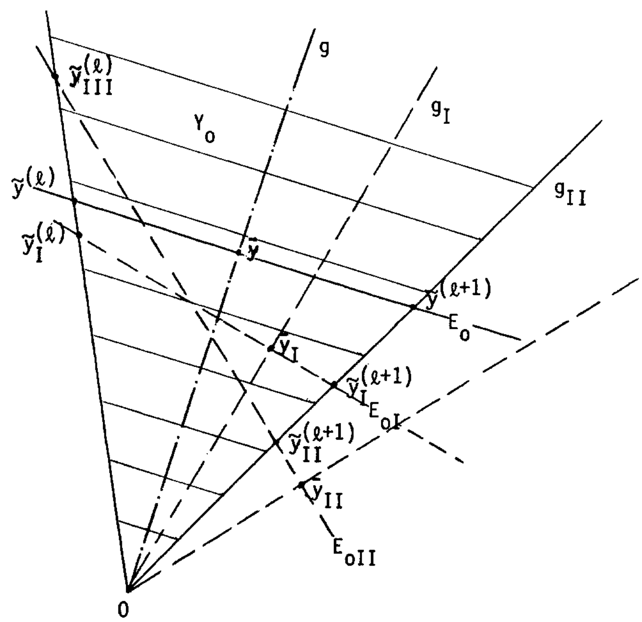

Fig. 4. Construction of a cone axis g of Y_o : Approximative generators $\overline{\mathbf{y}}_I, \overline{\mathbf{y}}_{II}$

Since the axis $g = \{\lambda\overline{\mathbf{y}} : \lambda \geq 0\}$ should pass through the centre of the cone Y_o, the devistions between $\overline{\mathbf{y}}$ and the points $\tilde{\mathbf{y}}^{(\ell)}$, $\ell = 1, \ldots, 2n$, should be well-balanced. Thus, $\overline{\mathbf{y}}$ is chosen such that the quantity

$$\max_{1\le\ell\le 2n} \|\tilde{\mathbf{y}}-\tilde{\mathbf{y}}^{(\ell)}\| = \max_{1\le\ell\le 2n} \left(\frac{\|\overline{\mathbf{y}}\|^4}{(\mathbf{y}^{(\ell)\prime}\overline{\mathbf{y}})^2}\|\mathbf{y}^{(\ell)}\|^2 - \|\overline{\mathbf{y}}\|^2 \right)^{1/2}, \tag{101}$$

cf. (79), is minimized, which yields, see (80), the optimization problem

$$\min \max_{1\le\ell\le 2n} \|\overline{\mathbf{y}} - \tilde{\mathbf{y}}^{(\ell)}\| \tag{102a}$$

$$\text{s.t. } \mathbf{y}^{(\ell)\prime}\overline{\mathbf{y}} > 0\,, \quad \ell = 1,\ldots,2n\,, \tag{102b}$$

$$\overline{\mathbf{y}} \in Y_o\,, \|\overline{\mathbf{y}}\| = 1\,, \tag{102c}$$

since only the direction of $\overline{\mathbf{y}}$ is relevant for our purposes, the norm constraint $\|\overline{\mathbf{y}}\| = 1$ is added. According to (101), problem (102) is equivalent to the program

$$\max t \tag{103a}$$

$$\text{s.t. } t - \frac{\mathbf{y}^{(\ell)\prime}\overline{\mathbf{y}}}{\|\mathbf{y}^{(\ell)}\|} \le 0\,, \quad \ell = 1,\ldots,2n\,, \tag{103b}$$

$$t \ge 0\,, \overline{\mathbf{y}} \in Y_o\,, \|\overline{\mathbf{y}}\| = 1\,. \tag{103c}$$

Note that

$$\frac{\mathbf{y}^{(\ell)\prime}\overline{\mathbf{y}}}{\|\mathbf{y}^{(\ell)}\|} = \cos \measuredangle\,(\mathbf{y}^{(\ell)}, \overline{\mathbf{y}})\,,$$

is the cosine of the angle between the vectors $\mathbf{y}^{(\ell)}$ and $\overline{\mathbf{y}}$.

Representing $\overline{\mathbf{y}} \in Y_o$ by $\overline{\mathbf{y}} = \sum_{k=1}^{2n} \alpha_k \mathbf{y}^{(k)}$, $\alpha_k \ge 0$, $k = 1,\ldots,2n$, we observe that (103) is closely related to (100).

8 Sensitivity analysis of probabilities of survival/failure

The parameter sensitivity factors in structural reliability have practical applications within reliability-based design, in optimization of structural design, construction maintenance and inspection under reliability constraints, in parameter studies of the reliability and in reliability updating (see Bjerager and Krenk 1987). Known methods for the approximative computation of sensitivities, i.e. the derivatives of p_s, p_f with respect to certain deterministic input or design variables are based mainly on the computation of the reliability or safety index β by means of first- or second-order reliability methods (FORM, SORM), (see Bjerager 1990; Bjerager and Krenk 1989; Frangopol 1985; Rackwitz and Cuntze 1987). By using the *Transformation method* in combination with the *Stochastic completion technique* developed by (Marti 1990, 1994, 1995a,b,c, 1996), in the following we derive simple representations of the derivatives of $p_s = P(\lambda, \mathbf{X})$, $\tilde{p}_s = \tilde{P}(\lambda, \mathbf{X}, \mathbf{z})$ - of arbitrary order - by means of certain expectations or multiple integrals (see also Breitung 1991). This enables then the direct computation of the sensitivities - of arbitrary order - by means of sampling techniques or asymptotic expansion techniques bassed on Laplace integral representation of multiple integrals (cf. Marti 1996).

8.1 The probability functions (24a)-(24c)

In order to show the differentiation of the probability functions given by (24a)-(24c) it is sufficient to consider probability functions of the type

$$P(\lambda, \mathbf{X}) := P\left\{\lambda\mathbf{R}_o(\omega) \in \left[\mathbf{A}(\mathbf{X})_d\boldsymbol{\sigma}^{\mathrm{L}}(\omega), \mathbf{A}(\mathbf{X})\boldsymbol{\sigma}^{\mathrm{U}}(\omega)\right]\right\}, \tag{104}$$

where we suppose that the random vectors $\mathbf{R}_o(\omega)$, $\left[\boldsymbol{\sigma}^{\mathrm{L}}(\omega), \boldsymbol{\sigma}^{\mathrm{U}}(\omega)\right]$ have sufficiently smooth probability densities $\varphi = \varphi(\mathbf{R}_o)$, $\psi = \psi(\boldsymbol{\sigma}^{\mathrm{L}}, \boldsymbol{\sigma}^{\mathrm{U}})$ on $I\!R^m, I\!R^{2n}$, respectively. Hence, using also (69), we have that

$$P(\lambda, \mathbf{X}) = P(\lambda, \mathbf{A})\,, \mathbf{A} = \mathbf{A}(\mathbf{X}) = [A_1(\mathbf{X}), \ldots, A_n(\mathbf{X})]'\,, \tag{105}$$

cf. (9), where

$$P(\lambda, \mathbf{A}) := \int\limits_{\lambda\mathbf{R}_o \in \mathbf{C}[\mathbf{A}_d\boldsymbol{\sigma}^{\mathrm{L}}, \mathbf{A}_d\boldsymbol{\sigma}^{\mathrm{U}}]} \varphi(\mathbf{R}_o)\psi(\boldsymbol{\sigma}^{\mathrm{L}}, \boldsymbol{\sigma}^{\mathrm{U}})\mathrm{d}\mathbf{R}_o\mathrm{d}\boldsymbol{\sigma}^{\mathrm{L}}\boldsymbol{\sigma}^{\mathrm{U}} = \int\limits_{\begin{pmatrix} \mathbf{C}\mathbf{A}_d\boldsymbol{\sigma}^{\mathrm{U}} - \lambda\mathbf{R}_o \\ \mathbf{A}_d\boldsymbol{\sigma}^{\mathrm{U}} - \mathbf{A}_d\boldsymbol{\sigma}^{\mathrm{L}} \end{pmatrix} \in Y_o} \varphi(\mathbf{R}_o)\psi(\boldsymbol{\sigma}^{\mathrm{L}}, \boldsymbol{\sigma}^{\mathrm{U}})\mathrm{d}\mathbf{R}_o\mathrm{d}\boldsymbol{\sigma}^{\mathrm{L}}\mathrm{d}\boldsymbol{\sigma}^{\mathrm{U}}\,. \tag{106}$$

Applying, for given variables $(\lambda, \mathbf{A})$ with $\lambda \ne 0, A_i > 0$, $i = 1,\ldots,n = n_o$ to (106) the transformation $T_{(\lambda,\mathbf{A})}$: $(\mathbf{R}, \mathbf{F}^{\mathrm{L}}, \mathbf{F}^{\mathrm{U}}) \longrightarrow (\mathbf{R}_o, \boldsymbol{\sigma}^{\mathrm{L}}, \boldsymbol{\sigma}^{\mathrm{U}})$ in $I\!R^m \times I\!R^{2n}$ defined by

$$\mathbf{R}_o := \frac{1}{\lambda}\mathbf{R},\ \boldsymbol{\sigma}^{\mathrm{L}} := \mathbf{A}_d^{-1}\mathbf{F}^{\mathrm{L}},\ \boldsymbol{\sigma}^{\mathrm{U}} := \mathbf{A}_d^{-1}\mathbf{F}^{\mathrm{U}}\,, \tag{107}$$

we obtain

$$P(\lambda, \mathbf{A}) = \int\limits_{\mathbf{R} \in \mathbf{C}[\mathbf{F}^{\mathrm{L}}, \mathbf{F}^{\mathrm{U}}]} \varphi\left(\frac{1}{\lambda}\mathbf{R}\right)\psi(\mathbf{A}_d^{-1}\mathbf{F}^{\mathrm{L}}, \mathbf{A}_d^{-1}\mathbf{F}^{\mathrm{U}}) \times \frac{1}{|\lambda|^m}\Pi_{j=1}^n\frac{1}{A_j^2}\mathrm{d}\mathbf{R}\mathrm{d}\mathbf{F}^{\mathrm{L}}\mathrm{d}\mathbf{F}^{\mathrm{U}}\,. \tag{108}$$

Under weak assumptions (Marti 1995a, 1996) we may interchange in (108) differentiation and integration, hence,

$$\frac{\partial P}{\partial\lambda}(\lambda, \mathbf{A}) = -\frac{1}{\lambda}\int\limits_{\mathbf{R} \in \mathbf{C}[\mathbf{F}^{\mathrm{L}}, \mathbf{F}^{\mathrm{U}}]} \left[\nabla\varphi\left(\frac{\mathbf{R}}{\lambda}\right)'\frac{\mathbf{R}}{\lambda} + m\varphi(\frac{\mathbf{R}}{\lambda})\right]\psi(\mathbf{A}_d^{-1}\mathbf{F}^{\mathrm{L}}, \mathbf{A}_d^{-1}\mathbf{F}^{\mathrm{U}}) \times \frac{1}{|\lambda|^m}\Pi_{j=1}^n\frac{1}{A_j^2}\mathrm{d}\mathbf{R}\mathrm{d}\mathbf{F}^{\mathrm{L}}\mathrm{d}\mathbf{F}^{\mathrm{U}}\,. \tag{109}$$

Using the inverse transformation $T_{(\lambda,\mathbf{A})}^{-1}$, (109) yields

$$\frac{\partial P}{\partial\lambda}(\lambda, \mathbf{A}) = -\frac{1}{\lambda}\int\limits_{\lambda\mathbf{R}_o \in \mathbf{C}[\mathbf{A}_d\boldsymbol{\sigma}^{\mathrm{L}}, \mathbf{A}_d\boldsymbol{\sigma}^{\mathrm{U}}]} \operatorname{div}\left[\mathbf{R}_o\varphi(\mathbf{R}_o)\right]\psi(\boldsymbol{\sigma}^{\mathrm{L}}, \boldsymbol{\sigma}^{\mathrm{U}})\mathrm{d}\mathbf{R}_o\mathrm{d}\boldsymbol{\sigma}^{\mathrm{L}}\mathrm{d}\boldsymbol{\sigma}^{\mathrm{U}}$$

$$= -\frac{1}{\lambda}\int \operatorname{div}\left[\mathbf{R}_o\varphi(\mathbf{R}_o)\right] P(\lambda, \mathbf{A}|\mathbf{R}_o)\mathrm{d}\mathbf{R}_o\,, \tag{110}$$

with the conditional probability function

$$P(\lambda, \mathbf{A}|\mathbf{R}_o) := P\left\{\lambda \mathbf{R}_o \in \mathbf{C}\left[\mathbf{A}_d\boldsymbol{\sigma}^{\mathrm{L}}(\omega), \mathbf{A}_d\boldsymbol{\sigma}^{\mathrm{U}}(\omega)\right]\right\},$$

$$\mathbf{R}_o \in \mathbb{R}^m. \tag{111}$$

Considering now the derivative $\frac{\partial P}{\partial X_k}(\lambda, \mathbf{X})$ with respect to a design variable X_k, $1 \leq k \leq r$, by means of (105) we find

$$\frac{\partial P}{\partial X_k}(\lambda, \mathbf{X}) = \sum_{j=1}^{n} \frac{\partial P}{\partial A_j}(\lambda, \mathbf{A})\frac{\partial A_j}{\partial X_k}(\mathbf{X}), \tag{112}$$

and using again (108), by interchanging differentiation and integration there we have

$$\frac{\partial P}{\partial A_j}(\lambda, \mathbf{A}) =$$

$$-\frac{1}{A_j} \int\limits_{\mathbf{R}\in\mathbf{C}[\mathbf{F}^{\mathrm{L}},\mathbf{F}^{\mathrm{U}}]} \varphi(\frac{\mathbf{R}}{\lambda}) \left[2\psi(\mathbf{A}_d^{-1}\mathbf{F}^{\mathrm{L}}, \mathbf{A}_d^{-1}\mathbf{F}^{\mathrm{U}}) + \right.$$

$$\frac{\partial \psi}{\partial \sigma_j^{\mathrm{L}}}(\mathbf{A}_d^{-1}\mathbf{F}^{\mathrm{L}}, \mathbf{A}_d^{-1}\mathbf{F}^{\mathrm{U}})\frac{F_j^{\mathrm{L}}}{A_j} +$$

$$\left.\frac{\partial \psi}{\partial \sigma_j^{\mathrm{U}}}(\mathbf{A}_d^{-1}\mathbf{F}^{\mathrm{L}}, \mathbf{A}_d^{-1}\mathbf{F}^{\mathrm{U}})\frac{F_j^{\mathrm{U}}}{A_j}\right] \times$$

$$\frac{1}{|\lambda|^m} \Pi_{j=1}^{n} \frac{1}{A_j^2} \mathrm{d}\mathbf{R}\mathrm{d}\mathbf{F}^{\mathrm{L}}\mathrm{d}\mathbf{F}^{\mathrm{U}}. \tag{113}$$

Using the inverse $T_{(\lambda,\mathbf{A})}^{-1}$ of (107) again, from (113) we have

$$\frac{\partial P}{\partial A_j}(\lambda, \mathbf{A}) =$$

$$-\frac{1}{Aj} \int\limits_{\lambda\mathbf{R}_o\in\mathbf{C}[\mathbf{A}_d\boldsymbol{\sigma}^{\mathrm{L}},\mathbf{A}_d\boldsymbol{\sigma}^{\mathrm{U}}]} \varphi(\mathbf{R}_o) \left[2\psi(\boldsymbol{\sigma}^{\mathrm{L}}, \boldsymbol{\sigma}^{\mathrm{U}}) + \right.$$

$$\left.\frac{\partial \psi}{\partial \sigma_j^{\mathrm{L}}}(\boldsymbol{\sigma}^{\mathrm{L}}, \boldsymbol{\sigma}^{\mathrm{U}})\sigma_j^{\mathrm{L}} + \frac{\partial \psi}{\partial \sigma_j^{\mathrm{U}}}(\boldsymbol{\sigma}^{\mathrm{L}}, \boldsymbol{\sigma}^{\mathrm{U}})\sigma_j^{\mathrm{U}}\right] \mathrm{d}\mathbf{R}_o\mathrm{d}\boldsymbol{\sigma}^{\mathrm{L}}\mathrm{d}\boldsymbol{\sigma}^{\mathrm{U}} =$$

$$-\frac{1}{A_j} \int \mathrm{div}\left[(\boldsymbol{\sigma}^{\mathrm{L}}, \boldsymbol{\sigma}^{\mathrm{U}})^{(j)}\psi(\boldsymbol{\sigma}^{\mathrm{L}}, \boldsymbol{\sigma}^{\mathrm{U}})\right] \times$$

$$P(\lambda, \mathbf{A}|\boldsymbol{\sigma}^{\mathrm{L}}, \boldsymbol{\sigma}^{\mathrm{U}})\mathrm{d}\boldsymbol{\sigma}^{\mathrm{L}}\mathrm{d}\boldsymbol{\sigma}^{\mathrm{U}}, \tag{114}$$

with the conditional probability function

$$P(\lambda, \mathbf{A}|\boldsymbol{\sigma}^{\mathrm{L}}, \boldsymbol{\sigma}^{\mathrm{U}}) := P\left\{\lambda \mathbf{R}_o(\omega) \in \mathbf{C}\left[\mathbf{A}_d\boldsymbol{\sigma}^{\mathrm{L}}, \mathbf{A}_d\boldsymbol{\sigma}^{\mathrm{U}}\right]\right\},$$

$$\boldsymbol{\sigma}^{\mathrm{L}}, \boldsymbol{\sigma}^{\mathrm{U}} \in \mathbb{R}^n, \tag{115}$$

and

$$(\boldsymbol{\sigma}^{\mathrm{L}}, \boldsymbol{\sigma}^{\mathrm{U}})^{(j)} := (0, \ldots, 0, \sigma_j^{\mathrm{L}}, 0, \ldots, 0, \sigma_j^{\mathrm{U}}, 0, \ldots, 0)', \tag{116}$$

where σ_j^{L}, σ_j^{U} are placed at the j-th, $(n+j)$-th position, respectively.

According to the representation (66) of the event $\left\{\mathbf{R} \in \mathbf{C}\left[\mathbf{F}^{\mathrm{L}}, \mathbf{F}^{\mathrm{U}}\right]\right\}$, the conditional probability functions $P(\lambda, \mathbf{A}|\mathbf{R}_o)$, $P(\lambda, \mathbf{A}|\boldsymbol{\sigma}^{\mathrm{L}}, \boldsymbol{\sigma}^{\mathrm{U}})$ can be represented, cf. (66)-(71) and (106), by

$$P(\lambda, \mathbf{A}|\mathbf{R}_o) = P\left[\mathbf{y}(\omega|\mathbf{R}_o) \in Y_o\right],$$

$$P(\lambda, \mathbf{A}|\boldsymbol{\sigma}^{\mathrm{L}}, \boldsymbol{\sigma}^{\mathrm{U}}) = \mathbf{P}\left[\mathbf{y}(\omega|\boldsymbol{\sigma}^{\mathrm{L}}, \boldsymbol{\sigma}^{\mathrm{U}}) \in Y_o\right], \tag{117}$$

where

$$y(\omega|\mathbf{R}_o) := \begin{pmatrix} \mathbf{C}\mathbf{A}_d\boldsymbol{\sigma}^{\mathrm{U}}(\omega) & - & \lambda\mathbf{R}_o \\ \mathbf{A}_d\boldsymbol{\sigma}^{\mathrm{U}}(\omega) & - & \mathbf{A}_d\boldsymbol{\sigma}^{\mathrm{L}}(\omega) \end{pmatrix},$$

$$\mathbf{y}(\omega|\boldsymbol{\sigma}^{\mathrm{L}}, \boldsymbol{\sigma}^{\mathrm{U}}) := \begin{pmatrix} \mathbf{C}\mathbf{A}_d\boldsymbol{\sigma}^{\mathrm{U}} & - & \lambda\mathbf{R}_o(\omega) \\ \mathbf{A}_d\boldsymbol{\sigma}^{\mathrm{U}} & - & \mathbf{A}_d\boldsymbol{\sigma}^{\mathrm{L}} \end{pmatrix}. \tag{118}$$

Hence, corresponding to the approximation $\tilde{P}(\lambda, \mathbf{A})$ of $P(\lambda, \mathbf{A})$, cf. (94)-(96), the above conditional probability functions (117) can be approximated by

$$\tilde{P}(\lambda, \mathbf{A}|\mathbf{R}_o) := P\left[\mathbf{y}(\omega|\mathbf{R}_o) \in \tilde{Y}_o\right],$$

$$\tilde{P}(\lambda, A|\boldsymbol{\sigma}^{\mathrm{L}}, \boldsymbol{\sigma}^{\mathrm{U}}) := P\left[y(\omega|\boldsymbol{\sigma}^{\mathrm{L}}, \boldsymbol{\sigma}^{\mathrm{U}}) \in \tilde{Y}_o\right], \tag{119}$$

where $\tilde{Y}_o$ is the approximation of the convex cone as described in Section 7.

By a slight modification in (110) and (115), the derivatives of $P = P(\lambda, \mathbf{A})$ can also be represented by means of *expectations*.

Theorem 8.1. Under the assumptions in Section 8 we have

$$\frac{\partial P}{\partial \lambda}(\lambda, \mathbf{A}) = \tag{120}$$

$$-\frac{1}{\lambda}E\frac{\mathrm{div}\left\{\mathbf{R}_o(\omega)\varphi\left[\mathbf{R}_o(\omega)\right]\right\}}{\varphi[\mathbf{R}_o(\omega)]}1_{\mathbf{C}[\mathbf{A}_d\boldsymbol{\sigma}^{\mathrm{L}}(\omega),\mathbf{A}_d\boldsymbol{\sigma}^{\mathrm{U}}(\omega)]} \times$$

$$[\lambda\mathbf{R}_o(\omega)]$$

$$= -\frac{1}{\lambda}E\frac{\mathrm{div}\left\{\mathbf{R}_o(\omega)\varphi\left[\mathbf{R}_o(\omega)\right]\right\}}{\varphi[\mathbf{R}_o(\omega)]}P\left[\lambda, \mathbf{A}|\mathbf{R}_o(\omega)\right],$$

$$\frac{\partial P}{\partial A_j}(\lambda, \mathbf{A}) =$$

$$-\frac{1}{A_j}E\frac{\mathrm{div}\left\{\left[\boldsymbol{\sigma}^{\mathrm{L}}(\omega), \boldsymbol{\sigma}^{\mathrm{U}}(\omega)\right]^{(j)}\psi\left[\boldsymbol{\sigma}^{\mathrm{L}}(\omega), \boldsymbol{\sigma}^{\mathrm{U}}(\omega)\right]\right\}}{\psi[\boldsymbol{\sigma}^{\mathrm{L}}(\omega), \boldsymbol{\sigma}^{\mathrm{U}}(\omega)]} \times$$

$$1_{\mathbf{C}}[\mathbf{A}_d\boldsymbol{\sigma}^{\mathrm{L}}(\omega), \mathbf{A}_d\boldsymbol{\sigma}^{\mathrm{U}}(\omega)][\lambda\mathbf{R}_o(\omega)] =$$

$$-\frac{1}{A_j}E\frac{\mathrm{div}\left\{\left[\boldsymbol{\sigma}^{\mathrm{L}}(\omega), \boldsymbol{\sigma}^{\mathrm{U}}(\omega)\right]^{(j)}\psi\left[\boldsymbol{\sigma}^{\mathrm{L}}(\omega), \boldsymbol{\sigma}^{\mathrm{U}}(\omega)\right]\right\}}{\psi[\boldsymbol{\sigma}^{\mathrm{L}}(\omega), \boldsymbol{\sigma}^{\mathrm{U}}(\omega)]} \times$$

$$P\left[\lambda, \mathbf{A}|\boldsymbol{\sigma}^{\mathrm{L}}(\omega), \boldsymbol{\sigma}^{\mathrm{U}}(\omega)\right]. \tag{121}$$

Remark 8.1. (a) Selecting fixed variables $(\overline{\lambda}, \overline{\mathbf{A}})$ such that $\overline{\lambda} \neq 0$, $\overline{A}_i > 0$, $i = 1, \ldots, n$, and applying then - instead of $T_{(\lambda,\mathbf{A})}^{-1}$ - the inverse transformation $T_{(\overline{\lambda},\overline{\mathbf{A}})}^{-1}$ to (109), we obtain

$$\frac{\partial P}{\partial \lambda}(\lambda, \mathbf{A}) = -\frac{1}{\lambda} \int\limits_{\overline{\lambda}\mathbf{R}_o\in\mathbf{C}[\overline{\mathbf{A}}_d\boldsymbol{\sigma}^{\mathrm{L}},\overline{\mathbf{A}}_d\boldsymbol{\sigma}^{\mathrm{U}}]} [\nabla\varphi\left(\frac{\overline{\lambda}}{\lambda}\mathbf{R}_o\right)'\frac{\overline{\lambda}}{\lambda}\mathbf{R}_o +$$

$$m\varphi(\frac{\overline{\lambda}}{\lambda}\mathbf{R}_o)] \times$$

$$\Psi(\mathbf{A}_d^{-1}\overline{\mathbf{A}}_d\boldsymbol{\sigma}^{\mathrm{L}}, \mathbf{A}_d^{-1}\overline{\mathbf{A}}_d\boldsymbol{\sigma}^{\mathrm{U}})\left|\frac{\overline{\lambda}}{\lambda}\right|^m \Pi_{j=1}^{n}(\frac{\overline{A}_j}{A_j})^2\mathrm{d}\mathbf{R}_o\mathrm{d}\boldsymbol{\sigma}^{\mathrm{L}}\mathrm{d}\boldsymbol{\sigma}^{\mathrm{U}},$$

and $\frac{\partial P}{\partial A_j}(\lambda, \mathbf{A})$ can be represented in the same way. Hence, the derivatives $\frac{\partial P}{\partial \lambda}$, $\frac{\partial P}{\partial A_j}$ may be represented by integrals over the *fixed* domain $B_{(\overline{\lambda},\overline{\mathbf{A}})} := \{(\mathbf{R}_o, \boldsymbol{\sigma}^{\mathrm{L}}, \boldsymbol{\sigma}^{\mathrm{U}}) : \overline{\lambda}\mathbf{R}_o \in \mathbf{C}[\overline{\mathbf{A}}_d\boldsymbol{\sigma}^{\mathrm{L}}, \overline{\mathbf{A}}_d\boldsymbol{\sigma}^{\mathrm{U}}]\}$ in the $(\mathbf{R}_o, \boldsymbol{\sigma}^{\mathrm{L}}, \boldsymbol{\sigma}^{\mathrm{U}})$-space.

(b) Since the domain of integration in the integral representation (109) and (113) of $\frac{\partial P}{\partial \lambda}$, $\frac{\partial P}{\partial A_j}$ is independent of the variables λ, $\mathbf{A} = (A_1, \ldots, A_n)'$, the *higher order derivatives - of arbitrary order* - of $P = P(\lambda, \mathbf{A})$ can be obtained by further differentiation of (109) and (113) with respect to λ, A_j, $1 \le j \le n$.

(c) Having the mean value representations (120) and (121), gradient estimates - as well as estimates of the probability function itself - can be obtained by a suitable sampling procedure.

8.2 The probability function (37)

Corresponding to (104)-(106) we note first that

$$\tilde{P}(\mathbf{X}, \mathbf{z}) = \tilde{P}(\mathbf{A}, \mathbf{z}), \mathbf{A} = \mathbf{A}(\mathbf{X}) = [A_1(\mathbf{X}), \ldots, A_n(\mathbf{X})]', \quad (122)$$

cf. (9), where

$$\tilde{P}(\mathbf{A}, \mathbf{z}) := P\begin{pmatrix} \mathbf{A}_{I,d}\boldsymbol{\sigma}_I^{\mathrm{L}}(\omega) & \le \mathbf{C}_I^{-1}[R_o - \mathbf{C}_{II}z] \\ & \le \mathbf{A}_{I,d}\boldsymbol{\sigma}_I^{\mathrm{U}}(\omega) \\ \mathbf{A}_{II,d}\boldsymbol{\sigma}_{II}^{\mathrm{L}}(\omega) & \le \mathbf{z} \\ & \le \mathbf{A}_{II,d}\boldsymbol{\sigma}_{II}^{U}(\omega) \end{pmatrix}. \quad (123)$$

Supposing that the random vectors $\mathbf{R}(\omega)$, $[\boldsymbol{\sigma}^{\mathrm{L}}(\omega), \boldsymbol{\sigma}^{\mathrm{U}}(\omega)]$ have sufficiently smooth densities $\varphi = \varphi(\mathbf{R})$, $\psi = \psi(\boldsymbol{\sigma}^{\mathrm{L}}, \boldsymbol{\sigma}^{\mathrm{U}})$ on $\mathbb{R}^m, \mathbb{R}^{2n}$, and defining here the integral transformation $T_{(z,\mathbf{A})} : (\mathbf{S}, \mathbf{F}_I^{\mathrm{L}}, \mathbf{q}_{II}^{\mathrm{L}}, \mathbf{F}_I^{\mathrm{U}}, \mathbf{q}_{II}^{\mathrm{U}}) \Longrightarrow (\mathbf{R}, \boldsymbol{\sigma}^{\mathrm{L}}, \boldsymbol{\sigma}^{\mathrm{U}})$ by

$$\mathbf{R} := \mathbf{C}_{II}z + \mathbf{S}$$

$$\boldsymbol{\sigma}_I^{\mathrm{L}} := \mathbf{A}_{I,d}{}^{-1}\mathbf{F}_I^{\mathrm{L}}, \quad \boldsymbol{\sigma}_{II}^{\mathrm{L}} := \mathbf{A}_{II,d}{}^{-1}(\mathbf{z} + \mathbf{q}_{II}^{\mathrm{L}})$$

$$\boldsymbol{\sigma}_I^{\mathrm{U}} := \mathbf{A}_{I,d}{}^{-1}\mathbf{F}_I^{\mathrm{U}}, \quad \boldsymbol{\sigma}_{II}^{\mathrm{U}} := \mathbf{A}_{II,d}{}^{-1}(\mathbf{z} + \mathbf{q}_{II}^{\mathrm{U}}), \quad (124)$$

where $A_j > 0$, $1 \le j \le n$, we find

$$\tilde{P}(\mathbf{A}, \mathbf{z}) = \int\limits_{\begin{pmatrix} \mathbf{F}_I^{\mathrm{L}} \le \mathbf{C}_I^{-1}\mathbf{S} \le \mathbf{F}_I^{\mathrm{U}} \\ \mathbf{q}_{II}^{\mathrm{L}} \le 0 \le \mathbf{q}_{II}^{\mathrm{U}} \end{pmatrix}} \varphi(\mathbf{C}_{II}\mathbf{z}+\mathbf{S}) \times$$

$$\psi\left[\begin{pmatrix} \mathbf{A}_{I,d}{}^{-1}\mathbf{F}_I^{\mathrm{L}} \\ \mathbf{A}_{II,d}{}^{-1}(\mathbf{z} + \mathbf{q}_{II}^{\mathrm{L}}) \end{pmatrix}, \begin{pmatrix} \mathbf{A}_{I,d}{}^{-1}\mathbf{F}_I^{\mathrm{U}} \\ \mathbf{A}_{II,d}{}^{-1}(\mathbf{z} + \mathbf{q}_{II}^{\mathrm{U}}) \end{pmatrix}\right] \times$$

$$\Pi_{j=1}^{n} \frac{1}{A_j{}^2} \mathrm{d}\mathbf{S}\mathrm{d}\mathbf{F}_I^{\mathrm{L}}\mathrm{d}\mathbf{q}_{II}^{\mathrm{L}}\mathrm{d}\mathbf{F}_I^{\mathrm{U}}\mathrm{d}\mathbf{q}_{II}^{\mathrm{U}}. \quad (125)$$

Having (125), the derivatives of $\tilde{P} = \tilde{P}(\mathbf{A}, \mathbf{z})$ - of various orders - follow again under weak assumptions (Marti 1995, 1996) by interchanging differentiation and integration in (125). Hence, corresponding to (113)-(116), for $j = 1, 2, \ldots, n$ we find

$$\frac{\partial \tilde{P}}{\partial A_j}(\mathbf{A}, \mathbf{z}) = -\frac{1}{A_j} \int\limits_{\mathbf{A}_{II,d}\boldsymbol{\sigma}_{II}^{\mathrm{L}} \le \mathbf{z} \le \mathbf{A}_{II,d}\boldsymbol{\sigma}_{II}^{\mathrm{U}}} \mathrm{div}\left[(\boldsymbol{\sigma}^{\mathrm{L}}, \boldsymbol{\sigma}^{\mathrm{U}})^{(j)}\psi(\boldsymbol{\sigma}^{\mathrm{L}}, \boldsymbol{\sigma}^{\mathrm{U}})\right] \times$$

$$P^I(\mathbf{A}, \mathbf{z}|\boldsymbol{\sigma}^{\mathrm{L}}, \boldsymbol{\sigma}^{\mathrm{U}})\mathrm{d}\boldsymbol{\sigma}^{\mathrm{L}}\mathrm{d}\boldsymbol{\sigma}^{\mathrm{U}}, \quad (126)$$

where $P^I = P^I(\mathbf{A}, \mathbf{z}|\boldsymbol{\sigma}^{\mathrm{L}}, \boldsymbol{\sigma}^{\mathrm{U}})$ is the conditional probability function given by

$$P^I(\mathbf{A}, \mathbf{z}|\boldsymbol{\sigma}^{\mathrm{L}}, \boldsymbol{\sigma}^{\mathrm{U}}) := P(\mathbf{A}_{I,d}\boldsymbol{\sigma}_I^{\mathrm{L}} \le \mathbf{C}_I^{-1}[\mathbf{R}(\omega) - \mathbf{C}_{II}\mathbf{z}] \le \mathbf{A}_{I,d}\boldsymbol{\sigma}_I^{\mathrm{U}}). \quad (127)$$

Moreover, for $j \in \{j_\ell : 1 \le \ell \le n - m\}$ we have

$$\frac{\partial \tilde{P}}{\partial z_j}(\mathbf{A}, \mathbf{z}) = \int \nabla\varphi(\mathbf{R})'\mathbf{c}_j\tilde{P}(\mathbf{A}, \mathbf{z}|\mathbf{R})\mathrm{d}\mathbf{R} +$$

$$\frac{1}{A_j} \int\limits_{\mathbf{A}_{II,d}\boldsymbol{\sigma}_{II}^{\mathrm{L}} \le z \le \mathbf{A}_{II,d}\boldsymbol{\sigma}_{II}^{\mathrm{U}}} [\frac{\partial \psi}{\partial \sigma_j^{\mathrm{L}}}(\boldsymbol{\sigma}^{\mathrm{L}}, \boldsymbol{\sigma}^{\mathrm{U}}) +$$

$$\frac{\partial \psi}{\partial \sigma_j^{\mathrm{U}}}(\boldsymbol{\sigma}^{\mathrm{L}}, \boldsymbol{\sigma}^{\mathrm{U}})]P^I(\mathbf{A}, \mathbf{z}|\boldsymbol{\sigma}^{\mathrm{L}}, \boldsymbol{\sigma}^{\mathrm{U}})\mathrm{d}\boldsymbol{\sigma}^{\mathrm{L}}\mathrm{d}\boldsymbol{\sigma}^{\mathrm{U}}, \quad (128)$$

where

$$\tilde{P}(\mathbf{A}, \mathbf{z}|\mathbf{R}) := P\begin{pmatrix} \mathbf{A}_{I,d}\boldsymbol{\sigma}_I^{\mathrm{L}}(\omega) \le & \mathbf{C}_I^{-1}(\mathbf{R} - \mathbf{C}_{II}\mathbf{z}) & \le \mathbf{A}_{I,d}\boldsymbol{\sigma}_I^{\mathrm{U}}(\omega) \\ \mathbf{A}_{II,d}\boldsymbol{\sigma}_{II}^{\mathrm{L}}(\omega) \le & \mathbf{z} & \le \mathbf{A}_{II,d}\boldsymbol{\sigma}_{II}^{\mathrm{U}}(\omega) \end{pmatrix} \quad (129)$$

Remark 8.2. Using the inverse transformation $T^{-1}_{(\overline{\mathbf{z}},\overline{\mathbf{A}})}$ with given fixed variables $\overline{\mathbf{z}}$, $\overline{\mathbf{A}}$, cf. Remark 8.1, we also obtain integral representations of the derivatives having a *fixed* domain of integration in the space of the orignal $(\mathbf{R}, \boldsymbol{\sigma}^{\mathrm{L}}, \boldsymbol{\sigma}^{\mathrm{U}})$-variables.

8.3 The probability function (54)

According to the definition of $\mathbf{F}^{\mathrm{L}}(\omega, \mathbf{X})$, $\mathbf{F}^{\mathrm{U}}(\omega, \mathbf{X})$ given in Section 1 for different cases, the probability function (54) can be represented by

$$P(\lambda, \mathbf{X}) = P[\mathrm{a}(\omega)'\mathbf{V}(\lambda, \mathbf{X})\boldsymbol{\delta} \le 0 \text{ for all } \boldsymbol{\delta} \in \Delta_o], \quad (130)$$

where

$$\mathbf{a}(\omega) := \begin{pmatrix} \mathbf{R}_o(\omega) \\ \boldsymbol{\sigma}^{\mathrm{U}}(\omega) \\ \boldsymbol{\sigma}^{\mathrm{L}}(\omega) \end{pmatrix}, \quad \boldsymbol{\delta} := \begin{pmatrix} \mathbf{u} \\ \tilde{\mathbf{u}}^+ \\ \tilde{\mathbf{u}}^- \end{pmatrix}, \quad (131)$$

Δ_o is the convex polyhedron of elements $\boldsymbol{\delta}$ represented by the system of linear equalities/inequalities (48b)-(48c) and

$$\mathbf{V} = \mathbf{V}(\lambda, \mathbf{X}) = \mathbf{V}(\lambda, \mathbf{A}(\mathbf{X}), W^{\overline{y}}(\mathbf{X}), W^{\overline{z}}(\mathbf{X}), W^{\overline{p}}(\mathbf{X})), \quad (132)$$

is an $(m + 2n) \times (m + 2n)$ matrix given analytically; in the case of trusses we have that

$$V(\lambda, \mathbf{X}) := \begin{pmatrix} \lambda\mathbf{I} & \vdots & & \vdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & \vdots & -\mathbf{A}_d(\mathbf{X}) & \vdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & \vdots & & \vdots & \mathbf{A}_d(\mathbf{X}) \end{pmatrix}. \quad (133)$$

Having the extreme points $\delta^{(\ell)}$, $\ell = 1, \ldots, \ell_o$, of Δ_o, we also have

$$P(\lambda, \mathbf{X}) = P[a(\omega)'\mathbf{V}(\lambda, \mathbf{X})\delta^{(\ell)} \le 0, \quad \ell = 1, \ldots, \ell_o]. \quad (134)$$

Since the number ℓ_o of extreme points of Δ_o may be very large, and the numerical computation of $\delta^{(\ell)}$, $\ell = 1, \ldots, \ell_o$, is very time-consuming in general, first we are looking for upper and lower bounds of $P = P(\lambda, \mathbf{X})$.

Considering an arbitrary sequence of elements
$\delta_1, \delta_2, \ldots, \delta_j, \ldots$ in Δ_o,
we find for any $\nu \in I\!N$ the upper bounds

$$P_\nu(\lambda, \mathbf{X}; \delta_1, \ldots, \delta_\nu) := P[\mathbf{a}(\omega)'\mathbf{V}(\lambda, \mathbf{X})\delta_j \leq 0, \quad j = 1, \ldots, \nu], \tag{135}$$

where

$$P_\nu(\lambda, \mathbf{X}; \delta_1, \ldots, \delta_\nu) \geq P_{\nu+1}(\lambda, \mathbf{X}; \delta_1, \ldots, \delta_{\nu+1}) \geq P(\lambda, \mathbf{X}). \tag{136}$$

for each $\nu = 1, 2, \ldots$. Obviously, *minimum upper bounds* $P_\nu^*(\lambda, \mathbf{X})$ are obtained by minimizing (135) with respect to $\delta_j \in \Delta_o$, $j = 1, \ldots, \nu$, hence, we put

$$P_\nu^*(\lambda, \mathbf{X}) := \min[P_\nu(\lambda, \mathbf{X}; \delta_1, \ldots, \delta_\nu) : \delta_j \in \Delta_o, \quad j = 1, \ldots, \nu]. \tag{137}$$

By means of optimum upper bounds the true probability $P(\lambda, \mathbf{X})$ can be reached in a finite number of steps:

Lemma 8.1. There is an integer $\nu_o = \nu_o(\lambda, \mathbf{X})$, $\nu \leq \ell_o$, such that

$$P_{\nu_o}^*(\lambda, \mathbf{X}) = P(\lambda, \mathbf{X}). \tag{138}$$

The assertion follows from the inequalities

$$P(\lambda, \mathbf{X}) = P_{\ell_o}(\lambda, X; \delta^{(1)}, \ldots, \delta^{(\ell_o)}) \leq P_\nu^*(\lambda, \mathbf{X}) \leq P_\nu(\lambda, \mathbf{X}; \delta^{(1)}, \ldots, \delta^{(\nu)}),$$

for each $\nu = 1, 2, \ldots, \ell_o$.

According to Zimmermann, Corotis and Ellis (1991), *suboptimum upper bounds* for $P(\lambda, \mathbf{X})$ can be obtained *interatively* as follows.

Stage 1. Define

$$\tilde{P}_1^*(\lambda, \mathbf{X}) := P_1^*(\lambda, \mathbf{X}) = \min\{P_1(\lambda, \mathbf{X}; \delta_1) : \delta_1 \in \Delta_o\}, \tag{139a}$$

and let $\delta_1^* = \delta_1^*(\lambda, \mathbf{X})$ denote an element of Δ_o such that $P_1^*(\lambda, \mathbf{X}) = P_1[\lambda, \mathbf{X}; \delta_1^*(\lambda, \mathbf{X})]$.

Stage ν. Having $\delta_j^* = \delta_j^*(\lambda, \mathbf{X}), j = 1, \ldots, \nu - 1$, for $\nu > 1$ define

$$\tilde{P}_\nu^*(\lambda, \mathbf{X}) := \min\{P_\nu(\lambda, \mathbf{X}; \delta_1^*(\lambda, \mathbf{X}), \ldots, \delta_{\nu-1}^*(\lambda, \mathbf{X}), \delta_\nu) : \delta_\nu \in \Delta_o\}, \tag{139b}$$

and denote by $\delta_\nu^* = \delta_\nu^*(\lambda, \mathbf{X}; \delta_1^*, \ldots, \delta_{\nu-1}^*)$ an optimal solution in (139b). Obviously we have that

$$\tilde{P}_\nu^*(\lambda, \mathbf{X}) \geq P_\nu^*(\lambda, \mathbf{X}) \geq P(\lambda, \mathbf{X}), \nu = 1, 2, \ldots.$$

Clearly, the advantage in (139b) is that we have only one single decision vector δ_ν, whereas in (137) we have to deal with ν decision vectors $\delta_1, \ldots, \delta_\nu$. On the other hand, with the suboptimal upper bounds $P_\nu^*(\lambda, \mathbf{X})$ the exact value $P(\lambda, \mathbf{X})$ can not be reached in general in a finite number of steps.

According to (134) we find, cf. (60) and (61),

$$P(\lambda, \mathbf{X}) = 1 - P(\mathbf{F}_1 \cup \ldots \cup \mathbf{F}_{\ell_o}),$$

with the failure domains $\mathbf{F}_\ell$ given by

$$\mathbf{F}_\ell := \left\{\omega \in \Omega : \mathbf{a}(\omega)'\mathbf{V}(\lambda, \mathbf{X})\delta^{(\ell)} > 0\right\}, \quad \ell = 1, \ldots, \ell_o.$$

Hence, *lower bounds* for $P(\lambda, \mathbf{X})$ follow by applying the bounds mentioned in Section 4 to $P(\bigcup_{\ell=1}^{\ell_o} \mathbf{F}_\ell)$. These bounds can be described by means of an estimate $\hat{\ell}_o$ of ℓ_o and by probability functions of the type

$$Q_\nu(\lambda, \mathbf{X}; \delta_1, \ldots, \delta_\nu) := P\left[\mathbf{a}(\omega)'\mathbf{V}(\lambda, \mathbf{X})\delta_j > 0, \quad j = 1, \ldots, \nu\right], \tag{140}$$

similar to $P_\nu(\lambda, \mathbf{X}; \delta_1, \ldots, \delta_\nu)$, $\delta_j \in \Delta_o$, $j = 1, \ldots, \nu$.
For example, for $\nu = 1$ we have

$$P(\lambda, \mathbf{X}) \geq 1 - \sum_{\ell=1}^{\ell_o} P[\mathbf{a}(\omega)'\mathbf{V}(\lambda, \mathbf{X})\delta^{(\ell)} > 0] = 1 - \sum_{\ell=1}^{\ell_o} Q_1(\lambda, \mathbf{X}; \delta^{(\ell)}) \geq 1 - \ell_o Q_1^*(\lambda, \mathbf{X}),$$

where, for $\nu = 1, 2, \ldots$,

$$Q_\nu^* := \max[Q_1(\lambda, \mathbf{X}; \delta_1, \ldots, \delta_\nu) : \delta_j \in \Delta_o, 1 \leq j \leq \nu]. \tag{141}$$

Consequently, for the approximative computation of the probability $P(\lambda, \mathbf{X})$ we must solve optimization problem of the type

$$\min_{\delta_j \in \Delta_o, 1 \leq j \leq \nu} P_\nu(\lambda, \mathbf{X}; \delta_1, \ldots, \delta_\nu). \tag{142}$$

and

$$\max_{\delta_j \in \Delta_o, 1 \leq j \leq \nu} Q_\nu(\lambda, \mathbf{X}; \delta_1, \ldots, \delta_\nu). \tag{143}$$

Moreover, for the approximative solution of reliability-oriented optimization problems of the type

$$\max_{\mathbf{X} \in D} P(\lambda_o, \mathbf{X}), \tag{144}$$

with a given load factor $\lambda_o \in I\!R$, we have the *maximin-problem*

$$\max_{\mathbf{X} \in D} \min_{\delta_j \in \Delta_o, 1 \leq j \leq \nu} P_\nu(\lambda_o, \mathbf{X}; \delta_1, \ldots, \delta_\nu). \tag{145}$$

Thus, in each case the derivatives of the probability functions P_ν and Q_ν are needed.

By the transformation

$$\theta_j := V(\lambda, \mathbf{X})\delta_j, j = 1, \ldots, \nu, \tag{146}$$

for $P_\nu = P_\nu(\lambda, \mathbf{X}; \delta_1, \ldots, \delta_\nu)$ we find the representation

$$P_\nu(\lambda, \mathbf{X}; \delta_1, \ldots, \delta_\nu) = W_\nu[V(\lambda, \mathbf{X})\delta_1, \ldots, V(\lambda, \mathbf{X})\delta_\nu], \tag{147}$$

where

$$W_\nu(\theta_1, \ldots, \theta_\nu) := P[\theta_j'\mathbf{a}(\omega) \leq 0, j = 1, \ldots, \nu], \tag{148}$$

and Q_ν can be represented in the same way. Since the derivatives of $\mathbf{V} = \mathbf{V}(\lambda, \mathbf{X})$ can be obtained analytically, the remaining problem is the differentiation of W_ν.

8.3.1 Exact differentiation formulae in the case of $\nu \leq \dim a(.)$

In many practical cases the stage number ν is small, hence, the assumption

$$\nu \leq d := \dim a(.) = m + 2n$$

is not too restrictive. For the computation of the partial derivative $\frac{\partial W_\nu}{\partial \theta_{jk}}$ for a given pair (j, k), $1 \leq j \leq \nu$, $1 \leq k \leq d$, we consider a partition $\Theta = (\Theta_I, \Theta_{II})$ of the $\nu \times d$ matrix

$$\Theta := \begin{pmatrix} \theta_1' \\ \theta_2' \\ \vdots \\ \theta_\nu' \end{pmatrix}, \tag{149}$$

such that
(i) θ_{jk} is an element of Θ_I and
(ii) $\text{rank}\Theta_I = \nu$. (150)

Supposing then that the random vector $\mathbf{a} = \mathbf{a}(\omega)$ has a probability density $f = f(\mathbf{a})$, and partitioning the vector $\mathbf{a} \in I\!R^d$ in the same way $\mathbf{a} = (\mathbf{a}_I, \mathbf{a}_{II})$ as the matrix Θ, by the integral transformation

$$\mathbf{a} = \begin{pmatrix} \mathbf{a}_I \\ \mathbf{a}_{II} \end{pmatrix} := \begin{pmatrix} \Theta_I^{-1}\mathbf{p}_I \\ \mathbf{p}_{II} \end{pmatrix}, \qquad (151)$$

we find

$$W_\nu(\theta_1, \ldots, \theta_\nu) = \int\limits_{\mathbf{p}_I + \Theta_{II}\mathbf{p}_{II} \le 0} f(\Theta_I^{-1}\mathbf{p}_I, \mathbf{p}_{II}) \frac{d\mathbf{p}_I}{|\det \Theta_I|}. \qquad (152)$$

Obviously, by means of this transformation, the domain of integration in (152) is now independent of the element θ_{jk} and - of course - also independent of all other elements $\theta_{\iota\kappa}$ constained in Θ_I. Thus, if the density f of $a(\omega)$ is sufficiently smooth the derivative follows (cf. Marti 1995a, 1996), by interchanging differentiation and integration.

Theorem 8.2. Under appropriate assumptions (Marti 1995a, 1996) on the density $f = f(a)$ of $a(\omega)$, the partial derivative $\frac{\partial}{\partial\theta_{jk}} W_\nu$, is given by

$$\frac{\partial W_\nu}{\partial \theta_{jk}}(\theta_1, \ldots, \theta_\nu) = - \int\limits_{\Theta\mathbf{a} \le 0} \left[\nabla_{\mathbf{a}_I} f(\mathbf{a})' (\Theta_I^{-1'})_j' \mathbf{a}_k + f(\mathbf{a})(\Theta_I^{-1'})_{jk} \right] d\mathbf{a}, \qquad (153a)$$

which can be represented also by the *expectation*

$$\frac{\partial W_\nu}{\partial \theta_{jk}}(\theta_1, \ldots, \theta_\nu) = -E\left[\left(\frac{\nabla_{\mathbf{a}_I} f[\mathbf{a}(\omega)]}{f[\mathbf{a}(\omega)]} \right)' (\Theta_I^{-1})_j' \mathbf{a}_k(\omega) + (\Theta_I^{-1'})_{jk} \right] 1_{[\Theta\mathbf{a} \le 0]}[\mathbf{a}(\omega)], \qquad (153b)$$

where $(\Theta_I^{-1'})_j$ denotes the j-th row of $\Theta_I^{-1'}$.

Corollary 8.1. The derivatives $\frac{\partial W_\nu}{\partial \theta_{\iota\kappa}}$ of W_ν with respect to elements $\theta_{\iota\kappa}$ of Θ_I have the same form.

8.3.2 Approximative derivatives of W_ν

If condition (150) cannot be fulfilled, e.g. in the case $\nu > d$, then approximative derivatives of W_ν - of an arbitrary high accuracy - can be obtained by the following *stochastic completion technique* (Marti 1994, 1996).

Let $z_j = z_j(\omega)$, $j = 1, \ldots, \nu$, denote real random variables such that
(i) $z_j = z_j(\omega)$, $j = 1, \ldots, \nu$, are stochastically independent,
(ii) $Ez_j(\omega) = 0$, and $z_j(\omega)$ has a continuous probability density $\psi_j = \psi_j(t)$, $j = 1, \ldots, \nu$. (154)

By means of the "stochastic completion terms" $z_j = z_j(\omega)$, $j = 1, \ldots, \nu$, for W_ν we obtain the approximative probability function

$$\tilde{W}_\nu(\theta_1, \ldots, \theta_\nu) := P(\theta_j'\mathbf{a}(\omega) + z_j(\omega) \le 0, j = 1, \ldots, \nu). \qquad (155)$$

We find

$$\tilde{W}_\nu(\theta_1, \ldots, \theta_\nu) = P[z_j(\omega) \le -\theta_j'\mathbf{a}(\omega), j = 1, \ldots, \nu] = E \prod_{j=1}^{\nu} \Psi_j[-\theta_j'\mathbf{a}(\omega)], \qquad (156)$$

where Ψ_j denotes the distribution function of $z_j(\omega)$; if $z_j(\omega)$ has the normal distribution $N(0, \sigma_j)$, then

$$\tilde{W}_\nu(\theta_1, \ldots, \theta_\nu) = E \prod_{j=1}^{\nu} \Phi[-\frac{1}{\sigma_j}\theta_j'\mathbf{a}((\omega))], \qquad (157)$$

where Φ is the distribution function of $N(0,1)$.

The partial derivatives of $\tilde{W}_\nu$ read

$$\frac{\partial \tilde{W}_\nu}{\partial \theta_{jk}} = -E \prod_{\substack{\ell=1 \\ \ell \ne j}}^{\nu} \Psi_\ell\left[-\theta_\ell'\mathbf{a}(\omega)\right] \psi_j\left[-\theta_j'\mathbf{a}(\omega)\right] \mathbf{a}_k(\omega), \qquad (158)$$

$j = 1, \ldots, \nu$, $k = 1, \ldots, d$, and higher order derivatives of $\tilde{W}_\nu$ can be obtained is the same way. If

$$\mathbf{z}(\omega) = [z_1(\omega), \ldots, z_\nu(\omega)]' \Longrightarrow 0 \ w.p.1 \qquad (159)$$

(with probability one), then under some regularity assumptions (Marti 1994, 1996)

$$\begin{aligned} \tilde{W}_\nu(\theta_1, \ldots, \theta_\nu) &\Longrightarrow W_\nu(\theta_1, \ldots, \theta_\nu), \\ \frac{\partial \tilde{W}_\nu}{\partial \theta_{jk}}(\theta_1, \ldots, \theta_\nu) &\Longrightarrow \frac{\partial W_\nu}{\partial \theta_{jk}}(\theta_1, \ldots, \theta_\nu). \end{aligned} \qquad (160)$$

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