Arch. Math., Vol. 59, 245–259 (1992) 0003-889X/92/5903-0245 $ 4.50/0


# On the use of time-maps for the solvability of nonlinear boundary value problems

By

ALESSANDRO FONDA and FABIO ZANOLIN

**1. Introduction.** In this article we are concerned with the solvability of the periodic boundary value problem associated to some nonautonomous scalar nonlinear second order differential equations of Duffing type. As a possible model for our investigation we consider, for instance, equation

(1.1) $$x'' + g(x) = p(t),$$

where $g: \mathbb{R} \to \mathbb{R}$ is continuous and $p: [0, T] \to \mathbb{R}$ is Lebesgue integrable. Solutions of (1.1) are intended in the generalized (i.e. Caratheodory) sense and are called $T$-periodic provided they are defined on $[0, T]$ and satisfy the boundary condition

(1.2) $$x(T) - x(0) = x'(T) - x'(0) = 0$$

($T > 0$ is a fixed positive constant).

The study of the periodic problem for equation (1.1) (or for some of its generalizations) represents a central subject in the qualitative theory of ordinary differential equations and it has been widely developed by the introduction of powerful tools from nonlinear functional analysis. See e.g. [22, 11, 9, 8, 16] and the references therein, for a source of various different techniques which can be used for this purpose.

A classical method to deal with problem (1.1)–(1.2) consists into the search of fixed points of the translation operator (Poincaré-Andronov map) $\psi: (x_0, y_0) \to (x(T; x_0, y_0), y(T; x_0, y_0))$ associated to the equivalent planar system

(1.3) $$x' = y, \quad y' = -g(x) + p(t),$$

where, in order to make all the subsequent discussion meaningful, we suppose that the function $p$ is continuous and that the solution $(x(\cdot; x_0, y_0), y(\cdot; x_0, y_0))$ of (1.3) satisfying the initial condition $x(0) = x_0$, $y(0) = y_0$ is unique and defined on $[0, T]$. In this setting, a useful approach, considered in [1, 10, 12] and extensively exploited in [13, 11], can be described as follows. At the beginning, a Jordan curve $\mathscr{J}$ is constructed, such that the origin and all the critical points of the vector field $v: (x, y) \mapsto (y, -g(x) + p(0))$ lie in the "interior" of $\mathscr{J}$ and the index of $v$, relatively to $\mathscr{J}$, is nonzero. As a second step, it is crucial to prove that all the points of $\mathscr{J}$ are of nonrecurrence (or $T$-irreversibility, according to [11]) for (1.3), that is, for any $(x_0, y_0) \in \mathscr{J}$, we have $(x(t; x_0, y_0), y(t; x_0, y_0)) \neq (x_0, y_0)$, for each $t \in ]0, T]$. This condition allows to perform an admissible homotopy, along the trajectories of (1.3), between $\psi - I_{\mathbb{R}^2}$ ($I_{\mathbb{R}^2}$ = identity in $\mathbb{R}^2$) and $v$. Hence, the index of $\psi - I_{\mathbb{R}^2}$, relatively to $\mathscr{I}$, is nonzero and the existence of fixed points of $\psi$ is ensured by a basic property of the Brouwer degree (Kronecker's existence theorem).

At the early sixties, Z. Opial [18, 20] introduced the use of estimates for the time-map in order to verify the property of "nonrecurrence" described above. In these papers, using a careful comparison with the solutions of the autonomous system

$$x' = y, \quad y' = -g(x), \tag{1.4}$$

he found lower estimates for any possible time $t_0 = t_0(x_0, y_0) > 0$ such that $(x(t_0; x_0, y_0), y(t_0; x_0, y_0)) = (x_0, y_0)$. Then the result was accomplished by suitable conditions ensuring that $t_0 > T$, for $(x_0^2 + y_0^2)$ sufficiently large.

In order to state the main theorem obtained by Opial in [18] through the above argument, we need the following notations. Set

$$G(x) = \int_0^x g(s)\, ds \tag{1.5}$$

and consider

$$\tau_g(x) := \sqrt{2} \left| \int_0^x \frac{ds}{\sqrt{G(x) - G(s)}} \right|$$

whenever it is defined. We note that a sufficient condition for $\tau_g(x)$ to be defined on neighborhoods of $\pm\infty$ is

$$\lim_{|x| \to \infty} g(x) \cdot \operatorname{sign}(x) = +\infty. \tag{$g_1$}$$

Indeed, in this case, every orbit $(x(t), y(t))$ of (1.4) lying sufficiently far from the origin is closed and its minimal period is $\tau_g(x^*) + \tau_g(x_*)$, where $x^*$ and $x_*$ are respectively the maximum and the minimum value of $x(t)$ along the orbit.

Define moreover

$$\tau^{\pm}(g) := \limsup_{x \to \pm\infty} \tau_g(x), \tag{1.7}$$

$$\tau_{\pm}(g) := \liminf_{x \to \pm\infty} \tau_g(x). \tag{1.8}$$

A slightly improved version of Opial's result is the following.

**Theorem A** ([18]). *Assume* ($g_1$) *and*

$$\tau_-(g) + \tau_+(g) > T. \tag{$g_2$}$$

*Then* (1.1)–(1.2) *has at least one solution, for every continuous function p.*

Sufficient conditions for ($g_2$) are obtained in [19] (see also [22]). In particular, if either

$$\limsup_{x \to \pm\infty} g(x)/x = k_{\pm}$$

or

$$\lim_{x \to \pm\infty} 2G(x)/x^2 = k_\pm ,$$

then

$$\tau_\pm(g) \geqq \pi/\sqrt{k_\pm} .$$

In this case, ($g_2$) holds provided that $1/\sqrt{k_+} + 1/\sqrt{k_-} > (T/\pi)$. One can in this way re-obtain some results in [2, 8].

Another case in which Theorem A can be applied deals with the so-called "one-sided growth restrictions" (see [23, 21, 24, 15]), that is, the problem of the solvability of (1.1)–(1.2) using conditions concerning the behaviour of the nonlinearity only for $x \geqq 0$ (the case $x \leqq 0$ is completely symmetric and will not be examined here). In this situation, since we have no way to control $\tau_-(g)$, we need to require

$$(1.9) \qquad \tau_+(g) > T ,$$

for the validity of ($g_2$). Then, by the estimates in [19], one gets the solvability of (1.1)–(1.2) provided that either

$$\limsup_{x \to +\infty} g(x)/x = k_+ < (\pi/T)^2$$

or

$$\lim_{x \to +\infty} 2G(x)/x^2 = k_+ < (\pi/T)^2$$

holds. (See also [22, p. 208].)

In some recent papers [6, 4] dealing with (1.1)–(1.2), the existence of solutions has been proved under more general one-sided growth restrictions like

$$(1.10) \qquad \liminf_{x \to +\infty} g(x)/x = 0, \quad xg'(x)/g(x) \leqq M < +\infty \quad \text{for } x \geqq d > 0$$

in [6], or

$$(1.11) \qquad \liminf_{x \to +\infty} 2G(x)/x^2 < (\pi/T)^2$$

in [4]. The main arguments in the proofs of the above results combine some estimates for the time-map of system (1.3) with a continuation lemma based on the use of topological degree in function spaces. Some examples can be easily produced in order to show that conditions (1.10) or (1.11) do not imply (1.9).

In this article, we find an improvement of Opial's theorem providing a general result which unifies and extends [6, 4]. In particular, we are able to prove the following

**Theorem B.** *Assume* ($g_1$) *and either*

$$(g_{2a}) \qquad \tau_-(g) + \tau^+(g) > T ,$$

*or*

$(g_{2b})$ $\qquad \tau^-(g) + \tau_+(g) > T.$

*Then* (1.1)–(1.2) *has at least one solution, for every integrable function* $p$.

The paper is organized as follows. In Section 2 we state our main existence result which is essentially an adaptation of Theorem B to a more general class of equations. The proof is then carried out in Section 3, making use of a continuation lemma based on topological degree arguments. In Section 4 we prove estimates for the time-map and show how our result includes the above mentioned theorems. Finally, in Section 5 we outline some possible applications to different types of boundary value problems.

This research started when the authors were visiting the Institute of Mathematics of Louvain-la-Neuve. The authors are very grateful for the kind hospitality.

**2. Statement of the general result.** In this section we consider the problem of the existence of periodic solutions with given period $T > 0$ of a second order differential equation of the form

(2.1) $$x'' + f(t, x) = p(t).$$

Here $f: [0, T] \times \mathbb{R} \to \mathbb{R}$ is supposed to satisfy the Caratheodory conditions, i.e. $f(\cdot, x)$ is measurable for all $x \in \mathbb{R}$, $f(t, \cdot)$ is continuous for almost every $t \in [0, T]$, and for each $r > 0$ there exists a Lebesgue integrable function $h_r: [0, T] \to \mathbb{R}$ such that $|f(t, x)| \leqq h_r(t)$ for almost every $t \in [0, T]$ and all $|x| \leqq r$. The function $p: [0, T] \to \mathbb{R}$ is only supposed to be Lebesgue integrable.

A $T$-periodic solution of (2.1) is meant to be a differentiable function $x: [0, T] \to \mathbb{R}$ whose derivative is absolutely continuous, satisfying (2.1) for almost every $t \in [0, T]$ and such that

(2.2) $$x(0) - x(T) = x'(0) - x'(T) = 0.$$

It is well-known that such a solution can be extended to a classical $T$-periodic solution of (2.1) on the whole real line when $f(t, x)$ and $p(t)$ are continuous and $T$-periodic in the variable $t$.

Let us denote by $\bar{p}$ the mean value of $p(t)$, i.e.

$$\bar{p} = \frac{1}{T} \int_0^T p(s)\, ds,$$

and assume the following condition:

(H) there exists $d \geqq 0$, $p_1, p_2 \in \mathbb{R}$ and a continuous function $\phi: \mathbb{R} \to \mathbb{R}$ satisfying the properties

$(h_1)$ $\lim_{|x| \to \infty} \phi(x) \operatorname{sign}(x) = +\infty$;

$(h_2)$ $p_2 \leqq f(t, x) \leqq \phi(x)$ for a.e. $t \in [0, T]$ and all $x \geqq d$;

$(h_3)$ $\phi(x) \leqq f(t, x) \leqq p_1$ for a.e. $t \in [0, T]$ and all $x \leqq -d$.

Let $\Phi$ denote a primitive of $\phi$. We can introduce the function

$$(2.3)\qquad \tau(x) = \sqrt{2}\left|\int_0^x \frac{ds}{\sqrt{\Phi(x) - \Phi(s)}}\right|$$

which, by assumption $(H)$ above, is defined when $|x|$ is sufficiently large. As mentioned in the introduction, $\tau(x)$ is the time-map associated to the autonomous planar system $x' = y$, $y' = -\phi(x)$. Finally, set

$$\tau^{\pm} = \limsup_{x \to \pm\infty} \tau(x),$$

$$\tau_{\pm} = \liminf_{x \to \pm\infty} \tau(x).$$

With the above assumptions, we have:

**Theorem 1.** *Equation* (2.1) *has a T-periodic solution if* $p_1 \leqq \bar{p} \leqq p_2$ *and, either* $\tau_- + \tau^+ > T$, *or* $\tau^- + \tau_+ > T$.

Clearly, the assumptions in Theorem 1 are satisfied if $p_1 \leqq \bar{p} \leqq p_2$ and, either $\tau^+ > T$, or $\tau^- > T$. This situation will be examined in more detail in the applications. Theorem 1 improves a result by Opial [18] where it was assumed that $\tau_- + \tau_+ > T$; indeed, a straightforward consequence of Theorem 1 is Theorem B stated in the introduction, where $f(t, x) = \phi(x) = g(x)$. The proof of Theorem 1 is carried out in Section 3. Explicit conditions under which the assumptions of Theorem 1 hold true will be given in Section 4.

**3. The proof.** Without loss of generality (using $(h_1)$) we can assume that $\phi(x) \neq \bar{p}$ for $|x| \geqq d$. Then we can use the following continuation lemma, whose proof, given for $f$ continuous in [17, 6], can easily be adapted to the Caratheodory case.

**Lemma 1.** *Assume there exist* $u_* \leqq -d$ *and* $u^* \geqq d$ *such that*:
$(h_4)$ *considered for* $\lambda \in ]0, 1[$ *the equation*

$$(3.1_\lambda)\qquad x'' + \lambda(\lambda f(t, x) + (1 - \lambda)\phi(x)) = \lambda p(t),$$

*either for any T-periodic solution* $x_\lambda$ *of* $(3.1_\lambda)$ *one has*

$$(3.2)\qquad \max x_\lambda \neq u^*$$

*or for any such solution one has*

$$(3.3)\qquad \min x_\lambda \neq u_*.$$

*Then equation* (2.1) *has a T-periodic solution.*

*Moreover the T-periodic solutions* $x_\lambda$ *of* $(3.1_\lambda)$ *have the following property*: *for every* $R \geqq d$, *there exists a constant* $M > 0$ *(depending only on* $R$*) such that, whenever* $\max x_\lambda \leqq R$ *(or* $\min x_\lambda \geqq -R$*), then* $|x_\lambda(t)| \leqq M$ *for every* $t \in [0, T]$. (See also [15].)

In order to apply Lemma 1, for any pair $(u_*, u^*)$, with $u_* \leqq -d$, $u^* \geqq d$, we need to evaluate the time-map of the solutions of $(3.1_\lambda)$ having maximum value $u^*$ and of those having minimum value $u_*$. To this aim, it is better to write $(3.1_\lambda)$ as an equivalent second order system. Defining the continuous $T$-periodic function

$$\tilde{P}(t) = \int_0^t (p(s) - \bar{p})\,ds,$$

equation $(3.1_\lambda)$ is equivalent to system

$$(3.4_\lambda) \qquad x' = y + \lambda \tilde{P}(t)$$

$$(3.5_\lambda) \qquad y' = -\lambda(\lambda f(t,x) + (1-\lambda)\phi(x) - \bar{p}).$$

Set $L = 2T\|\tilde{P}\|_\infty$ and let $T(x)$ be an auxiliary function defined outside a sufficiently large interval $]-r, r[$ as follows:

$$T(x) = \sqrt{2}\int_d^{x-L} \left(\sqrt{2}\,\|\tilde{P}\|_\infty + \sqrt{\Phi(x) - \Phi(s) + |\bar{p}|(x-s)}\right)^{-1} ds \quad \text{for } x \geqq r,$$

$$T(x) = \sqrt{2}\int_{x+L}^{-d} \left(\sqrt{2}\,\|\tilde{P}\|_\infty + \sqrt{\Phi(x) - \Phi(s) + |\bar{p}|(x-s)}\right)^{-1} ds \quad \text{for } x \leqq -r.$$

The function $T(x)$ permits us to evaluate the time-map in the following way.

**Lemma 2.** *Let $x_\lambda$ be a $T$-periodic solution of $(3.1_\lambda)$, for $\lambda \in ]0,1[$, such that* $\max x_\lambda = u^* \geqq r$ *and* $\min x_\lambda = u_* \leqq -r$. *Then, if $r$ is sufficiently large,*

$$T(u^*) + T(u_*) \leqq T.$$

Proof of Lemma 2. Let $(x_\lambda, y_\lambda)$ be the corresponding solution of system $(3.4_\lambda)$–$(3.5_\lambda)$. Integrating $(3.5_\lambda)$ over $[0, T]$ and exploiting $(h_2)$, $(h_3)$, the assumption on $\bar{p}$ and the remark at the beginning of this section, we get:

$$\exists \tilde{t} \in [0, T]: |x_\lambda(\tilde{t})| \leqq d.$$

Moreover, it follows from $(3.5_\lambda)$ that $y_\lambda(t)$ is decreasing when $x_\lambda(t) \geqq d$, and increasing when $x_\lambda(t) \leqq -d$. Assume

$$r > d + 2T\|\tilde{P}\|_\infty = d + L.$$

Extending our functions by $T$-periodicity if necessary, there will be $\alpha_1 < t_1 \leqq t_2 < \alpha_2$ such that

$$x_\lambda(\alpha_i) = d, \quad x_\lambda(t_i) = u^* \quad (i = 1, 2)$$

and

$$d < x_\lambda(t) < u^* \quad (t \in ]\alpha_1, t_1[ \cup ]t_2, \alpha_2[).$$

Let us now concentrate on the interval $[\alpha_1, t_1]$. Integration of $(3.4_\lambda)$ gives

$$u^* - d = \int_{\alpha_1}^{t_1} (y_\lambda(s) + \lambda\tilde{P}(s))\,ds < T[y_\lambda(\alpha_1) + \|\tilde{P}\|_\infty]$$

and so

$$y_\lambda(\alpha_1) > T^{-1}(r-d) - \|\tilde{P}\|_\infty > \|\tilde{P}\|_\infty.$$

On the other hand, evaluating $(3.4_\lambda)$ in $t_1$, we get $|y_\lambda(t_1)| \leqq \|\tilde{P}\|_\infty$. Hence, there exists $\beta_1 \in ]\alpha_1, t_1]$ such that $y_\lambda(\beta_1) = \|\tilde{P}\|_\infty$. Integration of $(3.4_\lambda)$ over $[\beta_1, t_1]$ yields

$$u^* - x_\lambda(\beta_1) = \int_{\beta_1}^{t_1} (y_\lambda(s) + \lambda \tilde{P}(s))\, ds \leqq 2T \|\tilde{P}\|_\infty$$

i.e.

$$x_\lambda(\beta_1) \geqq u^* - L > d. \tag{3.6}$$

From system $(3.4_\lambda)$–$(3.5_\lambda)$, for all $t \in [\alpha_1, \beta_1]$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left[\Phi(x_\lambda(t)) - \bar{p}\, x_\lambda(t) + \frac{1}{2}(y_\lambda(t) - \|\tilde{P}\|_\infty)^2\right] \\
&\quad = (\phi(x_\lambda) - \bar{p})\, x'_\lambda + (y_\lambda - \|\tilde{P}\|_\infty)\, y'_\lambda \\
&\quad \geqq (y_\lambda - \|\tilde{P}\|_\infty)\,[\phi(x_\lambda) - \bar{p} - \lambda(\lambda f(t, x_\lambda) + (1-\lambda)\phi(x_\lambda) - \bar{p})] \geqq 0.
\end{aligned}$$

Hence

$$\begin{aligned}
\Phi(x_\lambda(t)) - \bar{p}\, x_\lambda(t) + \frac{1}{2}(y_\lambda(t) - \|\tilde{P}\|_\infty)^2 &\leqq \Phi(x_\lambda(\beta_1)) - \bar{p}\, x_\lambda(\beta_1) \\
&\leqq \Phi(u^*) - \bar{p} u^*,
\end{aligned}$$

by (3.6) and the fact that $\phi(x) > \bar{p}$ for $x \geqq d$. Using again $(3.4_\lambda)$, we get

$$1 \geqq \frac{x'_\lambda(t)}{2\|\tilde{P}\|_\infty + \sqrt{2[\Phi(u^*) - \Phi(x_\lambda(t)) + |\bar{p}|(u^* - x_\lambda(t))]}}. \tag{3.7}$$

Integrating over $[\alpha_1, \beta_1]$ and taking into account (3.6) we obtain

$$\beta_1 - \alpha_1 \geqq \int_d^{x(\beta_1)} \frac{ds}{2\|\tilde{P}\|_\infty + \sqrt{2[\Phi(u^*) - \Phi(s) + |\bar{p}|(u^* - s)]}} \geqq \frac{1}{2} T(u^*).$$

A similar procedure (see [6]) can be used to establish that there exists $\beta_2 \in [t_2, \alpha_2[$ such that $y_\lambda(\beta_2) = -\|\tilde{P}\|_\infty$, and then $\alpha_2 - \beta_2 \geqq \frac{1}{2} T(u^*)$. Therefore we have

$$\alpha_2 - \alpha_1 \geqq T(u^*).$$

In a symmetric way one can prove that the time needed for $x_\lambda(t)$ to reach its minimum $u_*$ starting from the level $-d$ and to come back has to be at least $T(u_*)$. The result then immediately follows. □

Next we need a result, showing how the auxiliary function $T(x)$ is a good estimate for $\tau(x)$ when $|x|$ is large.

**Lemma 3.** $\lim_{x \to \pm\infty} [T(x) - \tau(x)] = 0.$

Proof of Lemma 3. Observe that, by definition, we have $\tau(x) \geqq T(x)$ for $|x|$ large. Let $\varepsilon$ be a small fixed positive number. We want to prove that for $|x|$ large enough one has $T(x) - \tau(x) \geqq -\varepsilon$. Let us consider the case $x \to +\infty$, the other one being treated similarly. By assumption ($h_1$), it is possible to find $d' \geqq d$ with the following properties:

$$\Phi(d') > \Phi(s) \quad (0 \leqq s < d'), \tag{3.8}$$

$$\phi(s) \geqq 4T|\bar{p}|/\varepsilon \quad (s \geqq d'). \tag{3.9}$$

For $x > d' + L$, we have

$$\tau(x) = \sqrt{2}\int_0^x \frac{ds}{\sqrt{\Phi(x) - \Phi(s)}} = \sqrt{2}\left[\int_0^{d'} + \int_{d'}^{x-L} + \int_{x-L}^{x}\right].$$

By (3.8),

$$\sqrt{2}\int_0^{d'} \frac{ds}{\sqrt{\Phi(x) - \Phi(s)}} \leqq \sqrt{2}\int_0^{d'} \frac{ds}{\sqrt{\Phi(x) - \Phi(d')}} \leqq \varepsilon/4$$

when $x \geqq M_1$, for a sufficiently large $M_1 = M_1(\varepsilon)$. Moreover

$$\sqrt{2}\int_{x-L}^{x} \frac{ds}{\sqrt{\Phi(x) - \Phi(s)}} \leqq \sqrt{2}\int_{x-L}^{x} \frac{ds}{\sqrt{L \min\limits_{[x-L,x]} \phi}} \leqq \varepsilon/4$$

when $x \geqq M_2$, for a sufficiently large $M_2 = M_2(\varepsilon)$. By (3.9) we have

$$\begin{aligned}
&\sqrt{2}\int_{d'}^{x-L} \frac{ds}{\sqrt{\Phi(x) - \Phi(s)}} \\
&= \sqrt{2}\sqrt{1 + \varepsilon/4T}\int_{d'}^{x-L}\left[\left(1 + \frac{\varepsilon}{4T}\right)\int_s^x \phi(\xi)\,d\xi\right]^{-1/2} ds \\
&\leqq \sqrt{2}\sqrt{1 + \varepsilon/4T}\int_{d'}^{x-L}\left[\int_s^x (\phi(\xi) + |\bar{p}|)\,d\xi\right]^{-1/2} ds \\
&\leqq \sqrt{2}\left(1 + \frac{\varepsilon}{4T}\right)^{3/2}\int_{d'}^{x-L}\left\{\left(1 + \frac{\varepsilon}{4T}\right)\left[\int_s^x (\phi(\xi) + |\bar{p}|)\,d\xi\right]^{1/2}\right\}^{-1} ds \\
&\leqq \sqrt{2}\left(1 + \frac{\varepsilon}{4T}\right)^{3/2}\int_{d'}^{x-L}\left\{\left[\int_s^x (\phi(\xi) + |\bar{p}|)\,d\xi\right]^{1/2}\right. \\
&\qquad\qquad \left. + \frac{\varepsilon}{4T}\cdot\left[\int_{x-L}^{x} (\phi(\xi) + |\bar{p}|)\,d\xi\right]^{1/2}\right\}^{-1} ds \\
&\leqq \sqrt{2}\left(1 + \frac{\varepsilon}{4T}\right)^{3/2}\int_{d'}^{x-L}\left\{\left[\int_s^x (\phi(\xi) + |\bar{p}|)\,d\xi\right]^{1/2}\right. \\
&\qquad\qquad \left. + \frac{\varepsilon}{4T}\cdot\left[L \min_{[x-L,x]} (\phi + |\bar{p}|)\right]^{1/2}\right\}^{-1} ds \\
&\leqq \sqrt{2}\left(1 + \frac{\varepsilon}{4T}\right)^{3/2}\int_{d'}^{x-L}\{[\Phi(x) - \Phi(s) + |\bar{p}|(x - s)]^{1/2} + \sqrt{2}\,\|\tilde{P}\|_\infty\}^{-1}\,ds \\
&\leqq \left(1 + \frac{\varepsilon}{4T}\right)^{3/2} T(x) \leqq \left(1 + \frac{\varepsilon}{2T}\right) T(x) \leqq T(x) + \varepsilon/2,
\end{aligned}$$

the above being true for $x \geqq M_3$, for a sufficiently large $M_3 = M_3(\varepsilon)$ ($\varepsilon$ small) since $T(x) \leqq T$ (by Lemma 2). Hence for $x \geqq M := \max\{M_1, M_2, M_3\}$, we have

$$T(x) \leqq \tau(x) \leqq T(x) + \varepsilon,$$

and the result is proved. □

Proof of Theorem 1. Let us consider the case $\tau_- + \tau^+ > T$, the other one being similarly treated. Let $\varepsilon > 0$ be such that

$$\tau_- + \tau^+ \geqq T + 3\varepsilon. \tag{3.10}$$

Consider a sequence $x_n^* \to +\infty$ such that $\tau(x_n^*) \to \tau^+$. We want to show that, for a sufficiently large $n$, for any $T$-periodic solution $x_\lambda$ of $(3.1_\lambda)$ one has $\max x_\lambda \neq x_n^*$. In this way, (3.2) is satisfied with $u^* = x_n^*$ and the first part of Lemma 1 can be applied. Suppose by contradiction that there exists a sequence $(x_{\lambda_n})$ of $T$-periodic solutions of $(3.1_\lambda)$ such that $\max x_{\lambda_n} = x_n^*$. For $n$ large enough we have, by Lemma 3,

$$T(x_n^*) \geqq \tau(x_n^*) - \varepsilon. \tag{3.11}$$

On the other hand, setting $x_{n*} = \min x_{\lambda_n}$, by the second part of Lemma 1 there will exist a subsequence – still denoted $x_{\lambda_n}$ – for which $x_{n*} \to -\infty$. Again by Lemma 3 we have

$$T(x_{n*}) \geqq \tau(x_{n*}) - \varepsilon. \tag{3.12}$$

Finally, by Lemma 2, (3.11) and (3.12), one has

$$T \geqq T(x_{n*}) + T(x_n^*) \geqq \tau(x_{n*}) + \tau(x_n^*) - 2\varepsilon,$$

which, for $n$ large, is in contradiction with (3.10). Lemma 1 then concludes the proof. □

**4. Some estimates for the time-maps.** We consider a continuous map $\phi: \mathbb{R} \to \mathbb{R}$ satisfying

$$\lim_{|x| \to \infty} \phi(x)\,\mathrm{sign}(x) = +\infty. \tag{h$_1$}$$

Denoting by $\Phi$ a primitive of $\phi$, we define $\tau(x)$ as in (2.3) and, correspondingly,

$$\tau_\pm = \liminf_{x \to \pm\infty} \tau(x), \quad \tau^\pm = \limsup_{x \to \pm\infty} \tau(x).$$

As mentioned in the introduction, Opial [19] found many useful estimates for $\tau_\pm$ and $\tau^\pm$. This section can be considered as a complement of [19]: we obtain some new estimates which can be combined with the ones in [19] and Theorem 1 in order to obtain existence results for our problem.

The proofs of the following results will be carried out only for the estimates of $\tau_+$ and $\tau^+$, as the ones for $\tau_-$ and $\tau^-$ are completely similar.

Define, for every $\eta > 0$ and $x \in \mathbb{R}$, the set $A_x^\eta$, as follows:

$$A_x^\eta = \{s \in [0, x] : \Phi(x) - \Phi(s) \leqq \tfrac{1}{2}\eta(x^2 - s^2)\}, \quad \text{for } x \geqq 0,$$

$$A_x^\eta = \{s \in [x, 0] : \Phi(x) - \Phi(s) \leqq \tfrac{1}{2}\eta(x^2 - s^2)\}, \quad \text{for } x \leqq 0.$$

We denote by $|A_x^\eta|$ the Lebesgue measure of $A_x^\eta$.

**Proposition 1.** *Assume*

$$\limsup_{x\to\pm\infty}(|A_x^\eta|/|x|) \geqq L^\pm \quad (\textit{resp.}\ \liminf_{x\to\pm\infty}(|A_x^\eta|/|x|) \geqq L_\pm).$$

*Then* $\tau^\pm \geqq \dfrac{2}{\sqrt{\eta}}\arcsin L^\pm$ (*resp.* $\tau_\pm \geqq \dfrac{2}{\sqrt{\eta}}\arcsin L_\pm$).

Proof. For $x \geqq 0$ sufficiently large, we have

$$\tau(x) = \sqrt{2}\int_0^x \frac{ds}{\sqrt{\Phi(x)-\Phi(s)}} \geqq \sqrt{2}\int_{A_x^\eta} \frac{ds}{\sqrt{\Phi(x)-\Phi(s)}} \geqq \frac{2}{\sqrt{\eta}}\int_{A_x^\eta}\frac{ds}{\sqrt{x^2-s^2}}.$$

Now, since $s \to (x^2-s^2)^{-1/2}$ is an increasing function, we have (see [3])

$$\tau(x) \geqq \frac{2}{\sqrt{\eta}}\int_0^{|A_x^\eta|}\frac{ds}{\sqrt{x^2-s^2}} = \frac{2}{\sqrt{\eta}}\arcsin(|A_x^\eta|/x).$$

Finally,

$$\tau^+ \geqq \frac{2}{\sqrt{\eta}}\arcsin\left\{\limsup_{x\to+\infty}(|A_x^\eta|/x)\right\} \geqq \frac{2}{\sqrt{\eta}}\arcsin L^+,$$

$$\tau_+ \geqq \frac{2}{\sqrt{\eta}}\arcsin\left\{\liminf_{x\to+\infty}(|A_x^\eta|/x)\right\} \geqq \frac{2}{\sqrt{\eta}}\arcsin L_+. \quad \square$$

**Corollary 1.** *Assume that for certain positive constants* $\varrho_+, \varrho_-$ *one has*

$$\limsup_{x\to\pm\infty}(\phi(x)/x) \leqq \varrho_\pm.$$

*Then* $\tau_\pm \geqq \pi/\sqrt{\varrho_\pm}$.

Proof. Take $\eta = \varrho_+ + \varepsilon$, with $\varepsilon > 0$. Since the function $\frac{1}{2}\eta\xi^2 - \Phi(\xi)$ is increasing and unbounded for $\xi$ positive and large, one has that $|A_x^\eta| = x$ for $x$ positive and large, and so $\liminf_{x\to+\infty}(|A_x^\eta|/x) = 1$. By Proposition 1, $\tau_+ \geqq \pi/\sqrt{\eta}$, and the result follows by letting $\varepsilon$ tend towards zero. $\square$

**Corollary 2.** *Assume that for certain positive constants* $\varrho_+, \varrho_-$ *one has*

$$\liminf_{x\to\pm\infty}(2\Phi(x)/x^2) \leqq \varrho_\pm.$$

*Then* $\tau^\pm \geqq \pi/\sqrt{\varrho_\pm}$.

Proof. As above, take $\eta = \varrho_+ + \varepsilon$, with $\varepsilon > 0$. Since $\limsup_{\xi\to+\infty}\{\frac{1}{2}\eta\xi^2 - \Phi(\xi)\} = +\infty$, there is an increasing sequence $x_n \to +\infty$ such that $|A_{x_n}^\eta| = x_n$, and so $\limsup_{x\to+\infty}(|A_x^\eta|/x) = 1$. By Proposition 1, $\tau^+ \geqq \pi/\sqrt{\eta}$, and the result follows by letting $\varepsilon$ tend towards zero. $\square$

R e m a r k. Opial [19] proved that when in the above the limit exists and

$$\lim_{x \to \pm\infty} (2\Phi(x)/x^2) \leqq \varrho_\pm,$$

then one has the stronger conclusion $\tau_\pm \geqq \pi/\sqrt{\varrho_\pm}$.

**Corollary 3.** *Assume that for certain positive constants $\varrho_+, \varrho_-$ one has*

$$\liminf_{x \to \pm\infty} (\phi(x)/x) \leqq \varrho_\pm.$$

*If moreover the function $\phi(\xi) - \varrho_\pm \xi$ is nondecreasing for $|\xi|$ large enough, then $\tau^\pm \geqq \pi/\sqrt{\varrho_\pm}$.*

P r o o f. We show that, in this case, $\liminf_{x \to +\infty} (2\Phi(x)/x^2) = \liminf_{x \to +\infty} (\phi(x)/x) := \varrho$, so that we are in the situation of Corollary 2. Indeed, let $x_n \to +\infty$ be such that $\phi(x_n)/x_n \to \varrho$. Since we always have that

$$\varrho = \liminf_{x \to +\infty} (\phi(x)/x) \leqq \liminf_{x \to +\infty} (2\Phi(x)/x^2),$$

it is sufficient to prove that $\liminf_{n \to \infty} (2\Phi(x_n)/x_n^2) \leqq \varrho$. Let $r \geqq 0$ be such that the map $x \to \phi(x) - \varrho x$ is nondecreasing for $x \geqq r$. Then, for $x_n \geqq r$, by the mean value theorem,

$$2\Phi(x_n) - \varrho x_n^2 \leqq (2\Phi(r) - \varrho r^2) + (x_n - r)(2\phi(x_n) - 2\varrho x_n).$$

Dividing by $x_n^2$ we finally get $\liminf_{n \to \infty} (2\Phi(x_n)/x_n^2) \leqq \varrho$. □

We observe that in the proof of Proposition 1, the configuration of the set $A_x^\eta$ does not play any role. Actually, better estimates for $\tau^\pm$ can be obtained whenever further information on the structure of the set $A_x^\eta$ is available. A result in this direction is the following (where we consider only the estimate for $\tau^+$, being the corresponding one for $\tau^-$ completely symmetrical).

**Proposition 2.** *Assume there exist two positive sequences $(a_n)$ and $(x_n)$ such that $a_n \leqq x_n$, $x_n \to +\infty$ and $[a_n, x_n] \subset A_{x_n}^\eta$. Then,*

$$\tau^+ \geqq \frac{2}{\sqrt{\eta}} \arccos \left\{ \liminf_{n \to \infty} (a_n/x_n) \right\}.$$

P r o o f. Arguing as in Proposition 1, we have

$$\tau(x_n) \geqq \frac{2}{\sqrt{\eta}} \int_{A_{x_n}^\eta} \frac{ds}{\sqrt{x_n^2 - s^2}} \geqq \frac{2}{\sqrt{\eta}} \int_{a_n}^{x_n} \frac{ds}{\sqrt{x_n^2 - s^2}} = \frac{2}{\sqrt{\eta}} \arccos(a_n/x_n),$$

and the result follows by considering the "limsup" and using the fact that the function "arccos" is decreasing. □

**Corollary 4.** *Assume $\phi(\xi)$ to be continuously differentiable for $\xi > 0$ sufficiently large, and such that, for a certain positive constant $\varrho_+$,*

$$\liminf_{x \to +\infty} (\phi(x)/x) \leqq \varrho_+ .$$

*If moreover there exists $M > 1$ such that*

$$\phi'(x) \leqq \varrho_+ + M\{(\phi(x)/x) - \varrho_+\}$$

*when $x > 0$ and $\phi(x) - \varrho_+ x > 0$ are sufficiently large, then*

$$\tau^+ \geqq (2/\sqrt{\varrho_+}) \arccos\{(M-1)/M\}.$$

Proof. Let $(a_n)$ be such that $a_n \to +\infty$ and $\phi(a_n)/a_n \to \liminf_{x \to +\infty} (\phi(x)/x)$. Fix $s > 1$, $\varepsilon > 0$, and set $k = \varrho_+ + Ms\varepsilon$, $x_n = a_n(k - (\varrho_+ + \varepsilon))/(k - (\varrho_+ + s\varepsilon))$. We want to show that the interval $[a_n, x_n]$ is contained in $A_{x_n}^{\varrho_+ + s\varepsilon}$, for sufficiently large $n$ (such that $(\phi(a_n)/a_n) < \varrho_+ + \varepsilon$). To this end, we prove that

$$(\phi(\xi)/\xi) \leqq \varrho_+ + s\varepsilon \quad \text{for all } \xi \in [a_n, x_n].$$

Indeed, assume by contradiction that there is $\xi_1 \in ]a_n, x_n]$ such that $(\phi(\xi_1)/\xi_1) > \varrho_+ + s\varepsilon$. By construction, the line $\mathscr{L}$ joining $(a_n, (\varrho_+ + \varepsilon)a_n)$ to $(x_n, (\varrho_+ + s\varepsilon)x_n)$ (which has slope $k$) lies below the line $y = (\varrho_+ + s\varepsilon)x$, for all $x \in [a_n, x_n]$. Then, there is a last point $\xi_0 \in ]a_n, \xi_1[$ such that $(x, \phi(x))$ lies above $\mathscr{L}$ for all $\xi_0 \leqq x \leqq \xi_1$. Hence $\phi'(\xi_0) \geqq k$, and $\varrho_+ + \varepsilon < (\phi(\xi_0)/\xi_0) < \varrho_+ + s\varepsilon$. From the hypothesis we then have $\phi'(\xi_0) \leqq \varrho_+ + M((\phi(\xi_0)/\xi_0) - \varrho_+) < \varrho_+ + Ms\varepsilon = k$, which is a contradiction.

Hence $[a_n, x_n]$ is contained in $A_{x_n}^{\varrho_+ + s\varepsilon}$ for sufficiently large $n$, and we have, by Proposition 2,

$$\begin{aligned}\tau^+ &\geqq \frac{2}{\sqrt{\varrho_+ + s\varepsilon}} \arccos\{\lim (a_n/x_n)\} \\ &= \frac{2}{\sqrt{\varrho_+ + s\varepsilon}} \arccos\{s(M-1)/(sM-1)\}.\end{aligned}$$

The result now follows by letting $\varepsilon$ tend towards zero and $s$ tend towards infinity, in such a way that $s\varepsilon \to 0$. □

**5. Periodic solutions under one-sided growth restrictions.** In this section we present some applications of Theorem 1 in which the explicit estimates for the time maps, obtained in Section 4, are exploited. For simplicity, we confine ourselves to the solvability of

$$x'' + g(x) = p(t) \tag{5.1}$$

$$x(T) - x(0) = x'(T) - x'(0) = 0. \tag{5.2}$$

We assume that $p \in L^1([0, T], \mathbb{R})$ and $g: \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying, for some $d \geqq 0$,

$$(\mathrm{g}_0) \qquad (g(x) - \bar{p})x \geqq 0 \quad \text{for } |x| \geqq d,$$

with $\bar{p} = \frac{1}{T}\int_0^T p(s)\,ds$.

Throughout the section we confine ourselves to the use of one-sided conditions, in order to show how several different results can be unified by our approach.

At first, we give a consequence of Proposition 1. Accordingly, we define, for $\eta > 0$ and $x \geqq 0$, the set

$$B_x^\eta = \{s \in [0, x]: G(x) - G(s) \leqq \tfrac{1}{2}\eta(x^2 - s^2)\},$$

where $G(x) = \int_0^x g(s)\,ds$.

**Theorem 2.** *Assume* $(\mathrm{g}_0)$ *and suppose that, for some* $\eta \in ]0, (\pi/T)^2[$,

$$(5.3) \qquad \limsup_{x \to +\infty} (|B_x^\eta|/x) > \sin(\tfrac{1}{2}T\sqrt{\eta}).$$

*Then problem* (5.1)–(5.2) *has a solution.*

Proof. We fix $\varepsilon > 0$ such that $\eta' := \eta + \varepsilon < (\pi/T)^2$ and

$$\limsup_{x \to +\infty} (|B_x^\eta|/x) > \sin(\tfrac{1}{2}T\sqrt{\eta'}).$$

Define $\phi(x) = g(x) + \varepsilon x$; correspondingly we have $\Phi(x) = G(x) + \frac{1}{2}\varepsilon x^2$, so that $B_x^\eta = A_x^{\eta'}$, with $A_x^{\eta'}$ defined as in Section 4. From Proposition 1, we then have

$$\tau^+ \geqq \frac{2}{\sqrt{\eta'}} \arcsin\left\{\limsup_{x \to +\infty} (|B_x^\eta|/x)\right\} > \frac{2}{\sqrt{\eta'}} \arcsin\left\{\sin\left(\frac{1}{2}T\sqrt{\eta'}\right)\right\} = T,$$

and the result follows from Theorem 1. □

In order to verify (5.3), one can proceed as in the proofs of Corollaries 2 and 3. In this way we immediately get:

**Corollary 5** ([6]). *Problem* (5.1)–(5.2) *has a solution if*

$$\liminf_{x \to +\infty} (2G(x)/x^2) < (\pi/T)^2.$$

**Corollary 6.** *Assume*

$$(5.4) \qquad \liminf_{x \to +\infty} (g(x)/x) = \varrho < (\pi/T)^2$$

*and that the map* $x \to g(x) - \varrho x$ *is nondecreasing for sufficiently large positive* $x$. *Then problem* (5.1)–(5.2) *has a solution.*

It has been shown in [5] by means of a counterexample that condition (5.4) alone is not sufficient to guarantee the existence of $T$-periodic solutions of (2.1). We will now give another sufficient condition to be added to (5.4) in order to have such an existence, which is an improvement of a theorem of Ding, Iannacci and Zanolin [4].

**Corollary 7.** *Assume that $g(\xi)$ is continuously differentiable for $\xi > 0$ sufficiently large, and there exists $\varrho \in ]0, (\pi/T)^2[$ such that*

$$\liminf_{x \to +\infty} (g(x)/x) \leqq \varrho .$$

*Assume moreover that there exists $M \in ]1, (1 - \cos(\frac{1}{2} T \sqrt{\varrho}))^{-1}[$ (where $0^{-1} = +\infty$) such that*

$$g'(x) \leqq \varrho + M\left(\frac{g(x)}{x} - \varrho\right),$$

*for $x > 0$ and $g(x) - \varrho x > 0$ large enough. Then problem (5.1)–(5.2) has a solution.*

The proof of Corollary 7 uses the following Theorem 3 together with the estimates of Corollary 4.

**Theorem 3.** *Assume* ($g_0$) *and suppose that there is $\eta \in ]0, (\pi/T)^2[$ and there are two positive sequences $(a_n)$ and $(x_n)$ such that $a_n \leqq x_n$, $x_n \to +\infty$ and $[a_n, x_n] \subset B^{\eta}_{x_n}$. If*

$$\lim (a_n/x_n) < \cos(\tfrac{1}{2} T \sqrt{\eta}),$$

*then problem (5.1)–(5.2) has a solution.*

**6. Related results for two-point BVP's.** We briefly outline one of the results which can be obtained through conditions on the time maps in the line of the preceding sections. We consider the two-point boundary value problem on the interval $[a, b]$

$$(6.1) \qquad x'' + f(t, x) = p(t), \quad x(a) = r_1, \quad x(b) = r_2,$$

with $r_1, r_2 \in \mathbb{R}$ and $f: [a, b] \times \mathbb{R} \to \mathbb{R}$ and $p: [a, b] \to \mathbb{R}$ satisfying the same regularity assumptions of Section 2. Then, we have the following.

**Theorem 4.** *Assume* (H) *holds. Then problem* (6.1) *has a solution provided that* $\min\{\tau^-, \tau^+\} > b - a$.

The proof follows the lines of [7, 14], showing that there exists a set $\Gamma = \{x \in C([a, b]) : -A < x(t) < B$ for every $t \in [a, b]\}$ such that no possible solution of the problem

$$(6.2_\lambda) \qquad x'' + \lambda f(t, x) = \lambda p(t), \quad x(a) = \lambda r_1, \quad x(b) = \lambda r_2,$$

with $\lambda \in ]0, 1[$, belongs to the boundary of $\Gamma$. The constant $B$ (and, in a similar way, the contant $-A$) is obtained by chosing $B = x_n^*$ for $n$ sufficiently large, where $x_n^* \to +\infty$ and $\tau(x_n^*) \to \tau^+$. Arguing as in the proof of Theorem 1, one has that no solution $x_\lambda$ of $(6.2_\lambda)$ is such that $\max x_\lambda = x_n^*$.

## References

[1] I. BERNSTEIN and A. HALANAY, The index of a critical point and the existence of periodic solutions to a system with small parameter. Dokl. Akad. Nauk. SSSR **111**, 923–925 (1956).

[2] E. N. DANCER, Boundary value problems for weakly nonlinear ordinary differential equations. Bull. Austral. Math. Soc. **15**, 321–328 (1976).

[3] D. De Figueiredo and J. P. Gossez, Nonresonance below the first eigenvalue for a semilinear elliptic problem. Math. Ann. **281**, 589–610 (1988).
[4] T. Ding, R. Iannacci and F. Zanolin, On periodic solutions of sublinear Duffing equations. J. Math. Anal. Appl. **158**, 316–322 (1991).
[5] T. Ding, R. Iannacci and F. Zanolin, Existence and multiplicity results for periodic solutions of semilinear Duffing equations. J. Differential Equations, to appear.
[6] M. L. Fernandes and F. Zanolin, Periodic solutions of a second order differential equation with one-sided growth restrictions on the restoring term. Arch. Math. **51**, 151–163 (1988).
[7] M. L. Fernandes, P. Omari and F. Zanolin, On the solvability of a semilinear two-point BVP around the first eigenvalue. Differential Integral Equations **2**, 63–79 (1989).
[8] S. Fucik, Solvability of Nonlinear Equations and Boundary Value Problems. Dordrecht, 1980.
[9] R. Gaines and J. Mawhin, Coincidence Degree and Nonlinear Differential Equations. LNM **568**, Berlin-Heidelberg-New York 1977.
[10] R. E. Gomory, Critical points at infinity and forced oscillations. In: Contributions to the theory of nonlinear oscillations, vol. 3. Ann. of Math. Stud. **36**, 85–126 (1956).
[11] M. A. Krasnosel'skii, The operator of Translation along Trajectories of Differential Equations. Providence, R.I. 1968.
[12] M. A. Krasnosel'skii and A. I. Perov, Existence of solutions for certain nonlinear operator equations. Dokl. Akad. Nauk. SSSR **126**, 15–18 (1959).
[13] M. A. Krasnosel'skii, A. I. Perov, A. I. Povolockii and P. P. Zabreiko, Plane Vector Fields. New York 1966.
[14] J. Mawhin, Nonlinear variational two-point boundary value problems. In: Proceedings of the conference "Variational Methods" Paris 1988, H. Beresticki, J.-H. Coron, I. Ekeland, ed., 209–219, Basel-Boston 1990.
[15] J. Mawhin and J. R. Ward, Periodic solutions of some forced Lienard differential equations at resonance. Arch. Math. **41**, 337–351 (1983).
[16] J. Mawhin and M. Willem, Critical Point Theory and Hamiltonian Systems. Berlin-Heidelberg-New York 1989.
[17] P. Omari, G. Villari and F. Zanolin, Periodic solutions of the Lienard equation with one-sided growth restrictions. J. Differential Equations **67**, 278–293 (1987).
[18] Z. Opial, Sur les solutions periodiques de l'equation differentielle $x'' + g(x) = p(t)$. Bull. Acad. Polon. Sci. Sér. Sci. Math., Astronom. Phys. **8**, 151–156 (1960).
[19] Z. Opial, Sur les periodes des solutions de l'equation differentielle $x'' + g(x) = 0$. Ann. Polon. Math. **10**, 49–72 (1961).
[20] Z. Opial, Sur l'existence des solutions periodiques de l'equation differentielle $x'' + f(x, x')x' + g(x) = p(t)$. Ann. Polon. Math. **11**, 149–159 (1961).
[21] R. Reissig, Periodic solutions of a second order differential equation including a one-sided restoring term. Arch. Math. **33**, 85–90 (1979).
[22] R. Reissig, G. Sansone und R. Conti, Qualitative Theorie Nichtlinearer Differentialgleichungen. Cremonese, Roma 1963.
[23] K. Schmitt, Periodic solutions of a forced nonlinear oscillator involving a one-sided restoring force. Arch. Math. **31**, 70–73 (1978).
[24] J. R. Ward, Periodic solutions for systems of second order ordinary differential equations. J. Math. Anal. Appl. **81**, 92–98 (1981).

Eingegangen am 3. 10. 1989

Anschriften der Autoren:

A. Fonda
Dipartimento di Scienze Matematiche
Università Trieste
P. le Europa 1
34100 Trieste
Italy

F. Zanolin
Dipartimento di Matematica e Informatica
Università
Via Zanon 6
33100 Udine
Italy