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Characteristic shocks of crossing velocities in Magnetohydrodynamics

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Abstract

We illustrate, in the framework of magnetohydrodynamics, an application of crossing shock formulas to a characteristic shock moving perpendicularly to the magnetic field.

1 Introduction

Consider the *conservative hyperbolic* system of N first order partial differential equations for the unknown *field* $\mathbf{u}(t, x^i)$

$$\partial_t \mathbf{u} + \partial_i \mathbf{f}^i(\mathbf{u}) = \mathbf{f}(\mathbf{u}), \qquad (i = 1, 2, 3) \tag{1}$$

with M *involutive constraints*

$$\partial_i \mathbf{c}^i(\mathbf{u}) = \mathbf{c}(\mathbf{u}) \tag{2}$$

Let (1) supplemented by an additional conservation law (energy or entropy law)

$$\partial_t h(\mathbf{u}) + \partial_i h^i(\mathbf{u}) = g(\mathbf{u}) \tag{3}$$

where $h(\mathbf{u})$ is a *convex* function of $\mathbf{u}$. The equation (3) can be obtained by multiplying (1) and (2) by suitable N and M vectors $\mathbf{u}'$ and $\mathbf{b}$ respectively.

When a wave velocity λ of (1) is exceptional (see § 3) a characteristic shock may propagate with this velocity. Independently of the number of equations, the jump of the field $\mathbf{u}$

$$[\mathbf{u}] \stackrel{\text{def}}{=} \mathbf{u} - \mathbf{u}_o$$

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i.e. the difference of its values on the back and on the front of the shock surface, can be expressed as a combination of at most $m+M+2$ known vectors (m multiplicity of λ, M number of constraints).

The simplest case is that of linear equations where the $\mathbf{f}^i$'s of (1) are linear functions of the field $\mathbf{u}$

$$\mathbf{f}^i = A^i\mathbf{u}$$

so that the Rankine-Hugoniot equations (see (14) in § 4) are just

$$(A_n - \lambda I)(\mathbf{u} - \mathbf{u}_o) = 0$$

which shows that the jump

$$[\mathbf{u}] = \alpha^I \mathbf{d}_I$$

depends linearly on m parameters α^I corresponding to the multiplicity of the eigenvalue.

When the conservative system (1) (with an additional law (3)) is non linear it is still possible to solve the Rankine-Hugoniot equations and to give a simple expression for the jump of the *main field* $\mathbf{u}'$ (§ 2)

$$[\breve{\mathbf{u}}'] \stackrel{\text{def}}{=} \breve{\mathbf{u}}' - \breve{\mathbf{u}}'_o = \alpha^I \mathbf{l}_{I^o} + w\mathbf{g}_o$$

where $\mathbf{g}$ and the left eigenvectors $\mathbf{l}_I$ are calculated for the value $\mathbf{u}_o$ of the field before the shock. The scalar w, a non linear function of the parameters α^I, represents the non linear part of the shock. In particular this formula appears each time a characteristic velocity $\lambda(\mathbf{u}, \vec{n})$ (then exceptional) with constant (i.e. independent of the direction $\vec{n}$) multiplicity exists.

Instead when multiplicity is variable that is when it occurs only for some values of $\vec{n}(\mathbf{u})$ (crossing velocities), a new relative vector, $\mathbf{g}^r$, has to be added and

$$[\breve{\mathbf{u}}'] = \alpha^I \mathbf{l}_{I^o} + w\mathbf{g}_o + \sigma\mathbf{g}_o^r$$

Another adjunction may even be necessary when the involutive constraints (2) are associated with the field equations (1). In this case one has the most complicated expression

$$[\breve{\mathbf{u}}'] = \alpha^I \mathbf{l}_{I^o} - (\mathbf{b} - \mathbf{b}_o)(M_{on} - \lambda_o I)^{-1}\nabla_o \mathbf{c}_{on} + w\mathbf{g}_o + \sigma{\mathbf{g}_o}^r, \quad I = 1, 2, \dots, m \quad (4)$$

where the second term is connected with the presence of constraints.

The aim of this paper is to illustrate this formula with an example taken from classical magnetohydrodynamics: a shock moving with the normal fluid velocity in a direction perpendicular to the magnetic field vector $\vec{B}$.

When the vector $\vec{n}$ is orthogonal to the magnetic field no less than five velocities coincide (§ 8). Further a constraint (div $\vec{B} = 0$) is present so that here eight vectors are necessary to describe the jump of the eight components of the main field. Surely it would have been better to find a physical example with less vectors than components. Nevertheless we show how to compute easily these various vectors without writing down matrices. Also we study in § 3, for a mathematical example, the behaviour of the radial velocities when the corresponding λ's coincide.

2 Some remarks on field equations

As it is well known the introduction of the main field

$$\mathbf{u}' = \frac{\partial h}{\partial \mathbf{u}} \tag{5}$$

allows the field equations (1) to be written in a Friedrichs-Lax-Godunov *symmetric form* [1], [2] by means of the Le Gendre transformation [3]-[8]

$$h'(\mathbf{u}') = \breve{\mathbf{u}}' \cdot \mathbf{u} - h(\mathbf{u}) \tag{6}$$

($\breve{\mathbf{u}}$ is the transpose of $\mathbf{u}$) and the introduction of the quantities

$$h'^i(\mathbf{u}') = \breve{\mathbf{u}}' \cdot \mathbf{f}^i + \mathbf{b} \cdot \mathbf{c}^i - h^i \tag{7}$$

In fact

$$\mathbf{u} = \breve{\nabla}' h', \quad \mathbf{f}^i = \breve{\nabla}' h'^i - \breve{\mathbf{c}}^i \breve{\nabla}' \breve{\mathbf{b}}$$

Further the involutive constraints must be such that [9]

$$\nabla \mathbf{c}_n A_n = M_n \nabla \mathbf{c}_n \tag{8}$$

with constant matrices $\nabla \mathbf{c}^i$ and M^i and

$$A^i \overset{\text{def}}{=} \nabla f^i, \quad A_n \overset{\text{def}}{=} \nabla \mathbf{f}_n, \quad \mathbf{f}_n \overset{\text{def}}{=} \mathbf{f}^i n_i$$

$$\nabla \overset{\text{def}}{=} \left(\frac{\partial}{\partial \mathbf{u}}\right)_{\vec{n}=const.}, \quad \nabla' \overset{\text{def}}{=} \left(\frac{\partial}{\partial \mathbf{u}'}\right)_{\vec{n}=const.}$$

3 Crossing velocities

The wave velocities of propagation in the direction of the unit vector $\vec{n}$ are given by the eigenvalues $\lambda(\mathbf{u}, \vec{n})$ of the matrix A_n. Suppose that some eigenvalue λ has a constant (i.e. independent on $\vec{n}$) multiplicity m and therefore m left and right eigenvectors

$$\mathbf{l}_I(\mathbf{u}, \vec{n})(A_n - \lambda I) = 0, \quad (A_n - \lambda I)\mathbf{d}_I(\mathbf{u}, \vec{n}) = 0, \qquad I = 1, 2, ..., m$$

When $m > 1$ it is known that λ is *exceptional* that is

$$\nabla \lambda \cdot \mathbf{d}_I \equiv 0 \tag{9}$$

As a consequence a *characteristic shock* exists which propagates with velocity λ.

On the other hand if the number of equations (1) is $N = \pm 2, \pm 3, \pm 4$ (mod 8) [10], [11] the eigenvalues of A_n cannot all be simple for every $\vec{n}$. Although these eigenvalues may still have constant multiplicity as for instance in the banal case of

$$A_n = a_n I$$

it may happen (in the so-called case of *variable multiplicity*) that two (or more) eigenvalues, say, $\lambda^{(1)}(\mathbf{u}, \vec{n})$ and $\lambda^{(2)}(\mathbf{u}, \vec{n})$ coincide only for some $\vec{n} = \vec{n}_o(\mathbf{u})$

$$\lambda^{(1)}(\mathbf{u}, \vec{n}_o) = \lambda^{(2)}(\mathbf{u}, \vec{n}_o)$$

For the corresponding radial velocities (propagating the weak disturbances)

$$\vec{\Lambda}^{(i)}(\mathbf{u}, \vec{n}) \stackrel{\text{def}}{=} \lambda^{(i)}\vec{n} + \frac{\partial \lambda^{(i)}}{\partial \vec{n}} - \left(\vec{n} \cdot \frac{\partial \lambda^{(i)}}{\partial \vec{n}}\right)\vec{n} \tag{10}$$

let us define

$$\vec{\Omega}(\mathbf{u}, \vec{n}_o) = \lim_{\vec{n} \to \vec{n}_o} \left\{\vec{\Lambda}^{(1)}(\mathbf{u}, \vec{n}) - \vec{\Lambda}^{(2)}(\mathbf{u}, \vec{n})\right\}$$

It follows that several cases are possible:

1) $\vec{\Omega}$ does not exist

2) $\vec{\Omega} = 0$

3) $\vec{\Omega} \neq 0$

which are illustrated with the following mathematical example.

Let

$$A^i = \begin{bmatrix} a^i & b^i & 0 \\ b^i & c^i & 0 \\ 0 & 0 & (a^i + c^i)/2 + b^i \end{bmatrix}$$

then

$$\lambda^{(1)/(2)} = \frac{1}{2}(a_n + c_n \pm \sqrt{\Delta}), \quad \Delta = (a_n - c_n)^2 + 4b_n^2, \quad a_n = a^i n_i = \vec{a} \cdot \vec{n}$$

$$\lambda^{(3)} = \frac{a_n + c_n}{2} + b_n$$

and

$$\vec{\Lambda}^{(1)/(2)} = \frac{1}{2}\left(\vec{a} + \vec{c} \pm \frac{(a_n - c_n)(\vec{a} - \vec{c}) + 4b_n\vec{b}}{\sqrt{\Delta}}\right) \tag{11}$$

$$\vec{\Lambda}^{(3)} = \frac{\vec{a} + \vec{c}}{2} + \vec{b} \tag{12}$$

Case 1) When $\vec{n}$ tends to $\vec{n}_o$ parallel to

$$(\vec{a} - \vec{c}) \wedge \vec{b}$$

the three velocities $\lambda^{(1)}$, $\lambda^{(2)}$, $\lambda^{(3)}$ tend to a_n but $\vec{\Lambda}^{(1)/(2)}$ have no limit and no $\vec{\Omega}$ exists.

Case 2) If

$$(\vec{a}-\vec{c})\cdot\vec{n}=0, \quad b_n>0$$

two velocities coincide

$$\lambda^{(1)}=\lambda^{(3)}=a_n+b_n$$

and since for such an $\vec{n}$, by (11), (12)

$$\vec{\Lambda}^{(1)}=\vec{\Lambda}^{(3)}=\frac{\vec{a}+\vec{c}}{2}+\vec{b}$$

$$\vec{\Omega}=\vec{\Lambda}^{(1)}-\vec{\Lambda}^{(3)}=0$$

In these first two cases the differences of velocities $\lambda^{(i)}-\lambda^{(j)}$ do not change sign when $\vec{n}$ goes over $\vec{n}_o$.

Case 3) Suppose $\vec{c}=\vec{a}$.

$$\lambda^{(1)}=\lambda^{(3)}=a_n+b_n, \qquad \lambda^{(2)}=a_n-b_n$$

$$\vec{\Lambda}^{(1)}=\vec{\Lambda}^{(3)}=\vec{a}+\vec{b}, \qquad \vec{\Lambda}^{(2)}=\vec{a}-\vec{b}$$

so that when b_n tends to zero , $\lambda^{(1)}$,$\lambda^{(2)}$, $\lambda^{(3)}$ tend to a_n and $\vec{\Omega}$ is different from zero

$$\vec{\Lambda}^{(1)}-\vec{\Lambda}^{(2)}=2\vec{b}$$

The velocities cross: $\lambda^{(1)}-\lambda^{(2)}=2b_n$ changes sign with b_n.

4 Explicit shock expression

A natural question to ask is whether the exceptional property (9) still holds for variable multiplicity i.e. when $\vec{n}=\vec{n}_o$. It turns out [12] then that the important exceptionality is not so much that of λ but rather that of the difference of the velocities

$$\nabla(\lambda^{(1)}-\lambda^{(2)})\cdot\mathbf{d}_I=0, \quad I=1,2,...,m \tag{13}$$

which implies the former one (9).

As already shown, a characteristic shock depending on m parameters propagating in a direction $\vec{n}_o$ is possible when the conditions (13) are satisfied. This shock is obtained by solving the Rankine-Hugoniot equations involving the fields $\mathbf{u}_o$ (*unperturbed* field) and $\mathbf{u}$ (*perturbed* field) ahead and behind the shock front

$$\mathbf{f}_n(\mathbf{u})-\lambda_o\mathbf{u}=\mathbf{f}_n(\mathbf{u}_o)-\lambda_o\mathbf{u}_o, \qquad \mathbf{c}_n(\mathbf{u})=\mathbf{c}_n(\mathbf{u}_o) \tag{14}$$

together with

$$\vec{n}(\mathbf{u})=\vec{n}(\mathbf{u}_o)=\vec{n}_o, \qquad \lambda(\mathbf{u},\vec{n}_o)=\lambda(\mathbf{u}_o,\vec{n}_o)=\lambda_o \tag{15}$$

Quite generally, when $\vec{\Omega} \neq 0$, an explicit solution of these equations is given in terms of the *jump* of the main field [13]

$$[\breve{\mathbf{u}}'] = \alpha^I \mathbf{l}_{I^o} - (\mathbf{b} - \mathbf{b}_o)(M_{on} - \lambda_o I)^{-1} \nabla_o \mathbf{c}_{on} + w\mathbf{g}_o + \sigma \mathbf{g}_o{}^r, \quad I = 1, 2, ..., m \quad (16)$$

All quantities with the subscript zero are calculated for $\mathbf{u} = \mathbf{u}_o$ and therefore depend only on the state before the shock.

In (16) the first term, where α^I are m arbitrary parameters, represents a linear part of the shock while the second term is connected with the presence of constraints. The positive scalar

$$w = [h'] - \breve{\mathbf{u}}_o[\mathbf{u}'] \quad (17)$$

vanishes only when the shock does, $\mathbf{g}$ and $\mathbf{g}^r$ are defined by the following formulas

$$\mathbf{g}(A_n - \lambda I) = -\nabla\lambda, \quad \mathbf{g}\mathbf{d}_I = 0, \quad \mathbf{g}^r(A_n - \lambda I) = -\nabla(\lambda^1 - \lambda^2), \quad \mathbf{g}^r\mathbf{d}_I = 0. \quad (18)$$

Thus to the usual expression of a characteristic shock with constant multiplicity, an additional term, the last one of (16) is added which is specific of crossing eigenvalues.

The scalar σ comes from the general equality

$$\phi^i \stackrel{\text{def}}{=} \breve{\mathbf{u}}'[\mathbf{f}^i] - [h^i] + \mathbf{b}[\mathbf{c}^i] - w\Lambda^i_o \equiv [h'^i] - [\breve{\mathbf{u}}']\mathbf{f}^i_o - [\mathbf{b}]\mathbf{c}^i_o - w\Lambda^i_o = \sigma\Omega^i_o \quad (19)$$

and, as well as w, is a function of the m parameters α^I.

In next sections we consider (16) in the case of magnetohydrodynamics by computing its termes.

5 The conservative form of the equations of magnetohydrodynamics

The equations describing the magnetohydrodynamic motions are [14], [15]

$$\partial_t \rho + \operatorname{div}(\rho\vec{v}) = 0 \quad (20)$$

$$\partial_t \vec{B} - \operatorname{rot}(\vec{v} \wedge \vec{B}) = 0 \quad (21)$$

$$\partial_t(\rho\vec{v}) + \operatorname{div}\vec{\Pi} = 0 \quad (22)$$

$$\partial_t E + \operatorname{div}\vec{q} = 0 \quad (23)$$

and express the conservation of the mass ρ (20), of the momentum $\rho\vec{v}$ (22), of the total energy E (23) and the evolution of the magnetic field $\vec{B}$ (21). To the system (20)-(23) must be added the involutive constraint

$$\operatorname{div}\vec{B} = 0 \quad (24)$$

The quantities $\vec{\Pi}$, E, $\vec{q}$, appearing in (22),(23) are defined:

$$\begin{aligned}\vec{\Pi} &= (p+\frac{B^2}{2})I - \vec{B}\otimes\vec{B} + \rho\vec{v}\otimes\vec{v}, \quad \vec{q} = \rho\vec{v}\left(e + \frac{p}{\rho} + \frac{v^2}{2}\right) + \vec{B}\wedge(\vec{v}\wedge\vec{B}) \\ E &= \rho\frac{v^2}{2} + \rho e + \frac{B^2}{2}\end{aligned}$$

where $\vec{v}$, p are respectively the velocity and the pressure of the fluid; e is the density of internal energy satisfying

$$de = TdS + pd\rho/\rho^2$$

with entropy S and temperature T.

The system (20)-(23) with (24) is of the form (1) and (2) without the second members and

$$\mathbf{u} = \begin{bmatrix}\rho \\ \vec{B} \\ \rho\vec{v} \\ E\end{bmatrix}, \qquad \mathbf{f}^i = \begin{bmatrix}\rho v^i \\ v^i\vec{B} - B^i\vec{v} \\ \left(p + \frac{B^2}{2}\right)\vec{e}^i - B^i\vec{B} + \rho v^i\vec{v} \\ \left(E + p + \frac{B^2}{2}\right)v^i - (\vec{v}\cdot\vec{B})B^i\end{bmatrix}, \qquad \mathbf{c}^i = B^i$$

By taking as supplementary equation (3) the conservation of entropy

$$\partial_t(\rho S) + \operatorname{div}(\rho S\vec{v}) = 0$$

one has

$$h(\mathbf{u}) = -\rho S, \qquad h^i(\mathbf{u}) = -\rho S v^i \qquad g(\mathbf{u}) = 0$$

and therefore [16] by (5)-(7)

$$\breve{\mathbf{u}}' = \frac{1}{T}\left(G - \frac{v^2}{2}, \vec{B}, \vec{v}, -1\right), \quad h' = \frac{1}{T}\left(p + \frac{B^2}{2}\right), \quad h'^i = \frac{v^i}{T}\left(p + \frac{B^2}{2}\right)$$

where $G = e + p/\rho - TS$ is the free enthalpy. The vector $\mathbf{b}$, in (7), reduces here to a scalar factor of the single constraint div $\vec{B} = 0$ and has already been calculated [16]. It is easy to see that

$$b = \frac{\vec{B}\cdot\vec{v}}{T}$$

6 Wave velocities

The formal substitution

$$\partial_t \to -\lambda\delta, \qquad \partial_i \to n_i\delta$$

permits us to write

$$\delta\mathbf{f}_n - \lambda\delta\mathbf{u} = (A_n - \lambda I)\delta\mathbf{u} = 0 \tag{25}$$

which shows immediately that $\delta\mathbf{u}$ is a linear combination of eigenvectors

$$\delta\mathbf{u} = \beta^I \mathbf{d}_I \tag{26}$$

Calculating the λ 's one finds the Alfvén velocities [14], [15]

$$\lambda^{(1)} = v_n + \frac{B_n}{\sqrt{\rho}}, \qquad \lambda^{(3)} = v_n - \frac{B_n}{\sqrt{\rho}}$$

the contact velocity

$$\lambda^{(2)} = v_n$$

the slow velocities

$$\lambda^{(4)/(5)} = v_n \pm \zeta, \qquad \zeta = \left\{ \frac{1}{2}(c^2 + \frac{B^2}{\rho} - \sqrt{\Delta}) \right\}^{\frac{1}{2}}$$

the fast velocities for which a plus sign precedes the square root of

$$\Delta = (c^2 + \frac{B^2}{\rho})^2 - 4\frac{B_n^2}{\rho}c^2, \qquad c = \sqrt{(\partial p/\partial \rho)_S}$$

and the null velocity.

By (10) the corresponding radial velocities are

$$\vec{\Lambda}^{(1)/(3)} = \vec{v} \pm \frac{\vec{B}}{\sqrt{\rho}}, \quad \vec{\Lambda}^{(2)} = \vec{v}, \quad \vec{\Lambda}^{(4)/(5)}(\vec{n}) = \vec{v} \pm \zeta\vec{n} \pm \frac{c^2 B_n}{\rho\zeta\sqrt{\Delta}}(\vec{B} - B_n\vec{n}) \tag{27}$$

7 Case $\vec{\Omega} = 0$

For

$$\vec{n}_o = \frac{\vec{B}}{|\vec{B}|}$$

the Alfvén velocities become double since

$$\lambda^{(4)/(5)}(\vec{n}_o) = \lambda^{(1)/(3)}(\vec{n}_o) = \vec{v} \cdot \frac{\vec{B}}{|\vec{B}|} \pm \frac{|\vec{B}|}{\sqrt{\rho}}$$

while (27) become

$$\vec{\Lambda}^{(4)/(5)}(\vec{n}_o) = \vec{v} \pm \frac{\vec{B}}{\sqrt{\rho}} = \vec{\Lambda}^{(1)/(3)}$$

and both $\vec{\Omega}$'s are zero

$$\vec{\Lambda}^{(4)} - \vec{\Lambda}^{(1)} = 0, \quad \vec{\Lambda}^{(5)} - \vec{\Lambda}^{(3)} = 0$$

8 Case of crossing velocities: $\vec{\Omega} \neq 0$

From now on we consider this possibility. When $\vec{\Omega}$ is not zero $\vec{n}_o$ is not unique and the velocities cross, that is, their difference changes sign when $\vec{n}$ goes over $\vec{n}_o$. This clearly occurs in magnetohydrodynamics when

$$\vec{B} \cdot \vec{n}_o = 0 \tag{28}$$

Then $\lambda^{(1)}$ to $\lambda^{(5)}$ are all equal to

$$\lambda = v_n \tag{29}$$

Observe that some of these velocities, for instance $\lambda^{(2)}$, $\lambda^{(4)}$, still do not cross; $\lambda^{(4)} - \lambda^{(2)} = \zeta \geq 0$ does not change sign in agreement with the fact that the corresponding $\vec{\Omega}$ is zero: $\vec{\Lambda}^{(4)}(\vec{n}_o) - \vec{\Lambda}^{(2)}(\vec{n}_o) = 0$. However $\lambda^{(1)}$, $\lambda^{(2)}$ do

$$\lambda^{(1)}(\vec{n}) - \lambda^{(2)}(\vec{n}) = \frac{B_n}{\sqrt{\rho}}, \qquad \vec{\Omega} = \vec{\Lambda}^{(1)}(\vec{n}_o) - \vec{\Lambda}^{(2)}(\vec{n}_o) = \frac{\vec{B}}{\sqrt{\rho}}$$

The non vanishing $\vec{\Omega}$'s are proportional to this one as for instance $\vec{\Lambda}^{(4)/(5)}(\vec{n}_o) - \vec{\Lambda}^{(1)/(3)}(\vec{n}_o)$.

It is also easy to verify (13) in magnetohydrodynamics. Although one could check that

$$\nabla(\lambda^{(1)} - \lambda^{(2)})(\vec{n}_o) = (\vec{0}, \frac{\vec{n}_o}{\sqrt{\rho}},, \vec{0}, 0)$$

is orthogonal to the five right eigenvectors $\mathbf{d}_I$ associated to (29) it is easier, by (26), to note that

$$\delta(\lambda^{(1)} - \lambda^{(2)}) = \delta\frac{B_n}{\sqrt{\rho}} = \frac{\delta B_n}{\sqrt{\rho}} = 0$$

according to the second equation (25) and (28)

$$\delta(v_n\vec{B} - B_n\vec{v}) - \lambda\delta\vec{B} = 0$$

multiplied by $\vec{n}$, for $v_n \neq 0$.

Equations (14), (15) read simply [5], [14], [15]

$$p + \frac{B^2}{2} = p_o + \frac{B^2{}_o}{2}, \quad \vec{B} \cdot \vec{n} = \vec{B}_o \cdot \vec{n} = 0, \quad \vec{v} \cdot \vec{n} = \vec{v}_o \cdot \vec{n} = \lambda_o \tag{30}$$

These are three conditions for eight field components leaving five arbitrary quantities.

We are now going to check (16) by computing its various components.

9 The left eigenvectors for $B_n = 0, \lambda = v_n$

To avoid the writing of the matrix A_n we set

$$\mathbf{l} \equiv (x, \vec{X}, \vec{Y}, y) \quad \text{and form the product} \quad \Phi = \mathbf{l}(\mathbf{f}_n - \lambda \mathbf{u}) \,.$$

By taking the variation, keeping $\mathbf{l}$ and λ constant, we get after setting $\lambda = v_n, B_n = 0$

$$(\delta\Phi)_{\lambda=v_n, B_n=0} = \mathbf{l}(A_n - v_n I)\delta\mathbf{u} = 0$$

for any $\delta\mathbf{u}$. Explicitly

$$\begin{aligned} \Phi &= (v_n - \lambda)\{\rho x + \vec{B}\cdot\vec{X} + \rho\vec{v}\cdot\vec{Y} + yE\} - B_n\{\vec{v}\cdot\vec{X} + \vec{B}\cdot\vec{Y} + y\vec{B}\cdot\vec{v}\} \\ &\quad + \left(p + \frac{B^2}{2}\right)(Y_n + yv_n) \end{aligned}$$

and so

$$\delta\Phi = \mathcal{V}\delta v_n + \mathcal{B}\delta B + \mathcal{P}\delta\left(p + \frac{B^2}{2}\right) = 0 \tag{31}$$

with

$$\mathcal{V} = \rho x + \vec{B}\cdot\vec{X} + \rho\vec{v}\cdot\vec{Y} + y(E + p + \frac{B^2}{2})$$

$$\mathcal{B} = -\vec{v}\cdot\vec{X} - \vec{B}\cdot(\vec{Y} + y\vec{v}) \qquad \mathcal{P} = Y_n + yv_n$$

But $\mathcal{V}, \mathcal{B}, \mathcal{P}$ must be zero and therefore yield the solutions

$$\begin{aligned} \mathbf{l}_i &= \left(\frac{v_i}{B^2}\vec{B}\cdot\vec{v} - \frac{1}{\rho}B_i, \quad \vec{e}_i, \quad -\frac{v_i}{B^2}\vec{B}, \quad 0\right) \qquad i = 1,2,3 \\ \mathbf{l}_4 &= \left(\frac{p + B^2}{\rho} + e - \frac{v^2}{2}, \quad \vec{0}, \quad \vec{v}, \quad -1\right) \\ \mathbf{l}_5 &= \{-(\vec{B}\wedge\vec{n})\cdot\vec{v}, \quad \vec{0}, \quad \vec{B}\wedge\vec{n}, \quad 0\} \end{aligned}$$

10 Determination of the absolute and relative $\mathbf{g}'s$

In the same manner by its very definition (18) the components of

$$\mathbf{g} \equiv (x, \vec{X}, \vec{Y}, y)$$

are determined by the conditions

$$\delta\Phi + \delta v_n = 0 \tag{32}$$

for all $\delta\mathbf{u}$ and

$$\mathbf{g}\cdot\delta\mathbf{u}=0 \tag{33}$$

for all $\delta\mathbf{u}$ satisfying

$$\delta v_n=0,\quad \delta B_n=0,\quad \delta\left(p+\frac{B^2}{2}\right)=0 \tag{34}$$

The conditions (32) are the same than (31) except for the first i.e.

$$\mathcal{V}=-1,\quad \mathcal{B}=0,\quad \mathcal{P}=0$$

By (33)

$$(x+\vec{Y}\cdot\vec{v})\delta\rho+\vec{X}\cdot\delta\vec{B}+\rho\vec{Y}\cdot\delta\vec{v}+y\delta E=0 \tag{35}$$

Since for a perfect polytropic fluid

$$p=\mathcal{R}\rho T,\quad p=a\rho^\gamma exp\left(\frac{\gamma-1}{\mathcal{R}}S\right),\quad E=p+\frac{B^2}{2}+\rho\frac{v^2}{2}+\frac{2-\gamma}{\gamma-1}p$$

(35) becomes by (34), with $\vec{v}_T=\vec{v}-v_n\vec{n}$

$$\left(x+\vec{Y}\cdot\vec{v}+y\frac{v^2}{2}\right)\delta\rho+\left(\vec{X}-\frac{2-\gamma}{\gamma-1}y\vec{B}\right)\cdot\delta\vec{B}_T+\rho(\vec{Y}+y\vec{v})\cdot\delta\vec{v}_T=0$$

so that

$$x+\vec{Y}\cdot\vec{v}+y\frac{v^2}{2}=0 \tag{36}$$

$$\vec{X}_T-\frac{2-\gamma}{\gamma-1}y\vec{B}_T=0 \tag{37}$$

$$\vec{Y}_T+y\vec{v}_T=0 \tag{38}$$

which, together with $(34)_1$, give

$$\mathbf{g}\doteq\frac{1-\gamma}{\gamma p+B^2}\left\{\frac{v^2}{2},\quad \frac{2-\gamma}{\gamma-1}\left(\vec{B}-\frac{\vec{B}\cdot\vec{v}}{v_n}\vec{n}\right),\quad -\vec{v},\quad 1\right\}$$

Now $\mathbf{g}^r\equiv(x,\vec{X},\vec{Y},y)$ satisfies (36) – (38) and

$$\mathcal{V}=0,\quad \mathcal{B}=-\frac{1}{\sqrt{\rho}},\quad \mathcal{P}=0$$

It appears immediately that

$$\mathbf{g}^r=\left(0,\quad \frac{\vec{n}}{v_n\sqrt{\rho}},\quad \vec{0},\quad 0\right)$$

It is to be observed that $\mathbf{g}$ and $\mathbf{g}^r$ by their definition (18) have the dimension of $1/\mathbf{u}$ or equivalently of $\mathbf{u}'/h'$.

Further $c_n=B_n$ and $\nabla c_n=(0,\vec{n},0,0)$ so that $\nabla c_n f_n\equiv 0$ $\nabla c_n A_n\equiv 0$ implying, by (8), $M_n=0$.

11 Jumps of the main field components

Expliciting eqs. (16) we get, putting $\tau = w/(\gamma p_o + B_o^2)$,

$$\begin{aligned}
\left[\frac{G-\frac{v^2}{2}}{T}\right] &= \frac{\vec{\alpha}\cdot\vec{v}_o}{B_o^2}(\vec{B}_o\cdot\vec{v}_o) - \frac{1}{\rho_o}\vec{B}_o\cdot\vec{\alpha} + \frac{\alpha_4}{\rho_o}\left(B_o^2 - \frac{\rho_o v_o^2}{2} + \frac{\gamma}{\gamma-1}p_o\right) + \\
&\quad -\alpha_5(\vec{B}_o\wedge\vec{n})\cdot\vec{v}_o + (1-\gamma)\frac{v_o^2}{2}\tau && (39)\\
\left[\frac{\vec{B}}{T}\right] &= \vec{\alpha} - (\vec{\alpha}\cdot\vec{n})\vec{n} - (2-\gamma)\tau\vec{B}_o && (40)\\
\left[\frac{\vec{v}}{T}\right] &= -\frac{(\vec{\alpha}\cdot\vec{v}_o)}{B_o^2}\vec{B}_o + \alpha_5(\vec{B}_o\wedge\vec{n}) + \left[\frac{1}{T}\right]\vec{v}_o && (41)\\
\left[\frac{1}{T}\right] &= \alpha_4 + (\gamma-1)\tau && (42)
\end{aligned}$$

taking account of the condition

$$v_{on}\vec{\alpha}\cdot\vec{n} + \left[\frac{\vec{B}\cdot\vec{v}}{T}\right] + (2-\gamma)\tau\vec{B}_o\cdot\vec{v}_o + \frac{\sigma}{\sqrt{\rho_o}} = 0 \qquad (43)$$

which comes, from (30) i.e. $\vec{n}\cdot\left[\frac{\vec{B}}{T}\right] = 0$.

12 Determination of σ and w

Since

$$\left[\frac{\vec{B}\cdot\vec{v}}{T}\right] = \frac{[\vec{B}]\cdot[\vec{v}]}{T} + \vec{B}_o\left[\frac{\vec{v}}{T}\right] + v_o\left[\frac{\vec{B}}{T}\right] - \vec{B}_o\cdot\vec{v}_o\left[\frac{1}{T}\right]$$

equation (43) can be rewritten

$$\sigma = -\sqrt{\rho_o}\frac{[\vec{B}]\cdot[\vec{v}]}{T} \qquad (44)$$

It is a matter of a simple calculation to check (19) with $\Lambda_o^i = v_o^i$.

Substituting eqs. (39)–(42) in (44) one gets the following relation between τ, σ, and the five parameters α

$$\frac{\sigma}{\sqrt{\rho_o}}\left\{\frac{1}{T_o} + \alpha_4 + (\gamma-1)\tau\right\} - \frac{\vec{\alpha}\cdot\vec{v}_o}{B_o^2}(\vec{\alpha}\cdot\vec{B}_o) + (\vec{\alpha}\cdot\vec{v}_o)(\alpha_4+\tau) + \alpha_5(\vec{B}_o\wedge\vec{n})\cdot\vec{\alpha} = 0$$

Another one is needed to determine w and σ. However if one inserts in the definition (17) of w the expressions (39)–(42) one merely obtains an identity.

In fact this new relation is obtained by writing

$$p + \frac{B^2}{2} = p_o + \frac{B_o^2}{2}$$

$$\text{For a polytropic fluid} \quad \left[\frac{G}{T}\right] = -[S]$$

and the relation determining τ results by replacing the jumps by their expressions (39)–(42) in

$$ln\left(1 - \frac{1}{2p_o}[B^2]\right) + \frac{\gamma}{\gamma - 1} ln\left(1 + T_o\left[\frac{1}{T}\right]\right) + \frac{1}{\mathcal{R}}[S] = 0$$

Here this relation is rather complicated. It is useless to write it explicitly since the shock is not bounded [16]: the transversal part of $\vec{v}$, for instance, is not limited as it is easily seen by (30). A different situation may occur when the field equations derive from a variational principle. In that case w, σ lie on a quadric surface [13].

In relativistic magnetohydrodynamics an additional question arises: the subluminal character of wave and shock velocities [5], [17], [18]. It is also interesting to note that, in the relativistic case, the constraint can be eliminated as proved in [19], [20].

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