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Random walk on the infinite cluster of the percolation model

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Summary. We consider random walk on the infinite cluster of bond percolation on $\mathbb{Z}^d$. We show that, in the supercritical regime when $d \geqq 3$, this random walk is a.s. transient. This conclusion is achieved by considering the infinite percolation cluster as a random electrical network in which each open edge has unit resistance. It is proved that the effective resistance of this network between a nominated point and the points at infinity is almost surely finite.

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1 Introduction

Let G be a connected subgraph of the d-dimensional hypercubic lattice $\mathbb{Z}^d$, and suppose that the origin 0 belongs to G. A particle performs a random walk on G, beginning at the origin; at each step it moves to one of its neighbours (in G), each such neighbour being picked with equal probability. Under what conditions on G is this random walk transient? It is clear that the walk is recurrent if $d \leqq 2$, since in this case the particle is constrained to a subset of $\mathbb{Z}^d$, and it is known that such a random walk is recurrent (see Doyle and Snell 1984). In contrast to the case $d \leqq 2$, the answer to the question is far from transparent when $d \geqq 3$. We prove in this paper that the walk is (a.s.) transient when G is a certain random subgraph of $\mathbb{Z}^d$, viz., the infinite cluster of a supercritical percolation process.

In principle, the analysis of this paper should be valid for any supercritical percolation process in three or more dimensions. For simplicity of exposition, we shall restrict ourselves to the special case of bond percolation on $\mathbb{Z}^3$. Extending the result to $\mathbb{Z}^d$, where $d \geqq 3$, poses no serious difficulty.

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Let $\mathbb{Z}^3$ be the set of vectors $x=(x_1, x_2, x_3)$ with integral components. For $x, y\in\mathbb{Z}^3$, we write

$$d(x, y)=\sum_{i=1}^{3} |x_i-y_i|,$$

and we place an edge between x and y if $d(x, y)=1$, in which case we write $x\sim y$. We denote by $\mathbb{L}^3=(\mathbb{Z}^3, \mathbb{E}^3)$ the corresponding graph.

Let $0<p<1$, and declare each edge of $\mathbb{L}^3$ *open* with probability p and *closed* otherwise, the state of each edge being independent of the states of all other edges. Denote by P_p the resulting measure on the configurations of open/closed edges. For $x, y\in\mathbb{Z}^3$, we write $x\leftrightarrow y$ if there exists a path of $\mathbb{L}^3$ with endvertices x and y, every edge of which is open. For $x\in\mathbb{Z}^3$, we denote by $C(x)$ the set of all vertices y such that $x\leftrightarrow y$, together with all open edges joining pairs of such vertices; $C(x)$ is called the 'open cluster at x'. Let

$$\theta(p)=P_p(|C(x)|=\infty).$$

Note that $\theta(p)$ does not depend on the choice of x, since the lattice is translation-invariant. It is a basic result of percolation theory that there exists a critical value p_c of the parameter p such that

$$\theta(p)\begin{cases}=0 & \text{if } p<p_c,\\ >0 & \text{if } p>p_c,\end{cases}$$

and furthermore $0<p_c<1$. It is generally believed (but not currently proved, if $d=3$) that $\theta(p_c)=0$. See Kesten (1982) and Grimmett (1989) for general accounts of percolation.

It is known that there is (a.s.) a unique infinite open cluster if $\theta(p)>0$, and we denote this cluster by I whenever it exists; later we shall make the (possibly more restrictive) assumption that $p>p_c$. We propose to study random walk on I.

Let ω be a configuration of edge-states, and let x be a vertex of $I=I(\omega)$. On $I(\omega)$ we construct a random walk in the following manner. First, $S_0=x$. Given $S_0, S_1, \ldots, S_n$, it is the case that S_{n+1} is chosen uniformly from the set of neighbours in $I(\omega)$ of S_n, this choice being independent of all earlier choices. We call ω a *transient configuration* if the random walk S is transient. Since $I(\omega)$ is connected, the transience/recurrence of S does not depend on the choice of starting point x. If ω is a transient configuration, we say that random walk on $I(\omega)$ is transient.

Theorem 1 *If $p>p_c$, then random walk on $I(\omega)$ is almost surely transient.*

There is an important and useful relationship between the theory of random walk and the theory of electrical resistance. In order to exploit this relationship, we construct a random electrical network in the following manner. Each edge of $\mathbb{Z}^3$ is replaced by an electrical resistor, such a resistor having unit resistance if the edge is open and having infinite resistance otherwise; that is to say, open edges conduct electricity at a fixed rate, while closed edges are insulators. Properties of the ensuing electrical network have been much studied. Of principal interest has been the effective resistance between opposite faces of the cube $[0, n]^3$. Writing R_n for this resistance, it is known that

$$0 < \liminf_{n\to\infty} \{nR_n\} \leqq \limsup_{n\to\infty} \{nR_n\} < \infty \quad \text{a.s.}$$

whenever $p > p_c$ (and $d=3$; see Grimmett and Marstrand 1990 and the references therein). It is conjectured that $R(p) = \lim_{n\to\infty} \{nR_n\}$ exists a.s., and that $R(p)$ behaves in the manner of $(p-p_c)^{-\sigma}$ in the limit as $p \downarrow p_c$, where σ is a critical exponent. Partial progress in this direction for the case of two dimensions was reported by Kesten (1982). Hammersley (1988) and Zhikov (1989) have made considerably greater progress with a related problem in which certain extra boundary conditions are imposed on the values of the potential function at points lying on the surface of the box $[0, n]^3$.

Of concern in the problem of random walk is the effective resistance of the infinite cluster I between a nominated source vertex and the points at infinity. For two vertices x and y lying in I, we write $\delta(x, y)$ for the number of edges of I in the shortest path joining x to y. For $x \in I$, we denote by $S_n(x)$ the set of all vertices y in I such that $\delta(x, y) \leqq n$, and we write $\partial S_n(x)$ for the set $S_n(x) \backslash S_{n-1}(x)$. We turn $S_n(x)$ into a graph by adding all induced edges of I, and we denote by $R_n(x)$ the effective resistance of the corresponding electrical network between the vertex x and the set $\partial S_n(x)$ of vertices. That is to say, each edge of $S_n(x)$ is replaced by a unit resistor, and all vertices in $\partial S_n(x)$ are 'shorted out'; $R_n(x)$ is the effective resistance between x and the composite vertex $\partial S_n(x)$.

There is a unique harmonic function ϕ on the vertex set of $S_n(x)$ with the boundary conditions $\phi(x)=1$, $\phi(y)=0$ for $y \in \partial S_n(x)$. The function ϕ may be seen as the potential function in the above electrical network, when the given boundary conditions are maintained by means of external electrical sources. Alternatively, $\phi(z)$ is the probability that a random walk, starting at z, hits x before it hits $\partial S_n(x)$. The consequent relationship between electrical networks and random walk has been discussed by many authors; see, for example, Nash-Williams (1959), Lyons (1983), and Doyle and Snell (1984). By an argument using monotonicity of effective resistance, the limit $R_\infty(x) = \lim_{n\to\infty} R_n(x)$ exists for all vertices x of I. It is a consequence of the above dual role of the potential function that random walk on I, beginning from x, is transient if and only if $R_\infty(x) < \infty$. It is by this route that we shall establish Theorem 1.

Theorem 2 *If $p > p_c$ then*

$$P_p(R_\infty(0) < \infty \mid 0 \in I) = 1.$$

In advance of moving to the proof proper, we sketch the required argument. As remarked by Doyle and Snell (1984), in order to prove the transience of random walk on $\mathbb{Z}^3$, it suffices to exhibit within $\mathbb{Z}^3$ a tree having finite resistance between its root and the points at infinity. We shall follow the same strategy here, and shall construct, within I, a (random) tree. We shall concentrate on a certain class of sub-trees of $\mathbb{Z}^3$ having the origin as root. Vertices in the nth generation of such trees are distributed on or near the surface of the β^n-ball of $\mathbb{Z}^3$, where β is a constant satisfying $3 < \beta < 4$. Each vertex in the nth generation is joined (with high probability) to each of four vertices in the $(n+1)$th generation, such connections being (nearly) disjoint and comprising open paths having length of order β^n. See Fig. 1 (a) for a sketch of such a tree.

Let α_n be the probability that a given member of the nth generation is connected in the desired way to its four descendants in the $(n+1)$th generation. We shall show that $1-\alpha_n \leqq e^{-\alpha(p)n}$ for all large n and some $\alpha(p)>0$, and furthermore that (for all large n) $\alpha(p)$ may be made arbitrarily large. In this way we shall show that the probability that some step of the construction fails, from generation N onwards, may be made arbitrarily small by a choice of N sufficiently large.

Having found such an N, we deduce that there exists within I a tree whose electrical properties are approximately those of the network in Fig. 1(b). The resistance of this network between the root and the points at infinity is of order

$$\sum_{k=0}^{\infty} \frac{\beta^{N+k}}{4^k},$$

which is finite since $\beta<4$. We shall add rigour to this argument in Sect. 3.

The proof has geometrical and probabilistic content. It is by a geometrical argument that one may see that the approach is feasible in three or more dimensions. The size of the nth generation is no larger than 4^n, and the surface of the β^n-ball has order $\beta^{n(d-1)}$ in d dimensions. Thus we require that $4^n<\beta^{n(d-1)}$, which is to say that $4<\beta^{d-1}$. Combining this with the assumption above that $\beta<4$, we deduce the necessary condition

$$4^{1/(d-1)}<\beta<4,$$

an inequality which is achievable only when $d\geqq 3$. In the proof of Sect. 3, we take $d=3$ and $\beta>3$.

In the next section, we state and prove an ancillary result concerning the approximation by a tree of a certain subgraph of the infinite cluster; in Sect. 3, we make use of this result in proving Theorem 2.

Finally, we note that random walk on the infinite open cluster, in the limit as the lattice spacing tends to 0, has been considered by DeMasi et al. (1985, 1989).

2 An estimate using trees

In this section we prove an auxiliary proposition about electrical networks which are ‘almost’ trees, in the sense that the connection between a vertex and its parent may intersect a bounded number of other such connections. The vertices will have labels which derive from the vertices of a true tree. To be more specific, we start with a rooted, labelled tree T, with root denoted by $\langle 0\rangle$. The vertices of T which are at distance k from $\langle 0\rangle$ (in T) constitute the kth generation of T, and have labels $x_k(i,j)$, with (i,j) ranging over a set Λ_k (in our application, Λ_k will be the set of integer pairs contained in $(-2^{k-1}, 2^{k-1})^2$). The children of each $x_k(i,j)$ will be a suitable subset $I_k(i,j)$ of Λ_{k+1}.

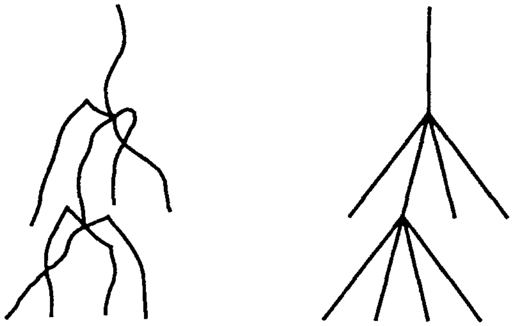

Fig. 1. a A tree-like subgraph of the lattice. **b** The graph obtained by the removal of common points and the replacement of component paths by single edges. The resistance of an edge emanating from a point in the kth generation is proportional to β^k

This tree T will be used only as a labelling device for the network of interest, which is actually part of the infinite open cluster I. We assume that to each vertex $x_k(i,j)$ of T there is assigned a vertex $z_k(i,j)$ of I. By the definition of I, any two $z_k(i,j)$ and $z_\ell(r,s)$ are then connected by an open path. Let there be selected a single open path $\pi_{k+1}(r,s)$ for each vertex $z_{k+1}(r,s)$, this path connecting $z_{k+1}(r,s)$ to its parent (i.e., the $z_k(i,j)$ which corresponds to the parent of $x_{k+1}(r,s)$ in T). Finally, each edge of I is regarded as a resistor of unit resistance.

Proposition 3 *Assume that there exist positive integers α and γ such that the following three conditions hold for a certain vertex $x_K(u,v)$ and all of its descendants $x_{K+\ell}(r,s)$:*

each such vertex has at least γ children, (2.1)

the length of $\pi_{K+\ell}(r,s)$ is at most $C\beta^{K+\ell}$, for each descendant $x_{K+\ell}(r,s)$ of $x_K(u,v)$, (2.2)

each edge of I belongs to at most α of the paths $\pi_{K+\ell}(r,s)$, with $x_{K+\ell}(r,s)$ a descendant of $x_K(u,v)$. (2.3)

Then the resistance between $z_K(u,v)$ and infinity in I is at most

$$\alpha \sum_{\ell=1}^{\infty} \frac{C\beta^{K+\ell}}{\gamma^\ell}. \tag{2.4}$$

Note that the last sum is finite if and only if $\beta<\gamma$.

Proof. Assume the conditions of the proposition hold. All resistances between points of I are unchanged if every edge of I is replaced by α parallel edges with the same endpoints, each such edge having resistance α instead of 1. We replace every edge of I in this way. Next, we need to see what becomes of the paths $\pi_{K+\ell}(r,s)$. These are no longer uniquely defined, but since each original edge of a typical π was used in at most α such π's, we can now replace the edges of all the π's by new edges in such a way that the resulting paths, denoted as $\hat{\pi}_{K+\ell}(r,s)$, are edge-disjoint for all descendants of $x_K(u,v)$.

Finally, we construct a new network $\hat{T}$, whose nodes are the $x_{K+\ell}(r,s)$ for $x_{K+\ell}(r,s)$ equal to, or a descendant of, $x_K(u,v)$. In $\hat{T}$, $x_{K+\ell}(r,s)$ is connected to its parent by a resistor of magnitude $\alpha|\pi_{K+\ell}(r,s)|$, where $|\pi|$ is the number of edges in a path π. These resistors are disjoint. Therefore, $\hat{T}$ is a tree, having root $x_K(u,v)$, and such that each vertex has at least γ children (by (2.1)).

Corresponding to this construction, there is a network N having as edges and vertices exactly those which belong to some $\hat{\pi}_{K+\ell}(r, s)$, as $(K+\ell, r, s)$ ranges over the set of triples corresponding to descendants of $x_K(u, v)$. In N, two distinct connections $\hat{\pi}_{K+\ell}(r, s)$ and $\hat{\pi}_{K+m}(r', s')$ have no edges in common, but may be connected at a number of single points, corresponding to vertices of $\mathbb{Z}^3$ which they have in common. We now view N as an electrical network, and we break these single-point connections whenever they occur at points other than $z_{K+\ell}(r, s)$ for some $(K+\ell, r, s)$. Since the removal of connections cannot decrease the effective resistance, we find that

$$R(N) \leqq R(\hat{N}) \tag{2.5}$$

where $\hat{N}$ is the resulting network, and $R(G)$ is the resistance in G between the vertex $z_K(u, v)$ and the points at infinity. Now $\hat{N}$ is a tree with the same resistance (from root to infinity) as $\hat{T}$. It is easily seen that this resistance is bounded above by (2.4). The easiest way to see this is to increase the resistance still further by making $\hat{T}$ into a homogeneous tree, in which each vertex has *exactly* γ children (thereby possibly ignoring some vertices), and by increasing each resistance between a remaining vertex $x_{K+\ell}(r, s)$ and a child $x_{K+\ell+1}(r', s')$ to the exact value $\alpha C \beta^{K+\ell+1}$. □

3 Proof of Theorem 2

In this proof we shall occasionally treat real-valued quantities as though they were integer-valued; this is for notational simplicity only, and has no essential significance for the proof.

For any set A of vertices, we write

$$\partial A = \{a \in A\colon \text{ there exists } b \notin A \text{ with } a \sim b\}$$

for the surface of A. Let $B(n)$ denote the box $[-n, n]^3$ of $\mathbb{Z}^3$. The surface of $B(n)$ may be expressed as the union of six faces, each being homeomorphic to the plane square $[-n, n]^2$; we shall concentrate on the face $F(n)$ containing all vertices x with $x_1 = n$, that is,

$$F(n) = \{x \in \mathbb{Z}^3\colon |x_2|, |x_3| \leqq n, \ x_1 = n\}.$$

We write $B_k = B(3^k)$ and $F_k = F(3^k)$ for $k \geqq 1$.

On F_k, we distinguish 4^k vertices, being the points

$$x_k(i, j) = (i d_k, j d_k), \qquad -2^{k-1} < i, j \leqq 2^{k-1}, \tag{3.1}$$

where $d_k = \lfloor (4/3)^k \rfloor$. Note that

$$d(x_k(i, j), x_k(r, s)) \geqq d_k \qquad \text{if } (i, j) \neq (r, s),$$

and that the $x_k(i, j)$ are, for fixed k, distributed within F_k in the manner of a $2^k \times 2^k$ rectangular grid.

For each $k \geqq 1$, and with each point $x_k(i,j)$, we associate exactly four points on F_{k+1}, these points being those belonging to the set

$$I_k(i,j) = \{x_{k+1}(r,s):\ r = 2i-1, 2i,\ s = 2j-1, 2j\}.$$

Note that

$$I_k(i,j) \cap I_k(r,s) = \varnothing \qquad \text{if } (i,j) \neq (r,s). \tag{3.2}$$

We write $\bar{I}_k(i,j) = d_{k+1}(2i-\frac{1}{2}, 2j-\frac{1}{2})$ ($\in \mathbb{R}^3$) for the 'centroid' of $I_k(i,j)$.

Associated with the set of all $x_k(i,j)$ is a tree T constructed as follows. The root is labelled $\langle 0 \rangle$, and the kth generation contains 4^k points labelled $x_k(i,j)$ for $-2^{k-1} < i, j \leqq 2^{k-1}$. The point $x_k(i,j)$ is adjacent to exactly the collection $I_k(i,j)$ of points in the $(k+1)$th generation.

For any two members u, v of $\mathbb{R}^3$, we let $L(u,v)$ be the set of vertices of $\mathbb{Z}^3$ which lie within euclidean distance $\sqrt{3}$ of some point of the straight line segment joining u to v. Let a be a positive constant. For each point $x_k(i,j)$, we define the region

$$T_k(i,j) = A_k(i,j) \cup C_k(i,j)$$

in the following manner. First,

$$A_k(i,j) = B(ak) + L(x_k(i,j), \bar{I}_k(i,j))$$

is the set of vertices that lie within (graph-theoretic) distance ak of some point belonging to $L(x_k(i,j), \bar{I}_k(i,j))$. Secondly,

$$C_k(i,j) = B(ak) + \bigcup_{x \in I_k(i,j)} L(\bar{I}_k(i,j), x)$$

is the set of vertices within distance ak of some point lying near the 'cross' with centre at $\bar{I}_k(i,j)$ and endpoints at each member of $I_k(i,j)$; points in this 'cross' are within distance $\sqrt{3}$ of F_{k+1}. We note that there exists $K(a)$ such that, for all $k \geqq K(a)$,

$$T_k(i,j) \cap T_k(r,s) = \varnothing \qquad \text{if } (i,j) \neq (r,s), \tag{3.3}$$

and

$$A_k(i,j) \cap T_{k+1}(r,s) = \varnothing \qquad \text{whenever } x_{k+1}(r,s) \in I_k(i,j). \tag{3.4}$$

In order to see this, note that each $A_k(i,j)$ is a tube having length of order 3^k and width of order ak; each $C_k(i,j)$ is the union of tubes having lengths of order $(4/3)^k$ and widths of order ak. For given k, the 'line segments' $L(x_k(i,j), \bar{I}_k(i,j))$, $-2^{k-1} < i, j \leqq 2^{k-1}$, are separated from one another by a distance of order $(4/3)^k$. Since the width of the tubes $T_k(i,j)$ is a fixed multiple of k, (3.3) and (3.4) are easily seen to hold for large k.

Within $T_k(i,j)$ we construct a sequence of vertices as follows. There exists a constant ν, and a sequence $y_1, y_2, \ldots, y_t$ of vertices of $A_k(i,j)$, such that

$$\text{each } y_u \text{ belongs to } L(x_k(i,j), \bar{I}_k(i,j)), \tag{3.5}$$

$$\tfrac{1}{3}ak \leqq d(y_u, y_{u+1}) \leqq \tfrac{2}{3}ak \quad \text{for } 1 \leqq u < t, \tag{3.6}$$

$$y_1 = x_k(i,j) \quad |y_t - \bar{I}_k(i,j)| \leqq 1, \quad t \leqq \nu 3^k. \tag{3.7}$$

Here, $|u-v|$ denotes the euclidean distance between u, v $(\in \mathbb{R}^3)$. Note that $t = t(k,i,j)$.

Turning to $C_k(i,j)$, for each $x \in I_k(i,j)$ we may find a sequence $y_1(x), \ldots, y_v(x)$ of vertices such that

$$\text{each } y_u(x) \text{ belongs to } L(\bar{I}_k(i,j), x), \tag{3.8}$$

$$\tfrac{1}{3}ak \leqq d(y_u(x), y_{u+1}(x)) \leqq \tfrac{2}{3}ak \quad \text{for } 1 \leqq u < v, \tag{3.9}$$

$$y_1(x) = y_t, \quad y_v(x) = x, \quad v \leqq \nu 3^k. \tag{3.10}$$

Note that $v = v(x)$.

We denote the set of such points by

$$Y_k(i,j) = \{y_1, \ldots, y_t\} \cup \Big\{ \bigcup_{x \in I_k(i,j)} \{y_1(x), \ldots, y_v(x)\} \Big\}. \tag{3.11}$$

We are interested in the existence of open paths within $T_k(i,j)$, joining points near $x_k(i,j)$ to points near each $x \in I_k(i,j)$. With a view to estimating the probability of the occurrence of such open paths, we define some events of interest.

Let b be a constant satisfying $0 < 7b < a$. Fix attention on a point $x_k(i,j)$, with associated set $Y_k(i,j)$ as in (3.11). For each $1 \leqq u < t$, we let $E_u = E_u(k,i,j)$ be the event that both of the following hold:

there exist $z_1 \in y_u + B(bk)$ and $z_2 \in y_{u+1} + B(bk)$ such that $z_i \leftrightarrow y_u + \partial B(ak)$ for $i = 1, 2$, (3.12)

any two points in $(y_u + B(bk)) \cup (y_{u+1} + B(bk))$ which are joined by open paths to $y_u + \partial B(ak)$ are also joined to each other by an open path lying entirely within $y_u + B(ak)$. (3.13)

Similarly, for $x \in I_k(i,j)$ and $1 \leqq u < v$ $(= v(x))$, we define the event $E_{x,u} = E_{x,u}(k,i,j)$ by replacing y_u and y_{u+1}, in (3.12) and (3.13) above, with $y_u(x)$ and $y_{u+1}(x)$. Finally, let

$$E_k(i,j) = \Big\{ \bigcap_{1 \leqq u < t} E_u \Big\} \cap \Big\{ \bigcap_{\substack{1 \leqq u < v(x) \\ x \in I_k(i,j)}} E_{x,u} \Big\}. \tag{3.14}$$

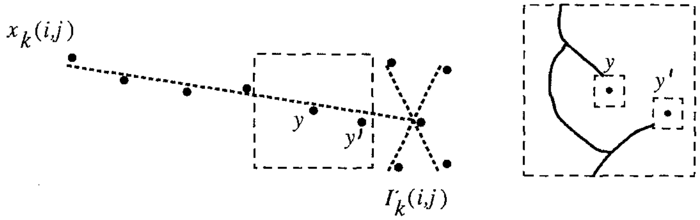

Fig. 2. **a** A diagram of the region $T_k(i,j)$, with the points in $Y_k(i,j)$ marked. **b** The larger box is an expanded view of the interior of the smaller. For each consecutive pair y, $y' \in Y_k(i,j)$, there are open paths joining $y+B(bk)$ and $y'+B(bk)$ to $y+\partial B(ak)$, and every such connection lies in the same open cluster of the box $y+B(ak)$. The union of such connections, for each such pair y, y', contains open paths from $x_k(i,j)+B(bk)$ to $x+B(bk)$ for all $x \in I_k(i,j)$

It may be seen (see Fig. 2) that, on the event $E_k(i,j)$, there exists $z \in x_k(i,j) + B(bk)$ and, for each $x \in I_k(i,j)$, there exists $z(x) \in x+B(bk)$, such that the following holds:

$$z \leftrightarrow z(x) \quad \text{in } T_k(i,j), \qquad \text{for each } x \in I_k(i,j).$$

Slightly more is valid: this relation holds for every $z \in x_k(i,j)+B(bk)$ with the property that $z \leftrightarrow x_k(i,j)+\partial B(ak)$, and similarly for every $z(x) \in x+B(bk)$ such that $z(x) \leftrightarrow x+\partial B(ak)$.

For any positive integer k, let $\mathscr{K}_k$ be the set of all triples $(k+\ell, r, s)$ for which $x_{k+\ell}(r,s)$ is a descendant (in the tree T) of $x_k(0,0)$; we assume the convention that $(k,0,0) \in \mathscr{K}_k$.

Proposition 4 *Suppose $p>p_c$. For all sufficiently large a and b satisfying $0<7b<a$, the following holds: there exists, with probability one, a (random) K such that*

$$E_k(r,s) \quad \textit{occurs for all} \quad (k,r,s) \in \mathscr{K}_K. \tag{3.15}$$

Before proving this, we note how it may be used in conjunction with Proposition 3 to obtain Theorem 2. We merely have to check conditions (2.1)–(2.3) for the sub-tree of T consisting of $x_K(0,0)$ and its descendants, where K is the earliest index for which (3.15) holds. Certainly (2.1) is valid with $\gamma=4$. Furthermore, as observed above, for each $(k,i,j) \in \mathscr{K}_K$ we may find a vertex $z_k(i,j)$ in $x_k(i,j) + B(bk)$ such that the following holds: if $(\ell, r, s) \in \mathscr{K}_K$ then there is an open path of $T_\ell(r,s)$ joining $z_\ell(r,s)$ to $z_{\ell+1}(r',s')$, for every $x_{\ell+1}(r',s') \in I_\ell(r,s)$. That is to say, within $\cup\{T_\ell(r,s): (\ell,r,s) \in \mathscr{K}_K\}$ there exists an infinite open cluster J such that

(a) $z_\ell(r,s) \in J$ for all $(\ell,r,s) \in \mathscr{K}_K$, and

(b) if $x_{\ell+1}(r',s') \in I_\ell(r,s)$, then there exists an open path $\pi_{\ell+1}(r',s')$ of J, lying entirely within $T_\ell(r,s)$, and joining $z_\ell(r,s)$ to $z_{\ell+1}(r',s')$.

The length of $\pi_{\ell+1}(r',s')$ is at most equal to the number of edges in $T_\ell(r,s)$, which is at most $C'k^3 3^k$ for some constant $C'=C'(a)$. Pick a constant C and a value of β satisfying $3<\beta<4$ such that

$$|\pi_{\ell+1}(r',s')| \leqq C\beta^{\ell+1} \qquad \text{for every } (\ell+1, r', s') \in \mathscr{K}_K, \quad \ell \geqq K. \tag{3.16}$$

This verifies condition (2.2).

Condition (2.3) with $\alpha=5$ follows from (3.3) and (3.4), so long as $K \geqq K(a)$; if $K<K(a)$, we replace K by $K(a)$.

Applying Proposition 3, we deduce from the fact that $\beta<\gamma$ $(=4)$ that I contains a.s. a tree having finite resistance between its root and infinity. Therefore, it is a.s. the case that $R_\infty(x)<\infty$ for all vertices x of I. Hence

$$P_p(R_\infty(0)<\infty \mid 0 \in I)=1.$$

This completes the proof that Theorem 2 follows from Proposition 4.

Proof of Proposition 4 If A and B are subsets of $\mathbb{Z}^3$, we write $A \leftrightarrow B$ if there exist $a \in A$ and $b \in B$ such that $a \leftrightarrow b$. We write $A \nleftrightarrow B$ if no such open path exists. The event that some vertex a in A has $|C(a)|=\infty$ is denoted by $\{A \leftrightarrow \infty\}$.

Let $p>p_c$. We denote by $p_c(k)$ the critical probability of bond percolation on the slab

$$S_k=\{x \in \mathbb{Z}^3 : 0 \leqq x_1<k\},$$

and we shall make use of the fact that

$$p_c(k) \rightarrow p_c \quad \text{as } k \rightarrow \infty, \tag{3.17}$$

a fact proved in Grimmett and Marstrand (1990). Pick a positive integer L satisfying $p_c<p_c(L)<p$, and let

$$\theta(p, L)=P_p(0 \leftrightarrow \infty \text{ in } S_L),$$

the probability that 0 is contained in an infinite open path of S_L.

Lemma 5 *There exists a positive constant $\gamma=\gamma(p)$ such that*

$$P_p(B(m) \leftrightarrow \infty) \geqq 1-e^{-\gamma m} \quad \textit{for all } m.$$

Proof. Cut $B(m)$ into 'slices' of thickness L. That is to say, $B(m)$ intersects the slices

$$S(i)=\{x \in \mathbb{Z}^3 : iL \leqq x_1<(i+1)L\}, \quad 0 \leqq i<\lfloor m/L \rfloor .$$

Let $y(i)$ be a vertex of $B(m) \cap \partial S(i)$. Then

$$\begin{aligned} P_p(y(i) \nleftrightarrow \infty \text{ for } 1 \leqq i<\lfloor m/L \rfloor) &\leqq \prod_{i=1}^{\lfloor m/L \rfloor} P_p(y(i) \nleftrightarrow \infty \text{ in } S(i)) \\ &=\{1-\theta(p, L)\}^{\lfloor m/L \rfloor}, \end{aligned}$$

whence

$$P_p(B(m) \leftrightarrow \infty) \geqq 1-\{1-\theta(p, L)\}^{\lfloor m/L \rfloor}$$

as required. □

Let $\sigma>1$ and let m be a positive integer. Let $A(m, \sigma)$ be the event that there exist two vertices inside $B(m)$ with the property that each is joined by open paths to vertices outside $\partial B(\sigma m)$, but that there is no open path of $B(\sigma m)$ joining this pair of vertices.

Lemma 6 *There exists a positive constant $\delta=\delta(p)$ such that*

$$P_p(A(m,\sigma))\leqq e^{-\delta m(\sigma-1)} \quad \textit{for all } m.$$

Proof. This is essentially Eq. (5.1) of Grimmett and Marstrand (1990); this reference is incorrect as stated, but is easily corrected. Another proof may be found within the proof of Proposition 1 of Kesten and Zhang (1990); see the estimate at (3.57). □

We turn to the event $E_k(i,j)$, defined in terms of the vertices belonging to $Y_k(i,j)$, as in (3.11). Since

$$t\leqq v3^k, \quad v=v(x)\leqq v3^k \quad \text{for } x\in I_k(i,j), \tag{3.18}$$

we have that

$$P_p(y+B(bk)\leftrightarrow\infty \text{ for all } y\in Y_k(i,j))\geqq 1-5v3^k e^{-\gamma bk}, \tag{3.19}$$

by Lemma 5.

Next we consider the requirement (3.13). Suppose there exist $z_1\in y_u+B(bk)$ and $z_2\in y_{u+1}+B(bk)$ such that $z_i\leftrightarrow y_u+\partial B(ak)$ for $i=1,2$, but that $z_1\not\leftrightarrow z_2$ in $y_u+B(ak)$. Then there exist two points of $y_u+B(ck)$ which are joined to $y_u+\partial B(ak)$ but which are not joined to each other inside $y_u+B(ak)$; here, $c=\frac{5}{6}a>\frac{2}{3}a+b$, and we have used (3.6). The probability of this is, by Lemma 6, smaller than $\exp(-\frac{1}{6}\delta ak)$. Again, we find that the probability that (3.13) fails for some $1\leqq u<t$ is smaller than $v3^k e^{-\delta ak/6}$. A similar argument is valid for each sequence $\{y_1(x),\ldots,y_v(x)\}$ for $x\in I_k(i,j)$. Combining this with (3.19), we deduce that

$$P_p(\overline{E_k(i,j)})\leqq 5v3^k(e^{-\gamma bk}+e^{-\delta ak/6}). \tag{3.20}$$

Therefore

$$\sum_{(k,i,j)\in\mathscr{K}_L} P_p(\overline{E_k(i,j)})\leqq \sum_{k=L}^{\infty} 4^{k-L}5v3^k(e^{-\gamma bk}+e^{-\delta ak/6}), \tag{3.21}$$

since there are 4^{k-L} terms of the form (k,i,j) lying in $\mathscr{K}_L$. We pick a and b such that $0<7b<a$ and $\max\{12e^{-\gamma b}, 12e^{-\delta a/6}\}<1$. With these choices for a and b, the right-hand side of (3.21) tends to 0 as $L\to\infty$, implying that $\max\{L: \mathscr{K}_L$ does not occur$\}$ is a.s. finite. This proves the proposition. □

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