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Axiomatization of identity-free equations valid in relation algebras

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Dedicated to the memory of Alan Day

Abstract. A finite axiom set for the identity-free equations valid in relation algebras is given. This is a simplification of the one given by Jónsson, and confirms a conjecture of Tarski. An axiom set for the identity-free equations valid in the representable relation algebras is given, too. We show that in the class of representable relation algebras, both the operation of taking converse and the identity constant are finitely axiomatizable (over the rest of the operations).

In [4], [5] Jónsson defined the class *SPA* of specification algebras as the 1′-free subreducts of relation algebras (*RA*'s). [4] and [5] give a finite axiomatization for *SPA*. This axiomatization consists of the 1′-free axioms of *RA* together with a finite set of equations called condition (iii) in [4]. It is asked in [4] whether condition (iii) can be simplified.

Independently, Tarski raised the problem whether either the equation

$$x \leq x \circ (y^{\cup} \circ y^{-})^{-} \quad \text{or} \tag{T1}$$

$$x \leq x \circ [(y^{\cup} \circ y^{-})^{-} \cdot (z^{\cup} \circ z^{-})^{-}] \tag{T2}$$

is sufficient in place of condition (iii). Tarski's original terminology of course was different, we quote Problem 21 from [10]:

"21. (Tarski) Can all relation algebraic identities not involving 1′ be derived from the axioms for relation algebras, with the identity $x = x \circ 1' = 1' \circ x$ replaced by $x \leq x \circ (y^{\cup} \circ y^{-})^{-}$ or by $x \leq x \circ [(y^{\cup} \circ y^{-})^{-} \cdot (z^{\cup} \circ z^{-})^{-}]$? Tarski believes that he once proved this for one or the other of the two identities. Note that the identity $1' \leq (y^{\cup} \circ y^{-})^{-}$ is true in every relation algebra."

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The present paper consists of two parts. The first part contains Theorem 1 and Proposition 1. Theorem 1 states that condition (iii) in the theorem of [4] can be replaced with the following simpler consequence of (T2)

$$x \leq x \circ [(y^{\smile} \circ y^{-})^{-} \cdot (y^{-\smile} \circ y)^{-}]$$

and Proposition 1 states that (iii) cannot be replaced with Tarski's shorter equation (T1). This way Tarski's problem receives a complete answer.

We note that independently of us, at the same time as we did, Bjarni Jónsson, Peter Jipsen and John Rafter also proved sufficiency of the simpler form of (T2) in place of (iii), see [7].

In the second part of the paper we show that the methods we use for proving Theorem 1 are suitable for proving results on representable relation algebras (*RRA*'s), too. At the end of the paper we show that given the Boolean operations, composition brings in an infinite number of valid equations, but both the operation of taking converses and the identity constant bring in only finitely many valid equations. So the sole cause for nonfinite axiomatizability of *RRA* is relation composition. This situation is radically different from algebras of n-ary relations, $n > 2$, where the operations share the blame for nonfinite axiomatizability: each of them brings in essentially infinitely many new valid equations modulo the rest of the operations, see [1].

We now briefly recall some of the definitions.

A relation algebra (an *RA*) is an algebra $\mathbf{A} = (\mathbf{A}_0, \circ, {}^{\smile}, 1')$, where $\mathbf{A}_0 = (A, +, \cdot, -, 0, 1)$ is a Boolean algebra, $\circ$, ${}^{\smile}$, $1'$ are binary, unary and zero-ary additive operations on $\mathbf{A}_0$, $(A, \circ, {}^{\smile}, 1')$ is an involuted monoid, and further the equation

$$x^{\smile} \circ (x \circ y)^{-} \leq y^{-} \tag{R}$$

holds.

The $1'$-free reduct of an *RA* $\mathbf{A} = (\mathbf{A}_0, \circ, {}^{\smile}, 1')$ is the algebra $(\mathbf{A}_0, \circ, {}^{\smile})$.

The canonical extension, or ultrafilter extension, or perfect extension, of an *RA* is defined e.g. in [8], Def. 2.14.

THEOREM 1. *Suppose* $\mathbf{A} = (\mathbf{A}_0, \circ, {}^{\smile})$ *is an algebra that satisfies the axioms for relation algebras with the exception of the axiom* $x \circ 1' = 1' \circ x = x$. *Then the following conditions are equivalent*:

(i) $\mathbf{A}$ *is a subalgebra of the* $1'$*-free reduct of an RA.*

(ii) *The canonical extension of* $\mathbf{A}$ *is a* $1'$*-free reduct of an RA.*

(iii) $\mathbf{A}$ *satisfies the identity* $x \leq x \circ [(y^{\smile} \circ y^{-})^{-} \cdot (y^{-\smile} \circ y)^{-}]$.

(iv) $\mathbf{A}$ *satisfies the identity* $x \leq x \circ [(y^{\smile} \circ y^{-})^{-} \cdot (z^{\smile} \circ z^{-})^{-}]$.

Proof. Assume that $\mathbf{A} = (\mathbf{A}_0, \circ, ^\smile)$ is as in the hypothesis of the theorem, i.e. $\mathbf{A}_0$ is a Boolean algebra, $\circ$, $^\smile$ are additive, $(A, \circ, ^\smile)$ is an involuted semigroup and (R) holds. First we show that the following also holds:

$^\smile$ is a Boolean homomorphism, in particular $x^{-\smile} = x^{\smile -}$.

Indeed, since $^\smile$ is additive, it is monotonic. Thus by $1^\smile \le 1$ we have $1^{\smile\smile} \le 1^\smile$, i.e. $1 \le 1^\smile$, hence $1 = 1^\smile$. Now $1 = 1^\smile = (x + x^-)^\smile = x^\smile + x^{-\smile}$, hence $x^{-\smile} \ge x^{\smile -}$. Then $(x^\smile)^{-\smile} \ge x^{\smile\smile -} = x^-$, and thus $x^{\smile -} = (x^{\smile -})^{\smile\smile} = (x^{\smile - \smile})^\smile \ge (x^-)^\smile$, i.e. $x^{-\smile} = x^{\smile -}$.

We now turn to proving Theorem 1. By substituting y, $1'$ in place of x, y in (R), we get $y^\smile \circ y^- \le 1'^-$, i.e. $1' \le (y^\smile \circ y^-)^-$, hence in every relation algebra the identity in (iv) holds. This shows (i) $\Rightarrow$ (iv). The implications (ii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (iii) are easy to check. Therefore it is enough to show that (iii) $\Rightarrow$ (ii). Assume (iii). For any $a \in A$ let

$$e(a) = (a^\smile \circ a^-)^- \quad \text{and} \quad \varepsilon(a) = e(a) \cdot e(a^-).$$

Then (iii) states that $x \le x \circ \varepsilon(y)$ holds in $\mathbf{A}$.

LEMMA 1. *$\varepsilon(a) \circ \varepsilon(a) \le \varepsilon(a) = \varepsilon(a)^\smile$ and $\varepsilon(a) \circ 1 = 1$, for any $a \in A$.*

Proof. Let $a \in A$ be arbitrary. First we show

$$e(a) \circ e(a) \le e(a). \tag{1}$$

By replacing x, y in the equation (R) with $a^\smile, a^-$ we immediately obtain that

$$a \circ e(a) \le a. \tag{2}$$

An easy computation shows

$$e(a)^\smile = e(a^-). \tag{3}$$

Indeed, $e(a)^\smile = (a^\smile \circ a^-)^{-\smile} = (a^\smile \circ a^-)^{\smile -} = (a^{-\smile} \circ a^{\smile\smile})^- = (a^{-\smile} \circ a)^- = (a^{-\smile} \circ a^{--})^- = e(a^-)$.

We will use the following consequence of (R):

$$x \circ y \le x \quad \text{implies} \quad y \le (x^\smile \circ x^-)^-. \tag{R1}$$

Indeed, assume $x \circ y \le x$, i.e. $x^- \le (x \circ y)^-$. By monotony of $\circ$ and by (R) we then have $x^\smile \circ x^- \le x^\smile \circ (x \circ y)^- \le y^-$, i.e. $y \le (x^\smile \circ x^-)^-$.

Now by (2) and monotony of $\circ$ we have $a \circ e(a) \circ e(a) \leq a$, hence by $(R1)$ we have $e(a) \circ e(a) \leq (a^{\smile} \circ a^{-})^{-} = e(a)$. Thus (1) is proved and this immediately implies $\varepsilon(a) \circ \varepsilon(a) \leq \varepsilon(a)$. By (3) we obtain $\varepsilon(a) = \varepsilon(a)^{\smile}$. By (iii) we have $1 \leq 1 \circ \varepsilon(a)$, thus $1 = 1 \circ \varepsilon(a)$. Then by applying converse to both sides we obtain $\varepsilon(a) \circ 1 = 1$. QED(Lemma 1)

LEMMA 2. *$a \circ a \leq a = a^{\smile}$ and $a \circ 1 = 1$ imply $\varepsilon(a) = a$, for any $a \in A$.*

Proof. Assume that $a \circ a \leq a = a^{\smile}$ *and* $a \circ 1 = 1$. By (3) it is enough to show that $e(a) = a$. By $a \circ a \leq a$ and $(R1)$ we have $a \leq (a^{\smile} \circ a^{-})^{-} = e(a)$. On the other hand, by $a \circ 1 = 1$ we have $a^{-} \leq 1 = a \circ 1 = a \circ a + a \circ a^{-}$, hence by $a^{-} \cdot (a \circ a) = 0$ we get $a^{-} \leq a \circ a^{-} = e(a)^{-}$. QED(Lemma 2)

LEMMA 3. *$\varepsilon(\varepsilon(a) \cdot \varepsilon(b)) = \varepsilon(a) \cdot \varepsilon(b)$ for any $a, b \in A$.*

Proof. It is enough to show that $m = \varepsilon(a) \cdot \varepsilon(b)$ satisfies the conditions of Lemma 2. Now $m \circ m \leq m = m^{\smile}$ holds since by Lemma 1, $\varepsilon(a) \circ \varepsilon(a) \leq \varepsilon(a) = \varepsilon(a)^{\smile}$ and the same holds for $\varepsilon(b)$. Thus it is enough to show $m \circ 1 = 1$.

By (R) and the first part of Lemma 1 we have $\varepsilon(a) \circ \varepsilon(a)^{-} \leq \varepsilon(a)^{-}$ (by replacing x, y in (R) with $\varepsilon(a), \varepsilon(a)$ respectively). By (iii) we have $\varepsilon(a) \leq \varepsilon(a) \circ \varepsilon(a) \leq \varepsilon(a) \circ (\varepsilon(a) \cdot \varepsilon(b)) + \varepsilon(a) \cdot (\varepsilon(a) \circ \varepsilon(a)^{-}) = \varepsilon(a) \circ (\varepsilon(a) \cdot \varepsilon(b)) = \varepsilon(a) \circ m$. Then by the second part of Lemma 1, $1 \leq 1 \circ \varepsilon(a) \leq 1 \circ \varepsilon(a) \circ m \leq 1 \circ m$, and by applying converse on both sides we get $m \circ 1 = 1$. Thus by Lemma 2 we have $m = \varepsilon(m)$. QED(Lemma 3)

By Lemma 3, the set $E = \{\varepsilon(a) : a \in A\} = \{a \in A : a \circ a \leq a = a^{\smile}, a \circ 1 = 1\}$ is closed under intersection, and clearly by (2) and (iii) we have $a \circ \varepsilon(a) = a$.

Let e be the meet of E in the canonical extension $\mathbf{A}^{\sigma}$ of $\mathbf{A}$, i.e. let $e = \Pi E$. This exists because $\mathbf{A}^{\sigma}$ is complete. We will show that $x \circ e = x$ for all $x \in A^{\sigma}$. Since $\circ$ is completely additive in $\mathbf{A}^{\sigma}$, it is enough to show this for atoms x of $\mathbf{A}^{\sigma}$. Every atom in $\mathbf{A}^{\sigma}$ is the meet of an ultrafilter of $\mathbf{A}$. Let x be an atom of $\mathbf{A}^{\sigma}$, and let $x = \Pi F, F \subseteq A$. Then $x \circ e \leq a \circ e \leq a$ for all $a \in F$, thus $x \circ e \leq \Pi F = x$. To show the other direction, recall that $b \leq b \circ a$ for all $b \in A$ and $a \in E$ by our equation (iii). Since $\mathbf{A}^{\sigma}$ is compact and E is closed under intersections, for any $b \in A$ we have that $e \leq b$ iff ($a \leq b$ for some $a \in E$). Since $x = \Pi F$, $e = \Pi E$, $F, E \subseteq A$, by the definition of a perfect extension we have

$$x \circ e = \prod \{y \circ b : x \leq y, e \leq b, y, b \in A\}$$

$$\leq \prod \{y \circ a : y \in F, a \in E\} \leq \prod \{y : y \in F\} = \prod F = x. \qquad \text{QED(Theorem 1)}$$

PROPOSITION 1. *In Theorem* 1, (iii) *cannot be replaced by* (iii)′ below:
(iii)′ **A** *satisfies the identity* $x \leq x \circ (y^{\cup} \circ y^{-})^{-}$.

Proof. Let $A = \{a, a^{\cup}, 0, 1\}$ and let $\mathbf{A}_0$ be the 4-element Boolean algebra with universe A (and atoms $a, a^{\cup}$). Let $\circ$ and $^{\cup}$ be additive, $a \circ a = a, a^{\cup} \circ a^{\cup} = a^{\cup}$, $a \circ a^{\cup} = a^{\cup} \circ a = 1$ and let $a^{\cup\cup} = a, 0^{\cup} = 0, 1^{\cup} = 1$.

We now check that $\mathbf{A} = (\mathbf{A}_0, \circ, ^{\cup})$ satisfies the equations defining RA, except those involving 1′.: For all x, y we have $x \circ y = a$ iff $x = y = a$, $x \circ y = a^{\cup}$ iff $x = y = a^{\cup}$, and $x \circ y = 0$ iff ($x = 0$ or $y = 0$). Using these we get $(x \circ y) \circ z = a$ iff $x = y = z = a$ iff $x \circ (y \circ z) = a$, $(x \circ y) \circ z = a^{\cup}$ iff $x = y = z = a^{\cup}$ iff $x \circ (y \circ z) = a^{\cup}$, and $(x \circ y) \circ z = 0$ iff $0 \in \{x, y, z\}$ iff $x \circ (y \circ z) = 0$. This shows that $\circ$ is associative.

Checking $(x \circ y)^{\cup} = y^{\cup} \circ x^{\cup}$: it is enough to check this for atoms. If $x = y$ then $(x \circ x)^{\cup} = x^{\cup} = x^{\cup} \circ x^{\cup}$, and if $x \neq y$, x, y are atoms, then $(x \circ y)^{\cup} = 1^{\cup} = 1 = y^{\cup} \circ x^{\cup}$.

Checking (R): If $0 \in \{x, y\}$ or if $(x \circ y) = 1$ then (R) holds trivially. So assume $0 \notin \{x, y\}$ and $(x \circ y) \neq 1$. Then $x \circ y = a$ or $x \circ y = a^{\cup}$. If $x \circ y = a$, then $x = y = a$, and $a^{\cup} \circ (a \circ a)^{-} = a^{\cup} = a^{-}$. If $x \circ y = a^{\cup}$, then $x = y = a^{\cup}$, and $a^{\cup\cup} \circ (a^{\cup} \circ a^{\cup})^{-} = a = a^{\cup -}$.

We have seen that **A** satisfies the 1′-free equations defining RA. If e denotes the term $e(x) = (x^{\cup} \circ x^{-})^{-}$, then we have $e(a) = (a^{\cup} \circ a^{-})^{-} = a, e(a^{\cup}) = a^{\cup}$ and $e(0) = 1 = e(1)$. Thus **A** satisfies $x \leq x \circ e(y)$ but **A** does not satisfy $x \leq x \circ (e(y) \cdot e(y^{-}))$. QED(Proposition 1)

Next we will use the methods used so far to axiomatize the identity-free equations valid in RRA and to describe the interconnections between the relation-theoretic operations: intersection, complementation, relation composition, converse and the identity constant.

It is not very difficult to finitely axiomatize converse over the rest of the operations, if we can use the identity constant. Namely, in any relation algebra $\mathbf{A} = (\mathbf{A}_0\, \circ, ^{\cup}, 1')$ we have $x^{\cup} = \Sigma\,\{y : y \circ x^{-} \leq 1'^{-}\}$, see e.g. [6]. The next theorem says that we can finitely axiomatize converse over relation composition without using the identity constant, and with using equations.

Let RRA° denote the class of all algebras, up to isomorphisms, whose universe consists of some subrelations of an equivalence relation and whose operations are the Boolean ones and relation composition. Similarly, let $RRA^{\circ,\cup}$ be the class of all algebras of binary relations, up to isomorphisms, whose operations are the Boolean ones, relation composition and taking converses.

Let Σ denote the set of the following equations:

$$x^{\cup\cup} = x, \qquad (x + y)^{\cup} = x^{\cup} + y^{\cup},$$

$$(x \circ y)^{\cup} = y^{\cup} \circ x^{\cup}, \qquad x \circ (x^{\cup} \circ y^{-})^{-} \leq y,$$

$$x \leq x \circ [(y^{\cup} \circ y^{-})^{-} \cdot (y^{-\cup} \circ y)^{-}].$$

THEOREM 2. *Σ is a finite axiomatization of $RRA^{\circ,\smile}$ over RRA°, i.e. for any algebra $(\mathbf{A}, \circ, {}^{\smile})$ such that $(\mathbf{A}, \circ) \in RRA^{\circ}$,*

$$(\mathbf{A}, \circ, {}^{\smile}) \in RRA^{\circ,\smile} \quad \textit{iff} \quad (\mathbf{A}, \circ, {}^{\smile}) \models \Sigma.$$

Proof. Let $\mathbf{A} = (\mathbf{A}_0, \circ, {}^{\smile})$ be an algebra such that $(\mathbf{A}_0, \circ) \in RRA^{\circ}$ and Σ is valid in $\mathbf{A}$. Then $\mathbf{A}$ satisfies the conditions of Theorem 1, hence the canonical extension $\mathbf{A}^{\sigma} = (\mathbf{A}_0^{\sigma}, \circ, {}^{\smile})$ of $\mathbf{A}$ is a reduct of an RA, say $(\mathbf{A}^{\sigma}, 1') \in RA$. In particular, $(\mathbf{A}_{\sigma}, 1')$ satisfies

$$x^{\smile} \circ x^{-} \leq 1'^{-}, \qquad x^{-\smile} = x^{\smile -}.$$

In order to show $(\mathbf{A}^{\sigma}, 1') \in RRA$, we first show that the "BA, $\circ$"-reduct $(\mathbf{A}_0^{\sigma}, \circ)$ of $\mathbf{A}^{\sigma}$ is representable as an algebra of subrelations of some equivalence relation.

Let $\mathbf{A}' = (\mathbf{A}_0, \circ)$. By our assumption, $\mathbf{A}' \in RRA^{\circ}$, hence $\mathbf{A}'$ is a subreduct of an RRA, say, $\mathbf{A}' \subseteq \mathbf{C}' = \mathbf{RdC}$ for some $\mathbf{C} \in RRA$. Let $\mathbf{A}'^{\sigma}, \mathbf{C}'^{\sigma}, \mathbf{C}^{\sigma}$ be the canonical extensions of $\mathbf{A}', \mathbf{C}', \mathbf{C}$. Then $\mathbf{A}'^{\sigma} \subseteq \mathbf{C}'^{\sigma} = \mathbf{RdC}^{\sigma}$. It is known that RRA is closed under canonical extensions. This was proved by J.D. Monk (announced in [9], for proofs see [11], or [2]). Thus $\mathbf{C}^{\sigma} \in RRA$ by $\mathbf{C} \in RRA$, and then $(\mathbf{A}_0^{\sigma}, \circ) \subseteq \mathbf{RdC}^{\sigma}$, showing that $(\mathbf{A}_0^{\sigma}, \circ)$ is a subreduct of an RRA.

Thus we may assume that the elements of A^{σ} are binary relations, with the greatest one, 1, being an equivalence relation, and the operations of $(\mathbf{A}_0^{\sigma}, \circ)$ are intersection, complementation, and real relation composition.

Now we show that we may assume that $1'$ (for which $(\mathbf{A}^{\sigma}, 1') \in RA$) is the identity relation on the field U of 1. Recall that the equation $x \circ 1' = 1' \circ x = x$ is satisfied. By $1' \circ 1' \subseteq 1'$ then we have that $1'$ is transitive. Next we show that $1'$ is reflexive, i.e. that $Id_U = \{(u, u) : u \in U\} \subseteq 1'$. Let $u \in U$ be arbitrary. Then $(u, u) \in 1$. Then by $1 \subseteq 1 \circ 1'$, there is a w with $(w, u) \in 1'$. If $(u, u) \notin 1'$ then $(u, u) \in 1'^{-}$ (by $(u, u) \in 1$), hence $(w, u) \in 1' \circ 1'^{-} \subseteq 1'^{-}$, contradicting $(w, u) \in 1'$. Thus $1'$ is reflexive. Now, $1'$ is also symmetric, because if $(u, w) \in 1'$ and $(w, u) \in 1'^{-}$, then $(u, u) \in 1' \circ 1'^{-} \subseteq 1'^{-}$, contradicting $(u, u) \in 1'$. Thus $1'$ is an equivalence relation on U. Let $(u, u'), (v, v') \in 1'$ and assume that $(u, v) \in x, x \in A^{\sigma}$. Then $(u', v') \in 1' \circ x \circ 1' \subseteq x$, thus $(u', v') \in x$. Therefore, we may "factorize out" with $1'$, i.e. we may assume that $1' = Id_U$.

Finally, we show that $x^{\smile} = x^{-1}$ for all $x \in A^{\sigma}$. By $x^{\smile} \circ x^{-} \leq 1'^{-}$ we have $x^{\smile} \circ x^{-} \subseteq Id_U^{-}$, hence $x^{\smile} \subseteq x^{-1}$. Similarly, $x^{-\smile} \subseteq (x^{-})^{-1}$, hence by $x^{-\smile} = x^{\smile -}$ we have $x^{\smile -} \subseteq (x^{-1})^{-}$, hence $x^{-1} \subseteq x^{\smile}$, i.e. $x^{\smile} = x^{-1}$ and we are done.

QED(Theorem 2)

We call an equation identity-free, or shortly $1'$-free, if the constant $1'$ does not occur in the equation.

COROLLARY 1. (i) *All the* 1′*-free equations valid in RA can be derived from* Σ *together with the Boolean equations, and associativity and additivity of* $\circ$.

(ii) *All the* 1′*-free equations valid in RRA can be derived from* Σ *together with those equations valid in RRA which contain only the Boolean operations and* $\circ$.

Proof. (i) Let Γ denote the set of equations consisting of the Boolean equations together with equations stating associativity and additivity of $\circ$. Le e be any 1′-free equation valid in RA, and let $\mathbf{A} = (\mathbf{A}_0, \circ, {}^{\smile})$ be any algebra such that $\mathbf{A} \models \Sigma \cup \Gamma$. Then by Theorem 1, $\mathbf{A}$ is a subalgebra of the 1′-free reduct of an RA, hence e holds in $\mathbf{A}$. We showed $\Sigma \cup \Gamma \models e$.

(ii) Let e be an 1′-free equation valid in RRA, and let $Eq(RRA^\circ)$ denote the set of all equations valid in RRA which contain only the Boolean operations and $\circ$. Then $Eq(RRA^\circ)$ is the set of all equations valid in RRA°. Let $\mathbf{A} = (\mathbf{A}_0, \circ, {}^{\smile})$ be any algebra in which $\Sigma \cup Eq(RRA^\circ)$ holds. It is proved in [12] that RRA° is a variety. Hence $(\mathbf{A}_0, \circ) \in RRA^\circ$ by $\mathbf{A} \models Eq(RRA^\circ)$. Then by $\mathbf{A} \models \Sigma$ and by Theorem 2 we have $\mathbf{A} \in RRA^{\circ,\smile}$, and then $\mathbf{A} \models e$ because 1′ does not occur in e and e is valid in RRA.

QED(Corollary 1)

REMARK 1. Theorem 2 and Corollary 1 are part of the following picture of the RRA-operations $\cup, -, \circ, {}^{\smile}, 1'$. If we take the Boolean operations $\cup, -$ for granted, then the only cause of nonfinite axiomatizability of RRA is $\circ$; namely ${}^{\smile}$ and 1′ are finitely axiomatizable over any other subsets of the operations, while $\circ$ is only infinitely axiomatizable over any set of others. This is shown in Figure 1, where also the finite axiom sets are given. In Figure 1 it is also indicated which reduct classes are only quasi-varieties and not varieties, and which of them are not finitely axiomatizable. With the exception of Theorem 2 we do not prove the above mentioned finite axiomatizability statements, because these proofs are easy, or are contained in the proof of Theorem 2.

How to read the following figure: The nodes represent classes of all subreducts of RRA's where the operations of the reducts are those indicated along the path leading to the node. In other words, a node represents the class of all algebras of binary relations, up to isomorphisms, where the greatest relation is an equivalence relation and the operations are those indicated along the path leading to the node.

The label ∞ on an edge between two nodes means that the smaller one of the two corresponding classes is not finitely axiomatizable over the other (in the same spirit as in Theorem 2). The set Δ of axioms on an edge between two nodes means that Δ is an axiomatization of the smaller one over the other. E.g. using the notation $RRA^{\circ,1'}$ analogously to RRA° etc, the figure states that

$$RRA^{\circ,1'} = \{\mathbf{A} = (\mathbf{A}_0, \circ, 1') \colon (\mathbf{A}_0, \circ) \in RRA^{\circ}, \mathbf{A} \models x \circ 1' = 1' \circ x = x\}, \text{ and}$$

$$RRA = RRA^{\circ,\smile,1'} = \{\mathbf{A} = (\mathbf{A}_0, \circ, {}^{\smile}, 1') \colon (\mathbf{A}_0, \circ, 1') \in RRA^{\circ,1'},$$
$$\mathbf{A} \models x^{\smile} \circ x^{-} \leq 1'^{-}, x^{\smile -} = x^{-\smile}\},$$

and these statements follow from the proof of Theorem 2. We note that $1'^{\smile} = 1'$ follows from $x \circ (x^{\smile} \circ y^{-})^{-} \leq y$ by substituting $1'$, $1'^{-}$ in place of x, y, and then using monotony and idempotence of ${}^{\smile}$.

On Figure 1, all classes represented by the nodes are varieties except the ones inside a box (those are only quasi-varieties), and the classes inside a double circle are not finitely axiomatizable, all others are. E.g. RRA° is not finitely axiomatizable, because it is known that RRA is not finitely axiomatizable, but by the proof of Theorem 2, as shown on the figure, RRA is finitely axiomatizable over RRA°.

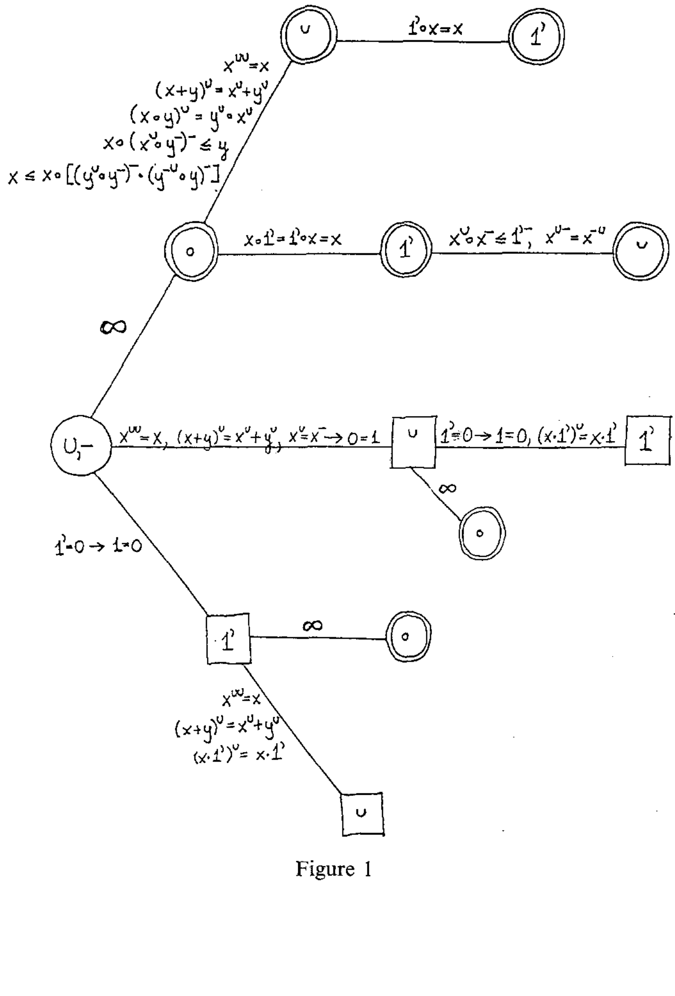

Figure 1

We note that the above picture is radically different for algebras of ternary relations (or n-ary with $2 < n < \omega$), where almost all the basic operations are nonfinitely axiomatizable over the rest, see [1] or [12].

We note that the axiomatization $x \circ 1' = 1' \circ x = x$ works only in the presence of complementation (see the proof of Thm. 2). For example, we know that $\cap, \circ$ is finitely axiomatizable (see [3]), but it is an open problem whether $\cap, \circ, Id$ is finitely axiomatizable or not. Related results about axiomatizability of some of the operations over the others without insisting on the presence of all the Booleans are in [12].

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