Algebra Universalis, 30 (1993) 469–478
0002-5240/93/040469-10$01.50 + 0.20/0

Relation algebras as residuated Boolean algebras

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1. Introduction

The arithmetic of relation algebras is more transparent, and appears less accidental, if it is placed within the general framework of Boolean algebras with residuated operators. The purpose of this paper is to substantiate this claim. Residuation, and the equivalent concept of conjugacy, have always played a central role in the axiomatic development of relation algebras (see e.g. Chin and Tarski [2] and Birkhoff [1]) but our aim is to make this more explicit. In the process we will obtain several characterizations of relation algebras and of non-associative relation algebras as special residuated Boolean algebras. For relation algebras, one such characterization already exists in Hoare and Jifeng [3, p. 234].

2. Residuated unary operators

The notion of residuation can be applied to unary operations on a partially ordered set, or to maps between partially ordered sets, but here it will only be needed for operations on Boolean algebras. In this setting it is convenient to consider also the equivalent notion of conjugacy.

A unary operation f on a Boolean algebra $\mathbf{A} = (A, +, 0, \cdot, 1, ^-)$ is said to be an *operator* provided it is additive (preserves binary joins). If, in addition, $f(0) = 0$, then f is said to be *linear*. By a *residual* and a *conjugate* of f we mean unary operations f^r and f^c on $\mathbf{A}$ satisfying the following equivalences for all $x, y \in A$.

$$f(x) \le y \qquad \text{iff } x \le f^r(y). \tag{1}$$

$$f(x)y = 0 \qquad \text{iff } f^c(y)x = 0. \tag{2}$$

Presented by R. McKenzie.
Received September 30, 1991; accepted in final form February 18, 1992.

The operations f^r and f^c are unique whenever they exist. If one of them exists, then so does the other, and they are related via the formulas

$$f^c(x) = f^r(x^-)^-, \qquad f^r(x) = f^c(x^-)^-. \tag{3}$$

If f^r and f^c exist, then f is said to be *residuated.* A residuated operation is linear; in fact, it preserves all existing joins. If f is residuated, then so is f^c, with $f^{cc} = f$, and therefore $f^{cr}(x) = f(x^-)^-$. On the other hand, f^r is residuated in the algebraic dual of $\mathbf{A}$. In particular, f^r preserves all existing meets.

Although residuals and conjugates can be used interchangeably, we tend to use conjugacy as the primary concept. This is partly due to the symmetry exhibited by conjugacy: If the operation f has a conjugate f^c, then f^c also has a conjugate and $f^{cc} = f$. Also, for a relation algebra the conjugate of the operator $x \mapsto a \circ x$ is $y \mapsto a^\cup \circ y$, while the residual is given by the more complicated formula $y \mapsto (a^\cup \circ y^-)^-$.

The following fact will be quite useful:

If f is a residuated unary operator on $\mathbf{A}$, then

$$f(x)y \leq f(xf^c(y)) \qquad \text{for all } x, y \in \mathbf{A}. \tag{4}$$

Indeed, let $z = f(xf^c(y))$. It is evident that $f(xf^c(y))z^- = 0$ and hence $xf^c(y)f^c(z^-) = 0$. Since f^c is isotone, this implies that $xf^c(yz^-) = 0$, $f(x)yz^- = 0$ and $f(x)y \leq z$.

We shall also make extensive use of the observation that the composition fg of two residuated unary operators f and g on $\mathbf{A}$ is residuated, with

$$(fg)^c = g^c f^c. \tag{5}$$

3. Residuated binary operators

Let f be an operation of rank $n > 1$ on a Boolean algebra $\mathbf{A}$. By a *translate* of f we mean a unary operation obtained by holding fixed all but one of the arguments of f. We say that f is an *operator* if all the translates of f are operators, that f is *linear* if all the translates of f are linear, and that f is *residuated* if all the translates of f are residuated.

The case of a binary operation $\circ$ is particularly important. If $\circ$ is residuated, then we denote by $\backslash$ and $/$ the operations such that, for all $x, y, z \in A$,

$$x \circ y \leq z \qquad \text{iff } y \leq x \backslash z \qquad \text{iff } x \leq z/y, \tag{1}$$

and by $\lhd$ and $\rhd$ the operations such that

$$(x \circ y)z = 0 \qquad \text{iff } (x \rhd z)y = 0 \qquad \text{iff } (z \lhd y)x = 0. \tag{2}$$

In other words, the operator $x \mapsto a \circ x$ has $y \mapsto a\backslash y$ as its residual and $y \mapsto a \rhd y$ as its conjugate, while for $x \mapsto x \circ a$ the residual is $y \mapsto y/a$ and the conjugate is $y \mapsto y \lhd a$. We refer to $\backslash$ and $/$ as the *right* and the *left residuals* of $\circ$, and to $\rhd$ and $\lhd$ as the *right* and *left conjugates of* $\circ$. From 2(3) we obtain

$$a \rhd y = (a\backslash y^{-})^{-}, \qquad a\backslash y = (a \rhd y^{-})^{-}, \tag{3}$$

$$y \lhd a = (y^{-}/a)^{-}, \qquad y/a = (y^{-} \lhd a)^{-}. \tag{4}$$

We will make frequent use of the fact that the opertors $\circ$, $\rhd$ and $\lhd$ are residuated. Indeed, by holding fixed one or the other of the arguments in each of the operators $\circ$, $\rhd$ and $\lhd$, we obtain three pairs of conjugate operator:

$$x \mapsto a \circ x \quad \text{and} \quad y \mapsto a \rhd y \quad \text{are conjugates.} \tag{5}$$

$$x \mapsto x \circ a \quad \text{and} \quad y \mapsto y \lhd a \quad \text{are conjugates.} \tag{6}$$

$$x \mapsto a \lhd x \quad \text{and} \quad y \mapsto y \rhd a \quad \text{are conjugates.} \tag{7}$$

These statements follow directly from (2).

It will be useful to have names for the translates of the operations $\circ$, $\rhd$ and $\lhd$. Let

$$L(a)(x) = a \circ x, \qquad R(a)(x) = x \circ a, \qquad Q(a)(x) = a \lhd x.$$

Then

$$L(a)^{c}(y) = a \rhd y, \qquad R(a)^{c}(y) = y \lhd a, \qquad Q(a)^{c}(y) = y \rhd a.$$

Applying 2(4) to these six operations, we obtain the following inclusions:

$$(a \circ x)y \le a \circ x(a \rhd y). \tag{8}$$

$$(x \circ a)y \le x(y \lhd a) \circ a. \tag{9}$$

$$(a \lhd x)y \le a \lhd x(y \rhd a). \tag{10}$$

$$(a \rhd x)y \le a \rhd x(a \circ y). \tag{11}$$

$$(x \lhd a)y \le x(y \circ a) \lhd a. \tag{12}$$

$$(x \rhd a)y \le x(a \lhd y) \rhd a. \tag{13}$$

Formula 2(5) can be used to rewrite various identities. As an illustration, we consider the associative law, $x \circ (y \circ z) = (x \circ y) \circ z$. Holding fixed two of the three variables, we write

$$a \circ (b \circ x) = (a \circ b) \circ x, \qquad a \circ (x \circ b) = (a \circ x) \circ b, \qquad x \circ (a \circ b) = (x \circ a) \circ b.$$

For fixed a and b, these identities are equivalent to the equations

$$L(a)L(b) = L(a \circ b), \qquad L(a)R(b) = R(b)L(a), \qquad R(a \circ b) = R(b)R(a),$$

and by 2(5) these yield

$$L(b)^c L(a)^c = L(a \circ b)^c, \qquad R(b)^c L(a)^c = L(a)^c R(b)^c, \qquad R(a \circ b)^c = R(a)^c R(b)^c.$$

This shows that the following four identities are equivalent

$$x \circ (y \circ z) = (x \circ y) \circ z. \tag{14}$$

$$x \rhd (y \rhd z) = (y \circ x) \rhd z. \tag{15}$$

$$(x \rhd y) \lhd z = x \rhd (y \lhd z). \tag{16}$$

$$x \lhd (y \circ z) = (x \lhd z) \lhd y. \tag{17}$$

4. Unital algebras

By a *residuated Boolean algebra*, or an *r-algebra*, we mean an algebra $\mathbf{A} = (\mathbf{A}_0, \circ, \rhd, \lhd)$ such that $\mathbf{A}_0 = (A, +, 0, \cdot, 1, ^-)$ is a Boolean algebra and $\circ$ is a residuated binary operation on $\mathbf{A}_0$ with $\rhd$ and $\lhd$ as its right and left conjugates. If $\circ$ has a unit e (that is, $x \circ e = e \circ x = x$ for all $x \in A$), the algebra $\mathbf{A}' = (\mathbf{A}_0, \circ, e, \rhd, \lhd)$ is called a *unital residuated Boolean algebra*, or a *ur-algebra*. If, furthermore, $\circ$ is associative, so that $(A, \circ, e)$ is a monoid, then $\mathbf{A}'$ is called a *residuated Boolean monoid*, or an *rm-algebra*.

Various generalizations of Tarski's notion of a relation algebra have been investigated by R. Maddux, see e.g. Maddux [4]. In our terminology, an algebra $\mathbf{A} = (\mathbf{A}_0, \circ, e, ^\cup)$ is a *non-associative relation algebra*, or an *NA*, if the algebra $\mathbf{A}' = (\mathbf{A}_0, \circ, e, \rhd, \lhd)$ with

$$a \rhd x = a^\cup \circ x, \qquad x \lhd a = x \circ a^\cup$$

is a *ur*-algebra, and $\mathbf{A}$ is a *relation algebra*, or an *RA*, if $\mathbf{A}'$ is an *rm*-algebra.

In order for a *ur*-algebra to be of the form $\mathbf{A}'$ for some *NA* $\mathbf{A}$, the operations $\rhd$ and $\lhd$ must be expressible in the form

$$a \rhd x = f(a) \circ x, \qquad x \lhd a = x \circ g(a)$$

with f and g operations on A, actually with f and g equal to the same operation, namely ${}^{\smile}$. The *ur*-algebras with this property will be characterized in the next section. Here we consider separately the two operations $\rhd$ and $\lhd$.

THEOREM 4.1. *For any ur-algebra* $\mathbf{A} = (\mathbf{A}_0, \circ, e, \rhd, \lhd)$, *the following conditions are equivalent*:

For some unary operation f on A,

$$a \rhd b = f(a) \circ b \quad \text{for all } a, b \in A. \tag{1a}$$

$$a \rhd b = (a \rhd e) \circ b \quad \text{for all } a, b \in A. \tag{1b}$$

$$a \circ b = (a \rhd e) \rhd b \quad \text{for all } a, b \in A. \tag{1c}$$

$$b \lhd a = e \lhd (a \lhd b) \quad \text{for all } a, b \in A. \tag{1d}$$

Proof. If (1a) holds, then $f(a) = f(a) \circ e = a \rhd e$, whence (1b) holds. Conversely, if (1b) holds, then (1a) holds with $f(a) = a \rhd e$.

Keeping fixed one or the other of the arguments a and b, we can write (1b) either as $L(a)^c = L(a \rhd e)$ or as $Q(b)^c = R(b)Q(e)^c$. Taking conjugates, we obtain the equivalent equations $L(a) = L(a \rhd e)^c$ and $Q(b) = Q(e)R(b)^c$, which are (1c) and (1d). □

Of course there is a symmetric result with $\rhd$ and $\lhd$ interchanged. For convenient reference, we state this result explicitly.

THEOREM 4.2. *For any ur-algebra* $\mathbf{A} = (\mathbf{A}_0, \circ, e, \rhd, \lhd)$, *the following conditions are equivalent*:

For some unary operation g on A,

$$a \lhd b = a \circ g(b) \quad \text{for all } a, b \in A. \tag{2a}$$

$$a \lhd b = a \circ (e \lhd b) \quad \text{for all } a, b \in A. \tag{2b}$$

$$a \circ b = a \lhd (e \lhd b) \quad \text{for all } a, b \in A. \tag{2c}$$

$$a \rhd b = (b \rhd a) \rhd e \quad \text{for all } a, b \in A. \tag{2d}$$

□

The next results shows that the map $a \mapsto e \lhd a$ is an involution whenever $a \mapsto a \rhd e$ is an involution, and conversely.

THEOREM 4.3. *For any ur-algebra* $\mathbf{A} = (\mathbf{A}_0, \circ, e, \rhd, \lhd)$, *the identities*

$$(a \rhd e) \rhd e = a \qquad \text{for all } a \in A, \tag{3a}$$

$$e \lhd (e \lhd a) = a \qquad \text{for all } a \in A \tag{3b}$$

are equivalent and imply the identity

$$a \rhd e = e \lhd a \qquad \text{for all } a \in A. \tag{3c}$$

The three identities hold whenever $\mathbf{A}$ *satisfies either Conditions* (1a)–(1d) *or Conditions* (2a)–(2d).

Proof. The identities (3a) and (3b) can be written as $Q(e)^c Q(e)^c = \tau$ and $Q(e)Q(e) = \tau$, respectively, where τ is the identity map. The two identities are therefore equivalent, since the conjugate of the identity map is the identity map.

We assume next that Condition (3a) holds. The condition asserts that the operation $a \mapsto a \rhd e$ is an involution and hence a bijection. Since it is also isotone, it is a Boolean automorphism. In particular,

$$ab \rhd e = (a \rhd e)(b \rhd e) \qquad \text{for all } a, b \in A.$$

Condition (3c) now follows from the fact that the conditions $(a \rhd e)x = 0$, $[(a \rhd e)x] \rhd e = 0$, $[(a \rhd e) \rhd e](x \rhd e) = 0$, $a(x \rhd e) = 0$, $(x \circ a)e = 0$ and $(e \lhd a)x = 0$ are equivalent.

To prove the last statement in the theorem, apply (1d) and (2d) with $a = e$ and use the identity $e \rhd x = x \lhd e = x$ to obtain (3b) and (3a), respectively. $\square$

We note that Condition (3c) does not imply (3a) and (3b). Indeed, let $\mathbf{A}$ be the complex algebra of a free monoid Λ^*, and let λ be the neutral element of Λ^*, that is, the empty word. Then $e = \{\lambda\}$. Hence $a \rhd e$ and $e \lhd a$ are both equal to e if $\lambda \in a$, but equal to 0 otherwise. Thus (3c) holds, but (3a) does not hold because $(a \rhd e) \rhd e \leq e$ for all $a \in A$.

5. Relation algebras

We first characterize those *ur*-algebras that come from *NA*'s. It has already been observed that the four equivalent conditions 4(1a)–4(1d), together with the four equivalent conditions 4(2a)–4(2d) are necessary. The next theorem includes the converse of that trivial observation.

THEOREM 5.1. *For any ur-algebra* $\mathbf{A} = (\mathbf{A}_0, \circ, e, \rhd, \lhd)$, *the following conditions are equivalent*:

For some unary operation $^\cup$ *on* A,

$$\mathbf{A}' = (\mathbf{A}_0, \circ, e, {}^\cup) \text{ is an } NA. \tag{1a}$$

For all $a, b \in A$,

$$a \rhd b = (a \rhd e) \circ b, \qquad a \lhd b = a \circ (e \lhd b). \tag{1b}$$

$$(\mathbf{A}_0, \circ, e, \_ \rhd e) \text{ is an } NA. \tag{1c}$$

$$(\mathbf{A}_0, \circ, e, e \lhd \_) \text{ is an } NA. \tag{1d}$$

Proof. If (1a) holds, then the operations

$$x \mapsto a \circ x \quad \text{and} \quad y \mapsto a^\cup \circ y$$

are conjugates of each other, and so are the operations

$$x \mapsto x \circ a \quad \text{and} \quad y \mapsto y \circ a^\cup.$$

Consequently, $a^\cup \circ y = a \rhd y$ and $y \circ a^\cup = y \lhd a$. Taking $y = e$, we obtain $a^\cup = a \rhd e = e \lhd a$, and (1b) follows.

Suppose (1b) holds. Then Conditions 4(1a)–4(1d), 4(2a)–4(2d) and 4(3a)–4(3c) hold. By 4(3c), $\_ \rhd e$ and $e \rhd \_$ are the same operation $^\cup$ and by 4(1b) and 4(2b), $a^\cup \circ x = a \rhd x$ and $x \circ a^\cup = x \lhd a$, so that (1c) and (1d) hold.

If (1c) holds, then (1a) holds with $x^\cup = x \rhd e$. Similarly, (1d) implies (1a). □

If **A** is an *rm*-algebra, then "*NA*" can be replaced by "*RA*" in Theorem 5.1, and we thus have a characterization of those *rm*-algebras that come from *RA*'s. However, in this case the two identities in (1b) turn out to be equivalent.

THEOREM 5.2. *For any rm-algebra* $\mathbf{A} = (\mathbf{A}_0, \circ, e, \rhd, \lhd)$ *the following conditions are equivalent*:

For some unary operation $^\cup$ *on* A,

$$\mathbf{A}' = (\mathbf{A}_0, \circ, e, {}^\cup) \text{ is an } RA. \tag{2a}$$

$$a \rhd b = (a \rhd e) \circ b \quad \text{for all } a, b \in A. \tag{2b}$$

$$a \lhd b = a \circ (e \lhd b) \quad \text{for all } a, b \in A. \tag{2c}$$

$$(\mathbf{A}_0, \circ, e, \_ \rhd e) \text{ is an } RA. \tag{2d}$$

$$(\mathbf{A}_0, \circ, e, e \lhd \_) \text{ is an } RA. \tag{2e}$$

Proof. It is to be shown that Conditions (2b) and (2c) are equivalent for rm-algebras. By symmetry, it suffices to show that (2c) implies (2b). Assume that (2c) holds. Then, in addition to 3(15), 4(2d) and 4(3a) hold. Hence for all $a, b \in \mathbf{A}$,

$$\begin{aligned} a \rhd b &= (b \rhd a) \rhd e = \{b \rhd [(a \rhd e) \rhd e]\} \rhd e \\ &= \{[(a \rhd e) \circ b] \rhd e\} \rhd e = (a \rhd e) \circ b, \end{aligned}$$

and (2b) holds.

The next result characterizes relation algebras as special ur-algebras. (The associative law is not explicitly assumed.) The backwards implication in (3b) arose first in a geometric context in the work of W. Prenowitz. He refers to it as the *Transposition Law*; see e.g. [5], p. 7.

THEOREM 5.3. *For any ur-algebra* $\mathbf{A} = (\mathbf{A}_0, \circ, e, \rhd, \lhd)$, *the following conditions are equivalent*:

For some unary operation $^\cup$ *on* A,

$$\mathbf{A}' = (\mathbf{A}_0, \circ, e, {}^\cup) \text{ is an } RA. \tag{3a}$$

For all $a, c, x, y \in A$,

$$(a \circ x)(y \circ c) \neq 0 \qquad \text{iff } (a \rhd y)(x \lhd c) \neq 0. \tag{3b}$$

$$\text{For all } a, b, c \in A, \qquad a \rhd (b \circ c) = (a \rhd b) \circ c. \tag{3c}$$

$$\text{For all } a, b, c \in A, \qquad (a \circ b) \lhd c = a \circ (b \lhd c). \tag{3d}$$

$$\text{For all } a, b, c \in A, \qquad a \lhd (b \lhd c) = (a \circ c) \lhd b. \tag{3e}$$

$$\text{For all } a, b, c \in A, \qquad (a \rhd b) \rhd c = b \rhd (a \circ c). \tag{3f}$$

Proof. Condition (3b) can be written

$$L(a)(x) \cdot R(c)(y) = 0 \qquad \text{iff } L(a)^c(y) \cdot R(c)^c(x) = 0$$

or equivalently,

$$[(L(a)^c R(c))(y)] \cdot x = 0 \qquad \text{iff } [(R(c)L(a)^c)(y)] \cdot x = 0.$$

For this to hold for all x and y means that

$$L(a)^c R(c) = R(c)L(a)^c.$$

This is (3c). Conditions (3d), (3e) and (3f) are obtained from (3c) by keeping fixed two of the variables and taking conjugates of both sides. Conditions (3b)–(3f) are therefore equivalent.

The identity (3c) holds in every *RA* (with $a \rhd x = a^{\smile} \circ x$). Hence (3a) implies the other five conditions. Finally, assuming (3c), we use Theorem 5.2 to prove (3a). Condition (2b), $a \rhd b = (a \rhd e) \circ b$, is obtained from (3c) by replacing b by e and c by b. Using this, we can write (3c) in the form $(a \rhd e) \circ (x \circ b) = ((a \rhd e) \circ x) \circ b$. From this it follows that the operation $\circ$ is associative, for by 4(3a), the map $a \mapsto a \rhd e$ is a bijection. Reference to Theorem 5.2 completes the proof. □

We conclude with an example showing that in Theorem 5.2, unlike Theorem 5.3, it was essential to postulate the associativity of the operation $\circ$. Indeed, without that assumption, Conditions (2b) and (2c) are not equivalent. Let S be a set with at least three elements, let $e \in S$, and define the poly-operation $\circ$ on S as follows:

$$a \circ b = b \qquad \text{whenever } a \neq b \neq e \neq a,$$

$$a \circ e = e \circ a = a \qquad \text{for all } a \in S,$$

$$a \circ a = \{a, e\} \qquad \text{whenever } a \neq e.$$

In the complex algebra of the unital poly-groupoid $(S, \circ)$, Condition (2c) fails, for if $a \neq b \neq e \neq a$, then

$$a \lhd b = 0, \qquad a \circ (e \lhd b) = b.$$

On the other hand, (2b) holds. For $a \neq b \neq e \neq a$ we have

$$a \rhd b = b = (a \rhd e) \circ b,$$

$$a \rhd a = \{a, e\} = (a \rhd e) \circ a,$$

and the remaining cases are equally trivial.

REFERENCES

[1] BIRKHOFF, G., *Lattice Theory*, Amer. Math. Soc. Colloq. Publ. vol. 25, 3rd ed., 1967.
[2] CHIN, L. H. and TARSKI, A., *Distributive and modular laws in the arithmetic of relation algebras*, Univ. Calif. Publ. Math. *1* (1951), 341–384.
[3] HOARE, C. A. R. and JIFENG, H., *The weakest prespecification*, Part II, Fundamenta Informaticae *9* (1986), 217–252.
[4] MADDUX, R., *Some varieties containing relation algebras*, Trans. Amer. Math. Soc. *272* (1982), 501–526.
[5] PRENOWITZ, W., *A contemporary approach to classical geometry*, The Amer. Math. Monthly *68* (No. 1, Part II) (1961), 1–67.

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