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On the supersolvability of finite groups

By

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1. Introduction. It is known that the product of two normal supersolvable subgroups of G is not necessarily supersolvable. In [1], Baer proved that if G is the product of two normal supersolvable subgroups and G' is nilpotent, then G is supersolvable. In [2], Friesen proved that if G is the product of two normal supersolvable subgroups of coprime indices, then G is supersolvable. If G is the product of two nilpotent subgroups, then G is not necessarily supersolvable. In [3], Kegel proved that if $G = HK = HL = KL$, H and K are nilpotent subgroups, and L is supersolvable, then G is supersolvable. The object of this paper is to continue the investigations of the above mentioned authors. Throughout, the groups are finite.

2. Basic definitions and lemmas. We first need the following definitions:

Definition 2.1. Subgroups H and K of the group G permute if

$$\langle H, K \rangle = HK = KH.$$

Definition 2.2. A subgroup H of G is said to be *quasinormal in* G if H permutes with every subgroup of G.

We indroduce the following definition which may be considered as a generalization of the concept of quasinormality.

Definition 2.3. Let H and K be subgroups of G. H is said to be *quasinormal in* K if H permutes with every subgroup of K.

Remark. It is very well-known that if $H \leqq G$ and H is quasinormal in G, then H is subnormal in G. If H and K are subgroups of G, and H is quasinormal in K, then H is not necessarily subnormal in G, as confirmed by S_3, the symmetric group of degree 3.

Lemma 2.4. *Suppose that* $G = HK$, H *and* K *are supersolvable subgroups of* G, $(|H|, |K|) = 1$, H *is quasinormal in* K *and* K *is quasinormal in* H. *Then* G *is supersolvable.*

Proof. Let p be a prime dividing $|G|$. Since $(|H|, |K|) = 1$, we can assume that p does not divide, say, $|K|$, in which case H contains a Sylow p-subgroup of G. Let P be a Sylow p-subgroup of H. Since H is supersolvable, it follows that H is solvable. Hence, by

[4, Theorem 4.5, p. 233], $H = PL$, where L is a p'-Hall subgroup of H. By hypothesis, LK is a subgroup of G. Then $|G:LK| = |P|$ and hence LK is a p'-Hall subgroup of G. It follows now, by [4, Theorem 4.5, p. 233], that G is solvable.

Let M be an arbitrary maximal subgroup of G. Since G is solvable, it follows, by [4, Theorem 1.5, p. 219], that $|G:M| = p^n$ for some prime p. Since $(|H|, |K|) = 1$, we can assume that p does not divide, say, $|K|$. Let M_1 be a p'-Hall subgroup of M. By [4, Theorem 4.1, p. 231], we have $K \leqq M_1^x$ for some x in G. Since M^x has the same properties as M, we can replace M by M^x and so we can assume without loss that $K \leqq M$. Since $G = HK$, it follows that $M = K(H \wedge M)$. Clearly, $(|K|, |M \wedge H|) = 1$. By hypothesis, K is quasinormal in H and so K is quasinormal in $H \wedge M$. We argue that $H \wedge M$ is quasinormal in K. Let K_1 be a subgroup of K. By hypothesis, $K_1 H$ is a subgroup of G. Then $K_1 H \wedge M = K_1(H \wedge M)$ and hence $K_1(H \wedge M)$ is a subgroup of G. Thus $H \wedge M$ is quasinormal in K. By induction on $|G|$, M is supersolvable. Hence all proper subgroups of G are supersolvable. Suppose that G is not supersolvable. Then it follows from Hilfssatz C of [5] that G has exactly one normal Sylow subgroup, say, P. So if $\phi(G) \neq 1$, then $G/\phi(G)$ is supersolvable by induction on $|G|$, which implies that G is supersolvable by a well-known theorem of Huppert, a contradiction. We may, therefore, assume that $\phi(G) = 1$. Then P is elementary abelian. By Satz 1 of [5], we have that P is a minimal normal subgroup of G. Set $|P| = p^n$, for some prime p dividing $|G|$. By Hilfssatz C of [5], G has a Sylow tower for the natural (descending) ordering of prime divisors of $|G|$, or G is nonnilpotent, each of whose proper subgroup is nilpotent. If G has a Sylow tower, then P is normal in G and p is the largest prime dividing $|G|$. We can assume that $P \leqq H$ and $p \nmid |K|$. Since H is supersolvable, it follows, by [6, Corollary 10.5.2, p. 159], that H has a normal subgroup L of order p. By hypothesis, LK is a subgroup of G. If $G = LK$, then G is supersolvable, a contradiction. Hence LK is a proper supersolvable subgroup of G and L is normal in LK. Now it follows easily that L is normal in G. This is impossible as P is a minimal normal subgroup of G. If G is nonnilpotent, each of whose proper subgroup is nilpotent, then, by [7, Satz 5.2, p. 281], $G = PQ$, where P is normal in G, P is a Sylow p-subgroup of G, Q is nonnormal cyclic Sylow q-subgroup of G, and $p \neq q$. Clearly, $|P| = p^n$, where $n \geqq 2$. We can assume that $P = H$ and $Q = K$. Let L be a proper subgroup of P. By hypothesis, LQ is a subgroup of G. Clearly, LQ is nilpotent. Then L is normal in LQ. Now it follows easily that L is normal in G. Th˙. is impossible as P is a minimal normal subgroup of G.

The argument which established our lemma can easily be adapted to yield the following result:

Theorem (Friesen [2]). *If H and K are normal supersolvable subgroups of G of coprime indices, then G is supersolvable.*

3. Main results. We prove the following theorems:

Theorem 3.1. *Suppose that H and K are supersolvable subgroups of G such that $G = HK$. Suppose further that each subgroup of H is quasinormal in K. Then G is supersolvable.*

Proof. Suppose that the theorem is false and let G be a counter-example of smallest order. Let p be the largest prime dividing $|G|$. Then, by [8, Theorem 13.2.5, p. 378], there exist a Sylow p-subgroup P_1 of H and a Sylow p-subgroup P_2 of K such that $P_1 P_2$ is a Sylow p-subgroup of G. Set $P = P_1 P_2$. We argue that $P \lhd G$. Since K is supersolvable, it follows that K is solvable. Then there exists a p'-Hall subgroup L_2 of K such that $K = P_2 L_2$. If $G = P_1 K$, then $G = P L_2$. By hypothesis, $P_1 L_2$ is a subgroup of G, P_1 is quasinormal in L_2 and L_2 is quasinormal in P_1. Lemma 2.4 implies that $P_1 L_2$ is supersolvable and so $P_1 \lhd P_1 L_2$. Since K is supersolvable, it follows that $P_2 \lhd K$. Hence $P \lhd G$. Now we can assume that $P_1 K$ and $P_2 H$ are proper subgroups of G. Our choice of G implies that $P_1 K$ and $P_2 H$ are supersolvable. Hence $P \lhd P_1 K$ and $P \lhd P_2 H$. Since $G = HK$, it follows that $P \lhd G$.

Assume that $\phi(G) \neq 1$. Then $G/\phi(G)$ is supersolvable by our choice of G, which implies G is supersolvable, a contradiction. Thus $\phi(G) = 1$. Then P is elementary abelian. We argue that P is a minimal normal subgroup of G. If not, by Maschke's Theorem, $P = R_1 \times R_2$, where $R_i \lhd G\,(i = 1, 2)$. Our choice of G implies that G/R_i is supersolvable $(i = 1, 2)$. Since $G = G/R_1 \wedge R_2 \gtrsim G/R_1 \times G/R_2$, it follows that G is supersolvable, a contradiction. Thus P is a minimal normal subgroup of G. Clearly, $P_1 L_2$ is a subgroup of G, where L_2 is a p'-Hall subgroup of K, P_1 is quasinormal in L_2 and L_2 is quasinormal in P_1. If $G = P_1 L_2$, then, by Lemma 2.4, G is supersolvable, a contradiction. Thus $P_1 L_2 < G$. Our choice of G implies that $P_1 L_2$ is supersolvable and so $P_1 \lhd P_1 L_2$. Also $P_1 \lhd H$. Hence $P_1 \lhd G$. Since P is a minimal normal subgroup of G, it follows that $P_1 = 1$ or $P_1 = P$. If $P_1 = 1$, then $P = P_2 \leqq K$ and H is a p'-subgroup of G. Since K is supersolvable, it follows that K has a normal subgroup P_3 of order p. By hypothesis, $P_3 H$ is a subgroup of G. If $G = P_3 H$, then, by Lemma 2.4, G is supersolvable, a contradiction. Thus $P_3 H$ is a proper subgroup of G. Our choice of G implies that $P_3 H$ is supersolvable and so $P_3 \lhd P_3 H$. Hence $P_3 \lhd G$ and so $P_3 = P$. Our choice of G implies that G/P is supersolvable and so G is supersolvable, a contradiction. Similarly, if $P_2 = 1$, we have a contradiction. If $P_1 = P$, then $1 < P_2 \leqq P$. If $P_2 = P$, then $P \leqq H$ and $P \leqq K$. Hence $H = P L_1$, where L_1 is a p'-Hall subgroup of H. Let P_3 be a subgroup of P of order p. By hypothesis, $P_3 L_1$ and $P_3 L_2$ are subgroups of G. Clearly, $P_3 L_1 \leqq H < G$ and $P_3 L_2 \leqq K < G$. Then $P_3 L_1$ and $P_3 L_2$ are supersolvable and so $P_3 \lhd P_3 L_1$ and $P_3 \lhd P_3 L_2$. Hence $P_3 \lhd G$ and so $P_3 = P$. Our choice of G implies that G/P is supersolvable and so G is supersolvable, a contradiction. If $1 < P_2 < P$, then, by hypothesis, $P_2 L_1$ is a subgroup of G. Our choice of G implies that $P_2 L_1$ is supersolvable and so $P_2 \lhd P_2 L_1$. Also, $P_2 \lhd K$. Hence $P_2 \lhd G$. This implies that $P_2 = P$. This is impossible as $P_2 < P$. This completes the proof of the theorem.

Remark. The alternating group of degree 4 shows that if $G = HK$, H and K are supersolvable subgroups of G, and H is quasinormal in K, then the group need not be supersolvable in general.

Our theorem may be considered as a generalization of the following well-known result:

Theorem. *If G/H and G/K are supersolvable, then $G/H \wedge K$ is supersolvable.*

Theorem 3.2. *Suppose that H is a nilpotent subgroup of G, K is a supersolvable subgroup of G and that $G = HK$. Suppose further that H is quasinormal in K and K is quasinormal in H. Then G is supersolvable.*

Proof. Suppose that the theorem is false and let G be a counter-example of smallest order. Let p be the largest prime dividing $|G|$. Then there exist Sylow p-subgroup P_1 of H and Sylow p-subgroup P_2 of K such that $P = P_1 P_2$ is a Sylow p-subgroup of G. By hypothesis, $P_1 K$ and $P_2 H$ are subgroups of G. Let M be a proper subgroup of G such that $H \leqq M$. Since $G = HK$ and $H \leqq M$, it follows easily that $M = H(K \wedge M)$. Clearly, H is quasinormal in $K \wedge M$. Let H_1 be a subgroup of H. By hypothesis, $H_1 K$ is a subgroup of G. Clearly, $H_1 K \wedge M = H_1(K \wedge M)$. Hence $K \wedge M$ is quasinormal in H. But now our choice of G implies that M is supersolvable. Similarly, if $K \leqq M$, then M is supersolvable. If $P_1 K$ is a proper subgroup of G, then, by our choice of G, $P_1 K$ is supersolvable and so $P \lhd P_1 K$. Since H is nilpotent, it follows that $P_1 \lhd H$. Hence if $y \in G$, $y = hk$ with $h \in H$ and $k \in K$, and consequently $P_1^y = P_1^{hk} = P_1^k \leqq P$. But then the normal closure $P_1^G = \langle P_1^y \mid y \in G \rangle \leqq P$. Clearly, G/P_1^G satisfies the conditions of the theorem. Hence by our choice of G, G/P_1^G is supersolvable. Since $P_1^G \leqq P$, it follows that $P \lhd G$. Similarly, if $P_2 H$ is a proper subgroup of G, then $P \lhd G$. Now we can assume that $G = P_1 K = P_2 H$. Then $G = PK = PH = KH$. Applying Kegel's Theorem [3], it follows that G is supersolvable, a contradiction. Therefore, $P \lhd G$, where P is a Sylow p-subgroup of G and p is the largest prime dividing $|G|$. If $\phi(G) \neq 1$, then $G/\phi(G)$ is supersolvable by our choice of G, which implies that G is supersolvable, a contradiction. We may, therefore, assume that $\phi(G) = 1$. Then P is elementary abelian. We argue that P is a minimal normal subgroup of G. If not, by Maschke's Theorem, $P = R_1 \times R_2$, where $R_i \lhd G (i = 1, 2)$. Since

$$G = G/R_1 \wedge R_2 \precsim G/R_1 \times G/R_2,$$

it follows that G is supersolvable, a contradiction. Thus P is a minimal normal subgroup of G. We consider the following cases:

Case 1. $P_1 = 1$. Then $P_2 = P \leqq K$. Since K is supersolvable, it follows that K has a normal subgroup P_3 of order p. Clearly, H is a p'-subgroup of G. By hypothesis, $P_3 H$ is a subgroup of G. If $G = P_3 H$, then it follows easily that G is supersolvable, a contradiction. Thus we may assume that $P_3 H$ is a proper subgroup of G. By our choice of G, $P_3 H$ is supersolvable and so $P_3 \lhd P_3 H$. Now it follows easily that $P_3 \lhd G$. Since P is a minimal normal subgroup of G, we have $P_3 = P$ and so G is supersolvable, a contradiction.

Case 2. $P_2 = 1$. Then we have a contradiction; see Case 1.

Case 3. $P_1 \neq 1$ and $P_2 \neq 1$. By our choice of G, it follows that G/P is supersolvable and so G is solvable. Consider $\operatorname{Fit}(G)$. Clearly, $P \leqq \operatorname{Fit}(G)$. Hence if $P < \operatorname{Fit}(G)$, there exists a Sylow q-subgroup Q of $\operatorname{Fit}(G)$, where $q \neq p$. Since Q char $\operatorname{Fit}(G) \lhd G$, it follows that $Q \lhd G$. By our choice of G, G/P and G/Q are supersolvable. Since

$$G = G/P \wedge Q \precsim G/P \times G/Q,$$

it follows that G is supersolvable, a contradiction. We may, therefore, assume that $P = \operatorname{Fit}(G)$. Since G is solvable, it follows from [8, Theorem 7.4.7, p. 167] that

$C_G(\mathrm{Fit}(G)) = C_G(P) \leqq \mathrm{Fit}(G) = P$. Let L be a p'-Hall subgroup of K. By hypothesis, HL is a subgroup of G. If $G = HL$, then $P_1 = P$. Since H is nilpotent, it follows that $H \leqq C_G(P) \leqq P$ and so $H = P$. If $P_2 = P$, then $H = P \leqq K$ and so $G = K$, a contradiction. We may assume that $P_2 < P$. Since K is supersolvable, it follows that $P_2 \lhd K$. Also $P_2 \lhd P = H$. Then $P_2 \lhd G$ and this is impossible as P is a minimal normal subgroup of G. Now assume that HL is a proper subgroup of G. By our choice of G, HL is supersolvable. Hence $P_1 \lhd HL$, and so $HL \leqq N_G(P_1)$. Now it follows easily that $P_1 \lhd G$. Since P is a minimal normal subgroup of G and $P_1 \neq 1$, it follows that $P_1 = P$. Then $P_2 \lhd P = H$. Since K is supersolvable, it follows that $P_2 \lhd K$. Hence $P_2 \lhd G$. Since P is a minimal normal subgroup of G, we have $P_2 = P$ and this is impossible as $P_2 < P$. This completes the proof of the theorem.

Remark. If we require that $G = HK$, H and K are supersolvable subgroups of G, H is quasinormal in K and K is quasinormal in H, then G is not necessarily supersolvable. Let

$$x = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

be matrices over J_5, where J_5 is the field of integers (mod 5). Then $S = \langle x, y \rangle$ is a quaternion group of order 8. Let $T = J_5 \times J_5$ and let $G = \mathrm{Hol}(T, S)$ (where S is interpreted as a group of automorphisms of T). Let $H = \langle T, x \rangle$ and $K = \langle T, y \rangle$. Then

(i) H and K are normal supersolvable subgroups of G;
(ii) $G = HK$;
(iii) G is not supersolvable.

(Huppert [9]; see also [8, Exercise 9.2.19, p. 219]).

As a corollary, we have

Corollary 3.3. *If H is a quasinormal nilpotent subgroup in G and K is a quasinormal supersolvable subgroup in G, then HK is a quasinormal supersolvable subgroup in G.*

As an immediate consequence of Corollary 3.3, we have the following well-known result:

Theorem. *If H is a normal nilpotent subgroup in G and K is a normal supersolvable subgroup in G, then HK is a normal supersolvable subgroup in G.*

Theorem 3.4. *Suppose that H and K are supersolvable subgroups of G of coprime indices and that for each pair of primes $\{p, q\}$ with $p > q$, where one of these primes divides $|G : H|$ and the other divides $|G : K|$, $p \not\equiv 1\ (q)$. Suppose further that H is quasinormal in K and that K is quasinormal in H. Then G is supersolvable.*

Proof. Suppose that the theorem is false and let G be a counter-example of smallest order. Let p be the largest prime dividing $|G|$. Let P be a Sylow p-subgroup of G. Hence if $p \nmid |G : H|$, we have $P \leqq H^x$ for some x in G. Since H^x has the same properties as H, we can replace H by H^x and so we can assume without loss that $P \leqq H$. Since H is supersolvable, it follows that $P \lhd H$. Similarly, if $p \nmid |G: K|$, we have $P \lhd K$. Hence if $p \nmid |G: H|$ and $p \nmid |G: K|$, we have $P \lhd G$. Our choice of G implies that G/P is supersolvable and so G is solvable. Now we can assume that $p \mid |G: K|$. Hence if $p \mid |K|$, $P_1 \lhd K$, where P_1 is a Sylow p-subgroup of K, we can assume without loss that $P_1 \leqq P \leqq H$. Since $P_1^{kh} = P_1^h \leqq H$, it follows that $P_1^G \leqq H$ and so P_1^G is normal supersolvable. Our choice of G implies that G/P_1^G is supersolvable and so G is solvable. If $p \nmid |K|$, then K is a p'-subgroup of G. Let Q be a Sylow q-subgroup of K, where q is the largest prime dividing $|K|$. Hence if $q \nmid |G: H|$, we can assume without loss that $Q \leqq H$ and so $Q^G \leqq H$. Our choice of G implies that G/Q^G is supersolvable. Since $Q^G \leqq H$, it follows that Q^G is supersolvable. Hence G is solvable. Thus we can assume that $q \mid |G: H|$. Let P_2 be a subgroup of P of order p. By hypothesis, $P_2 K$ is a subgroup of G. If $G = P_2 K$, then $N_G(Q) = K$ or $Q \lhd G$. If $Q \lhd G$, then it follows easily that G is solvable. If $N_G(Q) = K$, then by Sylow's Theorem $p \equiv 1\,(q)$ and this is impossible as $p \not\equiv 1\,(q)$. Thus $P_2 K$ is a proper subgroup of G. Let M be a proper subgroup of G such that $K \leqq M$. Since $G = HK$, it follows that $M = K(M \wedge H)$. Clearly, $(|M: K|, |M: M \wedge H|) = 1$, K and $M \wedge H$ are supersolvable subgroups of M, and K is quasinormal in $M \wedge H$. Let K_1 be a subgroup of K. By hypothesis, $K_1 H$ is a subgroup of G. Clearly, $M \wedge K_1 H = K_1 (M \wedge H)$. Then $M \wedge H$ is quasinormal in K. Our choice of G implies that M is supersolvable. Hence $P_2 K$ is supersolvable and so $P_2 \lhd P_2 K$. Now it follows that $P_2^G \leqq H$ and so G is solvable. Therefore, we have that G is solvable.

Let M be a maximal subgroup of G. Since G is solvable, we have $|G: M|$ is a power of prime, say r^e. If $r \nmid |G: H|$ and $r \nmid |G: K|$, then $G = MH = MK$. Clearly, $|G: H| = |M: M \wedge H|$, $|G: K| = |M: M \wedge K|$, $M \wedge H$ and $M \wedge K$ are supersolvable subgroups of M, $M \wedge H$ is quasinormal in $M \wedge K$, and $M \wedge K$ is quasinormal in $M \wedge H$. Hence M is supersolvable by our choice of G. Now we can assume that $r \mid |G: K|$. Then $G = MH$ and so $|G: H| = |M: M \wedge H|$. Let R be a Sylow r-subgroup of K. Then $K = RK_1$, where K_1 is an r'-Hall subgroup of K. Since G is solvable, we can assume without loss that $K_1 \leqq M$. Since

$$|G: K_1| = |G: M|\,|M: K \wedge M|\,|K \wedge M: K_1| = |G: K|\,|K: K_1|,$$

we have $\pi(M: K \wedge M) \leqq \pi(G: K)$. Since H and K are of coprime indices in G, it follows that $M \wedge H$ and $M \wedge K$ are of coprime indices in M. Hence M is supersolvable by our choice of G. Thus all proper subgroups of G are supersolvable. By Hilfssatz C of [5], G has exactly one normal Sylow subgroup P. So if $\phi(G) \neq 1$, then $G/\phi(G)$ is supersolvable by our choice of G, which implies that G is supersolvable, a contradiction. We may, therefore, assume that $\phi(G) = 1$. Then P is elementary abelian. By Satz 1 of [5], P is a minimal normal subgroup of G. Let $|G| = p^a q^b$, where $p \neq q$. Since H and K are of coprime indices, we can assume that $P \leqq H$ and $Q \leqq K$. If $Q < K$, then $P_1 \lhd K$, where P_1 is a Sylow p-subgroup of K. Since $P_1 \lhd P$, we have $P_1 \lhd G$ and this is impossible as P is a minimal normal subgroup of G. Thus $Q = K$. By hypothesis, KP_2 is a subgroup of G, where P_2 is a subgroup of P of order p. Clearly, if $P_2 K = G$, then G is supersolvable, a contradiction. Hence $P_2 K$ is a proper subgroup of G. Clearly, $P_2 \lhd P_2 K$ and $P_2 \lhd P$. Hence $P_2 \lhd G$ and this is impossible as P is a minimal normal subgroup of G. If $|G|$ is divisible by at least four different primes, then, by Satz 4 of [5], G is supersolvable, a contradiction. Thus we can assume that $|G|$ is divisible by three different primes. Set $\pi(G) = \{p, q, r\}$, where $p > q > r$. We deal with the following cases:

Case 1. $|G:H| = p^{e_1}$. We argue that H is a p'-Hall subgroup of G. If not, $H = P_3 H_1$, where P_3 is a Sylow p-subgroup of H and H_1 is a p'-Hall subgroup of H. Clearly, $G = PH$, $P_3 \lhd P$ and $P_3 \lhd H$. Hence $P_3 \lhd G$ and this is impossible as P is a minimal normal subgroup of G. Thus H is a p'-Hall subgroup of G. Since H is quasinormal in K, it follows that $P_1 H$ is a subgroup of G, where $1 < P_1 < P \leqq K$. Since $P_1 H$ is supersolvable, we have $P_1 \lhd P_1 H$ and $P_1 \lhd G$ and this is impossible as P is a minimal normal subgroup of G.

Case 2. $|G:H| = r^{e_3}$. Then $|G:K| = p^{e_1}$ or q^{e_2} or $p^{e_1} q^{e_2}$. If $|G:K| = p^{e_1}$, then, as in Case 1, we have a contradiction. If $|G:K| = q^{e_2}$ or $p^{e_1} q^{e_2}$, then by hypothesis, $q \not\equiv 1\,(r)$. By Satz 2 of [5], there exists a p'-Hall subgroup L of G such that L is non-abelian and all its proper subgroups are abelian. Clearly, L is supersolvable and $q > r$. By [7, p. 285, 14), b)], Q is a minimal normal subgroup of L. Since L is supersolvable, we have $|Q| = q$. Now by Sylow's Theorem $q \equiv 1\,(r)$, a contradiction.

Case 3. $|G:H| = q^{e_2}$. Then $|G:K| = p^{e_1}$ or r^{e_3} or $r^{e_3} p^{e_1}$. If $|G:K| = p^{e_1}$, then, as in Case 1, G is supersolvable, a contradiction. If $|G:K| = r^{e_3}$, then, as in Case 2, G is supersolvable, a contradiction. If $|G:K| = r^{e_3} p^{e_1}$, then, by hypothesis, $q \not\equiv 1\,(r)$. On the other hand $q \equiv 1\,(r)$ as G contains a non-abelian subgroup L and all proper subgroup of L are abelian and $Q \lhd L$, where Q is a Sylow q-subgroup of L of order q. Hence $q \not\equiv 1\,(r)$ and $q \equiv 1\,(r)$, a contradiction. Therefore, G is supersolvable and the theorem is proved.

As a corollary of the proof of Theorem 3.4, we have

Corollary 3.5. *Suppose that H and K are subgroups of G of coprime indices and that H and K have the Sylow tower property. Suppose further that H is quasinormal in K and K is quasinormal in H. Then G satisfies the Sylow tower property.*

We can now prove the following generalization of Corollary 3.5:

Corollary 3.6. *Suppose that H and K are subgroups of G such that $G = HK$ and that H and K satisfy the Sylow tower property. Suppose further that H is quasinormal in K and K is quasinormal in H. Then G satisfies the Sylow tower property.*

Proof. Suppose that the theorem is false and let G be a counter-example of smallest order. Let p be the largest prime dividing $|G|$. Then there exist a Sylow p-subgroup P_1 of H and a Sylow p-subgroup P_2 of K such that $P = P_1 P_2$ is a Sylow p-subgroup of G. We consider the following cases:

Case 1. $P_1 = P$ and $P_2 = 1$. Then $P \lhd H$. By hypothesis, PK is a subgroup of G. If $PK = G$, then $|G:K| = |P|$. Clearly, $p \nmid |G:H|$. Hence $(|G:H|, |G:K|) = 1$. Applying Corollary 3.5, it follows that G has a Sylow tower, a contradiction. If PK is a proper subgroup of G, then, by our choice of G, PK has a Sylow tower and so $P \lhd PK$. Since $P \lhd H$ and $P \lhd PK$, we have $P \lhd G$. By our choice of G, G/P has a Sylow tower. Hence G has a Sylow tower, a contradiction. Similarly, if $P_2 = P$ and $P_1 = 1$, we have a contradiction.

Case 2. $P_1 = P$ and $1 < P_2 < P$. Then $P_2 \lhd K$ and $P_1 \lhd H$. Since $P_2^{kh} = P_2^h \leqq P^h = P \leqq H$, it follows that $P_2^G \leqq P$. By our choice of G, G/P_2^G has a Sylow tower and so $P \lhd G$. Hence G has a Sylow tower, a contradiction. Similarly, if $P_2 = P$ and $1 < P_1 < P$, we have a contradiction.

Case 3. $1 < P_1 < P$ and $1 < P_2 < P$. By hypothesis, $P_2 H$ is a subgroup of G. If $G = P_2 H$, then $G = HP$. Hence if $y \in G$, $y = hk$, with $h \in H$ and $k \in P$, and consequently $P_1^y = P_1^{hk} = P_1^k \leqq P$. But then the normal closure $P_1^G = \langle P_1^y \mid y \in G \rangle \leqq P$. By our choice of G, G/P_1^G has a Sylow tower and so $P \lhd G$. Hence G has a Sylow tower, a contradiction. Similarly, if $G = P_1 K$, we have a contradiction. Thus we can assume that $P_2 H$ and $P_1 K$ are proper subgroups of G. By our choice of G, $P_2 H$ and $P_1 K$ satisfy the Sylow tower property. Hence $P \lhd P_2 H$ and $P \lhd P_1 K$ and so $P \lhd G$. Therefore, G has a Sylow tower, a final contradiction.

The proof of Theorem 3.4 can easily be adapted to yield the following corollary:

Corollary 3.7. *Suppose that H and K are supersolvable subgroups of G of coprime indices and that G' is nilpotent. Suppose further that H is quasinormal in K and K is quasinormal in H. Then G is supersolvable.*

The following result may be considered as an improvement of Baer's Theorem [1].

Theorem 3.8. *Suppose that H and K are supersolvable subgroups of G, G' is nilpotent and $G = HK$. Suppose further that H is quasinormal in K and K is quasinormal in H. Then G is supersolvable.*

Proof. Suppose the theorem is false and let G be a counterexample of smallest order. If G contain two normal subgroups R_1 and R_2 such that $(|R_1|, |R_2|) = 1$, then G/R_1 and G/R_2 are supersolvable by our choice of G. Hence $G = G/R_1 \wedge R_2 \precsim G/R_1 \times G/R_2$ is supersolvable, a contradiction. This implies that G' is a q-group for some prime q. Corollary 3.6 implies that $P \lhd G$, where P is a Sylow p-subgroup of G and p is the largest prime dividing $|G|$. Hence $q = p$ and $G' \leqq P$. So if $\phi(G) \neq 1$, then $G/\phi(G)$ is supersolvable by our choice of G, which implies that G is supersolvable, a contradiction. We may, therefore, assume that $\phi(G) = 1$. We argue that P is a minimal normal subgroup of G. If not, by Maschke's Theorem, $P = R_1 \times R_2$, where $R_i \lhd G\,(i = 1, 2)$. By our choice of G, G/R_i is supersolvable $(i = 1, 2)$. Since $G = G/R_1 \wedge R_2 \precsim G/R_1 \times G/R_2$, it follows that G is supersolvable, a contradiction. Thus P is a minimal normal subgroup of G. Since $G' \leqq P$, we have $G' = P$. If $G' \leqq H$ and $G' \leqq K$, then it is clear that H and K are normal subgroups of G. From Baer's Theorem [1], we conclude that G is supersolvable, a contradiction. Hence if $G' = P \nleqq H$, we have PH is a subgroup of G. Assume that $G = PH$. Hence if $P \leqq K$, we have $(|G : H|, |G : K|) = 1$ and so Corollary 3.7 can be applied to yield that G is supersolvable, a contradiction. Thus $P \nleqq K$. Since $G = HK$, it follows that there exist a Sylow p-subgroup P_1 of H and a Sylow p-subgroup P_2 of K such that $P = P_1 P_2$. Since $P \nleqq H$ and $P \nleqq K$, it follows that $1 < P_i < P\,(i = 1, 2)$. Since $G = PH$ and $P_1 \lhd H$, it follows easily that $P_1 \lhd G$ and this is impossible as P is a minimal normal subgroup of G. Thus we can assume that PH and PK are proper subgroups of G. Now it follows easily that PH and PK are normal subgroups of G. Our choice of G implies that PH and PK are supersolvable subgroups of G. Again Baer's Theorem implies that G is supersolvable, a final contradiction.

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