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Two Basic Problems in Reliability-Based Structural Optimization

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Abstract: Optimization of structures with respect to performance, weight or cost is a well-known application of mathematical optimization theory. However optimization of structures with respect to weight or cost under probabilistic reliability constraints or optimization with respect to reliability under cost/weight constraints has been subject of only very few studies. The difficulty in using probabilistic constraints or reliability targets lies in the fact that modern reliability methods themselves are formulated as a problem of optimization. In this paper two special formulations based on the so-called first-order reliability method (FORM) are presented. It is demonstrated that both problems can be solved by a one-level optimization problem, at least for problems in which structural failure is characterized by a single failure criterion. Three examples demonstrate the algorithm indicating that the proposed formulations are comparable in numerical effort with an approach based on semi-infinite programming but are definitely superior to a two-level formulation.

Key Words: Reliability-oriented optimization, structural reliability, reliability optimization, cost optimization, one-level optimization.

1 Introduction

Optimization of structures with respect to weight or cost has been one of the prominent applications and challenges of mathematical optimization. Observation of reliability constraints in terms of "safety factors" has always been a natural part of many studies. More recently, optimization under reliability constraints in terms of restrictions on stochastic quantities such as the variance of some structural performance quantity or the failure probability itself has been under study. Unfortunately, in most cases the reliability part was dealt with on a somewhat elementary level. However, it is not more than about 20 years ago that the theory of structural reliability experienced a breakthrough in that it could reduce the task of solving high dimensional volume integrals – still a numerically rather impractical problem by standard methods – into an optimization problem plus some simple algebra. That theory is of asymptotic nature and based on so-called Laplace integrals (Hasofer/Lind (1974), Rackwitz/Fiessler (1978), Breitung (1984), Hohenbichler et al. (1987)). It is known by Second-Order-Reliability-Method (SORM) and has a natural, significantly more practical first order version, the First-Order-Reliability-Method (FORM). Numerous developments in part making use of importance sampling methods and response surface methods have turned the initial ideas into a powerful tool for practical reliability analysis. Its use in structural optimization, however, resulted in serious numerical problems. Therefore, whereas the reliability analysis of structures now is well known and computationally efficient the inverse problem of optimal probabilistic design of structures is still under development. Various attempts have been made by formulating a 2-step-algorithm, one for the design parameters and a second for the reliability part and which is called by the first (see, for example, Enevoldsen, Sørensen (1993, 1994)). If, for example, sequential quadratic programming methods are used on both optimization levels second order derivatives, mostly evaluated numerically, are required at the second level for FORM and even third order derivatives for SORM. Moreover, a mathematical proof is still missing under which such a two level approach is converging. The difficulty of proof lies in the fact that the failure domains themselves depend on the design parameters. The numerical difficulties lead some authors to develop interactive SQP-algorithms (Pederson/Thoft-Christensen (1994, 1996)), apparently with some success. Another promising approach in the framework of FORM making use of semi-infinite programming was recently proposed by Kirjner, Polak and Der Kiureghian (1995).

In the following yet another approach will be developed based on an idea proposed by Madsen/Friis Hansen (1992) which is also based on FORM. Two formulations will be developed. The first optimizes structural weight or cost under reliability and performance constraints. As a generalization cost will be understood as expected cost, i.e. including the failure cost multiplied by the failure probability. Hence, the objective itself contains reliability. The second formulation will optimize structures for reliability under cost and performance constraints. Both types of formulations will be demonstrated at simple examples.

2 Structural Reliability Methods-Optimality Conditions for β-Points

Let $\mathbf{X} = (X_1, \ldots, X_n)^T$ be a n-dimensional vector of random variables with continuously differentiable distribution function $F_{\mathbf{X}}(\mathbf{x})$. Let further $G(\mathbf{X}, \mathbf{p}) \leq 0$ be the failure domain and $G(\mathbf{X}, \mathbf{p}) = 0$ the so-called limit state which is assumed to be at least twice differentiable. $\mathbf{p}$ is a d-dimensional vector of design parameters. It can involve deterministic parameters but also parameters of the distribution function $F_{\mathbf{X}}(\mathbf{x})$. Then the time-invariant probability of failure is given by

$$P_f(\mathbf{p}) = \int_{G(\mathbf{x},\mathbf{p}) \leq 0} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \tag{1}$$

where $f_{\mathbf{X}}(\mathbf{x})$ is the probability density of $\mathbf{X}$. Analytical results for this integral are almost absent. However, let a probability distribution transformation $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$ exist which maps an arbitrary n-dimensional random vector $\mathbf{X} = (X_1, \dots, X_n)^T$ into an independent standard normal vector $\mathbf{U} = (U_1, \dots, U_n)^T$ (Hohenbichler/Rackwitz, 1981, Der Kiureghian/Liu, 1986, Winterstein/ Bjerager, 1987). With $G(\mathbf{x}, \mathbf{p}) = G(\mathbf{T}(\mathbf{u}), \mathbf{p}) = g(\mathbf{u}, \mathbf{p})$ and the failure domain $\mathscr{F}_{\mathbf{p}} = \{\mathbf{u} : g(\mathbf{u}, \mathbf{p}) \leq 0\}$, it is:

$$P_f(\mathbf{p}) = \int_{\mathscr{F}_p} P_{\mathbf{U}}(d\mathbf{u}) = \int_{g(\mathbf{u},\mathbf{p}) \leq 0} \varphi_U(\mathbf{u}) d\mathbf{u} \tag{2}$$

where $P_U(.)$ is the standard normal distribution law and $\varphi_U(\mathbf{u})$ is the standard normal density. Now, if $g(\mathbf{u}, \mathbf{p}) = \boldsymbol{\alpha}^T \mathbf{u} + \beta$ an exact result is $P_f(\mathbf{p}) = \Phi(-\beta)$. $\Phi(.)$ is the standard normal integral. If $g(\mathbf{u}, \mathbf{p}) \approx \boldsymbol{\alpha}^T \mathbf{u} + \beta$ with $\beta = -\boldsymbol{\alpha}^T \mathbf{u}^*$ and where $\mathbf{u}^*$ is the solution of the following optimization problem

$$(\beta P) \qquad \textbf{minimize} \quad \|\mathbf{u}\| \qquad \text{subject to} \quad g(\mathbf{u}, \mathbf{p}) \leq 0 \,,$$

there is $P_f(\mathbf{p}) \approx \Phi(-\beta_{\mathbf{p}})$ (Rackwitz/Fiessler, 1978). The solution point $\mathbf{u}^*$ of the optimization problem (βP), the so called design point or β-point, defines the reliability index

$$\beta_{\mathbf{p}} = \|\mathbf{u}^*\| \tag{3}$$

$\boldsymbol{\alpha}$ is the vector of direction cosines of the solution point. Reference to the parameter vector $\mathbf{p}$ is omitted here and in the following whenever this is possible without loosing clarity. Breitung (1984) established the following asymptotic result. For $\beta_{\mathbf{p}} \to \infty$ there is:

$$P_f(\mathbf{p}) = \int_{g(\mathbf{u},\mathbf{p}) \leq 0} \varphi_{\mathbf{U}}(\mathbf{u}) d\mathbf{u} \approx \Phi(-\beta_{\mathbf{p}}) \cdot \prod_{i=1}^{n-1} (1 - \beta_{\mathbf{p}} \kappa_i)^{-1/2} \tag{4}$$

where κ_i are the main curvatures of the limit state function in the solution point. This result indicates that $P_f(\mathbf{p}) \approx \Phi(-\beta_{\mathbf{p}})$ in fact is a first order approximation which is sufficiently accurate for most practical cases. Note that, the first-order approximation only requires simple differentiability of $g(\mathbf{u}, \mathbf{p}) = 0$.

All subsequent considerations will be based on the first order theory. Hence, reliability analysis involves a probability distribution transformation, the search for the "*β-point*" and the evaluation of the standard normal integral. Hereby, the search for the "*β-point*" is the numerically most challenging task. Most more recent applications use a SQP-algorithm specialized for the task in optimization problem (βP) (see, for example Abdo/Rackwitz (1991)).

For FORM the first-order reliability index $\beta_{\mathbf{p}}$, i.e. the minimum distance from the origin to the limit state surface in standard normal space, can alternatively be used as a measure of reliability. If $\mathbf{u}^*$ is an optimal point for (βP), the β-point is a Kuhn-Tucker-point.

Theorem 1: If $\mathbf{u}^*$, with $\mathbf{u}^* \neq \mathbf{0}$, is the solution point of optimization problem (βP), then the following two statements hold for each $\mathbf{p}$:

a) $g(\mathbf{u}^*, \mathbf{p}) = 0$,
b) $\mathbf{u}^{*T} \nabla_{\mathbf{u}} g(\mathbf{u}^*, \mathbf{p}) + \|\mathbf{u}^*\| \|\nabla_{\mathbf{u}} g(\mathbf{u}^*, \mathbf{p})\| = 0$. □

Proof: Because of the assumption, that $\mathbf{u}^*$ is the solution point of the optimization problem (βP), the Kuhn-Tucker-condition is fulfilled.
There exists $\lambda > 0$ with:

i) $\nabla_{\mathbf{u}}(\|\mathbf{u}^*\|) + \lambda \nabla_{\mathbf{u}} g(\mathbf{u}^*, \mathbf{p}) = 0$,
ii) $\lambda g(\mathbf{u}^*, \mathbf{p}) = 0$.

Since $\mathbf{u}^* \neq \mathbf{0}$ it follows from *i)* and *ii)* with $\mathbf{u}^* = -\lambda \|\mathbf{u}^*\| \nabla_{\mathbf{u}} g(\mathbf{u}^*, \mathbf{p})$ and $\lambda \neq 0$ that there is

$$g(\mathbf{u}^*, \mathbf{p}) = 0 \,.$$

Further from *i)* we have

$$\|\mathbf{u}^*\| = \lambda \|\mathbf{u}^*\| \|\nabla_{\mathbf{u}} g(\mathbf{u}^*, \mathbf{p})\|$$

and

$$\lambda = \frac{1}{\|\nabla_{\mathbf{u}} g(\mathbf{u}^*, \mathbf{p})\|} \tag{5}$$

With (5) it is easily verified that

$$\nabla_{\mathbf{u}} g(\mathbf{u}^*, \mathbf{p}) + \frac{\|\nabla_{\mathbf{u}} g(\mathbf{u}^*, \mathbf{p})\|}{\|\mathbf{u}^*\|} \mathbf{u}^* = 0 .$$

Using simple vector multiplication with $\mathbf{u}^*$ the final result follows:

$$\mathbf{u}^{*^T} \nabla_{\mathbf{u}} g(\mathbf{u}^*, \mathbf{p}) + \|\mathbf{u}^*\| \|\nabla_{\mathbf{u}} g(\mathbf{u}^*, \mathbf{p})\| = 0 .$$ ■

3 Reliability-Oriented Structural Optimization

Many practical applications of structural optimization pursue at least three conflicting aims:

- low cost or weight of the structure
- high reliability
- good structural performance

The third option will not be dealt with explicitely, however. The cost can or cannot include the expected failure cost. Therefore two principally different types of optimization can be defined, i.e. where cost (or weight) or reliability is optimized:

(CRP) a constrained minimization problem where the total cost, possibly including initial cost and expected cost of failure, are minimized subject to a given minimum reliability and other structural performance requirements,

and

(RCP) a constrained maximization problem where the reliability of a structure is maximized subject to a given maximum cost and other structural performance requirements.

3.1 Cost Optimization with Reliability Constraints (CRP)

The structural optimization problem **(CRP)** is a problem where cost, including initial cost of design and expected cost of failure are minimized with constraints on structural performance, design parameters, and on reliability. The reliability is obtained using first-order reliability (FORM) techniques. In principle, the solution is a problem with two levels of optimization. The first problem (top-level) is cost optimization. The second problem (sub-level) determines the reliability index which is needed in the objective function (failure cost) and in at least one constraint. Instead of using a two-level approach the two optimizations can be combined into one optimization problem.

The necessary first-order optimality condition for design points from Theorem 1 are inserted into the cost optimization problem. More precisely, the optimization problem **(CRP)** must fullfil the necessary optimality conditions for the reliability index problem (βP):

$$
\begin{aligned}
&\textbf{CRP} \quad \textbf{minimize} && C(\mathbf{p}, \beta_{\mathbf{p}}) \\
&\qquad\qquad \text{subject to} && g(\mathbf{u}, \mathbf{p}) = 0 \\
&&& \mathbf{u}^T \nabla_{\mathbf{u}} g(\mathbf{u}, \mathbf{p}) + \|\mathbf{u}\| \|\nabla_{\mathbf{u}} g(\mathbf{u}, \mathbf{p})\| = 0 \\
&&& \textit{constraints on design and cost parameter} \\
&&& \textit{constraint for reliability} \\
&&& \textit{simple bounds for design and cost parameter},
\end{aligned}
$$

The constraint related to reliability in **(CRP)** is specified by $\Phi(-\beta_{\mathbf{p}}) \leq P_f^{max}$, where P_f^{max} is the maximum allowable failure probability.

The objective function $C(\mathbf{p}, \beta_{\mathbf{p}})$ can be given as

$$
\begin{aligned}
C_t(\mathbf{p}, \mathbf{u}) &= C(\mathbf{p}, \beta_{\mathbf{p}}) \\
&= C_i(\mathbf{p})(1 - P_f(\mathbf{p})) + C_f(\mathbf{p}) P_f(\mathbf{p}) \\
&\approx C_i(\mathbf{p}) + C_f(\mathbf{p}) P_f(\mathbf{p}) \ ,
\end{aligned} \tag{6}
$$

where $C_t(\cdot, \cdot)$ is the objective function of total expected cost, $C_i(\cdot)$ is the initial cost of design and construction, $C_f(\cdot)$ is the cost of failure and $P_f(\cdot)$ is the failure of probability. The simplification is admissible because structural failure probabilities should be small numbers.

Thus, the complete optimization problem for **(CRP)** is:

$$
\begin{aligned}
&\textbf{(CRP)} \quad \textbf{minimize} && C_t(\mathbf{p}, \mathbf{u}) = C_i(\mathbf{p}) + C_f(\mathbf{p}) \Phi(-\|\mathbf{u}\|) \\
&\qquad\qquad \text{subject to} && g(\mathbf{u}, \mathbf{p}) = 0 \\
&&& \mathbf{u}^T \nabla_{\mathbf{u}} g(\mathbf{u}, \mathbf{p}) + \|\mathbf{u}\| \|\nabla_{\mathbf{u}} g(\mathbf{u}, \mathbf{p})\| = 0 \\
&&& h_i(\mathbf{T}(\mathbf{u}), \mathbf{p}) = 0 \ , \quad i = 1, \ldots, m' \\
&&& \tilde{h}_j(\mathbf{T}(\mathbf{u}), \mathbf{p}) \leq 0 \ , \quad j = m' + 1, \ldots, m \\
&&& \Phi(-\|\mathbf{u}\|) \leq P_f^{max} \\
&&& (\mathbf{x}^l, \mathbf{p}^l) \leq (\mathbf{T}(\mathbf{u}), \mathbf{p}) \leq (\mathbf{x}^u, \mathbf{p}^u) \ ,
\end{aligned}
$$

where $h_i(\cdot,\cdot)$ denote m' equality constraints and $\tilde{h}_j(\cdot,\cdot)$ denote $m - m'$ inequality constraints for the design vector and the parameters. $(\mathbf{x}^l, \mathbf{p}^l)$, $(\mathbf{x}^u, \mathbf{p}^u)$ are simple lower and upper bounds for the random vector $\mathbf{x} = \mathbf{T}(\mathbf{u})$ and the parameter $\mathbf{p}$. The vector relation "$\leq$" is defined by the ordinary order relation for the components of a vector.

3.2 Reliability Optimization with Cost Constraints (RCP)

The inverse optimization problem **(RCP)** is a problem where the reliability is maximized, i.e. the failure probability is minimized under constraints on structural performance and design parameters. The total cost, including cost of design and expected cost of failure, are bounded by maximum total cost.

The necessary optimality condition of the reliability index problem (βP) is fulfilled by each solution point of the following problem **(RCP)**:

(RCP) **minimize** $P_f(\mathbf{p})$

subject to $g(\mathbf{u}, \mathbf{p}) = 0$

$$\mathbf{u}^T \nabla_{\mathbf{u}} g(\mathbf{u}, \mathbf{p}) + \|\mathbf{u}\| \|\nabla_{\mathbf{u}} g(\mathbf{u}, \mathbf{p})\| = 0$$

constraints on design and cost parameter
constraint for total costs
simple bounds for design and cost parameter,

where $P_f(\mathbf{p})$ is defined by the FORM-approximation of the failure probability given in section 2. Clearly, the following optimization problems

$$\textbf{minimize}\ P_f(\mathbf{p}) \quad \text{and} \quad \textbf{maximize}\ \beta_{\mathbf{p}} = \|\mathbf{u}^*\|$$

are equivalent. It is then easy to verify that the following optimization problem (**RCP**) will maximize the reliability of a structure subject to a given maximum cost:

$$
\begin{array}{lll}
\textbf{(RCP)} & \textbf{maximize} & \|\mathbf{u}\| \\
& \text{subject to} & g(\mathbf{u},\mathbf{p}) = 0 \\
& & \mathbf{u}^T \nabla_{\mathbf{u}} g(\mathbf{u},\mathbf{p}) + \|\mathbf{u}\| \|\nabla_{\mathbf{u}} g(\mathbf{u},\mathbf{p})\| = 0 \\
& & h_i(\mathbf{T}(\mathbf{u}),\mathbf{p}) = 0\ , \quad i = 1,\ldots,m' \\
& & \tilde{h}_j(\mathbf{T}(\mathbf{u}),\mathbf{p}) \leq 0\ , \quad j = m'+1,\ldots,m \\
& & C_i(\mathbf{p}) + C_f(\mathbf{p})\Phi(-\|\mathbf{u}\|) \leq C_t^{max} \\
& & (\mathbf{x}^l,\mathbf{p}^l) \leq (\mathbf{T}(\mathbf{u}),\mathbf{p}) \leq (\mathbf{x}^u,\mathbf{p}^u)\ ,
\end{array}
$$

where $C_i(\cdot) + C_f(\cdot)\Phi(-\| \cdot \|)$ are the total expected cost defined by (6) and C_t^{max} is a maximum total cost.

4 Sensitivity Analysis for CRP and RCP

It is well known that sensitivity analysis of a structural optimization problem can help to formulate it appropriately and collect information about suitable starting values. It further can provide insight into the causes of possible non-convergence.

4.1 Sensitivity Analysis for Parameters in the β-Points

In a first step the Lagrange function for the optimization problem (βP) is differentiated with respect to a parameter element p_j. From the Kuhn-Tucker optimality condition of the problem (βP), especially equation (5), follows for the optimal β-point $\mathbf{u}^*$:

$$
\frac{\partial \beta_{\mathbf{p}}}{\partial p_j} = \frac{\partial \|\mathbf{u}^*\|}{\partial p_j} + \left(\frac{\partial g(\mathbf{u}^*,\mathbf{p})}{\partial p_j} \cdot \|\nabla_{\mathbf{u}} g(\mathbf{u}^*,\mathbf{p})\| - g(\mathbf{u}^*,\mathbf{p}) \cdot \frac{\partial \|\nabla_{\mathbf{u}} g(\mathbf{u}^*,\mathbf{p})\|}{\partial p_j} \right) \cdot \|\nabla_{\mathbf{u}} g(\mathbf{u}^*,\mathbf{p})\|^{-2}\ .
$$

Because the β-point $\mathbf{u}^*$ is the solution for the reliability index problem, it can be shown by theorem la), that the first derivative of $\beta_{\mathbf{p}}$ with respect to a parameter element p_j has the following form:

$$\frac{\partial \beta_{\mathbf{p}}}{\partial p_j} = \frac{1}{\|\nabla_{\mathbf{u}} g(\mathbf{u}^*, \mathbf{p})\|} \cdot \frac{\partial g(\mathbf{u}^*, \mathbf{p})}{\partial p_j} \ . \tag{7}$$

The non-dimensional elasticities $S^{\beta}_{p_j}$, $j = 1, \ldots, d$, of the reliability index with respect to an element of the parameter are obtained by use of equation (7):

$$\begin{aligned} S^{\beta}_{p_j} &= \frac{\partial \beta_{\mathbf{p}}}{\partial p_j} \cdot \frac{p_j}{\beta_{\mathbf{p}}} \\ &= \frac{1}{\|\nabla_{\mathbf{u}} g(\mathbf{u}^*, \mathbf{p})\|} x \cdot \frac{p_j}{\|\mathbf{u}^*\|} \cdot \frac{\partial g(\mathbf{u}^*, \mathbf{p})}{\partial p_j} \ . \end{aligned} \tag{8}$$

Of course this definition makes sense only if $p_j \neq 0$ and $\beta_{\mathbf{p}} \neq 0$. Equation (8) can be calculated easily. It can be used to investigate the importance of the parameter elements in $\mathbf{p}$. Knowing the elasticities enables to determine a good starting vector $(\mathbf{u}^0, \mathbf{p}^0)$, which may be essential for convergence.

In particular, starting values $u_i \neq 0, i = 1, \ldots, n$, can be selected such that $u_i^0 = g(0,p) + s \cdot \frac{\partial g(0,p)}{\partial u_i}$ with $s > 0$ depending on the value of $\frac{\partial g(0,p)}{\partial u_i}$.

4.2 Sensitivity Analysis of the Cost Function and Importance of Sensitivities

The first derivative of the cost function C_t with respect to the d cost parameter elements p_j and the n elements u_i of the transformed vector $\mathbf{u}$ can be written as:

$$\frac{\partial C_t(\mathbf{p}, \mathbf{u})}{\partial p_j} = \frac{\partial C_i(\mathbf{p})}{\partial p_j} + \Phi(-\|\mathbf{u}\|) \cdot \frac{\partial C_f(\mathbf{p})}{\partial p_j} \tag{9}$$

and

$$\frac{\partial C_t(\mathbf{p}, \mathbf{u})}{\partial u_i} = -C_f(\mathbf{p}) \cdot \phi(\|\mathbf{u}\|) \cdot \frac{u_i}{\|\mathbf{u}\|} \ , \tag{10}$$

where ϕ is the 1-dimensional density function of standard normal distribution. From equation (9) follow the elasticities $S_{p_j}^{C_t}$, $j = 1, \ldots, d$, of the total cost function C_t with respect to the parameter element p_j:

$$S_{p_j}^{C_t} = \left(\frac{\partial C_i(\mathbf{p})}{\partial p_j} + \Phi(-\|\mathbf{u}\|) \cdot \frac{\partial C_f(\mathbf{p})}{\partial p_j} \right) \cdot \frac{p_j}{C_t(\mathbf{p}, \mathbf{u})} \ . \tag{11}$$

For values much less than zero the standard normal distribution converges very quickly to zero, e.g. $\Phi(-5) = 2.87 \cdot 10^{-7}$ or $\Phi(-7) = 1.29 \cdot 10^{-12}$. Therefore, for large reliability indices $\beta = \|\mathbf{u}\|$, for example, $\|\mathbf{u}\| \geq 7.0$, the two products involving the failure probability P_f or, more precisely, its FORM-approximation $\Phi(-\|\mathbf{u}\|) \cdot C_f(\mathbf{p})$ and $\Phi(-\|\mathbf{u}\|) \cdot \partial C_f(\mathbf{p}) / \partial p_j$, respectively, are small. In this case the elasticities $S_{p_j}^{C_t}$ of the total cost function C_t with respect to the parameter elements p_j from equation (11) can be approximated by:

$$S_{p_j}^{C_t} \approx \frac{\partial C_i(\mathbf{p})}{\partial p_j} \cdot \frac{p_j}{C_i(\mathbf{p})}$$

The elasticities $S_{u_i}^{C_t}$, $i = 1, \ldots, n$, of the total cost function C_t with respect to the n elements of the transformed vector $\mathbf{u}$ follow from equation (10):

$$S_{u_i}^{C_t} = -\left(C_f(\mathbf{p}) \cdot \phi(\|\mathbf{u}\|) \cdot \frac{u_i}{\|\mathbf{u}\|} \right) \cdot \frac{u_i}{C_t(\mathbf{p}, \mathbf{u})} \ . \tag{12}$$

Again, the standard normal density ϕ decays rather rapidly to zero for values far from zero, e.g. $\phi(-5) = 1.49 \cdot 10^{-6}$ or $\phi(-7) = 9.13 \cdot 10^{-12}$. Therefore, for large values of reliability index $\beta = \|\mathbf{u}\|$ the standard normal density function ϕ can be set equal to zero and, therefore,

$$S_{u_i}^{C_t} \approx 0$$

Since the elasticities $S_{u_i}^{C_t}$ can be approximated by zero, a change of an element u_i of the vector $\mathbf{u}$ leads to a "ZERO-change" of the total cost function including initial cost and expected cost of failure.

4.3 Optimality Conditions for Solution Points

The Lagrange function for the optimization problem **(CRP)** can be written with the Lagrange multipliers $(\nu, \lambda) \in \mathbb{R}^{m'+2} \times \mathbb{R}_{+}^{m-m'+2n+2d+1}$ in the following form:

$$\begin{aligned} L_C(\mathbf{u},\mathbf{p},v,\lambda) =& C_t(\mathbf{p},\mathbf{u}) + v_1 g(\mathbf{u},\mathbf{p}) + v_2(\mathbf{u}^T \nabla_{\mathbf{u}} g(\mathbf{u},\mathbf{p}) + \|\mathbf{u}\| \|\nabla_{\mathbf{u}} g(\mathbf{u},\mathbf{p})\|) \\ &+ \sum_{i=1}^{m'} v_{i+2} h_i(\mathbf{T}(\mathbf{u}),\mathbf{p}) + \sum_{j=1}^{m-m'} \lambda_j \tilde{h}_j(\mathbf{T}(\mathbf{u}),\mathbf{p}) \\ &+ \lambda_{m-m'+1}(\Phi(-\|\mathbf{u}\|) - P_f^{max}) \\ &+ \lambda^{l^T}((\mathbf{x}^l,\mathbf{p}^l) - (\mathbf{T}(\mathbf{u}),\mathbf{p})) + \lambda^{u^T}((\mathbf{T}(\mathbf{u}),\mathbf{p}) - (\mathbf{x}^u,\mathbf{p}^u)) , \end{aligned} \tag{13}$$

where the Lagrange-subvectors $\lambda^l, \lambda^u \in \mathbb{R}_+^{n+d}$ for simple bounds of design vector and cost parameter are defined by $\lambda^l = (\lambda_{m-m'+2}, \ldots, \lambda_{m-m'+n+d+1})^T$ and $\lambda^u = (\lambda_{m-m'+n+d+2}, \ldots, \lambda_{m-m'+2n+2d+1})^T$, respectively. The gradient of the Lagrange function uses the gradient of the second equality term in **(CRP)** and **(RCP)**. But with Theorem 1b) it is easy to verify that this gradient is equal to zero in an optimal point, i.e. a Kuhn-Tucker-point. Then:

$$\nabla(\mathbf{u}^T \nabla_{\mathbf{u}} g(\mathbf{u},\mathbf{p}) + \|\mathbf{u}\| \|\nabla_{\mathbf{u}} g(\mathbf{u},\mathbf{p})\|)|_{(\mathbf{u}^*,\mathbf{p}^*)} = \mathbf{0} . \tag{14}$$

where the operator $\nabla(\ )$ defines the gradient vector of a function with respect to the $(n+d)$-dimensional vector $(\mathbf{u},\mathbf{p})$.

From the equations (13) and (14) the first order Kuhn-Tucker-conditions for the reliability-based optimization problem (CRP) follow directly:

Theorem 2: (KUHN-TUCKER-condition for (CRP)-formulation)
If the $(n+d)$-dimensional vector $(\mathbf{u}^*,\mathbf{p}^*)$ is a solution point of optimization problem (CRP), then a Kuhn-Tucker-vector $(v^*,\lambda^*) \in \mathbb{R}^{m'+2} \times \mathbb{R}_+^{m-m'+2n+2d+1}$ exists with:

i$_C$) $\nabla L^C(\mathbf{u}^*,\mathbf{p}^*,v^*,\lambda^*)$

$$= \begin{pmatrix} -C_f(\mathbf{p}^*)\phi(\|\mathbf{u}^*\|)\frac{\mathbf{u}^*}{\|\mathbf{u}^*\|} \\ \nabla_{\mathbf{p}} C_i(\mathbf{p}^*) + \Phi(-\|\mathbf{u}^*\|)\nabla_{\mathbf{p}} C_f(\mathbf{p}^*) \end{pmatrix}$$

$$+ v_1^* \begin{pmatrix} \nabla_{\mathbf{u}} g(\mathbf{u}^*,\mathbf{p}^*) \\ \nabla_{\mathbf{p}} g(\mathbf{u}^*,\mathbf{p}^*) \end{pmatrix} + \sum_{i=1}^{m'} v_{i+2}^* \begin{pmatrix} \nabla_{\mathbf{T}} h_i(\mathbf{T}(\mathbf{u}^*),\mathbf{p}^*) \cdot \nabla_{\mathbf{u}} \mathbf{T}(\mathbf{u}^*) \\ \nabla_{\mathbf{p}} h_i(\mathbf{T}(\mathbf{u}^*),\mathbf{p}^*) \end{pmatrix}$$

$$+\sum_{j=1}^{m'} \lambda_j^* \begin{pmatrix} \nabla_{\mathbf{T}} \tilde{h}_j(\mathbf{T}(\mathbf{u}^*), \mathbf{p}^*) \cdot \nabla_{\mathbf{u}} \mathbf{T}(\mathbf{u}^*) \\ \nabla_{\mathbf{p}} \tilde{h}_j(\mathbf{T}(\mathbf{u}^*), \mathbf{p}^*) \end{pmatrix} + \lambda_{m-m'+1}^* \begin{pmatrix} -\phi(\|\mathbf{u}^*\|) \cdot \frac{\mathbf{u}^*}{\|\mathbf{u}^*\|} \\ \mathbf{0} \end{pmatrix}$$

$$+ (\lambda^{*u} - \lambda^{*l})^T \begin{pmatrix} \nabla_{\mathbf{u}} \mathbf{T}(\mathbf{u}^*) \\ \mathbf{1} \end{pmatrix}$$

$$= \mathbf{0}$$

ii$_C$) *Complementary Condition for (CRP) :*

$$\lambda_j^* \tilde{h}_j(\mathbf{T}(\mathbf{u}^*), \mathbf{p}^*) = 0 \,, \qquad j = 1, \ldots, m - m'$$

$$\lambda_{m-m'+1}^* (\Phi(-\|\mathbf{u}^*\|) - P_f^{max}) = 0$$

$$\lambda^{*^{lT}} ((\mathbf{x}^l, \mathbf{p}^l) - (\mathbf{T}(\mathbf{u}^*), \mathbf{p}^*)) = 0$$

$$\lambda^{*uT} ((\mathbf{T}(\mathbf{u}^*), \mathbf{p}^*) - (\mathbf{x}^u, \mathbf{p}^u)) = 0$$

□

The Lagrange function for the reliability optimization problem (RCP) is formulated as:

$$\begin{aligned} L_R(\mathbf{u}, \mathbf{p}, \nu, \lambda) = & -\|\mathbf{u}\| + \nu_1 g(\mathbf{u}, \mathbf{p}) + \nu_2 (\mathbf{u}^T \nabla_{\mathbf{u}} g(\mathbf{u}, \mathbf{p}) + \|\mathbf{u}\| \|\nabla_{\mathbf{u}} g(\mathbf{u}, \mathbf{p})\|) \\ & + \sum_{i=1}^{m'} \nu_{i+2} h_i(\mathbf{T}(\mathbf{u}), \mathbf{p}) + \sum_{j=1}^{m-m'} \lambda_j \tilde{h}_j(\mathbf{T}(\mathbf{u}), \mathbf{p}) \\ & + \lambda_{m-m'+1} (C_t(\mathbf{p}, \mathbf{u}) - C_t^{max}) \\ & + \lambda^{l^T} ((\mathbf{x}^l, \mathbf{p}^l) - (\mathbf{T}(\mathbf{u}), \mathbf{p})) + \lambda^{uT} ((\mathbf{T}(\mathbf{u}), \mathbf{p}) - (\mathbf{x}^u, \mathbf{p}^u)) \,. \end{aligned} \quad (15)$$

where $(\nu, \lambda) \in \mathbb{R}^{m'+2} \times \mathbb{R}_+^{m-m'+2n+2d+1}$ are the Lagrange multipliers of the problem.

From the above formulation of the Lagrange-function for the reliability-based optimization problem (CRP) the first order Kuhn-Tucker-conditions give:

Theorem 3: (KUHN-TUCKER-condition for (RCP)-formulation)
If the $(n + d)$-dimensional vector $(\mathbf{u}^*, \mathbf{p}^*)$ is a solution point of optimization problem (RCP), then a Kuhn-Tucker-vector $(\nu^*, \lambda^*) \in \mathbb{R}^{m'+2} \times \mathbb{R}_+^{m-m'+2n+2d+1}$

exists with:

$\mathrm{i_R}$) $\nabla L^C(\mathbf{u}^*, \mathbf{p}^*, \nu^*, \lambda^*)$

$$
\begin{aligned}
&= \begin{pmatrix} \frac{\mathbf{u}^*}{\|\mathbf{u}^*\|} \\ 0 \end{pmatrix} + \nu_1^* \begin{pmatrix} \nabla_{\mathbf{u}} g(\mathbf{u}^*, \mathbf{p}^*) \\ \nabla_{\mathbf{p}} g(\mathbf{u}^*, \mathbf{p}^*) \end{pmatrix} \\
&+ \sum_{i=1}^{m'} \nu_{i+2}^* \begin{pmatrix} \nabla_{\mathbf{T}} h_i(\mathbf{T}(\mathbf{u}^*), \mathbf{p}^*) \cdot \nabla_{\mathbf{u}} \mathbf{T}(\mathbf{u}^*) \\ \nabla_{\mathbf{p}} h_i(\mathbf{T}(\mathbf{u}^*), \mathbf{p}^*) \end{pmatrix} \\
&+ \sum_{j=1}^{m'} \lambda_j^* \begin{pmatrix} \nabla_{\mathbf{T}} \tilde{h}_j(\mathbf{T}(\mathbf{u}^*), \mathbf{p}^*) \cdot \nabla_{\mathbf{u}} \mathbf{T}(\mathbf{u}^*) \\ \nabla_{\mathbf{p}} \tilde{h}_j(\mathbf{T}(\mathbf{u}^*), \mathbf{p}^*) \end{pmatrix} \\
&+ \lambda_{m-m'+1}^* \begin{pmatrix} C_f(\mathbf{p}^*)\phi(\|\mathbf{u}^*\|)\frac{\mathbf{u}^*}{\|\mathbf{u}^*\|} \\ -\nabla_{\mathbf{p}} C_i(\mathbf{p}^*) + \Phi(-\|\mathbf{u}^*\|)\nabla_{\mathbf{p}} C_f(\mathbf{p}^*) \end{pmatrix} \\
&+ (\lambda^{*u} - \lambda^{*l})^T \begin{pmatrix} \nabla_{\mathbf{u}} \mathbf{T}(\mathbf{u}^*) \\ \mathbf{1} \end{pmatrix} \\
&= \mathbf{0}
\end{aligned}
$$

$\mathrm{ii_R}$) *Complementary Condition for* (RCP) :

$$\lambda_j^* \tilde{h}_j(\mathbf{T}(\mathbf{u}^*), \mathbf{p}^*) = 0\ , \qquad j = 1, \ldots, m - m'$$

$$\lambda_{m-m'+1}^*(C_t(\mathbf{p}^*, \mathbf{u}^*) - C_t^{max}) = 0$$

$$\lambda^{*l^T}((\mathbf{x}^l, \mathbf{p}^l) - (\mathbf{T}(\mathbf{u}^*), \mathbf{p}^*)) = 0$$

$$\lambda^{*u^T}((\mathbf{T}(\mathbf{u}^*), \mathbf{p}^*) - (\mathbf{x}^u, \mathbf{p}^u)) = 0$$

□

5 Asymptotic Equivalence of $\Phi(-\beta_{\mathbf{p}})$ and $P_f(\mathbf{p})$

The following theorem states that asymptotically a quadratic approximation of failure surface is sufficient for failure probability estimation.

Theorem 4: (Breitung, 1984)
If $0 < \mathcal{P}(\mathcal{F}_{\mathbf{p}})$, *then for each* $\mathbf{p}$*:*

$$\lim_{b\to\infty} \frac{\mathcal{P}(b\mathcal{F}_{\mathbf{p}})}{\mathcal{P}(b\mathcal{F}_{SORM,\mathbf{p}})} = 1 . \qquad \square$$

Hohenbichler (1984) proved the following (weaker) asymptotic result by using the central scaling of the failure domain $b\mathcal{F}_{\mathbf{p}} = \{b\mathbf{u} : \mathbf{u} \in \mathcal{F}_{\mathbf{p}}\}$. The important theorem shows for $b \to \infty$, that the reliability index $\beta_{\mathbf{p}}$ converges "relatively" to the exact reliability index $\beta_{\mathbf{p}}^E$

$$\beta_{\mathbf{p}}^E = \beta^E(\mathcal{F}_{\mathbf{p}}) = -\Phi^{-1}(P_f(\mathbf{p})) .$$

Theorem 5: (Hohenbichler, 1984)
If $0 < \beta(\mathcal{F}_{\mathbf{p}}) < \infty$, *then for each* $\mathbf{p}$:

$$\lim_{b\to\infty} \frac{\beta(b\mathcal{F}_{\mathbf{p}})}{\beta^E(b\mathcal{F}_{\mathbf{p}})} = 1 . \qquad \square$$

In other words:

For "large" reliability indices $\beta_{\mathbf{p}}^E$ or for "small" failure probabilities $P_f(\mathbf{p})$ the geometrical reliability index $\beta_{\mathbf{p}}$ computed by optimization problem (βP) is a good approximation of the (exact) reliability index $\beta_{\mathbf{p}}^E$.

The following consideration will be based on a cost function which depends on the cost parameters only, i.e. $C_t = C(\mathbf{p})$. The (compact) subset $T_C =_{def} \{\mathbf{p} : C(\mathbf{p}) \le C^{maximum}\}$ of the $\mathbf{p}$-space defines the admissible set of cost parameters. Then the following corollary of the Hohenbichler-Theorem can be proved.

Corollary 6: If $0 < \beta(\mathcal{F}_{\mathbf{p}}) < \infty$, *then for each* $\mathbf{p}$:

$$\lim_{b\to\infty} \frac{\max(\beta(b\mathcal{F}_{\mathbf{p}}) : \mathbf{p} \in T_C)}{\max(\beta^E(b\mathcal{F}_{\mathbf{p}}) : \mathbf{p} \in T_C)} = 1 . \qquad \square$$

From corollary 6 follows for the reliability optimization problem with a cost constraint:

For "small" failure probabilities $P_f(\mathbf{p})$ the optimal reliability index β^*, computed by maximization of reliability index subject to the cost constraint $C(\mathbf{p}) \le C^{maximum}$, is a good approximation of the (exact) maximum reliability index $\beta^{E^*} = \max(\beta_{\mathbf{p}}^E : \mathbf{p} \in T_C)$.

Proof:

i) The reliability index $\beta_{\mathbf{p}} = \beta(\mathscr{F}_{\mathbf{p}})$ fulfills

$$\beta(b\mathscr{F}_{\mathbf{p}}) = b\beta(\mathscr{F}_{\mathbf{p}}) \,, \tag{16}$$

because

$$\begin{aligned}\beta(b\mathscr{F}_{\mathbf{p}}) &= \min\{\|\mathbf{u}\| : \mathbf{u} \in b\mathscr{F}_{\mathbf{p}}\} \\ &= \min\{\|b\mathbf{u}\| : \mathbf{u} \in \{\mathbf{u} : g(\mathbf{u}, \mathbf{p}) \leq 0\}\} \\ &= \min\{b\|\mathbf{u}\| : \mathbf{u} \in \mathscr{F}_{\mathbf{p}}\} \\ &= b\min\{\|\mathbf{u}\| : \mathbf{u} \in \mathscr{F}_{\mathbf{p}}\} \\ &= b\beta(\mathscr{F}_{\mathbf{p}}) \,.\end{aligned}$$

ii) If the reliability index $\beta_{\mathbf{p}}$ exists for all $\mathbf{p}$, i.e. $0 < \beta(\mathscr{F}_{\mathbf{p}}) < \infty$, then the limit $\lim_{\mathbf{x}\downarrow\mathbf{0}} x\beta^E(\frac{1}{x}\mathscr{F}_{\mathbf{p}})$ exists with:

$$\forall \mathbf{p} : \quad \lim_{\mathbf{x}\downarrow\mathbf{0}} x\beta^E\left(\frac{1}{x}\mathscr{F}_{\mathbf{p}}\right) = \beta(\mathscr{F}_{\mathbf{p}}) \,.$$

The reliability index $\beta_{\mathbf{p}}$ fulfills the assumption $0 < \beta(\mathscr{F}_{\mathbf{p}}) < \infty$. It follows from theorem 5

$$\lim_{b\to\infty} \frac{\beta(b\mathscr{F}_{\mathbf{p}})}{\beta^E(b\mathscr{F}_{\mathbf{p}})} = 1 \,,$$

From equation (16) further follows

$$\begin{aligned}\lim_{b\to\infty} \frac{\beta(b\mathscr{F}_{\mathbf{p}})}{\beta^E(b\mathscr{F}_{\mathbf{p}})} &= \lim_{b\to\infty} \frac{b\beta(\mathscr{F}_{\mathbf{p}})}{\beta^E(b\mathscr{F}_{\mathbf{p}})} \\ &= \beta(\mathscr{F}_{\mathbf{p}}) \cdot \lim_{b\to\infty} \frac{b}{\beta^E(b\mathscr{F}_{\mathbf{p}})} = 1\end{aligned}$$

and

$$\lim_{b\to\infty} \frac{\beta^E(b\mathscr{F}_{\mathbf{p}})}{b} = \lim_{x\downarrow 0} x\beta^E\left(\frac{1}{x}\mathscr{F}_{\mathbf{p}}\right) = \beta(\mathscr{F}_{\mathbf{p}}) \tag{17}$$

Lemma 7: Let $A \subset \mathbb{R}^d$ be a compact subset of the d-dimensional real space. The function $f : (0, \infty) \times A \to (0, \infty)$ is continuous on $[0, \infty)$. It holds for each $\mathbf{p} \in A$:

$$f(0, \mathbf{p}) = \lim_{\mathbf{x}\downarrow \mathbf{0}} f(x, \mathbf{p}) .$$

and therefore

$$\max_{\mathbf{p}\in A}\{f(0,\mathbf{p})\} = \lim_{\mathbf{x}\downarrow \mathbf{0}} \max_{\mathbf{p}\in A}\{f(x,\mathbf{p})\} . \qquad \square$$

iii) Equation (18) follows with $f(x, \mathbf{p}) = x\beta^E(\frac{1}{x}\mathscr{F}_{\mathbf{p}})$ and $f(0, \mathbf{p}) = \beta(\mathscr{F}_{\mathbf{p}})$

$$\max_{\mathbf{p}\in \mathbf{T}_c}\{\beta(\mathscr{F}_{\mathbf{p}})\} = \lim_{x\downarrow 0} \max_{\mathbf{p}\in \mathbf{T}_c}\left\{x\beta^E\left(\frac{1}{x}\mathscr{F}_{\mathbf{p}}\right)\right\} . \tag{18}$$

With the equations (18) and (16) it is easily verified that

$$1 = \frac{\max_{\mathbf{p}\in \mathbf{T}_c}\{\beta(\mathscr{F}_{\mathbf{p}})\}}{\lim_{x\downarrow 0} \max_{\mathbf{p}\in \mathbf{T}_c}\{x\beta^E(\frac{1}{x}\mathscr{F}_{\mathbf{p}})\}} = \frac{\max_{\mathbf{p}\in \mathbf{T}_c}\{\beta(\mathscr{F}_{\mathbf{p}})\}}{\lim_{b\to\infty} \frac{1}{b}\cdot \max_{\mathbf{p}\in \mathbf{T}_c}\{\beta^E(b\mathscr{F}_{\mathbf{p}})\}}$$

$$= \lim_{b\to\infty} \frac{\max_{\mathbf{p}\in \mathbf{T}_c}\{\beta(\mathscr{F}_{\mathbf{p}})\}}{\frac{1}{b}\cdot \max_{\mathbf{p}\in \mathbf{T}_c}\{\beta^E(b\mathscr{F}_{\mathbf{p}})\}}$$

$$= \lim_{b\to\infty} \frac{\frac{1}{b}\cdot \max_{\mathbf{p}\in \mathbf{T}_c}\{\beta(b\mathscr{F}_{\mathbf{p}})\}}{\frac{1}{b}\cdot \max_{\mathbf{p}\in \mathbf{T}_c}\{\beta^E(b\mathscr{F}_{\mathbf{p}})\}} .$$

Finally we get

$$\lim_{b\to\infty} \frac{\max(\beta(b\mathscr{F}_{\mathbf{p}}) : \mathbf{p} \in T_C)}{\max(\beta^E(b\mathscr{F}_{\mathbf{p}}) : \mathbf{p} \in T_C)} = 1 . \qquad \blacksquare$$

This means that the **(RCP)**-problem can be solved asymptotically exact, whereas the **(CRP)**-problem can be solved only approximately in our formulation.

6 Numerical Examples

In the following three examples for reliability-based optimization, which use the solution of the problems (**CRP**) and (**RCP**), are presented.

The reliability-based structural optimization is carried out by a non-commercial PC/DOS program package based on tools of SYSREL 9.0 routines (see SYSREL 9.0, RCP GmbH, 1994), and the non-linear SQP-optimization algorithm NLPQL by Schittkowski (1985) both written in the programming language FORTRAN.

The first example compares our own results with an example using semi-infinite programming which was outlined by Kirjner, Polak and Der Kiureghian (1995). The problem is to determine the depth h and width b of a short column with rectangular cross section with a minimal total mass bh. The uncertain vector $\mathbf{X} = (P, M, Y)$, the stochastic parameters and the correlations of the vector elements are given by:

Variable	Symbol	Distribution	Mean/St. dev.	Unit	Corr. P	Corr. M	Corr. Y
Yield Stress	P	Normal	500/100	MPa	1	0.5	0
Bending Moments	M	Normal	2000/400	MNm	0.5	1	0
Axial Force	Y	Lognormal	5/0.5	MPa	0	0	1

The limit state function in terms of the vector $\mathbf{x} = (P, M, Y)$ and the parameter vector $\mathbf{p} = (b, h)$ is given by:

$$G(\mathbf{x}, \mathbf{p}) = 1 - \frac{4M}{bh^2 Y} - \frac{P^2}{(bhY)^2} .$$

The cost (or mass) function is

$$C_t(\mathbf{p}) = b \cdot h\,.$$

No constraints on parameters are given. The depth h and the width b of the section had to satisfy $15 \leq h \leq 25$ and $5 \leq b \leq 15$. The allowable failure probability is 0.00621 or in other word a reliability index less than or equal to 2.5. Starting from the initial point $(\mathbf{u}^0, \mathbf{p}^0) = ((1,1,-1),(5,15))$ the NLPQL-algorithm converged for the problem (**CRP**) to $(b^*, h^*) = (8.668, 25.0)$. The optimization algorithm took 6 iterations with 83 evaluations of the limit state function and 56 evaluations of its gradient. Kirjner-Neto et al. report 14 iterations with 98 limit state function evaluations and 77 gradient evaluations for the semi-infinite method and 277 limit state evaluations for a nested optimization algorithm. This shows that our algorithm is comparable in numerical efficiency with semi-infinite programming having in mind the different formulation but also the differences in the algorithms used in both cases.

The second example compares our the results of reliability optimization with several maximum cost. The example is a steel column with cost parameter $\mathbf{p} = (b, d, h)$:

Variable	Symbol	Unit
Mean of Flange Breadth	b	mm
Mean of Flange Thickness	d	mm
Mean of Height of Steel Profile	h	mm

The steel column has a constant length of 7500 *mm*. The function of total cost $C_t(\mathbf{p}, \mathbf{u})$ includes no failure cost, i.e. $C_f = 0$, and has the following form:

$$C_t(\mathbf{p}, \mathbf{u}) = C_i(\mathbf{p}) = (bd + 5[\text{mm}] \cdot h) \cdot [CU/\text{mm}^2] \quad (CU = \text{currency unit}).$$

The independent uncertain vector $\mathbf{X} = (F_s, P_1, P_2, P_3, B, D, H, F_0, E)$ and its stochastic characteristics are given by:

Variable	Symbol	Distribution	Mean/Standard derivation	Unit
Yield Stress	F_s	LogN	400/35	MPa
Dead Weight Load	P_1	N	500000/50000	N
Variable Load	P_2	Gumbel	600000/90000	N
Variable Load	P_3	Gumbel	600000/90000	N
Flange Breadth	B	LogN	$b/3$	N
Flange Thickness	D	LogN	$d/2$	mm
Height of Profile	H	LogN	$h/5$	mm
Initial Deflection	F_0	N	30/10	mm
Youngs Modulus	E	Weibull	21000/4200	MPa

The limit state function in terms of the random vector **X**, the parameter (b, d, h) and auxiliary functions $\mathscr{A}_s$, $\mathscr{M}_s$, $\mathscr{M}_i$, $\mathscr{E}_b$, $\mathscr{P} = P_1 + P_2 + P_3$ is defined by:

$$G(\mathbf{x}, \mathbf{p}) = F_s - \mathscr{P}\left(\frac{1}{\mathscr{A}_s} + \frac{F_0}{\mathscr{M}_s} \cdot \frac{\mathscr{E}_b}{\mathscr{E}_b - \mathscr{P}}\right) .$$

where

$$\mathscr{A}_s = 2BD \,, \quad \text{(area of section)}$$

$$\mathscr{M}_s = BDH \,, \quad \text{(modulus of section)}$$

$$\mathscr{M}_i = \frac{1}{2} BDH^2 \,, \quad \text{(moment of iteria)}$$

$$\mathscr{E}_b = \frac{\pi^2 E \mathscr{M}_i}{s^2} \,, \quad \text{(Euler buckling load)}$$

No other constraints on cost and design parameters are given in the example. The elements of the transformed standard normal vector must be within the interval $(-30.00, 30.00)$. The means of flange breadth b and flange thickness d have the lower bounds 200 *mm* resp. 10 *mm* and the upper bounds 400 *mm* resp. 30 *mm*. The interval (100[*mm*], 500[*mm*]) defines the admissible mean height h of the T-shaped steel profile.

The results of the reliability optimization of steel column with various maximal permitted (total) cost are given in the following table.

Maximal cost	Optimal cost vector $\mathbf{p}^*$			Reliability index
$C^{maximum}$	b^*	d^*	h^*	$\beta^*_{\mathbf{p}}$
4 000.00	200.00	17.50	100.00	3.132
5 000.00	200.00	22.50	100.00	4.961
6 000.00	200.00	27.50	100.00	6.369
7 000.00	216.67	30.00	100.00	7.427
8 000.00	250.00	30.00	100.00	8.249
9 000.00	283.33	30.00	100.00	8.967
10 000.00	316.67	30.00	100.00	9.605
11 000.00	350.00	30.00	100.00	10.180
12 000.00	383.33	30.00	100.00	10.709
13 000.00	400.00	30.00	200.00	11.065

The increase of the parameters b, h and h depends on the maximum cost $C^{maximum}$ and the importance of the individual parameters for the reliability of the structure.

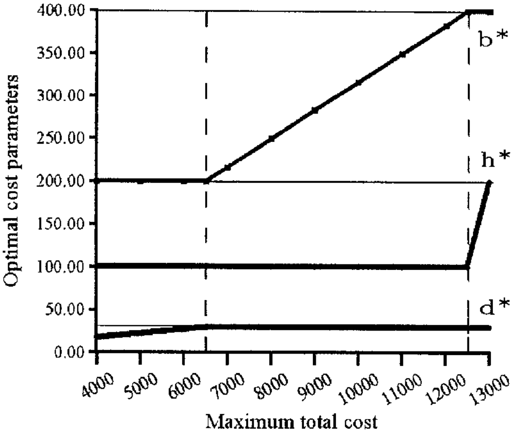

Fig. 6.1. Dependence of optimal cost vector elements b, h and h from maximal total cost of the steel column

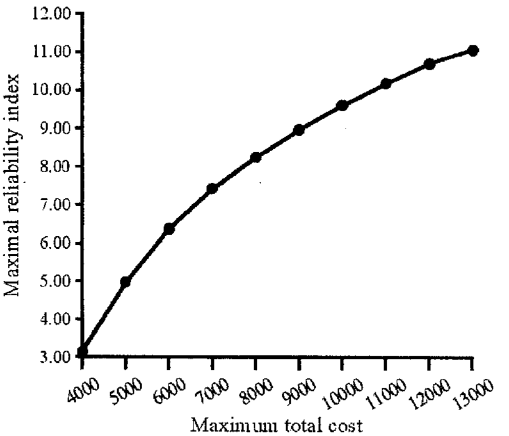

Fig. 6.2. Dependence of reliability index on maximum admissible total cost of the steel column

It is seen, that at first the mean of flange thickness and only subsequently the mean of flange breadth increases to the upper bounds 30.00 and 400.00, respectively. The mean height of the steel column remains at the lower limit up to a maximum cost of 12 500 CU. This is illustrate in figure 6.1.

It is further seen that higher maximum cost $C^{maximum}$ lead to an increase of the maximum reliability index. Figure 6.2 shows also that the reliability indices β_{p}^* from the (RCP)-problem decrease exponentially for lower maximum cost.

In the third example a rectangular reinforced concrete beam with parameters, $\mathbf{p} = (w, d, a_s)$, is considered (see Madsen/Friis Hansen, 1992, but with modified parameters), different cost parameters and two constraints on the parameters w, d and a_s:

Variable	Symbol	Unit
Width of concrete beam	w	m
Effective depth of concrete beam	d	m
Reinforcement area of beam	a_s	m^2

The independent uncertain design vector $\mathbf{X} = (T_s, T_c, M_b, K)$ and its stochastic characteristics are given in the following table:

Variable	Symbol	Distribution	Mean/Standard derivation (Parameters)	Unit
Steel yield stress	T_s	Normal	360.0/36.0	MPa
Concrete comp. strength	T_c	Lognormal	40.0/6.0	MPa
Applied bending moment	M_b	Gumbel	0.01/0.003	MNm
Model uncertainty	K	Rectangular	(0.5, 0.667)	–

The limit state function dependent on the random vector (T_s, T_c, M_b, K) and on the parameter (w, d, a_s) is:

$$G(\mathbf{x}, \mathbf{p}) = \left(1 - K\frac{a_s T_s}{w d T_c}\right) a_s d T_s - M_b \ .$$

The reinforced concrete beam has a fixed span of 5 m. The initial cost is given by

$$C_i(\mathbf{p}) = 5\,[\mathrm{m}](800\,[\mathrm{CU/m^3}] \cdot wd + 2000\,[\mathrm{CU/m^3}] \cdot a_s) \ .$$

The failure cost is estimated as:

$$C_f = 50000 \ CU \ .$$

Two constraints on parameters are given, a lower bound for the area of the beam and a maximum admissible area of reinforcement in relation to the total area of the concrete section.

$$0.01 \leq wd$$

$$a_s \leq 5\% \cdot wd$$

The maximal total cost of the beam is 55.00 CU and the allowable failure probability is 10^{-5}.

The transformed standard normal vector elements are bounded by −30.0 and 30.0. The width w and the effective depth d of the beam have the lower bound 0.05 m and the upper bound 0.5 m. The area of the steel reinforcement a_s must be within the interval $(10^{-4}, 10^{-2})$.

The results of the optimization of the cost minimization and the reliability maximization are:

	Total cost minimization (CRP)	Reliability maximization (RCP)
Starting values		
$(u^0_{T_s}, u^0_{T_c}, u^0_{M_b}, u^0_K)$	$(-0.25, -0.25, 1.00, 0.25)$	$(-0.25, -0.25, 1.00, 0.25)$
(w^0, d^0, a^0_s)	(0.050, 0.172, 0.0001)	(0.050, 0.172, 0.0001)
Optimization results		
Final total cost	54.39 CU	55.00 CU
Final Failure Probab.	10^{-5}	$6.60 \cdot 10^{-6}$
Final Reliability Index	4.265	4.357
$(u^*_{T_s}, u^*_{T_c}, u^*_{M_b}, u^*_K)$	$(-0.89, -0.80, 4.06, 0.51)$	$(-0.90, -0.85, 4.14, 0.53)$
(w^*, d^*, a^*_s)	(0.050, 0.239, 0.00060)	(0.050, 0.243, 0.00061)
Number of calls		
Function-calls	11	76
Gradient-calls	11	44

It is seen that the (**RCP**) problem requires considerably more numerical effort which is expected. It should also be mentioned that in both cases convergence could not be reached for all admissible starting values. In fact, sensitivity analysis is necessary to select suitable values. Also, some suitable transformations of the constraints which made their gradients more homogeneous have made convergence much faster. Other, in part more complicated examples confirmed the conclusions from this example.

7 Summary and Conclusion

Two one-level structural optimization algorithms, (**CRP**) and (**RCP**), respectively, based on reliability and expected total cost were derived. The advantages of our formulations and their numerical implementation are (see also Madsen and Friis Hansen, 1992):

- a well-known standard non-linear optimization algorithm, e.g. a SQP-algorithm, can be used,
- scaling problems for complicated problems are handled by standard optimization routines,
- the methods appear locally stable and robust.

The disadvantages of our formulation of the two structural optimization problems (**CRP**) and (**RCP**) are:

- the usually numerical calculation of second derivatives of the limit state function is required,
- the transformation from the standard normal **u**-variables to the physical variables **x** and vice versa must be included explicitly. Those probability distribution transformations may require some additional numerical effort and can cause numerical problems in extreme cases (numerical inversion of distribution functions).
- for both problems good starting values usually are required
- monotonic transformations of both objectives and constraints sometimes may be necessary in order to achieve convergence.

The one-level formulations proposed here are limited to FORM as the reliability part is concerned. The asymptotic correctness of reliability optimization with cost constraints is proved by a corollary showing the asymptotic equivalence of first-order and exact reliability indices for small failure probabilities. Extension to SORM does not seem to be straightforward. Only one algorithm, preferably a SQP-algorithm is necessary. However, several tricks (transformations and/or good starting values) must be applied in order to achieve convergence, because both optimization problems can have objectives and/or constraints with widely varying gradients or objectives and/or constraints with numerically zero gradients in extreme cases. With these tricks it can be stated that the algorithms work quite efficiently. Further substantial improvements of the numerics are still possible.

The formulations are restricted to one state function only and thus to one failure mode so far. It appears possible to generalize the approach to multiple failure modes (unions of failure modes) but not necessarily to the case where intersections of failure modes form system failure.

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