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COMPONENTS IN THE SPACE OF COMPOSITION OPERATORS

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We consider the topological space of all composition operators, acting on certain Hilbert spaces of holomorphic functions on the unit disc, in the uniform operator topology. A sufficient condition is given for the component of a composition operator to be a singleton. A necessary condition is given for one composition operator to lie in the component of another. In addition, we prove analogous results for the component of the image of a composition operator in the Calkin algebra. Finally, we obtain some related results on the essential norm of a linear combination of composition operators.

1. INTRODUCTION

We consider here the set of composition operators acting on certain Hilbert spaces H of holomorphic functions on the unit disc $\mathbb{D}$ in the complex plane, considered as a subset of $\mathcal{B}(H)$, the bounded linear operators on H in the uniform operator topology. The basic problem we are interested in is that of determining the components in the topological space of all composition operators on H. While we do not give a complete solution to this problem, we give a sufficient condition on φ for the component containing the composition operator C_φ to be the singleton $\{C_\varphi\}$ (Corollary 2.3), and we give a necessary condition for C_ψ to be in the component of C_φ if this component is not a singleton (Theorem 2.4). We also give the analogous results for the component of the image of C_φ in the Calkin algebra.

In this section, we describe the Hilbert spaces H under consideration and give some necessary background information. Many of our results involve the notion of the angular derivative of a mapping $\varphi: \mathbb{D} \to \mathbb{D}$; for completeness we summarize the relevant

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results on angular derivatives which comprise the Julia-Caratheodory theory.

For $\alpha > 1$, let D_α denote the set of holomorphic functions f in $\mathbb{D}$ for which

$$\frac{\alpha-1}{\pi}\int_{\mathbb{D}}|f(z)|^2(1-|z|^2)^{\alpha-2}dA(z) \equiv \|f\|_\alpha^2 < \infty.$$

When $\alpha = 1$, we define D_α to be the Hardy space

$$H^2(\mathbb{D}) = \{f: \sup_{0<r<1}\int_{\partial\mathbb{D}}|f(re^{i\theta})|^2\,\frac{d\theta}{2\pi} \equiv \|f\|^2 < \infty\}.$$

For any $\alpha \geq 1$, D_α is a Hilbert space of holomorphic functions on $\mathbb{D}$, with reproducing kernel functions $k_w(z) = (1-\bar{w}z)^{-\alpha}$. Note that $\alpha = 2$ gives the familiar Bergman space $A^2(\mathbb{D})$.

If $\varphi: \mathbb{D} \to \mathbb{D}$ is holomorphic, define C_φ on D_α by $C_\varphi = f\circ\varphi$. It is a consequence of Littlewood's subordination principle [L] that C_φ is a <u>bounded</u> linear operator on D_α for all $\alpha \geq 1$. We use the notation $\mathcal{C}(D_\alpha)$ for the set of composition operators on D_α, in the uniform operator topology.

An easy and important observation about the adjoint of C_φ is the action of C_φ^* on the reproducing kernel functions k_z:

$$C_\varphi^*\, k_z = k_{\varphi(z)}.$$

Verification of this fact is left to the reader.

A holomorphic $\varphi: \mathbb{D} \to \mathbb{D}$ is said to have an angular derivative at a point $\zeta \in \partial\mathbb{D}$ if there exists $w \in \partial\mathbb{D}$ so that the non-tangential limit

$$\lim_{z\to\zeta}\frac{\varphi(z)-w}{z-\zeta}$$

exists; we write $\varphi'(\zeta)$ for this limit. It is a consequence of the Julia-Caratheodory theory that this limit exists (in the finite sense) if and only if

$$(1) \qquad \liminf_{z\to\zeta}\frac{1-|\varphi(z)|}{1-|z|} < \infty,$$

where z approaches ζ unrestrictedly in $\mathbb{D}$. Moreover, the value of this lim inf is $|\varphi'(\zeta)|$ and $\varphi'(\zeta) = \lim_{z \to \zeta} \varphi'(z)$ (non-tangentially). We will write $|\varphi'(\zeta)| = \infty$ if the lim inf in (1) is infinite. It is a consequence of the definition that if $|\varphi'(\zeta)| < \infty$, φ has radial limit of modulus 1 at ζ. In the case where $\lim_{r\to 1} \varphi(r\zeta) \equiv \varphi^*(\zeta) = \zeta$ and $|\varphi'(\zeta)| < \infty$, $\varphi'(\zeta)$ will be positive. The details of these results can be found in ([N],[R]).

The notion of the angular derivative of φ has played an important role in other results on composition operators on the spaces D_α. The connection with questions on the compactness of C_φ is relevant to our discussion here. In particular, we have the following results.

THEOREM 1.1 ([S-T],[M-S]). *If φ has a finite angular derivative at any point of $\partial\mathbb{D}$, then C_φ is not compact on D_α for any $\alpha \geq 1$.*

THEOREM 1.2 ([M-S]). *If φ has no finite angular derivative at any point of $\partial\mathbb{D}$, then C_φ is compact on D_α for all $\alpha > 1$.*

The exact necessary and sufficient conditions for compactness of C_φ on $D_1 = H^2$ are more complicated and do not concern us here; see [S1] for the complete solution to the compactness question in the H^2 case. Theorems 1.1 and 1.2 motivate the direction we take in the next section.

Previous work on the question of which composition operators on H^2 are isolated has been done by Berkson [B], who showed that if φ has radial limits of modulus 1 on a set of positive measure in $\partial\mathbb{D}$, then C_φ is isolated. Several questions suggested by Berkson's work were posed by Joel Shapiro in a problem session on composition operators at the AMS Summer Research Institute in Durham, New Hampshire (July 1988). In this paper we consider some of these questions. Shapiro and Sundberg [S-S] recently answered some related questions, and we discuss some of their results at the end of Section 2.

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2. COMPONENTS IN $\mathcal{C}(D_\alpha)$ FOR $\alpha \geq 1$

We begin with an easy observation which is well known.

PROPOSITION 2.1. *The set of compact composition operators on D_α is arcwise connected.*

SKETCH OF PROOF. We leave the details of the proof to the reader. One shows that the map $\Gamma_\varphi\colon [0,1] \to \mathcal{C}(D_\alpha)$ given by $\Gamma_\varphi(r) = C_{r\varphi}$ is continuous whenever C_φ is compact. Note that $\Gamma_\varphi(0)$ is simply the operator of evaluation at 0. The continuity of Γ_φ follows from the following characterization of compact composition operators on D_α: C_φ is compact if and only if whenever $\{f_n\}$ is a bounded sequence in D_α with $f_n \to f$ uniformly on compact subsets of $\mathbb{D}$, then $C_\varphi f_n \to C_\varphi f$ in D_α. ∎

We next want to consider C_φ in a situation when C_φ is not compact. Theorem 1.2 motivates the hypothesis of the next result. Part of this result (giving the estimate for $\|C_\varphi - C_\psi\|_e^2$ in the case that $\varphi^*(e^{i\theta}) \neq \psi^*(e^{i\theta})$) has been independently obtained by Joel Shapiro [S2].

THEOREM 2.2. *Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ and suppose that φ has finite angular derivative at a point $e^{i\theta} \in \partial\mathbb{D}$. Let $\psi\colon \mathbb{D} \to \mathbb{D}$ be holomorphic and consider C_φ and C_ψ acting on D_α. Then, unless both*

$$\psi^*(e^{i\theta}) = \varphi^*(e^{i\theta})$$

and

$$\psi'(e^{i\theta}) = \varphi'(e^{i\theta}),$$

we have $\|C_\varphi - C_\psi\|_e^2 \geq |\varphi'(e^{i\theta})|^{-\alpha}$, where $\|\ \|_e$ denotes the essential norm of an operator.

PROOF: Without loss of generality, we may assume $e^{i\theta} = 1$, $\varphi^*(1) = 1$ and $\varphi'(1) = s < \infty$. Let $\psi: \mathbb{D} \to \mathbb{D}$ be holomorphic and assume that either $\psi^*(1) \neq 1$ (we include here the possibility that ψ does not have a radial limit at 1) or that $\psi^*(1) = 1$, but $\psi'(1) \neq s$ (where here we include the possibility that $|\psi'(1)| = \infty$). If k_z is a reproducing kernel function, we have

$$\|(C_\varphi^* - C_\psi^*)k_z\|^2 = \|k_{\varphi(z)} - k_{\psi(z)}\|^2$$

$$= \|k_{\varphi(z)}\|^2 + \|k_{\psi(z)}\|^2 - 2 \operatorname{Re} k_{\varphi(z)}(\psi(z)).$$

Thus

$$\|C_\varphi - C_\psi\|^2 \geq \frac{\|k_{\varphi(z)}\|^2}{\|k_z\|^2} + \frac{\|k_{\psi(z)}\|^2}{\|k_z\|^2} - \frac{2\operatorname{Re} k_{\varphi(z)}(\psi(z))}{\|k_z\|^2}$$

$$\geq \frac{\|k_{\varphi(z)}\|^2}{\|k_z\|^2} - \frac{2\operatorname{Re} k_{\varphi(z)}(\psi(z))}{\|k_z\|^2}$$

for any point z in $\mathbb{D}$.

Now $\|k_{\varphi(z)}\|^2/\|k_z\|^2 = (1-|z|^2)^\alpha/(1-|\varphi(z)|^2)^\alpha$ and by the Julia-Caratheodory theory, $[1-|\varphi(z)|]/(1-|z|)$ has non-tangential limit $\varphi'(1) = s$ at 1. Thus

$$\frac{\|k_{\varphi(z)}\|^2}{\|k_z\|^2} \to \left(\frac{1}{s}\right)^\alpha$$

as $z \to 1$ non-tangentially.

Next we consider the term $2\operatorname{Re} k_{\varphi(z)}(\psi(z))/\|k_z\|^2$ and distinguish two cases.

(i) If $\psi^*(1) \neq 1$, then there exists $r_n \uparrow 1$ so that $\lim_{n\to\infty} \psi(r_n) = \beta \neq 1$. Setting $z = r_n$ in $2 \operatorname{Re} k_{\varphi(z)}(\psi(z))/\|k_z\|^2$ yields

$$2 \operatorname{Re} \left[\frac{(1-r_n^2)}{1-\overline{\varphi(r_n)}\psi(r_n)} \right]^{\alpha},$$

which has limit zero as $n \to \infty$, since $\varphi(r_n)\psi(r_n) \to \beta \neq 1$. This, in conjunction with the above estimate on $\|k_{\varphi(z)}\|^2/\|k_z\|^2$ for $z = r_n$ shows that, in this case, $\|C_\varphi - C_\psi\|^2 \geq s^{-\alpha}$.

(ii) If $\psi^*(1) = 1 = \varphi^*(1)$, but $\varphi'(1) \neq \psi'(1)$ (we emphasize that we include here the possibility that $|\psi'(1)| = \infty$), then to estimate $2 \operatorname{Re} k_{\varphi(z)}(\psi(z))/\|k_z\|^2$ we first consider

$$(*) \qquad \frac{1-\overline{\varphi}(z)\psi(z)}{1-|z|^2} = \frac{1-\overline{\varphi}(z)\varphi(z)+\overline{\varphi}(z)\varphi(z)-\overline{\varphi}(z)\psi(z)}{1-|z|^2}$$

$$= \frac{1-|\varphi(z)|^2}{1-|z|^2} + \overline{\varphi}(z) \frac{1-z}{1-|z|^2} \left[\frac{1-\psi(z)}{1-z} - \frac{1-\varphi(z)}{1-z} \right].$$

Consider $\Gamma_M = \{z \in \mathbb{D}: |1-z|/(1-|z|^2) = M\}$, the boundary of a non-tangential approach region at 1. As $z \to 1$ along Γ_M, the Julia-Caratheodory theory shows that

(a) $$\frac{1-|\varphi(z)|^2}{1-|z|^2} \to s,$$

(b) $$\frac{1-\varphi(z)}{1-z} \to s,$$

(c) $$\frac{1-\psi(z)}{1-z} \to \psi'(1) \text{ if } \psi'(1) < \infty, \text{ or}$$

$$\left| \frac{1-\psi(z)}{1-z} \right| \to \infty \text{ as } z \to 1 \text{ nontangentially if } |\psi'(1)| = \infty.$$

Given any $N > 0$ we may, by choosing M sufficiently large, find a sequence $z_n \to 1$ along Γ_M so that for n sufficiently large

$$\left| \overline{\varphi}(z_n) \frac{1-z_n}{1-|z_n|^2} \left[\frac{1-\psi(z_n)}{1-z_n} - \frac{1-\varphi(z_n)}{1-z_n} \right] \right| > N.$$

In other words, given $\delta > 0$ we may find a sequence $z_n \to 1$

nontangentially so that for n sufficiently large

$$\left| 2 \text{ Re } \frac{k_{\varphi(z_n)}(\psi(z_n))}{\|k_{z_n}\|^2} \right| < \delta .$$

This, together with our previous estimate on $\|k_{\varphi(z)}\|^2/\|k_z\|^2$ shows that

$$\|C_\varphi - C_\psi\|^2 \geq s^{-\alpha} - \delta .$$

Since δ is arbitrary, we have $\|C_\varphi - C_\psi\|^2 \geq s^{-\alpha}$.

To obtain the estimate $\|C_\varphi - C_\psi\|_e^2 \geq s^{-\alpha}$, recall first that the essential norm of an operator $A \in \mathcal{B}(H)$ is defined by $\|A\|_e = \inf\{\|A+K\| : K \text{ is compact on } H\}$. Now consider

$$\left\| (C_\varphi - C_\psi + K)^* \frac{k_z}{\|k_z\|} \right\| \geq \left\| (C_\varphi - C_\psi)^* \frac{k_z}{\|k_z\|} \right\| - \left\| K^* \left[\frac{k_z}{\|k_z\|} \right] \right\|$$

where K, and hence K^*, is a compact operator. Now $k_z/\|k_z\| \to 0$ weakly as $z \to \partial \mathbb{D}$. By compactness, $\|K^*(k_z/\|k_z\|)\| \to 0$, as $z \to \partial \mathbb{D}$. This, together with the above estimate on $\|(C_\varphi - C_\psi)^* (k_z/\|k_z\|)\|$, gives the desired result. ∎

As a corollary to Theorem 2.2, we get a sufficient condition for the component of C_φ to be a singleton.

COROLLARY 2.3. *If φ has a finite angular derivative on a set of positive measure, then C_φ is isolated in the space of composition operators on D_α, with $\|C_\varphi - C_\psi\|_e^2 \geq s^{-\alpha}$ where*

$$s = \text{ess inf } \{ |\varphi'(e^{i\theta})| : |\varphi^*(e^{i\theta})| = 1 \}$$

and $\psi \neq \varphi$.

PROOF. Let $\psi : \mathbb{D} \to \mathbb{D}$. If $\psi \neq \varphi$, then $\{e^{i\theta} : \varphi^*(e^{i\theta}) = \psi^*(e^{i\theta})\}$ has Lebesgue measure zero.

Choose $\eta \in \partial\mathbb{D}$ so that $\psi^*(\eta) \neq \varphi^*(\eta)$, and $|\varphi^*(\eta)| = 1$. By Theorem 2.2, $\|C_\varphi - C_\psi\|_e^2 \geq |\varphi'(\eta)|^{-\alpha}$. Thus

$$\|C_\varphi - C_\psi\|_e^2 \geq \sup\left\{\left[\frac{1}{|\varphi'(\eta)|}\right]^\alpha : |\varphi^*(\eta)| = 1 \text{ and } \varphi^*(\eta) \neq \psi^*(\eta)\right\},$$

which is the desired result. ■

REMARKS. (1) If we let π be the quotient map from $\mathcal{B}(D_\alpha) \to \mathcal{B}(D_\alpha)/\mathcal{K}(D_\alpha)$ where $\mathcal{K}(D_\alpha)$ denotes the compact operators on D_α, then Corollary 2.3 actually shows that if φ has finite angular derivative on a set of positive measure, then $\{\pi(C_\varphi)\}$ is a component in $\pi(\mathcal{C}(D_\alpha))$.

(2) Joel Shapiro [S2] has also has obtained a version of Corollary 2.3

Next we use Theorem 2.2 to give a necessary condition for C_ψ to be in the same component as C_φ. Let us say that φ and ψ *have the same data* at $e^{i\theta} \in \partial\mathbb{D}$ if φ and ψ have radial limits of modulus 1 at $e^{i\theta}$,

(i) $$\varphi^*(e^{i\theta}) = \psi^*(e^{i\theta}),$$

and

(ii) $$|\varphi'(e^{i\theta})| = |\psi'(e^{i\theta})|.$$

We remark that in the presence of (i), (ii) actually implies $\varphi'(e^{i\theta}) = \psi'(e^{i\theta})$ when $|\varphi'(e^{i\theta})| < \infty$.

THEOREM 2.4. *If C_ψ is in the component containing C_φ, then φ and ψ must have the same data at any point $e^{i\theta}$ where $|\varphi'(e^{i\theta})| < \infty$. Moreover, if $\pi(C_\varphi)$ and $\pi(C_\psi)$ are in the same component in $\pi(\mathcal{C}(D_\alpha))$, then φ and ψ have the same data at any point where $|\varphi'(e^{i\theta})| < \infty$.*

PROOF. Suppose $|\varphi'(e^{i\theta})| < \infty$. Without loss of generality we may take $e^{i\theta} = 1$, $\varphi(1) = 1$, and $\varphi'(1) = s < \infty$. At each point C_τ in the component containing C_φ, let $U(C_\tau,s)$ denote the intersection of the ball (in $\mathcal{B}(D_\alpha)$) of radius $\frac{1}{2}s^{-\alpha/2}$ centered at C_τ with the component of C_φ in $\mathcal{C}(D_\alpha)$. From the collection $\{U(C_\tau,s)\}$, we extract a simple chain $\{U(C_{\tau_j},s)\}_{j=1}^n$ from C_φ to C_ψ [H-Y; p. 108]. To simplify the notation, write $U_j = U(C_{\tau_j},s)$. Thus $C_\varphi \in U_1$, $C_\psi \in U_n$, and $U_j \cap U_k \neq \emptyset$ if and only if $|j-k| \leq 1$. Consider a point C_{γ_1} in $U_1 \cap U_2$. By Theorem 2.2, γ_1 must have the same data as φ at 1, since $\|C_\varphi - C_{\gamma_1}\|^2 < \frac{1}{s^\alpha}$. Similarly, since $C_{\gamma_1} \in U_2$, γ_1 and τ_2 must have the same data at 1 (where C_{τ_2} is the "center" of U_2). If C_{γ_2} is in $U_2 \cap U_3$, γ_2 and τ_2 have the same data at 1, since $\|C_{\gamma_2} - C_{\tau_2}\|^2 < \frac{1}{s^\alpha}$ and $\tau_2'(1) = s$. Continuing along the chain to C_ψ, we conclude ψ has the same data as φ at 1. ■

Recall that if φ is not the identity map on $\mathbb{D}$, then there is a unique point ζ in $\overline{\mathbb{D}}$ with the properties that $\varphi^*(\zeta) = \zeta$, and $|\varphi'(\zeta)| \leq 1$. If ζ is in $\mathbb{D}$, we interpret this as $\varphi(\zeta) = \zeta$ and $|\varphi'(\zeta)| \leq 1$ where $\varphi'(\zeta)$ has the ordinary meaning. Moreover, in the case $|\zeta| = 1$, $0 < \varphi'(\zeta) \leq 1$. This point ζ is called the Denjoy-Wolff point of φ; we denote it by $DW(\varphi)$. The next result follows immediately from Theorem 2.4.

COROLLARY 2.5. *Suppose* $DW(\varphi) \in \partial\mathbb{D}$. *Then if* C_ψ *is in the same component as* C_φ, $DW(\psi) = DW(\varphi)$, *and the angular derivatives at the Denjoy-Wolff point are the same.*

In view of the theorems characterizing compact composition operators on the spaces D_α and Theorem 2.2, a natural question to ask is whether every noncompact composition operator must be isolated in $\mathcal{C}(D_\alpha)$. Recent work of Shapiro and Sundberg has answered this in the negative, in the setting of H^2. They consider mappings φ for which $\varphi(\mathbb{D})$ contacts $\partial\mathbb{D}$ only finitely often and with "finite order of contact" at each such point. The prototypical example of such a map is $\varphi(z) = \frac{1}{2}(z + 1)$. (Here $\varphi(1) = 1$, $\varphi'(1) = \frac{1}{2}$ and C_φ is not compact on H^2). They show that for such φ, C_φ is not isolated and moreover lies in an arc in $\mathcal{C}(H^2)$ such that if C_ψ is in this arc, then $C_\varphi - C_\psi$ is compact. It is interesting to note that in the example $\varphi(z) = \frac{1}{2}(z + 1)$, the composition operators in the component containing C_φ do not include those C_ψ arising from $\psi(z) = sz + 1 - s$, $s \neq \frac{1}{2}$. (This follows from Theorem 2.4.) However, the methods of [S-S] show that the non-compact operators corresponding to the maps

$$\varphi_t(z) = \frac{z+1}{2} + t\left[\frac{1-z}{2}\right]^3,$$

t small, form an arc in $\mathcal{C}(H^2)$.

3. EXTENSIONS TO LINEAR COMBINATIONS OF COMPOSITION OPERATORS

In this section we fix $e^{i\theta} \in \partial\mathbb{D}$ and consider a linear combination

$$\sum_{j=1}^{N} a_j C_{\varphi_j},$$

where $\varphi_j^*(e^{i\theta})$ exists for each $j = 1,\dots,N$.

THEOREM 3.1. *Let $\mathcal{R}$ be a set of holomorphic self maps of $\mathbb{D}$ and suppose that $e^{i\theta} \in \partial\mathbb{D}$ has the property that $\varphi^*(e^{i\theta})$ exists for all φ in $\mathcal{R}$. Let $\mathcal{S} \subset \mathcal{R}$ be such that if $\varphi \in \mathcal{S}$, $\varphi^*(e^{i\theta})$ has modulus 1 and no other ψ in $\mathcal{R}$ has the same data as φ at $e^{i\theta}$. Then, given $\varphi_1,\dots,\varphi_N \in \mathcal{R}$ and complex numbers $a_1,\dots,a_N$, we have*

$$\Big\|\sum_{j=1}^{N} a_j C_{\varphi_j}\Big\|_e^2 \geq \sum_{\varphi_j \in \mathcal{S}} |a_j|^2 \frac{1}{|\varphi_j'(e^{i\theta})|^\alpha}.$$

PROOF. Without loss of generality, we may take $e^{i\theta}$ to be 1. As in the proof of Theorem 2.2, we will consider $\|(\sum_1^N a_j C^*_{\varphi_j})\nu_z\|^2$ where $\nu_z = k_z/\|k_z\|$, so that

$$\left\|\left(\sum_{j=1}^{N} a_j C^*_{\varphi_j}\right)\nu_z\right\|^2 = \frac{1}{\|k_z\|^2}\left\|\sum_{j=1}^{N} a_j k_{\varphi_j(z)}\right\|^2.$$

We may assume $\mathcal{S}$ is nonempty, else there is nothing to prove. Relabeling if necessary, suppose $\varphi_1, \varphi_2, \ldots, \varphi_m \in \mathcal{S}$. Partition the remaining members of $\varphi_1, \ldots, \varphi_N$ into disjoint sets $D_1, \ldots, D_r$, where D_1 consists of those φ_j with $|\varphi^*_j(1)| < 1$ and $D_2, \ldots, D_r$ are equivalence classes arising from the equivalence relation $\varphi_j \sim \varphi_k$ if φ_j and φ_k have the same data at 1. Write

$$\frac{1}{\|k_z\|^2}\left\|\sum_{j=1}^{N} a_j k_{\varphi_j(z)}\right\|^2$$

as

$$\frac{1}{\|k_z\|^2}\left\|a_1 k_{\varphi_1(z)} + \cdots + a_m k_{\varphi_m(z)} + \sum_{\varphi_j \in D_1} a_j k_{\varphi_j(z)} + \cdots + \sum_{\varphi_j \in D_r} a_j k_{\varphi_j(z)}\right\|^2.$$

A calculation shows this is equal to

$$(*) \qquad \frac{1}{\|k_z\|^2}\sum_{j=1}^{m} |a_j|^2 \|k_{\varphi_j(z)}\|^2 + \frac{1}{\|k_z\|^2}\sum_{k=1}^{r}\left\|\sum_{\varphi_j \in D_k} a_j k_{\varphi_j(z)}\right\|^2$$

$$+ \text{"cross-terms"},$$

where the cross-terms are one of the following

(i) $\dfrac{1}{\|k_z\|^2}\, 2\operatorname{Re}\langle a_j k_{\varphi_j(z)}, a_\ell k_{\varphi_\ell(z)}\rangle, \quad 1 \le j < \ell \le m;$

(ii) $\frac{1}{\|k_z\|^2} 2\mathrm{Re}\left\langle a_j k_{\varphi_j(z)}, \sum_{\varphi_\ell \in D_k} a_\ell k_{\varphi_\ell(z)} \right\rangle$, $\quad 1 \le j \le m,\ 1 \le k \le r$;

(iii) $\frac{1}{\|k_z\|^2} 2\mathrm{Re}\left\langle \sum_{\varphi_j \in D_k} a_j k_{\varphi_j(z)}, \sum_{\varphi_j \in D_{k'}} a_j k_{\varphi_j(z)} \right\rangle$, $\quad k < k'$.

Given $\delta > 0$, we may, as in the proof of Theorem 2.2, find a non-tangential sequence $z_n \to 1$ so that *each* cross-term, with $z = z_n$ and n sufficiently large, is less than δ/N^2, say. Now for $j = 1,\dots,m$,

$$\frac{\|k_{\varphi_j(z_n)}\|^2}{\|k_{z_n}\|^2} = \left[\frac{1-|z_n|^2}{1-|\varphi_j(z_n)|^2}\right]^\alpha,$$

which has limit $|\varphi_j'(1)|^{-\alpha}$ as $n \to \infty$. Thus, setting $z = z_n$ and letting $n \to \infty$ in (*) gives

$$(**) \qquad \left\|\sum_{j=1}^N a_j C_{\varphi_j}\right\|^2 \ge \sum_{\varphi_j \in \mathcal{S}} |a_j|^2 \frac{1}{|\varphi_j'(1)|^\alpha}.$$

Just as in the proof of Theorem 2.2, we use the fact that $\nu_z \to 0$ weakly as $z \to \partial\mathbb{D}$ to show that we can replace $\|\Sigma_1^N a_j C_{\varphi_j}\|^2$ by the essential norm $\|\Sigma_1^N a_j C_{\varphi_j}\|_e^2$. ∎

We give two corollaries to Theorem 3.1. If $e^{i\theta} \in \partial\mathbb{D}$ and $\varphi: \mathbb{D} \to \mathbb{D}$ is a map for which $|\varphi^*(e^{i\theta})| = 1$ and $|\varphi'(e^{i\theta})| < \infty$, let

$$\mathcal{L}(\varphi, e^{i\theta}) = \{\psi: \mathbb{D} \to \mathbb{D}: \psi^*(e^{i\theta}) \text{ exists and } \psi \text{ and } \varphi \text{ do not have the same data at } e^{i\theta}\}.$$

Let $\mathcal{M}$ be the set of all finite linear combinations of composition operators induced by maps $\psi \in \mathcal{L}(\varphi, e^{i\theta})$. Then, using $\mathcal{S} = \{\varphi\}$ and $\mathcal{B} = \mathcal{L}(\varphi, e^{i\theta}) \cup \{\varphi\}$ in Theorem 3.1, we have

COROLLARY 3.2. *For $\mathfrak{M}$ as described above,*

$$\inf_{A \in \mathfrak{M}} \|C_\varphi - A\|_e^2 \geq \frac{1}{|\varphi'(e^{i\theta})|^\alpha}.$$

In the next corollary, we have $\mathcal{S} = \mathcal{P} = \{\varphi_1, \ldots \varphi_n\}$.

COROLLARY 3.3. *Suppose $\{\varphi_1, \varphi_2, \cdots, \varphi_n\}$ are distinct holomorphic self-maps of $\mathbb{D}$ and $e^{i\theta} \in \partial\mathbb{D}$ is such that $\varphi_j^*(e^{i\theta})$ exists and has modulus 1 for $j = 1, 2, \ldots, n$. If no pair $\{\varphi_i, \varphi_j\}$ $(i \neq j)$ has the same data at $e^{i\theta}$, then*

$$\Big\| \sum_{j=1}^n a_j C_{\varphi_j} \Big\|_e^2 \geq \sum_{j=1}^n |a_j|^2 \frac{1}{|\varphi_j'(e^{i\theta})|^\alpha}.$$

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