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On the Nucleolus of NTU-Games Defined by Multiple Objective Linear Programs

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Abstract: In this article we derive a class of cooperative games with non-transferable utility from multiple objective linear programs. This is done in order to introduce the nucleolus, a solution concept from cooperative game theory, as a solution to multiple objective linear problems.

We show that the nucleolus of such a game is a singleton, which is characterized by inclusion in the least core and the reduced game property. Furthermore the nucleolus satisfies efficiency, anonymity and strategic equivalence.

We also present a polynomially bounded algorithm for computation of the nucleolus. Let n be the number of objective functions. The nucleolus is obtained by solving at most $2n$ linear programs. Initially the ideal point is computed by solving n linear programs. Then a sequence of at most n linear programs is solved, and the nucleolus is obtained as the unique solution of the last program.

Key Words: Multiple Objective Linear Programs, Nucleolus, NTU-games.

1 Introduction

Since the relationship between linear programming and game theory was established in Gale, Kuhn and Tucker (1951) an ever increasing body of literature has been concerned with the description of and investigation into the relationship between various fields of operations research on the one side and game theory, both cooperative and non-cooperative, on the other side. To mention all but an extract of this line of research is naturally beyond the scope of this paper. However, as an example of authors investigating the relationship between a field

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of operations research, namely the measurement of efficiency, and game theory one may find Banker (1980) and Banker, Charnes, Cooper and Clarke (1989).

Now, the result in Gale et al. (1951) establishing the equivalence of solving a linear programming problem and finding a certain maxmin solution to a non-cooperative game is certainly a most celebrated technical description of the linkage between operations research and game theory. On the other hand it can be argued, that this result may not always lend itself to intuition. because at the outset there seems to be little reason, why the action of players in a game choosing strategies should be the same as solving a linear programming problem.

There is, however, one field of operations research, namely that of multiple objective decision making, where the linkage to game theory is particularly intuitive. As such, multiple objective decision making is concerned with the description of how to maximize multiple, possibly conflicting objectives simultaneously. Here, the linkage to game theory is clear, since game theory attempts to describe ways of making a joint action, when the players have opposed views. Hence, one can relate the two simply by interchanging the words objectives and players.

The idea of relating multiple objective decision making to game theory is by no means new. Among the earlier contributions one finds Belenson and Kapur (1974), Bergstresser and Yu (1977), Blackwell (1956) and Zeleny (1975) to mention a few. But despite the fact, that the interface between multiple objective decision making and game theory in general has been investigated for quite some years, there seems to have been little emphasis on applying methods from the so-called cooperative part of game theory to multiple objective decision making. Among the exceptions one finds Forgó (1984). Rather, it seems as if the emphasis has been in the reverse direction by applying multiple objective methods to cooperative games (see e.g. Steuer (1986) for a short bibliography).

Therefore, it shall be the main aim of this article to demonstrate that methods from cooperative game theory can be applied successfully to problems of multiple objective decision making. In order to do so we shall make the simplifying assumption, that each multiple objective decision problem is linear, that is each objective function is linear, and the set of feasible alternatives is a polytope. This class of problems is also known as multiple objective linear programs (MOLP, for short).

To each MOLP we associate a cooperative game with non-transferable utility (NTU), and as it turns out, these associated games have a relatively simple structure, which allows us to apply the nucleolus, a solution concept from cooperative game theory, which was introduced by Schmeidler (1969). This approach enables us to show, that the nucleolus as a solution to any MOLP is efficient and unique in terms of objective function values, and it can be computed by an algorithm, which consists of at most $2n$ fairly simple linear programs, where n is the number of objectives. Moreover, it is shown that the nucleolus on the class of NTU-games associated with MOLPs can be characterized by two axioms. This proves that the characterization in Maschler, Potters and Tijs (1992) can be generalized to some extent to cover a larger class of games.

Thus, the outline of the article is as follows. In section 2 we introduce the necessary concepts from cooperative game theory and define the nucleolus, and in section 3 we derive a NTU-game from each MOLP. In sections 4 and 5 an algorithm for the computation of the nucleolus is presented along with an example. Finally, section 6 presents some of the properties of the nucleolus, and section 7 contains a characterization of the nucleolus.

2 Concepts from Game Theory

Let $N = \{1, \ldots, n\}$ be a finite set of players. A coalition is a non-empty subset of N. For every $x \in \mathbb{R}^N$ the restriction of x to $\mathbb{R}^S$, $S \subseteq N$, is denoted by x_S. The vector in $\mathbb{R}^N$ with all coordinates equal to one is denoted by e. Also let e_S denote the vector in $\mathbb{R}^S$ with all elements equal to 1.

A cooperative game with non-transferable utility (*NTU-game*) is a pair (N, V), where N is the set of players and V is a mapping which for each coalition, S, defines a characteristic set, V_S, satisfying:

1. V_S is a non-empty, closed, real subset of $\mathbb{R}^S$.
2. V_S is comprehensive,
   i.e. if $x, y \in \mathbb{R}^S$, $x \in V_S$ and $x \geq y$ then $y \in V_S$.
3. The set $\{y_S \in V_S | y_S \geq x_S\}$ is compact, $\forall x_S \in \mathbb{R}^S$.

In particular, a NTU-game (N, V) is called a game with transferable utility (*TU-game*) if the characteristic set for every coalition, S, is a closed half space with normal e_S.

Let $\Gamma = (N, V)$ be a NTU-game. We say that X is a *payoff set* for Γ if X is a non-empty, closed subset of V_N. A payoff $x \in X$ is said to be *efficient*, if there does not exist some $\tilde{x} \in X$ with $\tilde{x} \geq x$ and $\tilde{x} \neq x$.

Next, by a *game* we will mean a pair (Γ, X) where X is a payoff set associated with a given NTU-game Γ. Furthermore, let $\mathcal{G}$ denote the set of all games and let $\mathcal{G}'$ denote a subset of $\mathcal{G}$. A *solution concept* on $\mathcal{G}'$ is a correspondence, that associates with each game $(\Gamma, X) \in \mathcal{G}'$ a non-empty subset of X.

Now, one may interpret the payoff set as the set of attainable payoffs for the society, i.e. N, and the characteristic set of each coalition $S \subseteq N$ as the set of payoffs, that the coalition S could obtain had it been on its own. Given this interpretation it becomes natural to try to measure a coalition's content or discontent with any payoff x. There are several ways to measure this content or discontent, termed the *excess*, see e.g. Kalai (1975) for a very general approach. However, we shall follow Christensen (1991) to define the excess for a coalition

S as

$$h_S(V_S, x_S) = \max\{t \in \mathbb{R} | x_S + te_S \in V_S\} \; . \tag{1}$$

Hence, the excess-function, h_S, measures the largest possible gain (loss if negative) to every member of S, if they form their own coalition instead of keeping x_S.

Let $M = 2^N \backslash \{\emptyset\}$ and let $m = |M|$. Next, let θ be a mapping from $\mathbb{R}^M$ to $\mathbb{R}^m$ that rearranges the coordinates in a non-increasing order and let $\leq_L$ denote the lexicographic order on $\mathbb{R}^m$. We define the *nucleolus* of the game $((N, V), X) = (\Gamma, X)$ by:

$$Nu_h(\Gamma, X) = \{x \in X | \forall y \in X: \theta((h_S(V_S, x_S))_{S \in M}) \leq_L \theta((h_S(V_S, y_S))_{S \in M})\} \; .$$

The coalition with the strongest objection to a given payoff has the greatest excess-function value. Consequently, the nucleolus consists of the payoffs providing the minimal excesses in the lexicographic order.

We define the *least core* of the game (Γ, X) by:

$$Lcore_h(\Gamma, X) = \left\{x \in X | \forall y \in X: \max_{S \in M} h_S(V_S, x_S) \leq \max_{S \in M} h_S(V_S, y_S)\right\} \; .$$

The least core contains the payoffs which give the minimal excesses in the ordering according to the largest coordinate, and therefore the nucleolus is contained in the least core.

The following theorem is not new, but it states some basic properties of the nucleolus of NTU-games. See Christensen (1991) and Kalai (1975).

Theorem 1: For every game $(\Gamma, X) \in \mathcal{G}$ the nucleolus, $Nu_h(\Gamma, X)$, is non-empty and consists of a finite number of efficient payoffs.

It is an immediate consequence of Theorem 1, that the least core also is non-empty.

3 NTU-Games Defined by Multiple Objective Linear Programs

Consider the multiple objective linear program (MOLP):

$$\begin{aligned} \text{"max"} \quad & c^i x \text{ for } i = 1, \ldots, n \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned} \tag{2}$$

where A is an $r \times l$ matrix, $b \in \mathbb{R}^r$, $c^i \in \mathbb{R}^l$ for $i = 1, \ldots, n$ and $x \in \mathbb{R}^l$. Finally, the set $\mathscr{A} = \{x \in \mathbb{R}^l | Ax \leq b, x \geq 0\}$ is assumed to be a non-empty polytope.

Let $N = \{1, \ldots, n\}$ be the set of players. Each player is associated with an objective function. We want to associate a NTU-game with each MOLP. Therefore, let $S \subseteq N$, $S \neq \emptyset$. Then the characteristic set, V_S, is defined by:

$$V_S = Z_S - \mathbb{R}^S_+$$

where

$$Z_S = \{z \in \mathbb{R}^S | z = C_S x, x \in \mathscr{A}\}$$

and $\mathbb{R}^S_+$ denotes the non-negative orthant of the Euclidean space $\mathbb{R}^S$. C_S is the $|S| \times l$ matrix where the rows contain the coefficients of the objective functions associated with the players in S. The players are listed in increasing index order. It is easy to verify, that for all coalitions the sets V_S in fact are characteristic sets in the sense of the previous section.

The payoff set X for $\Gamma = (N, V)$ is assumed to be a non-empty polytope satisfying $X \subseteq V_N$. A natural example of a payoff set is the feasible region in criterion space, hence $X = Z_N$. Z_N is a payoff set since $\mathscr{A}$ is assumed to be a non-empty polytope.

Let $\mathscr{G}^{MOLP}$ denote the set of games defined by multiple objective linear programs. The following lemma demonstrates, that the games in $\mathscr{G}^{MOLP}$ all share the property, that coalition size and excess function value are inversely related. A property, which will prove itself central for the computation of the nucleolus of games in $\mathscr{G}^{MOLP}$.

Lemma 2: Let $((N, V), X) \in \mathscr{G}^{MOLP}$ *and* $x \in \mathbb{R}^N$. *Then* $h_S(V_S, x_S) \leq h_T(V_T, x_T)$, $\forall T \subseteq S \subseteq N, T \neq \emptyset$.

Proof: For $S = T$ there is nothing to prove. Let $T \subset S$, $T \neq \emptyset$. We claim that $V_S \subseteq V_T \times \mathbb{R}^{S \setminus T} = \{x \in \mathbb{R}^S | x_T \in V_T\}$. To see this let $y_S \in V_S$. This implies that there exists a $z_S \in Z_S$ and $w_S \in \mathbb{R}^S_+$ such that $y_S = z_S - w_S$. Hence $y_T = z_T - w_T$. Notice $z_S = (z_T, z_{S \setminus T}) = (C_T \bar{x}, C_{S \setminus T} \bar{x})$ for some $\bar{x} \in \mathscr{A}$. Hence $z_T \in Z_T$. Since $w_T \in \mathbb{R}^T_+$ it follows that $y_T \in V_T$. Then the proof follows since

$$\begin{aligned} h_S(V_S, x_S) &= \max t & &\leq \max t \\ &\quad \text{s.t.} & &\quad \text{s.t.} \\ &\quad x_S + te_S \in V_S & &\quad x_S + te_S \in V_T \times \mathbb{R}^{S \setminus T} \\ &= \max t & &= h_T(V_T, x_T). \\ &\quad \text{s.t.} & & \\ &\quad x_T + te_T \in V_T & & \end{aligned}$$

□

4 Computation of the Nucleolus of Games in $\mathscr{G}^{MOLP}$

For a game in $\mathscr{G}^{MOLP}$ the excess-functions $h_S(V_S, \cdot)\colon X \to \mathbb{R}$ as defined by (1) are concave polyhedral functions. However the relationship between the characteristic sets (see Lemma 2) implies that we only need to know the explicit value of the characteristic sets for the singleton coalitions. Then it is possible to compute the nucleolus using a fairly simple algorithm. The algorithm presented here is related to the one suggested by Peleg for computation of the nucleolus of TU-games. Peleg's algorithm is described in Kopelowitz (1967) and it is used in Maschler, Peleg and Shapley (1979) to provide a geometric characterization of the nucleolus of TU-games.

Let $(\Gamma, X) \in \mathscr{G}^{MOLP}$.

Initialization step: Compute the ideal point $(z_1^*, \ldots, z_n^*)$, i.e. solve

$$z_i^* = \max c^i x \quad \text{s.t.} \quad x \in \mathscr{A}$$

for $i = 1, \ldots, n$.

Start: Solve

$$\begin{aligned} P_1\colon & \min w_1 \\ & \text{s.t.} \quad w_1 + z_i \geq z_i^* \quad \text{for} \quad i = 1, \ldots, n \\ & \qquad z \in X \ . \end{aligned} \tag{3}$$

If (3) gives a unique optimal solution it will be the nucleolus of Γ. However (3) does not always gives a unique solution. See section 5 for an example. Put $k = 1$.

Step k: Let w_k^* denote the optimal value of w_k in P_k and let

$$A_k = \{z \in X \mid z \text{ is an optimal solution to } P_k\}$$

$$E_k = \{j \in N \mid z_j + w_k^* = z_j^* \forall z \in A_k\} \ .$$

If $N\backslash\bigcup_{j=1}^{k} E_j = \varnothing$ then let $q = k$. **Stop.** Otherwise let $k = k + 1$. Solve

$$P_k\text{: min} \quad w_k$$

$$\text{s.t.} \quad w_k + z_i \geq z_i^* \forall i \in N\backslash\bigcup_{j=1}^{k-1} E_j$$

$$w_j^* + z_i = z_i^* \forall i \in E_j, j = 1, \ldots, k-1$$

$$z \in X.$$

Repeat.

In order to implement the algorithm it can be noted that for any given $k \in \{1, \ldots, q\}$ a non-empty subset of E_k can be found by the optimal dual solution to P_k which is obtained in the solution process if t he simplex method is used. Let $\mathscr{E}_k$ denote such a subset. Let α be the dual variable associated with $w_k + z_i \geq z_i^*$ for some $i \in N\backslash\bigcup_{j=1}^{k-1} E_j$. Then it follows from linear programming duality theory that if the optimal value of α is strictly positive then $i \in \mathscr{E}_k$ and otherwise $i \in N\backslash\bigcup_{j=1}^{k} \mathscr{E}_j$. In the algorithm we can replace E_k with $\mathscr{E}_k$ and still produce the nucleolus. However, it may take a few steps more.

Next, in order to prove that the algorithm computes the nucleolus of (Γ, X) we need the following two lemmas, where the first lemma is obtained by an easy modification of Lemma 6.3 in Maschler et al. (1979). Its formal proof is therefore omitted.

Lemma 3: q is finite.

1. w_i^* *is finite*, $i = 1, \ldots, q$.
2. A_i *is a non-empty polytope*, $i = 1, \ldots, q$.
3. $E_i \neq \varnothing, i = 1, \ldots, q$.
4. $w_{i+1}^* < w_i^*, i = 1, \ldots, q-1$.

Lemma 4: Let $A_0 = X$. *For any* k, $1 \leq k \leq q$, *if* $x \in A_k$ *and* $y \in A_{k-1}\backslash A_k$, *then* $\theta((h_S(V_S, x_S))_{S \in M}) <_L \theta((h_S(V_S, y_S))_{S \in M})$.

Proof: Define the coalition $T = \bigcup_{j=1}^{k-1} E_j$. By the definition of $E_1, \ldots, E_{k-1}$ it follows that z_T is fixed for $z \in A_{k-1}$. Since $A_k \subset A_{k-1}$ we have

$$h_S(V_S, x_S) = h_S(V_S, y_S), \quad \forall S \subseteq T.$$

Next, for coalitions not in T we obtain by Lemma 2, that

$$\max_{S \in 2^N \setminus \{2^T, \varnothing\}} h_S(V_S, y_S) = \max_{i \in N \setminus T} h_i(V_i, y_i) = \max_{i \in N \setminus T} z_i^* - y_i > w_k^*,$$

where the last inequality follows from $y \in A_{k-1} \setminus A_k$. Applying the same argument for $x \in A_k$ gives

$$\max_{S \in 2^N \setminus \{2^T, \varnothing\}} h_S(V_S, x_S) = \max_{i \in N \setminus T} h_i(V_i, x_i) = \max_{i \in N \setminus T} z_i^* - x_i = w_k^*,$$

and hence, $\theta((h_S(V_S, x_S))_{S \in M}) <_L \theta((h_S(V_S, y_S))_{S \in M})$. □

We can now state the main theorem of this section.

Theorem 5: The algorithm computes the nucleolus of (Γ, X)*, which is a singleton.*

Proof: By Lemma 3 A_q is not empty. By the definition of $E_1, \ldots, E_q$, z_j is fixed for all $j \in N$ and all $z \in A_q$. Hence A_q contains a single vector.

Since the nucleolus is contained in X, it follows from Lemma 4 that the nucleolus of Γ is contained in A_q. □

By Theorem 5 the algorithm computes the nucleolus. To comment on the algorithm, it is noted, that the ideal point is initially computed by solving n linear programs. Thereby the explicit values of the characteristic sets for the singleton coalitions are obtained. Then the lexicographic minimization procedure is started. Theorem 5 implies that we can stop the algorithm whenever A_k, $1 \leq k \leq q$, consists of a single vector. Hence, the algorithm stops when $n + 1$ linear independent constraints of program P_k define the optimal solution. This will certainly be the case if $N \setminus \bigcup_{j=1}^{k} E_j = \varnothing$. This happens when at most n linear programs are solved. Hence the algorithm has polynomial complexity, since there exists polynomially bounded algorithms to solve linear programs. See e.g. Karmarkar (1984).

The simplicity of the algorithm is highlightened in Lemma 2. The excess for any coalition at a given payoff is less than or equal to the excess for any individual player in that coalition. Consequently, one only has to be concerned with the excesses of the singleton coalitions.

Finally, it should be noted that P_1 is an unweighted Tchebycheff program, where the ideal criterion vector is the ideal point. Particularly, if there is an unique solution to P_1, then it will be the nucleolus, so we may apply some of the properties of the nucleolus (sections 6 and 7) to grasp a more detailed and qualified description of such Tchebycheff programs.

5 An Example

To illustrate the details of the algorithm consider the following simple MOLP:

$$
\begin{aligned}
\text{"max"} \quad & x_i \ \text{ for } \ i = 1, 2, 3 \\
\text{s.t.} \quad & x_2 + x_3 \le 8 \\
& x_1 + x_2 + x_3 \le 10 \\
& x_1 \le 5 \\
& x_1 \ge 0, x_2 \ge 0, x_3 \ge 0.
\end{aligned}
$$

Let the payoff set be the feasible region in criterion space. $X = Z_N = \{z \in \mathbb{R}^3 | z_i = x_i, i = 1, 2, 3, x_2 + x_3 \le 8, x_1 + x_2 + x_3 \le 10, 5 \ge x_1 \ge 0, x_2 \ge 0, x_3 \ge 0\}$.

Initialization step: The ideal point $z^* = (5, 8, 8)$.

Start: Solve P_1:

$$
\begin{aligned}
\min \quad & w_1 \\
\text{s.t.} \quad & w_1 + z_1 \ge 5 \\
& w_1 + z_2 \ge 8 \\
& w_1 + z_3 \ge 8 \\
& z_2 + z_3 \le 8 \\
& z_1 + z_2 + z_3 \le 10 \\
& z_1 \le 5 \\
& z_1 \ge 0, z_2 \ge 0, z_3 \ge 0.
\end{aligned}
$$

Step 1: The optimal solutions to P_1 are:

$$(w_1^*, z_1, z_2, z_3) = (4, 2, 4, 4) + t(0, -1, 0, 0), \quad t \in [0, 1].$$

The optimal dual solution is $(0, 1/2, 1/2, -1/2, 0, 0)$ hence $E_1 \supseteq \{\{2\}, \{3\}\}$. In fact $E_1 = \{\{2\}, \{3\}\}$.

Solve P_2:

$$
\begin{array}{lllll}
\min & w_2 & & & \\
\text{s.t.} & w_2 + & z_1 & & \geq 5 \\
& z_2 & & & = 4 \\
& z_3 & & & = 4 \\
& z_2 + & z_3 & & \leq 8 \\
& z_1 + & z_2 + & z_3 & \leq 10 \\
& z_1 & & & \leq 5 \\
& \multicolumn{4}{l}{z_1 \geq 0, z_2 \geq 0, z_3 \geq 0.}
\end{array}
$$

which has the unique optimal solution $(w_2^*, z_1, z_2, z_3) = (3, 2, 4, 4)$. The algorithm stops according to the remark after Theorem 5. The nucleolus for this game is $z = (2, 4, 4)$ and can be obtained by choosing the solution $x = (2, 4, 4)$. The lesson to be learned from this simple example is that solving the program P_1 may not be sufficient for the computation of the nucleolus, even if only three objectives are involved.

6 Properties of the Nucleolus of Games in $\mathscr{G}^{MOLP}$

In this section we show that the nucleolus of games in $\mathscr{G}^{MOLP}$ satisfies three reasonable properties. These properties have all proved themselves useful in axiomatic characterizations of the nucleolus of different classes of games, see e.g. Maschler et al. (1992) and Sobolev (1975).

6.1 *Anonymity*

For every permutation $\pi: N \to N$ and every set $B \subseteq \mathbb{R}^S \forall S \subseteq N, S \neq \emptyset$ we define

$$B^\pi = \{y \in \mathbb{R}^{\pi(S)} | \exists \bar{y} \in B: \forall i \in S, y_{\pi(i)} = \bar{y}_i\}.$$

A solution concept, σ, on $\mathcal{G}'$ is said to satisfy *anonymity* if the following holds:

For every pair of games $((N, V), X), ((N, W), X) \in \mathcal{G}'$, if there exists a permutation $\pi: N \to N$ such that $(V_S)^\pi = W_{\pi(S)}$ for all coalitions $S \subseteq N$, then

$$(\sigma((N, V), X))^\pi = \sigma((N, W), X^\pi).$$

For solution concepts on $\mathcal{G}^{MOLP}$ this axiom says that the decision should be governed solely by the values of the objective functions. Thus, the decision should not be influenced by the names or the indexation of the objectives.

The nucleolus on $\mathcal{G}$ satisfies anonymity, since for every game $((N, V), X) \in \mathcal{G}$ and every permutation $\pi: N \to N$

$$h_S(V_S, x_S) = h_{\pi(S)}((V_S)^\pi, x^\pi_{\pi(s)}), \forall x \in X, S \in M.$$

This implies, that if we rearrange the objective functions the nucleolus will rearrange in a corresponding way due to anonymity. Hence, we do not have to worry about the order in which the objective functions are listed in (2). This also implies that all the objective functions are given equal weight a priori. This distinguishes the nucleolus from the solution obtained by lexicographic maximization, where the objective functions are ordered by importance a priori. For a description of this method see e.g. Steuer (1986).

6.2 *Strategic Equivalence*

A solution concept, σ, on $\mathcal{G}'$ is said to satisfy *strategic equivalence* if the following holds:

For every pair of games $((N, V), X), ((N, W), X) \in \mathcal{G}'$, if there exists a scalar $a > 0$ and $b \in \mathbb{R}^N$ such that $V_S = aW_S + b_S$ for all coalitions $S \subseteq N$, then

$$\sigma((N, V), aX + b) = a(\sigma((N, W), X)) + b.$$

For solution concepts on $\mathcal{G}^{MOLP}$ this axiom says that making the same change in the scaling of objectives should not influence the actual decision. Actually, the axiom says more than that. Namely, that the decision should not be influenced by whether some objectives attain only positive values or not. Hence, the decision is zero independent in terms of objective function values.

The nucleolus on $\mathscr{G}$ satisfies strategic equivalence, since for every game $((N, V), X) \in \mathscr{G}$ and every choice of $a > 0$ and $b \in \mathbb{R}^N$

$$h_S(aV_S + b_S, ax_S + b_S) = ah_S(W_S, x_S), \forall x \in X, S \in M.$$

Notice, that in order to compare the excesses of different coalitions at a given payoff, we have to measure the objective functions in some common unit. This is of course a major drawback, but necessary since the nucleolus is based upon comparisons of excesses. However, due to strategic equivalence it does not matter, what this common unit is.

6.3 *The Reduced Game Property*

Let (N, V) be a NTU-game with payoff set X. For every coalition $S \subset N$ and every $x \in X$ we define the *reduced game* with respect to S and x as the game $((S, V|S), X|x_{N\setminus S})$ where $V|S$ is the restriction of the domain of V to coalitions within S and the payoff set is given by $X|x_{N\setminus S} = \{\tilde{x}_S \in \mathbb{R}^S | (\tilde{x}_S, x_{N\setminus S}) \in X\}$.

This definition is closely related to the reduced game as defined in Maschler et al. (1992) for TU-games with permissible coalitions and permissible imputations. For TU-games where all coalitions are allowed to form the two definitions are equivalent.

Moreover, it is easily checked, using the same argument as in the first part of the proof of Lemma 2, that if $((N, V), X)$ is a game in $\mathscr{G}^{MOLP}$, then $((S, V|S), X|x_{N\setminus S})$ is also a game in $\mathscr{G}^{MOLP}$ for all coalitions $S \subset N$ and for all $x \in X$. Hence, it may be concluded, that the definition of the reduced game is internal consistent in the sense, that we do not move outside the class $\mathscr{G}^{MOLP}$.

A solution concept, σ, on $\mathscr{G}'$ is said to satisfy the *reduced game property* if the following holds:

For every game $((N, V), X) \in \mathscr{G}'$ and every coalition $S \subset N$, if $x \in \sigma((N, V), X)$, then

$$x_S \in \sigma((S, V|S), X|x_{N\setminus S}).$$

This axiom is a requirement for consistency of a solution concept. Approximately, it says that at the solution point every coalition, who looks at its payment and at the same time examine its "own game" (the reduced game), will not want to move away. The coalition will find that the payment is equal to the solution of the reduced game.

For solution concepts on $\mathscr{G}^{MOLP}$ this axiom says, that the decision maker should be able to decompose the original problem into smaller problems. That is, if for some reason only some of the objective function values are known at an optimal decision, then the method employed should allow the decision maker to consider only alternatives yielding the known optimal objective function values and to contemplate only the remaining objectives. Such an approach should allow the decision maker to still find the optimal decision.

Theorem 6: The nucleolus on $\mathscr{G}^{MOLP}$ satisfies the reduced game property.

Proof: Let $(\Gamma, X) = ((N, V), X) \in \mathscr{G}^{MOLP}$. Furthermore, let $x = Nu_h(\Gamma, X)$ and $S \subset N$, $S \neq \emptyset$. Suppose that there exists some $y_S \in X|x_{N\setminus S}$ such that $y_S \neq x_S$ and $y_S = Nu_h((S, V|S), X|x_{N\setminus S})$. Define $y \in X$ by $y = (y_S, x_{N\setminus S})$.

Let $\psi\colon \mathbb{R}^N \to \mathbb{R}^n$ be a mapping, that rearranges the coordinates in a non-increasing sequence. It is an immediate consequence of Theorem 5, that x is the nucleolus of (Γ, X), if and only if

$$\psi((h_i(V_i, x_i))_{i\in N}) \leq_L \psi((h_i(V_i, \tilde{x}_i))_{i\in N}),\ \forall \tilde{x} \in X.$$

But, since $y_{N\setminus S} = x_{N\setminus S}$ and y is the nucleolus of $((S, V|S), X|x_{N\setminus S})$, by an analogous argument we must also have

$$\psi((h_i(V_i, y_i))_{i\in N}) <_L \psi((h_i(V_i, x_i))_{i\in N}).$$

A contradiction, and hence $x_S = y_S$. □

So far we have established, that the nucleolus on $\mathscr{G}^{MOLP}$ satisfies three properties, and it has been argued that these properties may seem very reasonable not only from a game theoretic perspective, but also from the view of multiple criteria decision making.

From the pure game theoretic perspective, it is interesting to note, that these properties are sufficient to obtain an axiomatization of the nucleolus on the class of TU-games, see Sobolev (1975), although admittedly a somewhat different version of the reduced game property is used there. Here, we shall not attempt to generalize this result. Rather, to some extent we shall generalize another axiomatization of the nucleolus of TU-games provided by Maschler et al. (1992).

7 A Characterization of the Nucleolus of Games in $\mathcal{G}^{MOLP}$

We can now state the main theorem of this article.

Theorem 7: A solution concept, σ, on $\mathcal{G}^{MOLP}$ satisfies inclusion in the least core, i.e. $\sigma \subseteq Lcore_h$, and the reduced game property, if and only if $\sigma = Nu_h$.

Proof: The "if" part follows from the definition of the least core and Theorem 6. The "only if" part is proved by induction in the number of players. Therefore, let $(\Gamma, X) = ((N, V), X)$ be any game in $\mathcal{G}^{MOLP}$. Next, let $z^* \in \sigma(\Gamma, X)$ and assume that σ satisfies the two properties.

If $|N| = 1$ the least core is the unique efficient point in X, and hence by inclusion in the least core, $Lcore_h(\Gamma, X) = \{z^*\}$. Since, Nu_h is also included in the least core, it follows that $\sigma(\Gamma, X) = Nu_h(\Gamma, X)$.

Now, assume that there exists some p, $p \geq 2$, such that the theorem holds whenever the number of players is less than or equal to $p - 1$.

For $|N| = p$ it is first noted that $Lcore_h(\Gamma, X)$ is nothing but the set A_1 in the algorithm that computes the nucleolus. Hence, it follows from inclusion in the least core and Lemma 3 that there exists $l \in E_1$ such that $y_l = z_l^*$ for all $y \in Lcore_h(\Gamma, X)$. Particularly, if we let $x^* = Nu_h(\Gamma, X)$ then $x_l^* = z_l^*$.

Now, define $S = N \setminus \{l\}$. From the reduced game property it is concluded that $z_S^* \in \sigma((S, V|S), X|z_l^*)$. Since $x_l^* = z_l^*$, it follows from the induction hypothesis that $z_S^* = Nu_h((S, V|S), X|x_l^*)$. Finally, since the nucleolus satisfies the reduced game property (Theorem 6), one has $x_S^* = Nu_h((S, V|S), X|x_l^*)$, and consequently $z^* = (z_S^*, z_l^*) = (x_S^*, x_l^*) = x^*$, or equivalently, $\sigma(\Gamma, X) = Nu_h(\Gamma, X)$.

□

Theorem 7 has some merits on its own. First of all, it demonstrates that the axiomatization in Maschler et al. (1992) can be generalized to some extent to cover a broader variety of games.

Secondly, it brings axiom systems into multiple criteria decision making. That is, if the decision maker believes that the axioms of inclusion in the least core and the reduced game property are meaningful and reasonable properties of any solution method, then the number of possible methods comes down to one, namely the nucleolus. Conversely, if the decision maker does not believe in these properties, then he should choose some other method. This may be seen as the normative aspect of Theorem 7.

Finally, from a more technical point of view the axiom of inclusion in the least core is nothing but a requirement of engaging in a Tchebycheff program. Moreover, in section 6 it was argued that the reduced game property really was a concept of consistency. Thus, by Theorem 7 the implication of combining the two axioms is that nucleolus is the only result of a consistent use of Tchebycheff programs.

8 Concluding Remarks

In this article we have studied the nucleolus of NTU-games defined by multiple objective linear programs. The reason for doing so was at least twofold.

First of all, it was demonstrated by the provision of a fairly simple algorithm yielding a unique efficient point, that concepts from cooperative game could successfully be applied to multiple criteria decision making.

Secondly, the NTU-games derived from multiple objective linear programs was shown to possess a particularly simple structure. A structure, that made a generalization of results from TU-games possible.

Finally, from a normative perspective the application of concepts from cooperative game theory enabled the introduction of an axiom system, and to our belief an axiom system is the ultimate key for choosing among a variety of potential and at the outset equally suitable methods.

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