Algebra Universalis, 17 (1983) 191–199
0002-5240/83/020191-09$01.50+0.20/0

The structure of saturated free algebras

J. T. BALDWIN* and S. SHELAH†

Dedicated to Alfred Tarski on his 80th birthday

Abstract. Requiring an algebra M to be both free (for the variety it generates) and $\aleph_1$-saturated imposes very strong conditions on M. In the simplest examples (see below) there exist a finite number of relatively free algebras $A_0, \ldots, A_{n-1}$ whose theories are $\aleph_1$-categorical such that M is generated (as an algebra) by the $\bigcup A_i$. In particular, this implies Th (M) has at most $(\alpha + \aleph_0)$ models of cardinality $\aleph_\alpha$. We will show a weaker structure theorem in the general case but deduce the same constraint on the spectrum of T.

This paper was prompted by the observation in [1] that in both an arbitrary free algebra of countable similarity type and an arbitrary model of an ω-stable theory, every uncountable set contains an uncountable subset which is indiscernible over the empty set. This superficial resemblance between saturated and free structures prompted the study of structures which are both free and saturated. This investigation of saturated free structures rapidly developed from the (comparatively) naive methods of [1] to a use of the sophisticated methods and definitions of [3]. Thus, this paper requires acquaintance with III of V of [3] even to state the following main result. In addition to obtaining this result, we hope this paper will clarify for the reader some of the finer distinctions from [2], e.g., Example 3 shows the difference between "regular" and "weight one".

Our principal results says that if M is a free algebra in the variety V and if M is saturated as a model of the complete theory of M (Th (M)) then every model of Th (M) is "generated" by the union of a finite set of indiscernible sequences. Moreover, these indiscernible sequences satisfy a strong technical condition. There are two forms of this result. In the first we take "generated" in the weak sense of "prime over" and the strong technical condition is regularity.

THEOREM 1. *Let V be a variety with a countable similarity type. Suppose M is a free algebra in V and M is $\aleph_1$-saturated. Then there exists a finite set $q_1, \ldots, q_k$*

* Partially supported by N.S.F. grant 77-01667.

† This research was partially supported by the United States Israel Binational Science Foundation grant 1110.

Presented by R. McKenzie. Received January 28, 1981. Accepted for publication in final form September 8, 1982.

of regular types, such that if $N \models \mathrm{Th}(M)$ *there are indiscernible sets* $Y_1, \ldots, Y_n$ *with each* Y_i *based on some* q_j *such that* N *is prime over* $Y_1 \cup \cdots \cup Y_n$.

This result already gives us substantial information on the spectrum of $\mathrm{Th}(M)$. the conclusion of Theorem 1 just spells out the assertion: $\mathrm{Th}(M)$ is a finite dimensional ω-stable theory. By [3: V.5.8], we have,

COROLLARY. *If* $\aleph_\alpha \geq 2^{\aleph_0}$ *then* $\mathrm{Th}(M)$ *has at most* $\alpha + \aleph_0$ *models of power* $\aleph_\alpha$.

This result is established by the first five lemmas below. The remainder of the paper is devoted to strengthening "generate" to generate in the usual algebraic sense. The penalty for this strengthening is to weaken the requirement that the q_i be regular to the assertion that each q_i has weight one. More precisely we have:

THEOREM 2. *Let* V *be a variety with countable similarity type. Suppose* M *is a free algebra in* V *and* M *is* $\aleph_1$*-saturated. There exists a finite set* $q_1, \ldots, q_k$ *of types, each with weight one, such that if* $N \models \mathrm{Th}(M)$ *there are* $Y_1, \ldots, Y_n$ *which are indiscernible sets in* N *with* Y_i *based on some* q_i *such that* $M = \langle Y_1 \cup \cdots \cup Y_n \rangle$.

Here and below $\langle A \rangle$ denotes the subalgebra generated by A. Definitions of such notions as "weight one" appear in [3].

With the following exceptions our notation follows that of [3]. References of the form III.x.y are to [3]. For typographical convenience letters such as $a, b, c, (x, y, z)$ range over finite sequences of elements (variables) and we place a bar over such a letter only when we want to emphasize the distinction between a single element and a sequence when that distinction is important. Similarly, we have blurred the distinction [Defn, III, 12] between $t(A; B)$ and $t_*(A; B)$.

We say $p \in S(M)$ is based on $A \subseteq M$ if p does not fork (d.n.f.) over A and $p \mid A$ is stationary. We frequently construct sequences by induction; e.g. $\{a_i : i < k\}$. In such case we denote by the capital letter subscript i, the first i elements of the sequence e.g. $A_i = \{a_j : j < i\}$.

Roughly the argument goes like this. We first recall from [1] that if A and B are subsets of a free algebra M with $|A| > \aleph_0$ and $|B| = \aleph_0$ then A contains an uncountable set of indiscernibles over B. If M is also $\aleph_1$-saturated this easily yields that $\mathrm{Th}(M)$ is ω-stable. Now let M_0 be free on countably many generators and let M_0^1 be free on one additional generator y. Note $M \equiv M_0 \equiv M_0^1$. Now in any superstable theory we can decompose $t(y, M_0)$ into a finite number of regular types. In our situation, we can show that M_0^1 is in fact the $F^a_{\aleph_0}$-saturated model containing $M_0 \cup y$. (Since after all $M_0^1 \approx M_0$.) Since one sufficiently saturated model of Th (M_0) is finite dimensional so is Th (M_0). At this stage (after Lemma 5) we have represented M_0^1 as the model $F^a_{\aleph_0}$-prime over $z_0, \ldots, z_{n-1}$ where the z_i realize regular types. In the remainder of the proof we replace the z_i's by y_i's such that the y_i's realize weight one types and y is in the algebra generated by $M_0 \cup \{y_0, \ldots, y_{n-1}\}$.

For simplicity, the results here will be presented for a countable language. In the case of an uncountable language we would have to replace ω-stability below by superstability but that would be a minor change for the arguments here.

We state all our results for free algebras. In fact, the arguments apply to any structure M which is generated (as an algebra) by a set of indiscernibles. A slight further improvement could be obtained by replacing "free" by M is the algebraic closure (in sense of [3]) of a set of indiscernibles.

Here are two examples of "nice" situations. Example 4 below shows that example 2 is more complicated than it first appears to be.

EXAMPLE 1. The language L has a single n ary function symbol f and M is free for the variety given by $f^n(x) = x$. Then the free algebra on $\aleph_1$-generators is a disjoint union of n-element cycles. M is $\aleph_1$-saturated and Th (M) is $\aleph_1$-categorical and $\aleph_0$-categorical.

EXAMPLE 2. The language L has a binary function symbol $+$ and constant symbol 0, V is the variety of Abelian groups of exponent n. If n is a prime power Th (M) is categorical in all powers. If n is a composite M is a direct product of a finite number of groups each of which is free on $\aleph_1$-generators for an $\aleph_1$ and $\aleph_0$-categorical variety of Abelian groups given by an equation $p^k x = 0$.

These examples are overly simple in several respects. In particular, in these cases we decompose not only the algebra M into more managable algebras but in effect we also decompose the variety V. Our theorem, however, only refers to the elementary theory of M. Examples 3 and 5 show that such a restricted conclusion is necessary.

To prevent the repetition of hypotheses and notational conventions we will list all such as H_1, H_2 etc. Once formulated, such hypotheses hold until the end of the proof of the main theorem.

H_0. V is a variety in a language with countable similarity type.

H_1. M is a free algebra on the generating set $Y = \{y_i : i < \aleph_1\}$ for the variety V. M is $\aleph_1$-saturated.

LEMMA 1. *Th (M) is ω-stable.*

Proof. Let B be a countable subset of M and let $A = \langle a_i : i < \kappa \rangle$ realize the distinct 1-types over B which are realized in M. If $\kappa > \aleph_0$, by [1] A contains an uncountable subset which is indiscernible over B. This, of course, is impossible so only countably many types over an countable subset of M are realized in M. Since M is $\aleph_1$-saturated this implies Th (M) is ω-stable.

No such conclusion can be drawn about the variety V. Here is an example of a variety V and a free algebra M which is saturated but V is not even stable.

EXAMPLE 3. Fix an uncountable cardinal α. Let the universe of M be $\alpha \cup \alpha^{[2]}$. There will be a single binary function g

$$g(a, b) = \begin{cases} \langle \beta, \alpha \rangle & \text{if } a = \alpha \text{ and } b = \beta \text{ and } \beta < \alpha \\ \langle \alpha, \beta \rangle & \text{if } a = \alpha \text{ and } b = \beta \text{ and } \alpha < \beta \\ 0 & \text{otherwise.} \end{cases}$$

$$\alpha^{[2]} = \{\langle \beta, \gamma \rangle : \beta < \gamma < \alpha\}$$

Now, M is free on $\{\beta : 0 < \beta < \alpha\}$ and M is $\aleph_1$-saturated if $\alpha > \aleph_1$. To see this note that Th (M) can be axiomatized as follows. (We indicate g by juxtaposition.)

$(M, \cdot)$ is a commutative semigroup with operation g.
Every product of 3 elements is zero.
Each element x is either an annihilator ($\forall y\, xy = 0$) or a creator ($\exists y\, x \cdot y \neq 0$).
The product of any two creators is an annihilator.
There is a 1–1 correspondence between annihilators z and pairs of creators by given by $x \cdot y = z$.

Clearly these axioms are categorical in all powers (since a model is determined by the cardinality of its set of creators).

But V is not stable since M does not satisfy the term condition (cf. [2]). An algebra M satisfies the term condition if for every term $t(u, v) \forall x\, \forall y\, (\exists z (t(x, z) = t(y, z)) \rightarrow \forall w\, (t(x, w) = t(y, w)))$. It is shown in [2] that if V is stable all algebras in V satisfy the term condition. But if a, b, c are creators and d is an annihilator $a \cdot d = b \cdot d = 0$ but $a \cdot c \neq b \cdot c$.

$H_2 : M_0$ is the subalgebra of M free on $Y_0 = \{y_1 : i < \omega\}$.

LEMMA 2. *M_0 is $F^a_{\aleph_0}$-saturated.*

Proof. It is routine by a Tarski–Vaught argument to show M_0 is an $\aleph_0$-saturated elementary submodel of M. Since $\aleph_0$-saturation is the same as $F^a_{\aleph_0}$-saturation for ω-stable theories we have the result.

LEMMA 3. *If $p \in S(M_0)$ there is a finite subset $A \subseteq M_0$ and an infinite set $W \subseteq M_0$ such that W is indiscernible over A and $Av(W, M_0) = p$.*

Proof. This holds in any $F^a_{\aleph_0}$-saturated model.

H_3. y realizes $Av(Y, M_0)$. $M^1 = \langle M \cup \{y\} \rangle$.
$M_0^1 = \langle M_0 \cup \{y\} \rangle$. $p = t(y; M_0)$ is based on A and $|A| < \aleph_0$.

LEMMA 4. *If $W \subseteq M_0$ is indiscernible over A and based on A and $q = Av(W, M_0)$ is regular then q is realized in M_0^1.*

Proof. Certainly, q is realized in M so for some n, q is realized in $M_0^{(n)} = \langle M_0 \cup Y^n \rangle$ where $Y^n = \{y_{i_0}, \ldots, y_{i_{n-1}}\} \subseteq Y - Y_0$. Note that the isomorphism type of $M_0^{(n)}$ over M_0 does not depend on the choice of the y_{i_j}. Choose n minimal such that q is realized in $M_0^{(n)}$ and let w be such a realization. If, for some $a \in Y^n$, $t(w; M \cup \{a\})$ does not fork over M, let $N = \langle M \cup \{a\} \rangle$. Then there is an isomorphism f from M_0 onto N which fixes A. By the minimality of n, $f(q)$ should not be realized in $N^{(n-1)} = M_0^n$. But, since $q \restriction A$ is stationary and $t(w, N)$ does not fork over A, w realizes $f(q)$. Thus for each $a \in Y^n$, $t(a; M_0 \cup \{w\})$ forks over M_0. If $n > 1$ this contradicts V.3.1, since q is regular; so we have proved the lemma.

Note that Lemma 4 depends essentially on the hypothesis that q is regular. For, if in Example 3 we let W be the set of annihilators in M_0 and let $q = Av(w, M_0)$, q is not realized in M_0^1 but only in $M_0^{(2)}$.

We thank Anand Pillay for pointing out an error in the original proof of Lemma 4.

H_4. $Z = \{z_i : i < \mu\} \subseteq M_0^1$. $q_i = t(z_i, M_0)$ is based on $A_i \subseteq M_0$ with $|A_i| < \omega$. Each q_i is regular and $i \neq j$ implies $q_i = q_j$. Moreover, Z is a maximal independent subset of M_0^1 satisfying these conditions.

Note that $|Z|$ may well be greater than μ.

LEMMA 5. *Z is finite.*

Proof. By Lemma 4 each $q_i \leq_s p$ (cf. Defn. V.2.1). Thus $|Z|$ is less than the weight of p is less than $\aleph_0$ (Defn. V.3.2. Thm. V.3.9).

H_5. $|Z| = n$. Each $z_i \in Z$ realizes $Av(Z_i, M_0)$ where Z_i is an infinite set of indiscernibles. $q_i = Av(Z_i; M_0)$ is based on the finite subset A of M_0. $q_1, \ldots q_k$ (where $k \leq n$) are the distinct regular types realized in Z.

We have established that Th (M) is "finite dimensional". That is each model of Th (M) is prime (actually $F^a_{\aleph_0}$-prime; but by Lemma 4, we have prime) over a finite sequence of indiscernible sets and the cardinalities of those indiscernible sets determine the model. In the remainder of the paper we make this information more precise. We replace "N is prime over $Y_1 \cup \cdots \cup Y_n$" by "$N = \langle Y_1 \cup \cdots \cup Y_n \rangle$". The price for this is to weaken "$Av(Y_i)$ is regular" to "$Av(Y_i)$ has weight one".

The distinction between these notions is clarified by the following example.

EXAMPLE 4. Let G be the direct sum of ω copies of Z_4. As a vector space over the field with 4 elements G has a basis of elements $A = \{a_i : i < \omega\}$. Let b_i denote $2a_i$. Let B be $\{b_i : i < \omega\}$. Now $G = \langle A \rangle$ and G is prime over B. Moreover $Av(B, M)$ is regular (since $2x = 0$ is a strongly minimal formula [cf. V.1.18]). But $Av(A, M)$ has weight one and is not regular. (It must have weight one since $\mathrm{Th}(G)$ is $\aleph_1$-categorical. To see it is not regular, consider elements a, b, c; $4a = 4b = 4c = 0$; $2a = 2b$; $2c = a \quad b$ and the characterization V.1.9(2) of regularity.)

In order to establish the theorem, we must replace the collection $q_1, \ldots, q_k$ of regular types which are realized in Z, by types of weight one so that if $y_0, \ldots, y_{n-1}$ realize these types then $y \in \langle M \cup \{y_0, \ldots, y_{n-1}\} \rangle$.

In the next lemma we show how the new types will be chosen. Then we prove a general result about forking which will be needed in the construction. Then we construct $y_0, \ldots, y_{n-1}$ so that $y \in \langle M \cup \{y_0, \ldots, y_{n-1}\} \rangle$. (More precisely, $y \in \langle \bar{M} \cup \{y_0, \ldots, y_{n-1}\} \rangle$ where $\bar{M}$ is chosen so that $t(y; \bar{M})$ is parallel to $t(y; M)$.) Then we construct an M^{**} so that $y \in \langle M^{**} \cup \{y_0, \ldots, y_{n-1}\} \rangle$, $t(y; M^{**})$ is parallel to $t(y; M)$ and $t(y_i; M^{**})$ has weight one. This establishes the theorem.

LEMMA 6. *Suppose $t(z, M)$ is regular. Fix a type $p \in S(M)$ and y realizing p. If M' is chosen so that $R(t(y; M'))$ is minimal among all ranks of $t(y; N)$ such that $N \supseteq M$ and $t(z; N)$* d.n.f. *over M then $wt(t(y; M')) = 1$.*

Proof. Let M^* be $F^a_{\aleph_0}$-prime over $M' \cup \{y\}$ and suppose for contradiction there exist $w_0, w_1 \in M^*$ which are independent over M' and realize regular types over M^*. Then by V.3.1, since $t(z; M')$ is $\|$ to $t(z; M)$ (and hence regular), one of $t(z; M' \cup \{w_0\})$, $t(z; M' \cup \{w_1\})$ d.n.f. over M', say $t(z; M' \cup \{w_0\})$. By transitivity, $t(z; M' \cup \{w_0\})$ d.n.f. over M. Then there exists an $M'' \supseteq M' \cup \{w_0\}$ such that $t(M''; M' \cup \{z\})$ d.n.f. over M. Thus M'' is one of the N minimized over in the definition of M'. But $w_0 \in M^*$ and M^* is $F^a_{\aleph_0}$-prime over $M' \cup \{y\}$ so $t(w_0; M' \cup \{y\})$ forks over M'. By symmetry and monotonicity $t(y; M'')$ forks over M' so $R(t(y; M'')) < R(t(y; M'))$ contrary to the choice of M'.

We pause now to prove a general lemma about forking which will be used twice in this paper. Note that although we state the lemma for finite sequences it holds for arbitrary sets. (We use this fact in the second application.)

This lemma could also be proved by regarding non-forking as an isolation relation ($F^f_{k(T)}$) and applying IV.3.3. It does not rely on any of the continuing hypotheses H_i.

LEMMA 7. *Let* $\langle a_i : i<n\rangle$ *and* $\langle b_i : i<n\rangle$ *satisfy:*

(i) $\{a_i : i<n\}$ is independent over M

(ii) $t(b_i; M\cup A_n\cup B_i)$ d.n.f. *over* $M\cup\{a_i\}$ *then* $\{a_ib_i; i<n\}$ *is independent over* M.

Proof. We first show

(∗) For each i $t(a_i; M\cup A_i\cup B_i)$ d.n.f. over M. For this, we show by induction on $m\le i$ that $t(a_i; M\cup A_i\cup B_m)$ d.n.f. over M. If $m=0$, this is just the assertion that $\{a_i : i<n\}$ is independent over M. Suppose $m=k+1$ and $t(a_i; M\cup A_i\cup B_k)$ d.n.f. over M. Now by (ii) $t(b_k; M\cup A_{i+1}\cup B_k)$ d.n.f. over $M\cup\{a_k\}$. So by symmetry $t(a_i; M\cup A_i\cup B_{k+1})$ d.n.f. over $M\cup\{a_k\}$. We can apply symmetry since, by induction, $t(a_i; M\cup A_i\cup B_k)$ d.n.f. over M (and by monotonicity not over $M\cup\{a_k\}$. But $t(a_i; M\cup\{a_k\})$ d.n.f. over M since the a_i are independent. Hence, by transitivity, $t(a_i; M\cup A_i\cup B_{k+1})$ d.n.f. over M.

Now we show that for each m, $t(a_mb_m; M\cup A_m\cup B_m)$ d.n.f. over M. We have $t(a_m; M\cup A_m\cup B_m)$ d.n.f. over M. By (ii) $t(b_m; M\cup A_{m+1}\cup B_m)$ d.n.f. over $M\cup\{a_m\}$. Thus by II.4.15(1) $t(a_m-b_m; M\cup A_m\cup B_m)$ d.n.f. over M.

We return to the main line of the argument, and construct $\bar{M}$.

Construction 8. Choose $\langle y_i : i<|M|^+\rangle$ as follows.

(i) For $i<n$

(a) $t(y_i; M\cup\{z_i\})=t(y : M\cup\{z_i\})$

(b) $t(y_i; M\cup\{z_i,\ldots,z_{n-1}\}\cup Y_i)$ d.n.f. over $M\cup\{z_i\}$.

(ii) For $n\le i$

(a) $t(y_i; M)=t(y; M)$

(b) $t(y_i; M\cup Y_i)$ d.n.f. over M.

CLAIM 9. $\{y_i : i<|M|^+\}$ is independent over M.

Proof of Claim 9. By (ii) of the construction, it suffices to show that for $m<n$ $t(y_m; M\cup Y_m)$ d.n.f. over M. But this follows from (Lemma 7 and monotonicity) taking z_i for a_i and y_i for b_i.

LEMMA 10. *There exists a model* $\bar{M}$ *such that*

(i) $y\in\langle\bar{M}\cup\{y_0,\ldots,y_{n-1}\}\rangle$

(ii) $t(y;\bar{M})$ d.n.f. *over* M.

(iii) $t(y_0,\ldots,y_{n-1};\bar{M})$ d.n.f. *over* M

Proof. Let M^+ be $\langle M \cup \{y_i : i < |M|^+\}\rangle$. Then M^+ is $|M|^+$ saturated (since it is free) so without loss of generality we can imbed $M \cup \{z_0, \ldots, z_{n-1}\}$ into M^+ by a map which fixes M. Then for some $y_n, \ldots, y_{n+m}$ (possibly reordering) $y \in \langle M \cup y_0, \ldots, y_{n+m}\rangle$. By claim 9, $\{y_i : i < |M|^+\}$ is independent over M. Thus by V.3.15, for some set $Y' \subseteq Y$ with $|Y'| \leq n = wt(t(y; M))$, $Y - Y'$ is independent over $M \cup \{y\}$. But each y_i for $i < n$ is in Y'. For, $t(z_i; M \cup y_i)$ forks over M and $t(z_i; M \cup y)$ forks over M (since $z_i \in \langle M \cup \{y\}\rangle$, $z_i \in \langle M \cup \{y_i\}\rangle$) and $t(z_t; M)$ is regular so by V.3.2 $t(y_i; M \cup \{y\})$ forks over M.

Choose $\bar{M}$ prime over $M \cup \{y_n, \ldots, y_{n+m}\}$. Then certainly $y \in \langle \bar{M} \cup \{y_0, \ldots, y_{n-1}\}\rangle$, $t(y; \bar{M})$ d.n.f. over M and for $i < n$ $t(y_i; \bar{M})$ d.n.f. over M. (The non-forking follows from the previous paragraph and V.3.2.(1)i).)

Now we construct M^{**}. First we construct for each i an M_i'' such that M_i'' is to $\{y_i\}$ as the M' constructed in Lemma 6 is to y and such that the M_i'' are as independent as possible. Then we find an M^{**} meeting the required conditions.

CONSTRUCTION 11. Choose $\langle M_i'' : i < n\rangle$ so that

(i) $t(M_i''; M \cup \{y_i, z_i\}) = g_i(t(M_i'; M \cup \{y, z_i\})$ (where g_i is a mapping fixing $M \cup \{z_i\}$ and taking y to y_i. Such a map exists by the definition of y_i).

(ii) $t(M_i''; M \cup \bigcup \{M_j'' : j \neq i\} \cup \{y_0, \ldots, y_{n-1}\} \cup \{z_0, \ldots, z_{n-1}\} \cup \{y\})$ d.n.f. over $M \cup \{y_i, z_i\}$.

CLAIM 12. $\{z_0, \ldots, z_{n-1}, M_0'', \ldots, M_{n-1}''\}$ is independent over M.

Proof of Claim. By Lemma 1 taking $\{y_i, z_i\}$ for a_i and M_i'' for b_i,

$$(*) \quad t\left(\{y_i, z_i\} \cup M_i''; M \cup \bigcup_{j<i} M_j'' \cup Y_i \cup Z_i\right) \text{ d.n.f. over } M.$$

So $t(\{y_i, z_i\}; M \cup \bigcup_{j \leq i} M_j'' \cup Y_i \cup Z_i)$ d.n.f. over $M_i'' \cup M = M_i''$. But $t(z_i; M_i'')$ d.n.f. over M (by (i) of construction 11), so by transitivity and montonicity $t(z_i; M \cup \bigcup_{j \leq i} M_j'' \cup Y_i \cup Z_i)$ d.n.f. over M. By (*) $t(M_i''; \cup \bigcup_{j<i} M_j'' \cup Y_i \cup Z_i)$ d.n.f. over M. This establishes the claim.

LEMMA 13. *There exists an* M^{**} *such that*

(i) $t(y; M^{**})$ d.n.f. *over* M

(ii) $y \in \langle M^{**} \cup \{y_0, \ldots, y_{n-1}\}\rangle$

(iii) $t(y_i; M^{**})$ *has weight* 1.

(iv) $\{y_0, \ldots, y_{n-1}\}$ *are independent over* M^{**}.

Proof. Let M^{**} be prime over $\bigcup \{M_i'' : i < n\}$. Now $W = \{z_0, \ldots, z_{n-1}, M_0'', \ldots, M_{n-1}''\}$ are independent over M so again by V.3.15 there exists a $W' \subseteq W$

with $|W'| \leq n = wt(t(y; M))$ so that $(W - W') \cup \{y\}$ is independent over M. But $t(z_i; M \cup \{y\})$ forks over M for $i < n$ so $\{M''_0, \ldots, M''_{n-1}\}$ is independent over M. Thus by V.3.2 $t(y; M^{**})$ d.n.f. over M. Similarly (iv) holds. (ii) is immediate from Lemma 10(i) since $M^{**} \supseteq M$. Finally $t(y_i; M^{**})$ has weight 1 since $t(y_i; M''_i)$ has weight 1 by Lemma 7 and $t(y_i; M^{**})$ d.n.f. over M''_i by V.3.2 (and monotonicity).

This establishes the theorem.

In order to get a wholly algebraic conclusion to the main theorem, we would have to be able to investigate more fully the structure of the subalgebras $\langle Y_i \rangle$ for $i < n$. The first hope is that M_{y_i} is also saturated and so we could continue by applying our earlier arguments to M_{y_i}. This hope is dashed by the following variant on Example 2.

EXAMPLE 4. Let L contain $+$, 0 and infinitely many constant symbols $\langle c_i : i < \omega \rangle$. Let V be the variety of Abelian groups of exponent 6 with no axioms on the c_i. Then a free algebra of power $\aleph_1$ for V is just the direct sum of $\aleph_1$-copies of z_6 with the c_i naming independent elements of order 6. Now as in Example 2 if $X(Y)$ is a set of independent elements of order 2(3), $M \approx M_X \times M_y$. But M_y is not $\aleph_1$-saturated since it contains only countably many elements of order 6.

Note also that while in Examples 1 and 2, the Y_i generate relatively free algebras, this fails in this example.

REFERENCES

[1] J. T. BALDWIN, J. BERMAN, A. M. W. GLASS and W. HODGES, A combinatorial fact about free algebras, (to appear).
[2] J. T. BALDWIN and R. MCKENZIE, Counting models in universal Horn classes.
[3] S. SHELAH, *Classification Theory and the Number of Non-Isomorphic Models*, North Holland, Amsterdam (1978).

Univ. of Illinois at Chicago.

Hebrew University of Jerusalem
Mount Scopus
Jerusalem