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All generating pairs of all two-generator Fuchsian groups

By

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Introduction. The problem of when two transformations A, B in $\mathrm{PSL}(2, \mathbb{R})$ generate a Fuchsian group was thought to have been solved completely in the paper [14] of Purzitsky, when taken together with the results in [6], [12], [13], [15] and [16].

However, in [9] Matelski pointed out an error in the main statements of [14] and showed geometrically that there is just one more possibility for two hyperbolic elements with axes intersecting in exactly one point to generate a (2, 3, 7)-triangle group $\langle A, B \mid A^2 = B^3 = (AB)^7 = 1 \rangle$.

He gave an algorithmic approach for deciding the discreteness of two-generator subgroups of $\mathrm{PSL}(2, \mathbb{R})$. His approach is not very effective and does not help very much for the original problem. Especially, Matelski did not describe all generating pairs of all two-generator Fuchsian groups and also did not present criteria for deciding the discreteness of all two-generator subgroups of $\mathrm{PSL}(2, \mathbb{R})$.

In this paper we give a complete correction of the paper [14], describe all generating pairs of all two-generator Fuchsian groups and present a short algebraic algorithm for deciding the discreteness of all two-generator subgroups of $\mathrm{PSL}(2, \mathbb{R})$. We think that this is necessary for three reasons. First the results of Purzitsky in [14] are used and mentioned in several papers, for instance in [1], [5], [7], [11] and [17]. Second these descriptions and criteria are important for some unanswered questions about the Nielsen-Thurston theory as I learned from the talk of Jane Gilman at the Alta conference on combinatorial groups theory and very low dimensional topology 1984. Third, last but not least, it is good to have a final version because the partial results are strewed about several papers.

1. Preliminary remarks. In this paper we use the terminology and notation of [1], [8] and [19]; here $\langle \cdots \mid \cdots \rangle$ indicates a description of a group in terms of generators and relations. By $\langle A, B \rangle$ we denote the group generated by A and B.

We write $\{A, B\} \overset{N}{\sim} \{U, V\}$ if there is a free (Nielsen)-transformation from $\{A, B\}$ to $\{U, V\}$ and call $\{U, V\}$ Nielsen-equivalent to $\{A, B\}$.

Let $G = \langle A, B \rangle$ be a group such that A has finite order $n \geqq 2$. We call a transformation $\{A, B\} \mapsto \{A^m, B\}$ with $1 \leqq m < n$ and $(m, n) = 1$ an E-transformation (here (m, n) denotes the greatest common divisor of m and n).

By an extended free transformation from $\{A, B\}$ to a pair $\{U, V\}$ we mean each finite sequence of free transformations and E-transformations. We write $\{A, B\} \overset{eN}{\sim} \{U, V\}$ if there is an extended free transformation from $\{A, B\}$ to $\{U, V\}$. It should be remarked that $\langle A, B\rangle = \langle U, V\rangle$ if $\{A, B\} \overset{eN}{\sim} \{U, V\}$. Frequently we obtain from a pair $\{A, B\}$ a new one by extended free transformations and then denote the latter by the same symbols.

We write $[A, B] .= ABA^{-1}B^{-1}$ for the commutator of $A, B \in G$ (G a group) and $\operatorname{tr} A$ for the trace of $A \in \mathrm{SL}(2, \mathbb{R})$; also E denotes the unit matrix in $\mathrm{SL}(2, \mathbb{R})$. We identify $\mathrm{PSL}(2, \mathbb{R})$ with the group of all automorphisms of the upper half plane $\mathfrak{H}$. It is $\mathrm{PSL}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})/\{E, -E\}$; that means $\mathrm{PSL}(2, \mathbb{R})$ consists of the pairs $\{W, -W\}$, $W \in \mathrm{SL}(2, \mathbb{R})$. There is no space for confusion if we abbreviate W for $\{W, -W\}$ and do not distinguish between W and $\{W, -W\}$. We name an element $W \in \mathrm{PSL}(2, \mathbb{R})$, $W \neq E$, hyperbolic if $|\operatorname{tr} W| > 2$, parabolic if $|\operatorname{tr} W| = 2$ and elliptic if $|\operatorname{tr} W| < 2$.

Now let G be a subgroup of $\mathrm{PSL}(2, \mathbb{R})$. We say that G is elementary if the commutator of any two elements of infinite order has trace 2; equivalently, G is elementary if any two elements of infinite order have at least one common fixed point. G is said to be discrete if it does not contain any convergent sequence of distinct elements.

A Fuchsian group is a non-elementary discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$.

Lemma 1 (cf. [10] and [14]). *A two-generator Fuchsian group G has one and only one of the following descriptions in terms of generators and relations:*

(1.1): $G = \langle A, B; \rangle$, *that means G is a free group of rank two.*

(1.2): $G = \langle A, B \mid A^p = 1\rangle$ *for* $2 \leqq p$.

(1.3): $G = \langle A, B \mid A^p = B^q = 1\rangle$ *for* $2 \leqq p \leqq q$ *and* $p + q \geqq 5$.

(1.4): $G = \langle A, B \mid A^p = B^q = (AB)^r = 1\rangle$ *for* $2 \leqq p \leqq q \leqq r$ *and* $\dfrac{1}{p} + \dfrac{1}{q} + \dfrac{1}{r} < 1$.

(1.5): $G = \langle A, B \mid [A, B]^p = 1\rangle$ *for* $2 \leqq p$.

(1.6): $G = \langle A, B, C \mid A^2 = B^2 = C^2 = (ABC)^p = 1\rangle$ *for* $p = 2k + 1$, $k \geqq 1$.

Remark. In (1.6) is $G = \langle AB, CA\rangle$ and $[AB, CA] = (ABC)^2$.

A group G of type (1.1), (1.2) or (1.3) is a free product of the two cyclic groups $\langle A\rangle$ and $\langle B\rangle$. We call a group of type (1.4) a (p, q, r)-triangle group; in some sense the triangle groups are the most complicated two generator Fuchsian groups.

In the following we use the notation of Lemma 1 and refer to the presentations and canonical generators in Lemma 1.

2. Generating pairs of two-generator Fuchsian groups and discreteness criteria. In [9] Matelski showed geometrically that there exists a generating pair $\{U, V\}$ of the $(2, 3, 7)$-triangle group with $[U, V] = (AB)^4$; that means especially the order of $[U, V]$ is 7 and $[U, V]$ rotates by $\dfrac{8\pi}{7}$. This possibility was omitted in [14]. Matelski did not describe U, V as words in A, B. He also did not answer the question when two elements U, V of $\mathrm{PSL}(2, \mathbb{R})$ with $[U, V]^7 = 1$ generate a $(2, 3, 7)$-triangle group. Therefore first we want to handle with these two problems.

Lemma 2. *Let* $G = \langle A, B \mid A^2 = B^3 = (AB)^7 = 1\rangle$ *be a* $(2, 3, 7)$-*triangle group. Let* $U .= AB^2ABAB^2AB^2AB$ *and* $V .= B^2ABAB^2ABABA$.

Then the following hold.

a) $[U, V] = (AB)^4$.
b) *U and V generate G.*
c) *If $\{R, S\}$ is any generating pair of G such that $[R, S]$ is conjugate to $(AB)^{4\varepsilon}$, $\varepsilon = \pm 1$, then $\{R, S\} \overset{N}{\sim} \{U, V\}$.*

Proof. It is a straightforward calculation that $[U, V] = (AB)^4$ by using the relations $A^2 = B^3 = (AB)^7 = 1$.

Let $H = \langle U, V \rangle$ be the subgroup of G generated by U and V. Because $[U, V] = (AB)^4$ we have $AB \in H$. Therefore $(AB)^{-2} UV = BAB^2 \in H$, $V \cdot AB \cdot BAB^2 = B^2ABAB^2A \in H$ and $AB \cdot B^2ABAB^2A \cdot AB = BA \in H$. Finally $BA \cdot AB = B^2 \in H$, $B \in H$ and $AB \cdot B^2 = A \in H$; that means $H = G$.

Now let $\{R, S\}$ be a generating pair of G such that $[R, S]$ is conjugate to $(AB)^{4\varepsilon}$, $\varepsilon = \pm 1$. Since two $(2, 3, 7)$-triangle groups are conjugate in $\mathrm{PGL}(2, \mathbb{R})$ and since a $(2, 3, 7)$-triangle group is not a proper subgroup of any other Fuchsian group, $\{R, S\}$ must be Nielsen-equivalent to $\{U, V\}$ (cf. also Lemma 1 of [5]). □

Lemma 3. *Let U, V be two elements of* $\mathrm{PSL}(2, \mathbb{R})$ *such that $0 \leqq \operatorname{tr} U$, $0 \leqq \operatorname{tr} V$ and $\operatorname{tr}[U, V] = \pm 2 \cos \frac{4\pi}{7}$.*

Assume that $G = \langle U, V \rangle$ does not contain an element of order 4. Then G is discrete if and only if $\operatorname{tr}[U, V] = -2 \cos \frac{4\pi}{7}$ and $\{U, V\}$ is Nielsen-equivalent to a pair $\{R, S\}$ which also satisfies $\operatorname{tr} R = \operatorname{tr} S$, $\operatorname{tr} RS = \operatorname{tr} R - 1$. Moreover, if G is discrete then G is a $(2, 3, 7)$-triangle group.

Remark. If $\{U, V\}$ is Nielsen-equivalent to $\{R, S\}$ then $\operatorname{tr}[U, V] = \operatorname{tr}[R, S]$.

Proof. From $0 \leqq \operatorname{tr} U$, $0 \leqq \operatorname{tr} V$ and $\operatorname{tr}[U, V] < 2$ we automatically get $2 < \operatorname{tr} U$, $2 < \operatorname{tr} V$ (cf. Lemma (2.2) of [4]); that means U and V are hyperbolic elements.

Let G be discrete. Then by [9] G is a $(2, 3, 7)$-triangle group $G = \langle A, B \mid A^2 = B^3 = (AB)^7 = 1 \rangle$ because G does not contain an element of order 4 (cf. also [14]). Moreover, $[U, V]$ is conjugate to $(AB)^{4\varepsilon}$, $\varepsilon = \pm 1$, because $\operatorname{tr}[U, V] = \pm 2 \cos \frac{4\pi}{7}$. By Lemma 2 $\{U, V\}$ is Nielsen-equivalent to the pair $\{R, S\}$ with $R = AB^2ABAB^2AB^2AB$ and $S = B^2ABAB^2ABABA$.

Since $2 < \operatorname{tr} U$, $2 < \operatorname{tr} V$ we have also $2 < \operatorname{tr} R$, $2 < \operatorname{tr} S$ in the languages of traces (cf. [5]).

We remark that $R = AB^2ABS^{-1}B^2ABA$ and $RS = ABAB^2AB^2$. A straightforward calculation gives $\operatorname{tr} R = \lambda^2 = \operatorname{tr} S$ and $\operatorname{tr} RS = \lambda^2 - 1 = \operatorname{tr} R - 1$ where $\lambda = 2 \cos \frac{\pi}{7}$.

This leads to

$$\operatorname{tr}[R, S] = 2\lambda^4 + (\lambda^2 - 1)^2 - \lambda^4(\lambda^2 - 1) - 2 = \lambda^2 - \lambda - 1 = -2 \cos \frac{4\pi}{7}.$$

Now let $\operatorname{tr}[U, V] = -2\cos\frac{4\pi}{7}$ and $\operatorname{tr} U = \operatorname{tr} V = .x$, $\operatorname{tr} UV = \operatorname{tr} U - 1 = x - 1$ (this we may assume without any loss of generality (cf. for instance [5])). Then

$$\begin{aligned} 0 &= \operatorname{tr}[U, V] + 2\cos\frac{4\pi}{7} = 2x^2 + (x-1)^2 - x^2(x-1) - 2 + 2\cos\frac{4\pi}{7} \\ &= -x^3 + 4x^2 - 2x - 1 + 2\cos\frac{4\pi}{7}. \end{aligned}$$

The polynomial $f(x) = -x^3 + 4x^2 - 2x - 1 + 2\cos\frac{4\pi}{7}$ has 3 real zeros and only one of these zeros has an absolute value greater than 2, namely the zero λ^2 where again $\lambda = 2\cos\frac{\pi}{7}$. Therefore $\operatorname{tr} U = \operatorname{tr} V = \lambda^2$ and $\operatorname{tr} UV = \lambda^2 - 1$; that means G is a $(2, 3, 7)$-triangle group $G = \langle A, B \mid A^2 = B^3 = (AB)^7 = 1\rangle$ because each subgroup $\langle R, S\rangle$ of $\mathrm{PSL}(2, \mathbb{R})$ with $\operatorname{tr} R = \operatorname{tr} S = \lambda^2$ and $\operatorname{tr} RS = \lambda^2 - 1$ is conjugate in $\mathrm{PGL}(2, \mathbb{R})$ to $G = \langle U, V\rangle$. □

Remark. In Lemma 3 we obtain the pair $\{RS, R^{-1}\}$ from $\{U, V\}$ in a complete trace minimizing manner (cf. for instance Lemma 1 of [5]).

Now Lemma 2 and Lemma 3 when taken together with the results in [4], [5], [6], [9], [12], [13], [14], [15], [16] and [17] give the following final results (we use the notation of Lemma 1).

Theorem 1. *Let $G = \langle U, V\rangle$ be a two-generator Fuchsian group.*

(2.1): *If G is of type* (1.1) *or* (1.5) *then* $\{U, V\} \overset{N}{\sim} \{A, B\}$.

(2.2): *If G is of type* (1.6) *then* $\{U, V\} \overset{N}{\sim} \{AB, CA\}$.

(2.3): *If G is of type* (1.2) *or* (1.3) *then* $\{U, V\} \overset{eN}{\sim} \{A, B\}$.

(2.4): *If G is of type* (1.4) *then one (and only one) of the following cases holds:*

a) $\{U, V\} \overset{eN}{\sim} \{A, B\}$.
b) *G is a $(2, 3, r)$-triangle group with $(r, 6) = 1$ and* $\{U, V\} \overset{N}{\sim} \{ABAB^2, B^2ABA\}$.
c) *G is a $(2, 4, r)$-triangle group with $(r, 2) = 1$ and* $\{U, V\} \overset{N}{\sim} \{AB^2, B^3AB^3\}$.
d) *G is a $(3, 3, r)$-triangle group with $(r, 3) = 1$ and* $\{U, V\} \overset{N}{\sim} \{AB^2, B^2A\}$.
e) *G is a $(2, 3, 7)$-triangle group and*
$\{U, V\} \overset{N}{\sim} \{AB^2ABAB^2AB^2AB, B^2ABAB^2ABABA\}$.

Remarks. 1. The results of [14] are used in the proof of Lemma 5 of [17]; that means Lemma 5 (and therefore also Lemma 6) contains one error for the $(2, 3, 7)$-triangle group; one has to add case (2.4e). The results of [17] are correct for the (p, q, r)-triangle groups which are not isomorphic to a $(2, 3, 7)$-triangle group; Theorem 1 of [17] naturally is also correct for the $(2, 3, 7)$-triangle group (one only has also to consider case (2.4e) in the proof) and in Theorem 4 of [17] one has to add case (2.4e). Theorem 4 of [17] (the situation 1 of this theorem) is generalized by Theorem 2 of [5]; one also has to add here case (2.4e).

2. If one is interested in the Nielsen-equivalence classes of generating pairs in (2.3) and (2.4a) one can get all Nielsen-equivalence classes of generating pairs – up to the obvious ones – by the following possibilities to generate (p, q, r)-triangle groups by two elements of finite order whose product is also of finite order (one can derive the completeness from [6]).

a) The $(2, 3, r)$-triangle group with $(r, 3) = 1$ can be generated by $U = ABABA$ and $V = B$ with $UV = (AB)^3$.

b) The $(2, 3, r)$-triangle group with $(r, 2) = 1$ can be generated by $U = A$ and $V = BAB$ with $UV = (AB)^2$.

c) The $(2, 3, r)$-triangle group with $(r, 2) = 1$ can be generated by $U = (AB)^{-1}$ and $V = BABABAB^2$ with $UV = B^2(AB)^4B$.

d) The $(2, 3, 7)$-triangle group can be generated by $U = ABAB^2ABA$ and $V = B$ with $UV = ABAB^2ABAB = ABAB^2ABAB(AB)^{-7} = ABABAB^2(AB)BAB^2AB^2A$.

e) The $(2, p, q)$-triangle group with $(2, q) = 1$ can be generated by $U = ABA$ and $V = B$ with $UV = (AB)^2$.

f) The $(2, p, q)$-triangle group with $(2, p) = 1$ can be generated by $U = BA$ and $V = AB$ with $UV = B^2$. □

Theorem 2. *Let $U, V \in \mathrm{PSL}(2, \mathbb{R})$ with $\mathrm{tr}\,[U, V] > 2$. Let $G = \langle U, V \rangle$ be non-elementary. Then G is discrete if and only if there is an extended free transformation from $\{U, V\}$ to a pair $\{R, S\}$ which satisfies (after a suitable choice of signs)*

$$
\begin{aligned}
&0 \leqq \mathrm{tr}\,R \leqq \mathrm{tr}\,S \leqq |\mathrm{tr}\,RS|,\\
&\mathrm{tr}\,R = 2\cos\frac{\pi}{p} \quad \text{or} \quad \geqq 2,\\
&\mathrm{tr}\,S = 2\cos\frac{\pi}{q} \quad \text{or} \quad \geqq 2 \quad \text{and}\\
&\mathrm{tr}\,RS = -2\cos\frac{\pi}{r} \quad \text{or} \quad \leqq -2
\end{aligned}
$$

where $p, q, r \in \mathbb{N}\backslash\{1\}$.

Moreover, if G is discrete then G is of type (1.1), (1.2), (1.3) *or* (1.4). □

Remark. In Theorem 2 we obtain the pair $\{R, S\}$ from $\{U, V\}$ in the following manner. Assume without any loss of generality that $0 \leqq \mathrm{tr}\,U$, $0 \leqq \mathrm{tr}\,V$. Then after applying free transformations in a trace minimizing manner we obtain a pair $\{P, Q\}$ with $0 \leqq \mathrm{tr}\,P \leqq \mathrm{tr}\,Q$ and $\mathrm{tr}\,PQ < 0$.

Now apply E-transformations as in [5] if necessary. Then (after a possible changing of signs) apply again free transformations in a trace minimizing manner and so on. This algorithm comes to the pair $\{R, S\}$ after finitely many steps (cf. also [18]).

Theorem 3. *Let* $U, V \in \mathrm{PSL}(2,\mathbb{R})$ *with* $0 \leqq \operatorname{tr} U$, $0 \leqq \operatorname{tr} V$ *and* $\operatorname{tr}[U,V] < 2$. *Let* $G = \langle U, V\rangle$. *Then G is discrete if and only if one of the following cases holds:*

(2.5): $\operatorname{tr}[U,V] \leqq -2$.

(2.6): $\operatorname{tr}[U,V] = -2\cos\frac{\pi}{p}$, $p \in \mathbb{N}\backslash\{1\}$.

(2.7): $\operatorname{tr}[U,V] = -2\cos\frac{2\pi}{p}$, $p \in \mathbb{N}\backslash\{1\}$, *with* $(p,2) = 1$.

(2.8): $\operatorname{tr}[U,V] = -2\cos\frac{6\pi}{r}$, $r \in \mathbb{N}$, $r \geqq 7$, *with* $(r,6) = 1$ *and* $\{U,V\}$ *is in a trace minimizing manner Nielsen-equivalent to a pair* $\{R,S\}$ *which satisfies* $\operatorname{tr} R = \operatorname{tr} S = \operatorname{tr} RS$.

(2.9): $\operatorname{tr}[U,V] = -2\cos\frac{4\pi}{r}$, $r \in \mathbb{N}$, $r \geqq 5$, *with* $(r,2) = 1$ *and* $\{U,V\}$ *is in a trace minimizing manner Nielsen-equivalent to a pair* $\{R,S\}$ *which satisfies* $\operatorname{tr} R = \operatorname{tr} S$ *and* $\operatorname{tr} RS = \frac{1}{2}(\operatorname{tr} R)^2$.

(2.10): $\operatorname{tr}[U,V] = -2\cos\frac{3\pi}{r}$, $r \in \mathbb{N}$, $r \geqq 4$, *with* $(r,3) = 1$ *and* $\{U,V\}$ *is in a trace minimizing manner Nielsen-equivalent to a pair* $\{R,S\}$ *which satisfies* $\operatorname{tr} R = \operatorname{tr} S = \operatorname{tr} RS$.

(2.11): $\operatorname{tr}[U,V] = -2\cos\frac{4\pi}{7}$ *and* $\{U,V\}$ *is in a trace minimizing manner Nielsen-equivalent to a pair* $\{R,S\}$ *which satisfies* $\operatorname{tr} S = \operatorname{tr} RS = \operatorname{tr} R + 1$.

Moreover, if G is discrete then G is of type (1.1) *in* (2.5), *of type* (1.5) *in* (2.6), *of type* (1.6) *in* (2.7), *a* $(2,3,r)$*-triangle group in* (2.8), *a* $(2,4,r)$*-triangle group in* (2.9), *a* $(3,3,r)$*-triangle group in* (2.10) *and a* $(2,3,7)$*-triangle group in* (2.11). □

Corollary. *Let* $G = \langle U, V\rangle \subset \mathrm{PSL}(2,\mathbb{R})$ *be a Fuchsian group. Then* $|\operatorname{tr}[U,V] - 2| \geqq 2 - 2\cos\frac{\pi}{7}$. □

Remark. In [18] this inequality was used to prove the following theorem (cf. also [2] and [3]; we mention that case (2.11) does not affect this inequality).

"Let G be a non-elementary subgroup of $\mathrm{PSL}(2,\mathbb{R})$. Then G is discrete if and only if each cyclic subgroup of G is discrete." But in fact this inequality is not necessary for the proof and it is very simple and straightforward to prove this theorem. Namely, let G be a non-elementary subgroup of $\mathrm{PSL}(2,\mathbb{R})$ with the property that each cyclic subgroup of G is discrete. Let U, V be any two elements of G with $\operatorname{tr}[U,V] \neq 2$ (such U, V exist because G is non-elementary). If $\operatorname{tr}[U,V] > 2$ and $\langle U, V\rangle$ is non-elementary then after applying extended free transformations we obtain easily $\operatorname{tr}[U,V] \geqq \left(2\cos\frac{\pi}{7}\right)^2 - 1$ (cf. the arguments in [18]). If $\operatorname{tr}[U,V] < 2$ then $\operatorname{tr}[U, VU^{-1}V^{-1}] > 2$. So G must be discrete.

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