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Two examples concerning hyperbolic quotients

By

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In this note we shall answer some questions raised by S. Kobayashi in connection with the notion of a hyperbolic quotient ([3], Problem A.8).

Let M be a complex variety (i.e. an irreducible reduced complex space) and denote by k_M its Kobayashi pseudodistance (cf. [2]). M is called *hyperbolic* (in the sense of Kobayashi), if k_M is a distance.

For compact M denote by R the equivalence relation $k_M^{-1}(0)$ and by $p: M \to M/R$ the projection onto the topological quotient. As holomorphic maps are distance-decreasing with respect to k, every holomorphic map $M \to N$ with hyperbolic N factors through p. On the other hand, there exists a holomorphic map $q: M \to M_h$ – the *hyperbolic quotient* of M – such that every holomorphic $M \to N$ with hyperbolic N factors through q.

Kobayashi's Problem A.8 stems from the imminent question whether $p = q$. In our first example we shall show that, in general, M/R does not admit a complex structure such that p becomes holomorphic. But strangely enough, even if p can be made holomorphic, M/R may still be far from hyperbolic as is demonstrated by a second example. This latter example also provides answers in the negative for two other questions in A.8, to wit:

(i) If k_M is not identically zero, must then M_h consist of more than one point?
(ii) Is k_M identically zero, if M carries the structure of a fibre space $M \to B$ with $k_B \equiv 0$ and $k_F \equiv 0$ for every fibre F?

Finally, Example 2 demonstrates the somewhat surprising fact that the hyperbolic quotients of locally isomorphic and – with respect to k – locally isometric compact complex manifolds need not even have the same dimension.

Whereas Example 2 does not require any background material beyond the basic facts about hyperbolic manifolds, Example 1 needs some preparational remarks.

For a complex manifold M with tangent bundle TM let $F_M: TM \to \mathbb{R}$ be defined by $F_M(\xi) := \inf\left\{\frac{1}{r}: r > 0 \text{ such that there exists a holomorphic } f: \Delta_r \to M \text{ with } f'(0) = \xi\right\}$, where $\Delta_r = \{z \in \mathbb{C}: |z| < r\}$ and $f'(0) = \mathrm{Df}(0)\left(\left(\frac{\partial}{\partial z}\right)_0\right)$. F_M is upper semicontinuous and k_M is the integrated form of F_M (see [4]).

For $x_0 \in M$, $\varepsilon > 0$ denote by $V_M(x_0, \varepsilon)$ the open ε-ball with respect to k_M around x_0 and let $r(\varepsilon) := \frac{e^{2\varepsilon} - 1}{e^{2\varepsilon} + 1}$ (i.e. $k_{\Delta_1}(0, r(\varepsilon)) = \varepsilon$).

Lemma 1. *Let* $V_j := V_M(x_0, r(j\varepsilon))$ *where* $j = 1, 2$, *and let* $r := r(\varepsilon)$.

Then $F_M | TV_1 \geqq r \cdot F_{V_2} | TV_1$ *and* $k_M | V_1 \times V_1 \geqq r \cdot k_{V_2} | V_1 \times V_1$.

Proof. Of course, the second part of the assertion follows immediately from the first one.

Let $\xi \in TV_1$ and let $f_n: \Delta_{r_n} \to M$, $n \in \mathbb{N}$, be holomorphic with $f_n'(0) = \xi$ and $\frac{1}{r_n} \to F_M(\xi)$ for $n \to \infty$. Then the sequence (g_n) with $g_n := f_n | \Delta_{r \cdot r_n}$ satisfies $g_n'(0) = \xi$ and $g_n(\Delta_{r \cdot r_n}) \subset V_2$, whence $\frac{1}{r} \cdot F_M(\xi) = \lim \frac{1}{rr_n} \geqq F_{V_2}(\xi)$.

Corollary 1. *Let* $\varphi: M \to N$ *be a holomorphic map from* M *into a hyperbolic manifold* N, *and let* $W \subset N$ *be open such that* $W' := \varphi^{-1}(W)$ *is hyperbolic.*

Then $k_M(x, x') > 0$ *for all* $(x, x') \in W' \times M$ *with* $x \neq x'$.

Proof. We may assume that $W' \neq \emptyset$. Let $x \in W'$ and let $\varepsilon > 0$ such that $V_N(\varphi(x), 2\varepsilon) \subset W$; then $V := V_M(x, 2\varepsilon) \subset W'$.

If $x' \in M \setminus V_M(x, \varepsilon)$, then $k_M(x, x') \geqq \varepsilon > 0$. Otherwise,

$$k_M(x, x') > r(\varepsilon) \cdot k_V(x, x') \geqq r(\varepsilon) \cdot k_{W'}(x, x') > 0.$$

Now assume that M is endowed with some hermitian metric $|\cdot|$. In [1], R. Brody proved the following

Lemma 2. *Let* $V \Subset M$ *be an open subset and let* $f_n: \Delta_{r_n} \to M$, $n \in \mathbb{N}$, *be holomorphic with* $r_n \to \infty$ *and* $f_n(\Delta_{r_n}) \subset V$ *such that* $(|f_n'(0)|)$ *does not tend to zero.*

Then there exists a non-constant holomorphic $f: \mathbb{C} \to M$ *with* $f(\mathbb{C}) \subset \bar{V}$.

Corollary 2. *Let* $\varphi: M \to N$ *be a proper holomorphic map between complex manifolds.*

If N *and all the fibres of* φ *are hyperbolic, then so is* M.

Proof. Evidently, there exists no non-constant holomorphic map $\mathbb{C} \to M$.

By Corollary 1, it suffices to show that every $\varphi^{-1}(V_N(y, \varepsilon))$, $y \in N$, $\varepsilon > 0$, is hyperbolic. Fix some $y \in N$ and $\varepsilon > 0$, let $V := \varphi^{-1}(V_N(y, \varepsilon))$, and assume that $\inf\{F_V(\xi): \xi \in TV, |\xi| = 1\} = 0$. Then $V \Subset M$ and there exist $f_n: \Delta_{r_n} \to V$, $n \in \mathbb{N}$, with $|f_n'(0)| = 1$ and $r_n \to \infty$. Therefore, by Lemma 2, there exists a non-constant holomorphic $f: \mathbb{C} \to M$, a contradiction.

Example 1. There exists a smooth compact complex surface X such that $p: X \to X/R$ cannot be made holomorphic.

Let $C \subset \mathbb{P}_2$ be given by the equation $x_0^4 + x_1^4 + x_2^4 = 0$, and let

$$X = \{(x, y) \in C \times \mathbb{P}_2: x_0 y_0^4 + x_1 y_1^4 + x_2 y_2^4 = 0\}.$$

C and the general fibre of $\pi: X \to C$, $(x, y) \to x$, are smooth curves of genus 3; X is smooth and the singular fibres $\pi^{-1}(s)$, where $s \in S := \{x \in C: x_0 x_1 x_2 = 0\}$, are unions of rational curves. Therefore, p factors through the identification map p' that collapses each of the singular fibres of π. Note that, on the other hand, every holomorphic map $X \to Y$ that factors through p' also factors through π; in particular, $\pi: X \to C$ is the hyperbolic quotient of X.

By Corollary 2, $\pi^{-1}(C \setminus S)$ is hyperbolic, and from Corollary 1 we infer $k_X(x, x') > 0$ for all $(x, x') \in \pi^{-1}(C \setminus S) \times X$ with $x \neq x'$. Thus p collapses at most the singular fibres of π and we conclude that $p = p'$ and hence cannot be made holomorphic.

Example 2. There exists a smooth compact complex surface X with the following properties:

a) k_X is not identically zero and $q: X \to X_h$ is constant.
b) $p: X \to X/R$ is holomorphic and $k_{X/R}$ is identically zero.
c) X admits a representation $X \to B$ as a fibre space with identically vanishing k_B and k_F for every fibre F.
d) There exists a finite unramified covering $X' \to X$ such that $\dim X'_h > 0$.

Let C be a hyperelliptic curve of genus $\geqq 2$ with involution σ and with corresponding quotient map $\alpha: C \to C/(\sigma) = \mathbb{P}_1$, and let T be a one-dimensional complex torus.

Fix some $t_0 \in T$ with $t_0 \neq 0 = 2t_0$ and denote by $\bar{\sigma}$ the involution $T \times C \to T \times C$ given by $(t, c) \to (t + t_0, \sigma(c))$. $\bar{\sigma}$ has no fixed points and hence gives rise to a two-sheeted unramified covering $\bar{\alpha}: T \times C \to X$ with a smooth surface X.

Evidently, the hyperbolic quotient of $T \times C$ coincides with the projection onto the second factor; thus, $\bar{\alpha}$ being locally isometric with respect to k, we conclude that $p: X \to X/R$ is given by $\bar{\alpha}(t, c) \to \alpha(c) \in \mathbb{P}_1$. Letting $(X' \to X) = \bar{\alpha}$ and $(X \to B) = p$, each of the postulated properties is verified.

References

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