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Resolutions of the prescribed volume form equation

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Abstract

For a given volume form fdx on a bounded regular domain Ω in $\mathbb{R}^n$, we are looking for a transformation u of Ω, keeping the boundary fixed and which sends the Lebesgue measure dx into fdx (i.e. we solve $\det(\nabla u) = f$). For f in various spaces, we propose two different constructions which ensure the existence of u with some gain of regularity. Our methods permit the recovery Dacorogna and Moser's results [4], but also, we prove the existence of such u in Hölder spaces for f in C^0, or even in L^∞.

1 Introduction

Let Ω be a regular bounded domain of $\mathbb{R}^n$ and let f be a positive function on Ω verifying the following hypothesis denoted by (H)

$$(H) \qquad \begin{cases} \exists\, c > 0 \text{ such that } f \geq c \text{ in } \Omega \\ \displaystyle\int_\Omega f\, dx = |\Omega|, \end{cases}$$

where $|\Omega|$ is the Lebesgue measure of Ω.

We are interested in finding an application u from $\overline{\Omega}$ into itself which transforms the Lebesgue measure $dx_1 \wedge \ldots \wedge dx_n$ into the measure $f(x)dx_1 \wedge \ldots \wedge dx_n$

and such that $u(x) = x$ on $\partial\Omega$. More precisely, we are looking for u satisfying

$$\begin{cases} u^{\#}(dx_1 \wedge \ldots \wedge dx_n) = f(x)dx_1 \wedge \ldots \wedge dx_n, \text{ in } \Omega \\ u(x) = x, \qquad \text{on } \partial\Omega \,; \end{cases} \tag{1.1}$$

this is equivalent to solving the following equation

$$\begin{cases} \det(\nabla u) = f, & \text{in } \Omega \\ u(x) = x, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

This equation, which can be called the prescribed volume form equation, has a lot of applications in physics, mainly in elasticity theory as for example, in the problem of equilibrium of gases (see [3]). On the other hand, this equation contains the construction of volume preserving mappings as indicated in [4].

The difficulties of solving (1.2) come, in particular, from the strong non-unique-ness and the strong non-linearity of the Jacobian determinant. One of the natural questions is how to find a solution u having the best regularity as possible for a given f. We observe that for any given $f \in C^\infty$, one can find $u \notin C^1$ solving (1.2), this illustrates the difficulties mentioned above.

In [4], Dacorogna and Moser have overcame those difficulties in the case where f belongs to the Hölder spaces $C^{k,\alpha}(\overline{\Omega})$ for $0 < \alpha < 1$ and $0 \leq k \leq \infty$. In fact, they proved the existence of u in $C^{k+1,\alpha}(\overline{\Omega})$ solving (1.2). Clearly, this is the best regularity that one can expect.

In [12], the optimal result, in the sense of regularity for u, has been obtained for f in $W^{m,p}(\overline{\Omega})$ with $\max(1, n/m) < p < \infty$.

For f in other spaces, the optimal regularity of u is not decided. For example, an interesting case would be to consider f in $C^k(\overline{\Omega})$ for k in $\mathbb{N}$, and to know whether one can expect to get a u in $C^{k+1}(\overline{\Omega})$ or at least in $C^{k,1}(\overline{\Omega})$.

Until now, the best regularity of u proved for f in $C^0(\overline{\Omega})$ was if u was a homeomorphism (see for example [8] or [4]), but the equation (2) is meaningless for such u and has to be replaced by a weaker form

$$\begin{cases} \forall\, E \text{ open set in } \Omega, & \displaystyle\int_E f\, dx = |u(E)|, \\ u(x) = x, \qquad \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

In the first part of our paper (section 2), we propose a constructive method based on dyadic decomposition of the domain Ω. This method, which permits the recovery of Oxtoby and Ulam's result for $f \in L^1(\Omega)$ in one hand, and gives on the other hand the existence of more regular solutions u of (1.3) under some control conditions on the oscillation of f.

Our main results in the first part are

Theorem 1 *Let $f \in C^0(\overline{\Omega})$ verifying (H), then there exists u solution of (1.3) such that u, $u^{-1} \in \cap_{\alpha<1} C^{0,\alpha}(\overline{\Omega})$.* ■

Theorem 2 *Let $f \in L^\infty(\Omega)$ verifying (H), then there exists u solution of (1.3) such that u, $u^{-1} \in C^{0,\beta}(\overline{\Omega})$ for any $\beta > 0$ satisfying*

$$\beta < \left[\min\left(\inf_\Omega f, \frac{1}{\sup_\Omega f}\right)\right]^c$$

where c is a positive constant independent of f. ■

Theorem 3 *Let f be in $BMO(\overline{\Omega})$ verifying (H), then there exists $u \in C^{0,\gamma}(\overline{\Omega})$ solution of (1.3) for any $\gamma > 0$ satisfying*

$$\gamma < c_1 \left(\frac{\inf_\Omega f}{1 + \|f\|_{BMO}}\right)^{c_2}$$

where c_1 and c_2 do not depend on f. ■

For f in $C^0(\overline{\Omega})$, one could expect to arrive at a $C^1(\overline{\Omega})$ or a $C^{0,1}(\overline{\Omega})$ solution of (1.3), but we have to add some stronger hypothesis on the oscillation of f for ensuring the existence of a Lipschitz solution. We have the following theorem:

Theorem 4 *Let f be a positive function on Ω verifying (H) and such that there exists an increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\int_0^1 \varphi(t) t^{-1} dt < +\infty$ for which*

$$\forall\, x \in \Omega, t > 0, \qquad \frac{1}{|\, \Omega_{x,t} \,|} \int_{\Omega_{x,t}} |\, f(y) - \overline{f}_{x,t} \,|\, dy \le \varphi(t), \tag{1.4}$$

where $\Omega_{x,t} = B_t(x) \cap \Omega$, $\overline{f}_{x,t} = \left(\int_{\Omega_{x,t}} f(y)dy\right) / |\, \Omega_{x,t} \,|$. Then there exists a Lipschitz homeomorphism solving (1.2). ■

Remark 1 *The previous hypothesis for f ensures that f is continuous but it is weaker than the Dini-continuous condition.* ■

Theorem 5 *Let f be a positive bounded function verifying (H) and such that there exists $\sigma \in (0,1)$, $p > 1$ for which $f \in W^{\sigma,p}(\Omega, \mathbb{R})$, then there exists $q \in \mathbb{R}_+$ and a homeomorphism $u \in W^{1,q}(\Omega)$ verifying*

$$u^{\#}(dx) = f(x)dx. \tag{1.5}$$

Moreover, for any neighborhood U of $\overline{\Omega}$, one may ensure $u(\overline{\Omega}) \subset U$. Precisely q is any positive number in $\left[1, \frac{\sigma \ln 2}{\ln(1+C_0\tau)}\right)$ where C_0 is a universal constant and τ_0 is the following constant:

$$\tau_0 = \lim_{t\to 0}\left(\sup_{|z|\leq t,\, y\in\Omega} \frac{\mid f(y+z)-f(y)\mid}{\inf_{\Omega} f}\right). \tag{1.6}$$

■

Remark 2 *For q sufficiently large (for example $q \geq n^2/(n+1)$), (1.5) is equivalent to $det(\nabla u) = f$.* ■

Finally, using the same method, we have the following theorem:

Theorem 6 *Let f be a positive function verifying (H) and (1.4). If f verifies*

$$\iint_{\Omega\times\Omega} \frac{\mid f(x)-f(y)\mid}{\mid x-y\mid^{n+\sigma}}dxdy < \infty$$

for some $\sigma \in (0,1)$, then there exists u solution of (1.2) such that $u \in W^{1,\infty}(\overline{\Omega})$ and

$$\iint_{\Omega\times\Omega} \frac{\mid \nabla u(x)-\nabla u(y)\mid}{\mid x-y\mid^{n+\sigma}}dxdy < \infty. \tag{1.7}$$

■

As it can be noted, employing using the details of the dyadic decompositions we use, this method is not adapted to the regular cases, but to the limiting ones ($f \in C^0, L^\infty, \ldots$). In the second part of our paper (section 3), we reconsider the flow method of Moser (see [7]) with regularizing paths in the forms space which interpolate $f(x)dx$ and dx. This idea leads, on one hand to a new and direct proof of Dacorogna and Moser's results with also new estimates which could be useful for numerical approaches. On the other hand, we establish new results for $f \in C^k(\bar{\Omega})$ ($k \geq 1$).

It is also interesting to note that this method, completely different from the dyadic decomposition proposed above, leads to the same gain of regularity in the limiting cases, $f \in C^0, L^\infty$ and BMO. (cf. § 3.3)

Our main results, with the second method, are the following (for simplicity we always assume Ω to be regular):

Theorem 7 *Let $k \in \mathbb{N}$, $0 < \alpha < 1$ and $f \in C^{k,\alpha}(\overline{\Omega})$ verifying (H), then there exists $u \in C^{k+1,\alpha}(\overline{\Omega})$ solving (1.2). Futhermore we have the following estimate, $\forall\, \delta \in (0, min(1, k+\alpha))$,*

$$\|u - Id\|_{C^{k+1,\alpha}(\overline{\Omega})} \leq C\|f-1\|_{C^{k,\alpha}(\overline{\Omega})}, \tag{1.8}$$

where C only depends on $\|f\|_{0,\delta}$, $\inf_{\Omega} f$, δ, α, k and Ω. ■

Remark 3 • *Some similar estimates have been obtained by Zehnder in [13] under more a restrictive hypothesis: for $\|f-1\|_{0,\delta}$ small enough.*

- *The method that we use for proving Theorem 7 is a direct one and could be useful for a numerical approach. Futhermore, estimate (1.8) illustrates the stability of the method.*

Theorem 8 *Let $k \in \mathbb{N}$ and let f be in $C^{k,1}(\overline{\Omega}) = W^{k+1,\infty}(\Omega)$ verifying (H), then there exists u, solution of (1.2) such that, for any $\alpha < 1$, u is in $C^{k+1,\alpha}(\overline{\Omega})$ and such that for any $0 < \delta < 1$,*

$$\|u - Id\|_{k+1,\alpha} \leq C\|f-1\|_{k,1}, \tag{1.9}$$

where C only depends on α, k, Ω, $\inf_{\Omega} f$, δ and $\|f\|_{0,\delta}$. ■

Remark 4 • *As a consequence of Theorem 1 in [4], for each $\alpha < 1$, there exists u_α in $C^{k+1,\alpha}(\overline{\Omega})$ solution of (1.2), but it is not clear whether u_α remains the same for all $\alpha < 1$ or not.*

- *The best regularity for u that one can expect, under the conditions of Theorem 8, is u in $C^{k+1,1}(\overline{\Omega})$. But we are not able to decide whether the u obtained here is in this space.*

2 A constructive method for prescribing a volume form $f(x)dx$

2.1 Presentation of the constructive method

Let $f \in L^1(\Omega)$ verifying (H). First of all, we consider the case $\Omega = (0,1)^n$. For $i \in \mathbb{N}$ and $k = (k_l)_{1\leq l \leq n} \in \mathbb{Z}^n$, we denote by $C(k,i)$ the following cube in the dyadic decomposition of $[0,1]^n$,

$$C(k,i) = \prod_{1\leq l\leq n} \left[\frac{k_l}{2^i}, \frac{k_l+1}{2^i}\right]. \tag{2.1}$$

Let $f_i \in L^\infty(\Omega)$ be the following approximation of f

$$f_i(x) = \frac{1}{\mid C(k,i) \mid} \int_{C(k,i)} f(y)dy \qquad \text{on } C(k,i) \tag{2.2}$$

for any $C(k,i) \subset [0,1]^n$. Note that f_i verifies always (H) and tends weakly to $f(x)$ in $L^1(\Omega)$. Our aim is to construct by induction a sequence (u_i) of Lipschitz homeomorphisms from $\overline{\Omega}$ into itself verifying

$$\begin{cases} \det(\nabla u_i) &= f_i \quad \text{in } \Omega \\ \quad u_i(x) &= x \quad \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

We use the following lemma which is the key point of our construction.

Lemma 1 *Let $D = [0,1]^n$, $\mathrm{A} = [0,1]^{n-1} \times [0,1/2]$ and $B = [0,1]^{n-1} \times [1/2,1]$. Let α, $\beta \in \mathbb{R}_+^*$ such that $\alpha + \beta = 1$. There exits a Lipschitz homeomorphism Φ from D into itself such that*

$$\begin{array}{ll} (i) & \Phi(x) = x, \qquad x \in \partial D; \\ (ii) & \left\{ \begin{array}{ll} det(\nabla\Phi) = \alpha & in\ \mathrm{A} \\ det(\nabla\Phi) = \beta & in\ B; \end{array} \right. \\ (iii) & \|\nabla(\Phi - Id)\|_{L^\infty(D)} \leq C_\eta \mid 1 - 2\alpha \mid \end{array} \tag{2.4}$$

where $0 < \eta \leq \alpha \leq (1-\eta) < 1$ and C_η only depends on η. ■

Take $u_{-1} = Id$ and $u_{i+1} = u_i \circ \Phi_i$, where Φ_i is constructed in the following way: Using the previous lemma: $\forall\ k = (k_l) \in \{0, 1, \dots, 2^i - 1\}^n$, Φ_i is a homeomorphism from $C(k,i)$ into itself such that $\Phi_i(x) = x$ on $\partial C(k,i)$. Take $\varepsilon = (\varepsilon_l) \in \{0,1\}^n$ and denote by $A_i^p(\varepsilon)$ and $B_i^p(\varepsilon)$ the following subsets of $C(k,i)$ (for $1 \leq p \leq n$)

$$A_i^p(\varepsilon) = \prod_{l \leq p-1} \left[\frac{k_l}{2^i}, \frac{k_l+1}{2^i}\right] \times \left[\frac{k_p}{2^i}, \frac{2k_p+1}{2^{i+1}}\right] \times \prod_{l \geq p+1} \left[\frac{k_l}{2^i} + \frac{\varepsilon_l}{2^{i+1}}, \frac{k_l}{2^i} + \frac{\varepsilon_l+1}{2^{i+1}}\right];$$

$$B_i^p(\varepsilon) =$$

$$\prod_{l \leq p-1} \left[\frac{k_l}{2^i}, \frac{k_l+1}{2^i}\right] \times \left[\frac{2k_p+1}{2^{i+1}}, \frac{k_p+1}{2^i}\right] \times \prod_{l \geq p+1} \left[\frac{k_l}{2^i} + \frac{\varepsilon_l}{2^{i+1}}, \frac{k_l}{2^i} + \frac{\varepsilon_l+1}{2^{i+1}}\right].$$

Denote by $\alpha_i^p(\varepsilon)$ and $\beta_i^p(\varepsilon)$ the following positive numbers:

$$\begin{aligned} \alpha_i^p(\varepsilon) &= \frac{\displaystyle\int_{A_i^p(\varepsilon)} f(x)dx}{\displaystyle\int_{A_i^p(\varepsilon) \cup B_i^p(\varepsilon)} f(x)dx} \\ \beta_i^p(\varepsilon) &= \frac{\displaystyle\int_{B_i^p(\varepsilon)} f(x)dx}{\displaystyle\int_{A_i^p(\varepsilon) \cup B_i^p(\varepsilon)} f(x)dx} = 1 - \alpha_i^p(\varepsilon). \end{aligned} \tag{2.5}$$

Consider Φ the homeomorphism given by Lemma 1 for $\alpha = \alpha_i^p(\varepsilon)$ and $\beta = \beta_i^p(\varepsilon)$ and using dilatations, translations and rotations, we get a homeomorphism $\Phi_i^p(\varepsilon)$ from $A_i^p(\varepsilon) \cup B_i^p(\varepsilon)$ into itself such that

$$\left\{ \begin{array}{lll} \det\nabla\Phi_i^p(\varepsilon) = 2\alpha_i^p(\varepsilon) & \text{in} & A_i^p(\varepsilon) \\ \det\nabla\Phi_i^p(\varepsilon) = 2\beta_i^p(\varepsilon) & \text{in} & B_i^p(\varepsilon), \end{array} \right. \tag{2.6}$$

and $\Phi_i^p(\varepsilon)(x) = x$ on $\partial(A_i^p(\varepsilon) \cup B_i^p(\varepsilon))$. Denote by Φ_i^p the homeomorphism from $C(k,i)$ into self such that $\Phi_i^p = \Phi_i^p(\varepsilon)$ in $A_i^p(\varepsilon) \cup B_i^p(\varepsilon)$. Take $\Phi_i = \Phi_i^n \circ \Phi_i^{n-1} \circ \ldots \circ \Phi_i^1$ and

$$u_{i+1} = u_i \circ \Phi_i, \qquad \text{in } C(k,i). \tag{2.7}$$

In constructing Φ_i, one has composed homeomorphisms whose Jacobian determinants are constant by parts in the way that one can easily calculate $\det(\nabla\Phi_i)$ on each sub-cube $C(k', i+1)$. More precisely, for $x \in C(k', i+1) \subset C(k,i)$,

$$\det\nabla\Phi_i(x) = \frac{\displaystyle\int_{C(k',i+1)} f(x)dx}{\displaystyle\int_{C(k,i)} f(x)dx},$$

and this proves the existence of u_{i+1} verifying (2.3).

The problem now is to understand under which condition for f the sequence u_i converges in a way that $\det(\nabla u_i)$ passes also to the limit at least in the weak sense (1.5).

Under the hypothesis that f belongs only in L^1, we have already that u_i^{-1} converges in $C^0(\overline{\Omega})$ to a continuous map $v : \Omega \to \Omega$ verifying

$$\forall\ E \text{ open set of } \Omega, \qquad \int_E f(x)dx = |\ v^{-1}(E)\ | \tag{2.8}$$

where $v^{-1}(E)$ denotes the co-image of E by v. In fact, since $\Phi_i(C(k,i)) = C(k,i)$, one has $\|u_{i+1}^{-1} - u_i^{-1}\|_\infty \leq \operatorname{diam}(C(k,i)) = C/2^i$.

On the other hand, if we suppose that $f \in L^\infty(\Omega)$ and $\|f-1\|_\infty \leq \eta$ where η will be chosen small enough, clearly using (2.4)(iii), one has $\|\nabla(\Phi_i - Id)\|_{L^\infty(D)} \leq C\eta$ and then

$$\|\nabla u_{i+1}\|_\infty = \|(\nabla u_i \circ \Phi_i)\nabla\Phi_i\|_\infty \leq \|\nabla u_i\|_\infty(1 + C\eta), \tag{2.9}$$

thus $\|\nabla u_i\| \leq (1+C\eta)^i$. Moreover, since $\Phi_i(C(k,i)) = C(k,i)$, we get $\|\Phi_i - Id\|_\infty \leq C/2^i$. Considering

$$\|u_{i+1} - u_i\|_\infty = \|u_i \circ \Phi_i - u_i\|_\infty \leq \|\nabla u_i\|_\infty \|\Phi_i - Id\|_\infty$$

and the two previous inequalities implies that

$$\|u_{i+1} - u_i\|_\infty \leq C\left(\frac{1+C\eta}{2}\right)^i. \tag{2.10}$$

For $(1 + C\eta) < 2$, the sequence u_i converges in $C^0(\overline{\Omega})$. Since now we have $f_i \to f$ in L^∞ weak-$*$, $u_i^{-1} \to u^{-1}$ in $C^0(\overline{\Omega})$ and $u_i \to u$ in $C^0(\overline{\Omega})$, u verifies (1.3). Indeed, let F be any compact set of Ω, $u(F)$ is compact. Consider $V_\epsilon = \{x \in \Omega,\ \text{s.t. } d(x, u(F)) < \epsilon\}$, since $u(F)$ is compact, we know that $|\ V_\epsilon\ | \to |\ F\ |$ as ϵ tends to 0. For i sufficiently great, $u_i(F) \subset V_\epsilon$ and we have $|\ u_i(F)\ | = \int_F f_i(x)dx \leq |\ V_\epsilon\ |$, thus

$$\lim_{i\to\infty} \int_F f_i(x)dx = \int_F f(x)dx \leq |\ u(F)\ |. \tag{2.11}$$

Since u is a homeomorphism, $u(F^c)$ is an open set, we prove exactly in the same way that

$$\lim_{i\to\infty}\int_{F^c} f_i(x)dx = \int_{F^c} f(x)dx \le | u(F^c) |. \tag{2.12}$$

Adding (2.11) and (2.12), we have

$$| \overline{\Omega} | = \int_{\overline{\Omega}} f(x)dx \le | u(F) | + | u(F^c) | = | u(\overline{\Omega}) | = | \overline{\Omega} |,$$

this proves that the two previous inequalities (2.11) and (2.12) are equalities.

In general cases, we solve by induction the following problems: $w_0 = Id$ and for any $k \le (p-1)$,

$$w^{\#}_{k+1}(dx) = \frac{\int_\Omega f^{k/p}}{\int_\Omega f^{(k+1)/p}} \times \left(f^{1/p} \circ v_k^{-1}\right) dx,$$

where $v_k = w_k \circ w_{k-1} \circ \ldots w_1$. For p sufficiently large such that $\|f^{1/p} - 1\|_\infty \le \eta$. $u = w_p \circ \ldots \circ w_1$ is a homeomorphism satisfying (1.3) (cf. § 2.2.2).

Under the hypothesis that f belongs only to L^1, we recall that u_i^{-1} converges to v in $C^0(\overline{\Omega})$ verifying (2.8), but if we want to ensure that v is a homeomorphism, we have to refine the previous process using Oxtoby and Ulam's idea (see [8]).

Here we are more interested in the case where f is more regular than L^1 (for example $f \in BMO(\overline{\Omega})$, $L^\infty(\Omega)$, $C^0(\overline{\Omega})$ etc.) for which the previous construction permits achieving a solution u having more regularity than $C^0(\overline{\Omega})$. Moreover, if $supp(f-1) \subset\subset \Omega$, we can assume that $supp(u - Id) \subset\subset \Omega$, since we can work in a smaller domain as $\Omega' = [\zeta, 1-\zeta]^n$ ($\zeta > 0$). Thus by the partition argument of Moser in [7], the proofs of our theorems for the case $\Omega = [0,1]^n$ are valid for general Lipschitz domains or manifolds.

2.2 Proof of Theorems 1 to 4

2.2.1 Proof of Theorem 1

Take $f \in C^0(\overline{\Omega})$ verifying (H) and follow the method in § 2.1. Let $0 < \alpha < 1$. Then there exists a sufficiently small η such that $\frac{1+C\eta}{2} = 2^{-\alpha}$, where C is the constant in Lemma 1. For i sufficiently great ($i \ge i_\alpha$), for any p and ε, $| \alpha_i^p(\varepsilon) - 1/2 | \le \eta$, since f is uniformly continuous in $\overline{\Omega}$.

Consider $x \ne y \in \overline{\Omega}$ such that $| x - y | \le 2^{-i_\alpha}$. Then there exists $i \ge i_\alpha$ such that $2^{-i-1} \le | x-y | \le 2^{-i}$. From (2.9), we have $\|\nabla u_i\|_\infty \le (1+C\eta)^{i-i_\alpha} \|\nabla u_{i_\alpha}\|_\infty$

and from the same argument for (2.10) we have

$$\|u - u_i\|_\infty \leq C_\alpha \left(\frac{1+C\eta}{2}\right)^i. \tag{2.13}$$

We get then the following estimate

$$\begin{aligned} |\, u(x) - u(y) \,| &\leq |\, u(x) - u_i(x) \,| + |\, u_i(x) - u_i(y) \,| + |\, u(y) - u_i(y) \,| \\ &\leq C_\alpha \left(\frac{1+C\eta}{2}\right)^i + C_\alpha (1+C\eta)^i |\, x - y \,| + C_\alpha \left(\frac{1+C\eta}{2}\right)^i \\ &\leq 2C_\alpha \left(\frac{1}{2^i}\right)^\alpha + C_\alpha \left(\frac{1+C\eta}{2}\right)^i \\ &\leq C_\alpha \left(\frac{1}{2^i}\right)^\alpha \leq 2C_\alpha |\, x - y \,|^\alpha. \end{aligned} \tag{2.14}$$

This proves that u is in $C^{0,\alpha}(\overline{\Omega})$.

On the other hand, by noting that $\det(\nabla u_i) \geq a > 0$ and that

$$\begin{cases} \|u_i^{-1} - u^{-1}\|_\infty \leq C2^{-i} \\ \|\nabla u_i^{-1}\|_\infty \leq \dfrac{C}{a} \|\nabla u_i\|_\infty, \end{cases} \tag{2.15}$$

one proves in the same way that $u^{-1} \in C^{0,\alpha}(\overline{\Omega})$ for any $0 < \alpha < 1$. ■

2.2.2 Proof of Theorems 2 and 3

a) Let $f \in L^\infty(\Omega)$ verifying (H) and let C be the constant in Lemma 1, for p sufficiently great such that $1 + C\|f^{1/p} - 1\|_\infty < 2$. Let

$$c_k = \left(\int_\Omega f^{\frac{k-1}{p}}\, dx\right) / \left(\int_\Omega f^{\frac{k}{p}}\, dx\right).$$

We will construct, by induction, a sequence of homeomorphisms w_k for $0 \leq k \leq (p-1)$ such that

$$\begin{cases} \forall\, E \text{ open set } \subset \Omega, & |w_{k+1}(E)| = \displaystyle\int_E c_{k+1} f^{1/p}(v_k^{-1})\, dx \\ w_{k+i}(x) = x, & \text{on } \partial\Omega, \end{cases} \tag{2.16}$$

where $v_0 = Id$ and $v_k = w_k \circ w_{k-1} \ldots \circ w_1$ for $1 \leq k \leq p$, with $w_k \in C^{0,1-\alpha}(\overline{\Omega}, \overline{\Omega})$ and $w_k(x) = x$ on $\partial\Omega$.

For $k = 1$, (2.16) is equivalent to

$$\forall\, E \text{ open set } \subset \Omega, \qquad |u_1(E)| = \int_E c_1 f^{1/p}\, dx.$$

Note that the problem is well posed since $c_1 f^{1/p}$ verifies (H) on Ω. More precisely, $\int_\Omega c_1 f^{1/p}\,dx = |\,\Omega\,|$ and $c_1 f^{1/p} \geq a^{1/p}$ (since $\int_\Omega f^{1/p}\,dx \leq |\,\Omega\,|$ by Hölder's inequality). Note also that $f^{1/p}$ and $c_1 f^{1/p}$ give the same estimates for α_i^p and β_i^p defined in § 2.1, thus w_1 constructed in § 2.1 for $c_1 f^{1/p}$ belongs to $C^{0,\alpha_p}(\overline{\Omega})$ where

$$\alpha_p = \left[-\ln\left(\frac{1+C\|f^{1/p}-1\|_\infty}{2}\right)/\ln 2\right]. \tag{2.17}$$

Suppose that w_l are constructed for $l \leq k$, we construct now w_{k+1} in the same way. Note first that the problem (2.16) is well posed. Indeed, by $\int_\Omega f^{\frac{k+1}{p}}\,dx \leq |\,\Omega\,|$, we get

$$c_{k+1} f^{1/p} \circ v_k^{-1} \geq \frac{\int_\Omega f^{\frac{k}{p}}\,dx}{\int_\Omega f^{\frac{k+1}{p}}\,dx} a^{1/p} \geq a^{\frac{k+1}{p}}. \tag{2.18}$$

Moreover, (2.16) implies, for $1 \leq l \leq k$,

$$\forall\, h \in L^\infty(\Omega), \qquad \int_\Omega h\,dx = \int_\Omega c_l\Big(h \circ w_l\Big)\Big(f^{1/p} \circ v_{l-1}^{-1}\Big)\,dx. \tag{2.19}$$

Thus

$$\begin{aligned}\int_\Omega c_{k+1} f^{1/p} \circ v_k^{-1}\,dx &= \int_\Omega c_{k+1}c_k f^{1/p} \circ v_{k-1}^{-1} \times \Big(f^{1/p}\circ v_k^{-1} \circ w_k\Big)\,dx \\ &= \int_\Omega c_{k+1}c_k f^{2/p} \circ v_{k-1}^{-1}\,dx \\ &\;\;\vdots \\ &= \left(\int_\Omega f^{\frac{k+1}{p}}\,dx\right) \prod_{1\leq l\leq k+1} c_l = |\,\Omega\,|.\end{aligned} \tag{2.20}$$

Since we always have $\|f^{1/p}\circ v_k^{-1} - 1\|_\infty = \|f^{1/p}-1\|_\infty$, w_k constructed in this way (for $c_k f^{1/p} \circ v_k^{-1}$), is always a homeomorphism in $C^{0,\alpha_p}(\overline{\Omega})$. Thus we get the sequence of w_k verifying (2.16).

Consider now $w = v_p = w_p \circ \ldots \circ w_1$ which is a homeomorphism from $\overline{\Omega}$ into itself. By (2.16), w is a solution of (1.3). Furthermore, $w \in C^{0,\beta}(\overline{\Omega})$ with $\beta = (\alpha_p)^p$. Making p tend to infinity, one easily proves that

$$\frac{\ln\beta}{c} \simeq \min\left\{\ln\left(\inf_\Omega f\right), -\ln\left(\sup_\Omega f\right)\right\}$$

where c does not depend on f. ■

b) Let f be in $BMO(\overline{\Omega})$ verifying (H). Let $M > \inf_\Omega f$ and note $T_M f$ the following truncated function $T_M f = \min(M, f)$. Denoting $h = f/T_M f$, we remark that $\|h\|_{BMO(\overline{\Omega})}$ tends to 0 as M tends to ∞. In fact, for any $x \in \mathbb{R}^n$ and $r > 0$, denoting $\Omega_{x,t} = B_t(x) \cap \Omega$ and $h_{x,t} = ⨍_{\Omega_{x,t}} h\,dz$, one has

$$\begin{aligned} \frac{1}{|B_{x,r}|}\int_{\Omega_{x,t}} |h(y) - h_{x,t}|\,dy = & \frac{1}{|B_{x,r}|}\int_{\Omega_{x,t}} \left| h(y) - \frac{1}{|B_{x,r}|}\int_{\Omega_{x,t}} h\,dz \right| dy \\ \leq & \frac{1}{|\Omega_{x,t}|^2}\int\int_{\Omega_{x,t}\times\Omega_{x,t}} |h(y) - h(z)|\,dydz. \end{aligned} \tag{2.21}$$

Clearly we have $|h(y) - h(z)| \leq \dfrac{|f(y) - f(z)|}{M}$, so (2.21) leads to

$$\begin{aligned} \frac{1}{|B_{x,r}|}\int_{\Omega_{x,t}} |h(y) - h_{x,t}|\,dy \leq & \frac{1}{M}\frac{1}{|\Omega_{x,t}|^2}\int\int_{\Omega_{x,t}\times\Omega_{x,t}} |f(y) - f(z)|\,dydz \\ \leq & \frac{1}{M}\frac{1}{|B_{x,r}|^2}\int\int_{B_{x,r}\times B_{x,r}} \Big(|f(y) - f_{x,r}| + |f(z) - f_{x,r}|\Big)\,dydz \\ \leq & \frac{2}{M}\|f\|_{BMO}. \end{aligned}$$

So we get the estimate $\|h\|_{BMO} \leq \frac{2}{M}\|f\|_{BMO}$. Let $c_h = \frac{1}{\Omega}\int_\Omega h(x)dx$ and write $f = (h/c_h) \times c_h T_M f$. Clearly $1 \leq c_h \leq (1 + 1/M)$. First we solve the problem for h/c_h. Note that we can apply the method in § 2.1, since we can estimate $|\,\alpha_i^p(\varepsilon) - 1/2\,|$ by $C\|h\|_{BMO(\overline{\Omega})}/a$, where $a = \inf_\Omega f$. Thus for $\|h\|_{BMO(\overline{\Omega})}$ small enough, we get a v in $C^{0,\alpha_M}(\overline{\Omega})$ satisfying $v^\#(dx) = h(x)dx/c_h$ where

$$\alpha_M = 1 - \frac{\ln\Big(1 + C\|h\|_{BMO(\overline{\Omega})}/a\Big)}{\ln 2}. \tag{2.22}$$

Let w be the solution given by § 2.2.2 a) for $c_h T_M f \circ v^{-1}$, then $u = w \circ v$ is a solution of (1.5) and $u \in C^{0,\gamma_M}(\overline{\Omega})$ where

$$\gamma_M < \alpha_M \left[\min\left(c_h a, \frac{1}{c_h M}\right)\right]^c.$$

By choosing a good value of M, one proves the assertion of Theorem 3. ■

2.2.3 Proof of Theorem 4

In view of § 2.1, we are interested in finding a condition on f which ensures the existence of a Lipschitz solution u since, in this case, (1.5) is equivalent to the following equation

$$\begin{cases} \det(\nabla u) = f & \text{in } \Omega \\ u(x) = x & \text{on } \partial\Omega. \end{cases} \tag{2.23}$$

The condition that we propose here is a kind of Dini decrease of the mean oscillation

$$(D) \quad \begin{cases} \exists\, \varphi \text{ an increasing positive function such that } \displaystyle\int_0^1 \varphi(t)t^{-1}dt < +\infty \\ \forall\, x \in \Omega \text{ and } t > 0, \qquad \displaystyle\frac{1}{\mid \Omega_{x,t} \mid}\int_{\Omega_{x,t}} \mid f(y) - \overline{f}_{x,t} \mid dy \leq \varphi(t), \end{cases}$$

where $\Omega_{x,t} = B_t(x) \cap \Omega$, $\overline{f}_{x,t} = \left(\int_{\Omega_{x,t}} f(y)dy\right) / \mid \Omega_{x,t} \mid$. This kind of condition (a strong one) was already considered in [9] and [11].

Under this hypothesis, we have the following estimation on $\mid \alpha_i^p(\varepsilon) - 1/2 \mid$,

$$\begin{aligned} \mid \alpha_i^p(\varepsilon) - 1/2 \mid \quad &\leq \quad \frac{1}{2}\left| \frac{\displaystyle\int_{A_i^p(\varepsilon)} f(x)dx - \int_{B_i^p(\varepsilon)} f(x)dx}{\displaystyle\int_{A_i^p(\varepsilon)\cup B_i^p(\varepsilon)} f(x)dx} \right| \\ &\leq \quad \frac{C}{a}\varphi(2^{-i}), \end{aligned} \tag{2.24}$$

where C is a universal constant. By Lemma 1, we have $\|\nabla(\Phi_i - Id)\|_\infty \leq C\varphi(2^{-i})$, and we get $\|\nabla u_{i+1}\|_\infty \leq (1 + C\varphi(2^{-i}))\|\nabla u_i\|_\infty$. By induction, this yields

$$\begin{aligned} \|\nabla u_i\|_\infty \quad &\leq \quad \Pi_{1\leq k\leq i-1} \left(1 + C\varphi(2^{-k})\right)\|\nabla u_0\|_\infty \\ &\leq C\left[\sum_{k=1}^{i-1}\varphi(2^{-k})\right]\|\nabla u_0\|_\infty \\ &\leq C\left[\sum_{k=1}^{i-1}\int_{2^{-k-1}}^{2^{-k}}\varphi(s)s^{-1}ds\right]\|\nabla u_0\|_\infty \\ &\leq C\left(\int_0^1 \varphi(s)s^{-1}ds\right)\|\nabla u_0\|_\infty. \end{aligned} \tag{2.25}$$

This implies that u is Lipschitz. Moreover, we can prove that u_i converges to u in $W^{1,p}(\overline{\Omega})$ for any $p \in [1, \infty)$, since u_i tends to u in $C^0(\overline{\Omega})$, u_i is bounded in $W^{1,\infty}(\overline{\Omega})$ and clearly $\|\nabla u_i\|_p$ tends to $\|\nabla u\|_p$ for any $p \in [1, \infty)$. ■

2.3 Proof of Theorems 5 and 6

2.3.1 Proof of Theorem 5

Consider the dyadic decomposition of $\Omega = [0,1]^n$ as in § 2. For $i \in \mathbb{N}$, $k = (k_l) \in \{0, 1, \ldots, 2^i - 1\}^n$ and $x \in \Omega$, denote by $C(k,i)(x)$, $A_i^p(\varepsilon)(x)$, $B_i^p(\varepsilon)(x)$, the $C(k,i)$, $A_i^p(\varepsilon)$ and $B_i^p(\varepsilon)$ which contain x. By $\alpha_i^p(\varepsilon)(x)$ and $\beta_i^p(\varepsilon)(x)$ we denote the following values

$$\begin{aligned} \alpha_i^p(\varepsilon)(x) &= \frac{\displaystyle\int_{A_i^p(\varepsilon)(x)} f(x)dx}{\displaystyle\int_{A_i^p(\varepsilon)(x)\cup B_i^p(\varepsilon)(x)} f(x)dx} \\ \beta_i^p(\varepsilon)(x) &= 1 - \alpha_i^p(\varepsilon)(x). \end{aligned} \tag{2.26}$$

By Lemma 1, we have

$$| \nabla(\Phi_i^p - Id) |(x) \leq C | \alpha_i^p(\varepsilon)(x) - 1/2 |,$$

thus we have

$$| \nabla(\Phi_i - Id) |(x) \leq C \sum_{p=1}^{n} | \alpha_i^p(\varepsilon)(x) - 1/2 |. \tag{2.27}$$

In fact, $\Phi_i = \Phi_i^n \circ \Phi_i^{n-1} \circ \ldots \circ \Phi_i^1$ and since $\alpha_i^p(\varepsilon)(x)$ (resp. $\beta_i^p(\varepsilon)(x)$) is constant in $A_i^p(\varepsilon)(x)$ (resp. $B_i^p(\varepsilon)(x)$) and $\Phi_i^l(A_i^p(\varepsilon)(x)) = A_i^p(\varepsilon)(x)$ (resp. $\Phi_i^l(B_i^p(\varepsilon)(x)) = B_i^p(\varepsilon)(x)$) for $l < p$, so $\forall\, 2 \leq p \leq n$,

$$\begin{cases} \alpha_i^p(\varepsilon)\left(\Phi_i^{p-1} \circ \ldots \circ \Phi_i^1(x)\right) = \alpha_i^p(\varepsilon)(x) \\ \beta_i^p(\varepsilon)\left(\Phi_i^{p-1} \circ \ldots \circ \Phi_i^1(x)\right) = \beta_i^p(\varepsilon)(x). \end{cases}$$

Moreover for, $x \in \Omega$, $u_{i+1}(x) = u_i(\Phi_i(x))$, this implies

$$\begin{aligned} | \nabla u_{i+1}(x) | &= | \nabla u_i(\Phi_i(x)) \nabla \Phi_i(x) | \\ &\leq | \nabla u_i(\Phi_i(x))(\nabla \Phi_i(x) - I) | + | \nabla u_i(\Phi_i(x)) | \\ &\leq \left(1 + C \sum_{p=1}^{n} | \alpha_i^p(\varepsilon)(x) - 1/2 |\right) | \nabla u_i(\Phi_i(x)) |. \end{aligned} \tag{2.28}$$

Using the previous inequality and the fact that $\forall\, i < j$, $\alpha_i^p(\varepsilon)(\Phi_j(x)) = \alpha_i^p(\varepsilon)(x)$ for any $1 \leq p \leq n$, we can prove by induction that

$$| \nabla u_i(x) | \leq \prod_{0 \leq j \leq i-1} \left(1 + C \sum_{p=1}^{n} | \alpha_j^p(\varepsilon)(x) - 1/2 |\right). \tag{2.29}$$

We ask how one can determine under which condition on f, in view of the previous estimate (2.29), ∇u_i remains bounded for some L^p norm. Take f bounded verifying

(H) and suppose that f is in $W^{\sigma,p}(\Omega,\mathbb{R})$ for some $0<\sigma<1$ and $1<p<\infty$. Furthermore, to avoid any problem with the boundary, we suppose $\mathrm{supp}(f-1)\subset\Omega$. Recall the definition of $W^{\sigma,p}(\Omega,\mathbb{R})$ (cf. [1], chapter 7)

$$W^{\sigma,p}(\Omega,\mathbb{R})=\left\{f\in L^p(\Omega)\text{ such that }\int\int_{\Omega\times\Omega}\frac{|\,f(x)-f(y)\,|^p}{|\,x-y\,|^{n+\sigma p}}dxdy<+\infty\right\}.$$

Consider a cube $\Omega'=[\zeta,1-\zeta]^n$ such that $\mathrm{supp}(f-1)\subset\Omega'$. We claim that we can find an orthonormal system $(e_1,e_2,\ldots,e_n)$ such that

$$\forall\,\sigma'<\sigma\text{ and }1\le i\le n,\qquad \sum_{k=0}^{\infty}(2^{\sigma'})^k\int_{\Omega}|\,f(x+2^{-k}e_i)-f(x)\,|dx<+\infty,\tag{2.30}$$

where we have defined $f(y)=1$ if $y\notin\overline{\Omega}$. Note that f extended in such a way, on all of $\mathbb{R}^n$, still belongs to $W^{\sigma,p}_{loc}(\mathbb{R}^n)$. So we have

$$\int_{B_1(0)}\int_{\Omega}\frac{|\,f(x+z)-f(x)\,|^p}{|\,z\,|^{n+\sigma p}}dxdz<\infty\,.$$

This implies (using Hölder's inequality)

$$\int_{B_1(0)}\int_{\Omega}\frac{|\,f(x+z)-f(x)\,|}{|\,z\,|^{n+\sigma'}}dxdz<\infty,\qquad\forall\,\sigma'<\sigma\tag{2.31}$$

and we have then

$$\sum_{k=0}^{\infty}\int_{B_{2^{-k}}(0)\setminus\overline{B_{2^{-k-1}}(0)}}\int_{\Omega}\frac{|\,f(x+z)-f(x)\,|}{|\,z\,|^{n+\sigma'}}dxdz<\infty,$$

yielding

$$\int_{B_1(0)\setminus B_{1/2}(0)}\int_{\Omega}\sum_{k=0}^{\infty}\frac{|\,f(x+2^{-k}u)-f(x)\,|}{(2^{\sigma'})^{-k}\,|\,u\,|^{n+\sigma'}}dxdu<\infty,\qquad\forall\,\sigma'<\sigma.\tag{2.32}$$

This means that

$$I(u)=\sum_{k=0}^{\infty}(2^{\sigma'})^k\int_{\Omega}|\,f(x+2^{-k}u)-f(x)\,|dx$$

is finite for almost every $u\in B_1(0)\setminus\overline{B_{1/2}(0)}$. Thus we can find an orthogonal basis $(e_1,e_2,\ldots,e_n)$ such that $\forall\ i$ and j, $|\,e_i\,|=|\,e_j\,|$, $I(e_i)<\infty$ and $(e_1,e_2,\ldots,e_n)$ can be chosen to be as close as we want to the canonical basis. We fix in particular $(e_1,e_2,\ldots,e_n)$ such that we can find a cube Ω_1 parallel to $(e_1,e_2,\ldots,e_n)$ of length $|\,e_i\,|$ and $\Omega'\subset\subset\Omega_1\subset\subset\Omega$. Modulo a small scale change transforming $\overline{\Omega}_1$ to $[0,1]^n$ we have (for $1\le i\le n$)

$$\sum_{k=0}^{\infty}(2^{\sigma'})^k\int_{\Omega}|\,f(x+2^{-k}\varepsilon_i)-f(x)\,|dx<\infty,\qquad\forall\,\sigma'<\sigma,\tag{2.33}$$

where (ε_i) is the canonical basis of $\mathbb{R}^n$. Note that (H) is still verified for f after this scale change. Define the global oscillation τ to be

$$\tau = \frac{\sup_\Omega f}{\inf_\Omega f} - 1.$$

Let $a_i^p(x)$ be the quantity $a_i^p(x) = \mid \alpha_i^p(\varepsilon)(x) - 1/2 \mid = \mid \beta_i^p(\varepsilon)(x) - 1/2 \mid$ where $\alpha_i^p(\varepsilon)(x)$ is defined by (2.26). We clearly have

$$\forall\, x \in [0,1]^n, \qquad a_i^p(x) \leq \frac{\tau}{4}. \tag{2.34}$$

We have also $\forall\, x \in [0,1]^n$,

$$a_i^p(x) \leq \frac{1}{4 \inf_\Omega f} \times \frac{1}{\mid A_i^p(\varepsilon)(x) \mid} \int_{A_i^p(\varepsilon)(x)} \left| f(y + 2^{-i}\varepsilon_p) - f(y) \right| dy. \tag{2.35}$$

By (2.29), we get

$$\begin{aligned} \mid \nabla u_i(x) \mid \quad &\leq \prod_{0 \leq j \leq i-1} \left(1 + C \sum_{p=1}^{n} \mid \alpha_j^p(\varepsilon)(x) - 1/2 \mid \right) \\ &= \prod_{0 \leq j \leq i-1} (1 + C a_j(x)) \end{aligned} \tag{2.36}$$

where $a_i(x) = \sum_{p=1}^{n} a_i^p(x)$. In view of (2.35), (2.33) implies ($a = \inf_\Omega f$)

$$\sum_{i=0}^{\infty} (2^{\sigma'})^i \int_\Omega a_i(x) dx \leq \frac{1}{2\alpha} \sum_{p=1}^{n} \sum_{i=0}^{\infty} (2^{\sigma'})^i \int_\Omega \mid f(x + 2^{-i}\varepsilon_p) - f(x) \mid dx < \infty. \tag{2.37}$$

We now use the following lemma whose proof is straightforward by induction.

Lemma 2 *Let a_i be a sequence of positive numbers, then we have*

$$\prod_{i \in \mathbb{N}} (1 + a_i) \leq 1 + \sum_{i=0}^{\infty} (1 + A)^i a_i \tag{2.38}$$

where $A = \sup_{i \in \mathbb{N}} a_i$. ∎

Apply the previous lemma with

$$a_i = (1 + C a_i(x))^p - 1 \tag{2.39}$$

where C is the constant in (2.36) and p will be fixed later. If $\left(1+\frac{nC\tau}{4}\right)^p - 1 \leq (2^{\sigma'}-1)$, we deduce from (2.37) and (2.38)

$$\begin{aligned}\int_\Omega |\nabla u_i(x)|^p dx &\leq \int_\Omega \left(1+\sum_{i=0}^{\infty}(2^{\sigma'})^i a_i(x)\right)dx \\ &\leq |\Omega| + \frac{1}{2a}\sum_{p=1}^{n}\sum_{i=0}^{\infty}(2^{\sigma'})^i \int_\Omega |f(x+2^{-i}\varepsilon_p)-f(x)|dx \\ &\leq C,\end{aligned} \tag{2.40}$$

where C does not depend on i. This implies that, if $p \geq 1$,

$$\int_\Omega |\nabla u|^p < \infty.$$

Thus if $\frac{\sigma \ln 2}{\ln(1+C_0\tau)} > 1$ (where C_0 is a universal constant), $u \in W^{1,p}(\Omega)$ for $1 \leq p < \frac{\sigma \ln 2}{\ln(1+C_0\tau)}$.

Remark that, instead of τ, we could take a global bound for the local oscillation τ_0 defined by (1.6). The proof is completed. ■

Corollary 1 *Let $f \in C^0(\overline{\Omega})$ verifying (H). Suppose that there exist $\sigma \in (0,1)$ and $p \in (1,\infty)$ such that $f \in W^{\sigma,p}(\Omega)$ and supp$(f-1) \subset \Omega$, then there exists $u \in \cap_{p<\infty} W^{1,p}(\Omega)$ satisfying (1.2).* ■

Proof of Corollary 1. Indeed, for $f \in C^0(\overline{\Omega})$, τ_0 defined by (1.6) is equal to zero. ■

Remark 5 *By the proof, we see that it suffices to suppose*

$$\iint_{\Omega\times\Omega} \frac{|f(x)-f(y)|}{|x-y|^{n+\sigma}}dxdy < \infty, \qquad \sigma \in (0,1)$$

for getting the same result in Theorem 5 and Corollary 1. ■

2.3.2 Proof of Theorem 6

As in the beginning of § 2.3.1, modulo some changes, we can suppose that $\Omega = (0,1)^n$ and (2.33) holds. That is,

$$\sum_{i=1}^{n}\sum_{k=0}^{\infty}(2^\sigma)^k \int_\Omega |f(x+2^{-k}\varepsilon_i)-f(x)|dx < \infty, \qquad \forall\, \sigma' < \sigma, \tag{2.41}$$

where (ε_i) is the canonical basis of $\mathbb{R}^n$. Define $a_i^p(x)$ and $a_i(x)$ in the same way as in § 2.3.1 and we have

$$\sum_{k=0}^{\infty}(2^\sigma)^k \int_\Omega a_k(x)dx < \infty. \tag{2.42}$$

Moreover, for u_i defined by (2.7), we have

$$\forall\, x \in \Omega, \qquad |\nabla u_i(x)| \leq \prod_{0\leq j\leq i-1} (1 + Ca_j(x)). \tag{2.43}$$

For any $h \in L^1(\Omega)$. Denote $F_\Omega(h)$ to be the following quantity:

$$F_\Omega(h) = \iint_{\Omega\times\Omega} \frac{|h(x) - h(y)|}{|x-y|^{n+\sigma}}dxdy.$$

We claim that, under the previous hypothesis, we have

$$F_\Omega(\nabla\Phi_i) \leq C\varphi(2^{-i}) \tag{2.44}$$

where Φ_i is defined in § 2 and φ is the increasing function for which f verifies (1.4). For proving (2.44), it suffices to prove the following upper bound in Lemma 1:

$$F_D(\nabla\Phi) \leq C_\eta|\, 1 - 2\alpha\, | \tag{2.45}$$

where C_η only depends on η such that $0 < \eta \leq \alpha \leq 1-\eta < 1$. To this aim, we use the following lemmas whose proofs are straightforward.

Lemma 3 *Let w be a Lipschitz homeomorphism of Ω and $v \in W^{1,\infty}(\Omega, \mathbb{R}^n)$. We then have*

$$F_\Omega(\nabla(v\circ w)) \leq F_\Omega(\nabla v)\left(\|\nabla w\|_\infty^{n+\sigma+1}\|det(\nabla w^{-1})\|_\infty^2\right) + F_\Omega(\nabla w)\|\nabla v\|_\infty. \tag{2.46}$$

■

Lemma 4 *Let Ω_1 and Ω_2 be two disjointed bounded Lipschitz domains of $\mathbb{R}^n$. Let f_i be in $L^\infty(\Omega_i)$ for $i = 1, 2$. Define $h = \sum_{i=1}^{2} f_i\chi_{\Omega_i}$, we then have*

$$F_\Omega(h) \leq \sum_{i=1}^{2} F_{\Omega_i}(f_i) + C\sum_{i=1}^{2}\|f_i\|_\infty \tag{2.47}$$

where C only depends on n, σ and Ω_i. ■

Recall that on $A \cup B$, in Lemma 1, $\Phi = \Psi \circ \varphi$ where Ψ is given by (A.1) and φ_t^{-1} is the solution of (A.21). From (A.4), we get

$$F_A(\nabla \Psi) = (1-2\alpha) F_A(\nabla(x_n h)) \tag{2.48}$$

where

$$x_n h = \int_0^{x_n} \operatorname{div}(w)(x', s) ds. \tag{2.49}$$

By definition, $\operatorname{div}(w)$ is C^∞ by parts in A, thus Lemma 4 ensures that $F_A(\nabla(x_n h)) < \infty$. On the other hand, from (A.21), we have

$$\frac{d}{dt}\varphi_t^{-1} = X_t(\varphi_t^{-1})$$

where X_t is in $W^{1,\infty}(A)$ and is C^∞ by parts, so $F_A(\nabla X_t) < \infty$. Thus we get (using Lemma 3 and the fact that α will be close to $1/2$ for i sufficiently great)

$$\frac{d}{dt} F_A\left(\nabla \varphi_t^{-1}\right) \leq C \frac{|\, 1-2\alpha \,|}{2\alpha} \left(1 + F_A\left(\nabla \varphi_t^{-1}\right)\right). \tag{2.50}$$

Remark that $F_A\left(\nabla \varphi_0^{-1}\right) = F_A(Id) = 0$, using Gronwall's Lemma, we have

$$F_A\left(\nabla \varphi^{-1}\right) = F_A\left(\nabla \varphi_1^{-1}\right) \leq C |\, 1-2\alpha \,| \tag{2.51}$$

and we easily deduce that

$$F_A(\nabla \varphi) \leq C |\, 1-2\alpha \,|. \tag{2.52}$$

Combining (2.48), (2.52) and (2.4)(iii), using Lemma 3, we obtain (α close to $1/2$)

$$F_A(\nabla \Phi) = F_A(\nabla(\Psi \circ \varphi)) \leq C |\, 1-2\alpha \,|. \tag{2.53}$$

Similarly we get

$$F_B(\nabla \Phi) = F_B(\nabla(\Psi \circ \varphi)) \leq C |\, 1-2\alpha \,|. \tag{2.54}$$

Using again Lemma 4 and (2.4)(iii), (2.45) is proved.

By (2.45), we deduce

$$F_{C(k,i)}(\nabla \Phi_i) \leq C (2^\sigma)^i a_i |_{C(k,i)}. \tag{2.55}$$

Thus

$$\begin{aligned} F_\Omega(\nabla \Phi_i) &= \iint_{\Omega \times \Omega} \frac{|\, \nabla \Phi_i(x) - \nabla \Phi_i(y) \,|}{|\, x-y \,|^{n+\sigma}} dx dy \\ &= \sum_{k,l=0}^{2^i-1} \iint_{C(k,i) \times C(l,i)} \frac{|\, \nabla \Phi_i(x) - \nabla \Phi_i(y) \,|}{|\, x-y \,|^{n+\sigma}} dx dy \\ &= \sum_{k,l=0,\ k \neq l}^{2^i-1} \iint_{C(k,i) \times C(l,i)} \frac{|\, \nabla \Phi_i(x) - \nabla \Phi_i(y) \,|}{|\, x-y \,|^{n+\sigma}} dx dy \\ &\quad + \sum_{k=0}^{2^i-1} F_{C(k,i)}(\nabla \Phi_i). \end{aligned} \tag{2.56}$$

On one hand

$$\begin{aligned}
&\sum_{k,l=0,\ k\neq l}^{2^i-1} \iint_{C(k,i)\times C(l,i)} \frac{|\nabla\Phi_i(x)-\nabla\Phi_i(y)|}{|x-y|^{n+\sigma}}dxdy \\
\leq\ &\sum_{k,l=0,\ k\neq l}^{2^i-1} \iint_{C(k,i)\times C(l,i)} \frac{\|\nabla(\Phi_i-Id)\|_{L^\infty(C(k,i))}}{|x-y|^{n+\sigma}}dxdy \\
&+\sum_{k,l=0,\ k\neq l}^{2^i-1} \iint_{C(k,i)\times C(l,i)} \frac{\|\nabla(\Phi_i-Id)\|_{L^\infty(C(l,i))}}{|x-y|^{n+\sigma}}dxdy \qquad (2.57)\\
\leq\ &\sum_{k,l=0,\ k\neq l}^{2^i-1} \iint_{C(k,i)\times C(l,i)} \frac{Ca_i(x)}{|x-y|^{n+\sigma}}dxdy \\
\leq\ &\sum_{k=0}^{2^i-1} \int_{C(k,i)} Ca_i(x) \int_{\Omega\setminus\overline{C(k,i)}} \frac{dxdy}{|x-y|^{n+\sigma}}.
\end{aligned}$$

Since $a_i(x)$ is constant on each $C(k,i)$, we are led to bound the following

$$\begin{aligned}
&\iint_{C(k,i)\times(\Omega\setminus\overline{C(k,i)})} \frac{dxdy}{|x-y|^{n+\sigma}} \\
\leq\ &\int_0^{2^{-i-1}} \left(\mathcal{H}^{\backslash-\infty}\left(\left\{\S\in\mathcal{C}(\|,\rangle);\ \lceil(\S,\partial\mathcal{C}(\|,\rangle))=\updownarrow\right\}\right)\int_\updownarrow^\infty \frac{\mathcal{C}\nabla^{\backslash-\infty}}{\nabla^{\backslash+\sigma}}\lceil\nabla\right)dl \qquad (2.58)
\end{aligned}$$

where $\mathcal{H}^{\backslash-\infty}$ stands for the $(n-1)$- dimensional Hausdorff measure, so

$$\mathcal{H}^{\backslash-\infty}\left(\left\{\S\in\mathcal{C}(\|,\rangle),\ \text{s.t.}\ \lceil(\S,\partial\mathcal{C}(\|,\rangle))=\updownarrow\right\}\right)=\in\backslash\left(\in^{-\rangle}-\in\updownarrow\right)^{\backslash-\infty}. \qquad (2.59)$$

Combining (2.57), (2.58) and (2.59), we get

$$\sum_{k,l=0,\ k\neq l}^{2^i-1} \iint_{C(k,i)\times C(l,i)} \frac{|\nabla\Phi_i(x)-\nabla\Phi_i(y)|}{|x-y|^{n+\sigma}}dxdy \leq C(2^{-i})^{n-\sigma}\sum_{k=0}^{2^i-1} a_k|_{C(k,i)}. \qquad (2.60)$$

Combining now (2.55), (2.56) and (2.60), we obtain

$$F_\Omega(\nabla\Phi_i)\leq C\int_\Omega (2^\sigma)^i a_i(x)dx. \qquad (2.61)$$

From (2.7) and Lemma 3, the following inequality holds

$$F_\Omega(\nabla u_i)\leq F_\Omega(\nabla u_{i-1})\Big(1+C\varphi(2^{-i})\Big)+F_\Omega(\nabla\Phi_i)\|\nabla u_{i-1}\|_\infty. \qquad (2.62)$$

Recall (2.25), that is

$$\|\nabla u_i\|_\infty\leq C\prod_{1\leq k\leq i-1}\Big(1+C\varphi(2^{-k})\Big). \qquad (2.63)$$

Using (2.62) and (2.63), by induction, we have

$$F_\Omega(\nabla u_i) \leq C \left[\prod_{1\leq k\leq i-1} \left(1 + C\varphi(2^{-k})\right) \right] \times \left[\sum_{k=1}^{i} (2^\sigma)^k \int_\Omega a_k(x)dx \right].$$

From (2.42) and (2.25), we get the following bound

$$F_\Omega(\nabla u_i) \leq C \int_0^1 \frac{\varphi(s)}{s} ds \tag{2.64}$$

and this implies that $F_\Omega(\nabla u) < \infty$, since u_i converges to u in $W^{1,p}(\overline{\Omega})$ for $p \in [1, \infty)$. ■

Remark 6 *In fact, $g \in L^\infty(\Omega)$ and $F_\Omega(g) < \infty$ imply that $g \in W^{\xi,\sigma/\xi}(\Omega)$ for any $\xi \in (0, \sigma)$.* ■

3 The path-regularizing flow method

3.1 Presentation of the method

3.1.1 The flow idea of Moser

Let f and g be two volume forms on Ω (i.e. two functions on Ω verifying (H)), let f_t for $t \in [0, T]$ be a family of volume forms connecting f and g. Any familly of transformations φ_t sending f to f_t (i.e. $\varphi_t^\# f_t = f$) verifies

$$\begin{cases} \dfrac{\partial}{\partial t}\left(f_t(\varphi_t)\det(\nabla\varphi_t)\right) \equiv 0, & \text{in } \Omega \\ f_0 = f,\ \varphi_0 = Id,\ f_T = g, & \text{in } \Omega. \end{cases} \tag{3.1}$$

This is equivalent to

$$\begin{cases} \dfrac{\partial}{\partial t}\det(\nabla\varphi_t) + \dfrac{\partial}{\partial t}\left(\ln f_t(\varphi_t)\right)\det(\nabla\varphi_t) = 0, & \text{in } \Omega \\ f_0 = f,\ \varphi_0 = Id,\ f_T = g, & \text{in } \Omega. \end{cases} \tag{3.2}$$

As J. Moser proposed in [7], we look for φ_t solution of a flow having the following form

$$\frac{\partial \varphi_t}{\partial t} = A_t(\varphi_t), \qquad \text{in } \Omega. \tag{3.3}$$

A classical computation yields

$$\frac{\partial}{\partial t}\det(\nabla\varphi_t) - \mathrm{div}A_t(\varphi_t)\det(\nabla\varphi_t) = 0. \tag{3.4}$$

Then it suffices to find A_t solution of the following transport equation which is equivalent to say that there is no volume creation.

$$\frac{\partial f_t}{\partial t} + \operatorname{div}(f_t A_t) = 0, \tag{3.5}$$

in view of solving (3.1). The difficulty in solving (1.2) by this method, remains in choosing an appropriate path f_t connecting f and 1, and a vector field A_t satisfying (3.5).

3.1.2 A construction of f_t and A_t

Let Ω be a bounded regular domain. Let f be in $C^\infty(\overline{\Omega})$ verifying (H). We first extend f to all of $\mathbb{R}^n$ into a function denoted by $\tilde{f}$. Let ξ be a cut-off function in $C_0^\infty(\mathbb{R}^n, [0,1])$ such that $\xi \equiv 1$ in $\overline{\Omega}$ and $\inf_{supp(\xi)} \tilde{f} \geq a > 0$. $\tilde{f}$ will be extended now by the following function in $\mathbb{R}^n$.

$$f = \xi\tilde{f} + (1-\xi).$$

Note that $f(x) \geq a > 0$, for any $x \in \mathbb{R}^n$. Following the ideas of section 2.1, we will construct f_t and A_t verifying

$$\begin{cases} \dfrac{\partial f_t}{\partial t} + \operatorname{div}(A_t f_t) = 0, & \text{in } \overline{\Omega} \times \mathbb{R}_+, \\ A_t \equiv 0, & \text{on } \partial\Omega \times \mathbb{R}_+, \\ f_0 = f, & \text{in } \overline{\Omega}, \end{cases} \tag{3.6}$$

and φ_t the solution of the following flow

$$\begin{cases} \dfrac{\partial \varphi_t}{\partial t} = A_t(\varphi_t), & \text{in } \overline{\Omega} \times \mathbb{R}_+, \\ \varphi_0 = Id & \text{in } \overline{\Omega}. \end{cases} \tag{3.7}$$

In view of § 3.1.1, under the previous notations, we have the following lemma.

Lemma 5 *For φ_t and f_t defined above, one has for any $t \geq 0$,*

$$\det(\nabla\varphi_t) \times f_t(\varphi_t) \equiv f, \qquad \text{in } \overline{\Omega}. \quad \blacksquare$$

Now we construct f_t and A_t in $C^\infty(\overline{\Omega} \times \mathbb{R}_+)$ verifying (3.6). Let η be a non negative function in $C_0^\infty(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} \eta \, dx = 1, \qquad \int_{\mathbb{R}^n} x\eta(x) \, dx = 0$$

and such that there exists some $r_0 >$ satisfying $B(0, r_0) \cap supp(\eta) = \emptyset$. Denote

$$\psi(x) = \eta(x)x, \;\; \eta_t(x) = \frac{1}{t^n}\eta\left(\frac{x}{t}\right) \;\text{ and }\; \psi_t(x) = \frac{1}{t^n}\psi\left(\frac{x}{t}\right) \;\text{ in } \mathbb{R}^n.$$

Define $g_t = f * \eta_t$ and $B_t = f * \psi_t$. One then has the following elementary lemma.

Lemma 6 *Let g_t and B_t be as above, one has*

$$\frac{\partial g_t}{\partial t} + divB_t = 0, \qquad in\ \mathbb{R}^n \times \mathbb{R}_+. \tag{3.8}$$

■

Proof of Lemma 6:

$$\begin{aligned} \frac{\partial g_t}{\partial t}(x) &= \frac{\partial}{\partial t}\left(\int_{\mathbb{R}^n} \frac{1}{t^n}\eta\left(\frac{x-y}{t}\right) f(y)dy\right) \\ &= -n\int_{\mathbb{R}^n} \frac{1}{t^{n+1}}\eta\left(\frac{x-y}{t}\right) f(y)dy \\ &\quad -\int_{\mathbb{R}^n} \frac{1}{t^n}\nabla\eta\left(\frac{x-y}{t}\right)\frac{x-y}{t^2} f(y)dy. \end{aligned} \tag{3.9}$$

On the other hand,

$$\begin{aligned} \mathrm{div}B_t &= \int_{\mathbb{R}^n} \frac{1}{t^n}\, div_x\left(\frac{x-y}{t}\eta\left(\frac{x-y}{t}\right)\right) f(y)dy \\ &= \int_{\mathbb{R}^n} \frac{1}{t^n} div_x\left(\frac{(x-y)}{t}\right)\eta\left(\frac{x-y}{t}\right) f(y)dy \\ &\quad +\int_{\mathbb{R}^n} \frac{1}{t^n}\frac{x-y}{t}\nabla\eta\left(\frac{x-y}{t}\right)\frac{1}{t} f(y)dy. \end{aligned} \tag{3.10}$$

We clearly have (3.9) + (3.10) = 0 and this proves the lemma. ■

One has $\int_\Omega g_t\, dx \geq |\Omega| \inf_\Omega g_t \geq |\Omega| \inf_\Omega f$. Then $c_t = \int_\Omega f\, dx / \int_\Omega g_t\, dx$ is well defined in $\mathbb{R}_+$. Since we have $\int_\Omega c_t g_t dx \equiv |\,\Omega\,|$, and by Lemma 6

$$\begin{aligned} \int_\Omega g_t\, \frac{\partial c_t}{\partial t} dx &= -\int_\Omega \frac{\partial g_t}{\partial t} c_t\, dx = c_t \int_\Omega \mathrm{div}B_t\, dx \\ &= c_t \int_{\partial\Omega} B_t.\nu d\sigma, \end{aligned} \tag{3.11}$$

the following linear problem admits a regular solution (see [4])

$$\begin{cases} \mathrm{div}D_t = \dfrac{1}{c_t}\dfrac{\partial c_t}{\partial t} \times g_t, & \text{in } \Omega \\ D_t = B_t, & \text{on } \partial\Omega. \end{cases} \tag{3.12}$$

Since $\frac{1}{c_t}\frac{\partial c_t}{\partial t} \times g_t$ and B_t depend regularly on t, the construction of solution of (3.12) in [4] ensures the regularity of D_t on $\overline{\Omega} \times \mathbb{R}_+$.

Let $f_t = c_t g_t$ and $A_t = (D_t - B_t)/g_t$, one verifies that (f_t, A_t) is a regular solution of (3.6). We turn now to the construction of u from $\overline{\Omega}$ into itself, solving

(1.2). First we remark that there exists some $0 < T < +\infty$ such that $f_T \equiv 1$ in $\overline{\Omega}$. Indeed, for $t \geq T = \dfrac{\operatorname{diam}(\overline{\Omega}) + \operatorname{diam}(supp(\xi))}{r_0}$, one has (recall that $B(0, r_o) \cap supp(\eta) = \emptyset$.)

$$\forall\, x \in \overline{\Omega}, \quad \forall\, y \in supp(\xi), \qquad \left|\frac{x-y}{t}\right| \leq r_0 \,.$$

This leads to $\forall\, x \in \overline{\Omega}$,

$$\begin{aligned} g_t(x) &= \int_{\mathbb{R}^n} \frac{1}{t^n}\eta\left(\frac{x-y}{t}\right) f(y)dy = \int_{\mathbb{R}^n \setminus supp(\xi)} \frac{1}{t^n}\eta\left(\frac{x-y}{t}\right) f(y)dy \\ &= \int_{\mathbb{R}^n \setminus supp(\xi)} \frac{1}{t^n}\eta\left(\frac{x-y}{t}\right) dy = \int_{\mathbb{R}^n} \frac{1}{t^n}\eta\left(\frac{x-y}{t}\right) dy = 1, \end{aligned} \tag{3.13}$$

and similarly, we get

$$\forall\, x \in \overline{\Omega}, \qquad \begin{aligned} B_t(x) &= \int_{\mathbb{R}^n} \frac{1}{t^n}\frac{x-y}{t}\eta\left(\frac{x-y}{t}\right) f(y)dy \\ &= \int_{\mathbb{R}^n} \frac{1}{t^n}\frac{x-y}{t}\eta\left(\frac{x-y}{t}\right) dy = 0. \end{aligned}$$

Moreover, for such t, $c_t = \dfrac{\int_\Omega dx}{\int_\Omega f\,dx} = \dfrac{|\Omega|}{\int_\Omega f\,dx} = 1$, and by definition $f_t \equiv 1$ in $\overline{\Omega}$.
Then for $t \geq T$, Lemma 1 implies that φ_t is a solution of (1.2). So we have proved the existence of $u \in C^\infty(\overline{\Omega})$ solution of (1.2) for any $f \in C^\infty(\overline{\Omega})$. ■

We will always follow this method in the proof of Theorems 1 to 5 but justifying each step more carefully.

3.2 Proofs of Theorem 7 and 8

3.2.1 Proof of Theorem 7

Let Ω be a bounded regular domain of $\mathbb{R}^n$ and f be a $C^{k,\alpha}(\overline{\Omega})$ function where $0 \leq k < \infty$ and $0 < \alpha < 1$. We extend f to all of $\mathbb{R}^n$ using the following lemma proved in the appendix.

Lemma 7 *Let Ω_1 be a tubular neighborhood of Ω (i.e. $\Omega \subset\subset \Omega_1$), $0 \leq m < \infty$, there exists $L_m : C^m(\overline{\Omega}) \to C^m(\mathbb{R}^n)$, a linear operator such that*

i) $L_m(f) = f$, in $\overline{\Omega}$.

ii) $L_m(1) \equiv 1$.

iii) $\forall\, \beta \in [0,1]$, $\forall\, l \le m$, *there exists* $C(l,\beta)$ *such that* $\forall\, f \in C^{l,\beta}(\overline{\Omega})$ *verifying (H)*

$$\|L_m f\|_{C^{l,\beta}(\mathbb{R}^n)} \le C(l,\beta)\|f\|_{C^{l,\beta}(\overline{\Omega})}. \tag{3.14}$$

iv) $L_m f \equiv 1$ *in* $\mathbb{R}^n \backslash \Omega_1$. ■

Denote $\tilde{f} = L_k(f)$. Then, there exists Ω_2 a regular domain containing $\overline{\Omega}$ such that

$$\inf_{\Omega_2} \tilde{f} \ge \frac{\inf_{\overline{\Omega}} f}{2} = \frac{a}{2} > 0. \tag{3.15}$$

More precisely, by choosing $\delta \in (0, \min(k+\alpha, 1))$

$$\Omega_2 = \left\{ y \in \Omega_1,\ d(y,\overline{\Omega}) \le \left(\frac{a}{2C(0,\delta)} \|f\|_{C^{0,\delta}} \right)^{1/\delta} \right\}, \tag{3.16}$$

where $C(0,\delta)$ is the constant given by (3.14) for $l = 0$, one verifies easily that (3.15) is true. Choose a cut-off function ζ as follows:

$$\forall\, x \in \mathbb{R}^n, \qquad \zeta(x) = \frac{d(x, \mathbb{R}^n \backslash \Omega_2)}{d(x, \mathbb{R}^n \backslash \Omega_2) + d(x,\Omega)}. \tag{3.17}$$

We replace now f by the following extension on all of $\mathbb{R}^n$ (still denoted by f)

$$\forall\, x \in \mathbb{R}^n, \qquad f(x) = \zeta(x)\tilde{f}(x) + 1 - \zeta(x). \tag{3.18}$$

Take ψ, η_t, ψ_t, g_t and B_t as in § 3.2.2, Lemma 6 still holds. Moreover, let c_t be equal to $\dfrac{\int_\Omega f\,dx}{\int_\Omega g_t\,dx}$, one has (3.11) :

$$\frac{\partial c_t}{\partial t} \int_\Omega g_t\,dx = c_t \int_{\partial\Omega} B_t.\nu\,d\sigma.$$

By definition

$$B_t = \psi_t * f = \psi_t * (f-1),$$

hence $\|B_t\|_0 \le C\|f-1\|_0$. Recall that

$$\int_\Omega g_t\,dx \ge |\Omega| \inf_\Omega g_t \ge |\Omega| \inf_{\mathbb{R}^n} f \ge \frac{|\Omega| a}{2},$$

thus by (3.11)

$$\frac{1}{c_t}\left|\frac{\partial c_t}{\partial t}\right| \le \left(\frac{2}{a|\Omega|}\right) |\partial\Omega| \|B_t\|_0 \le C(\Omega)\frac{\|f-1\|_0}{a}. \tag{3.19}$$

As in § 3.1, we solve the following linear problem

$$\begin{cases} \text{div} D_t = \dfrac{1}{c_t}\dfrac{\partial c_t}{\partial t} \times g_t, & \text{in } \Omega \\ D_t = B_t, & \text{on } \partial\Omega. \end{cases} \tag{3.20}$$

By the method in [4], we derive a solution of (3.20) with the following estimates: $\forall\, l \in \mathbb{N}$, $0 < \beta < 1$,

$$\|D_t\|_{l+1,\beta} \le C\left[\left|\frac{1}{c_t}\frac{\partial c_t}{\partial t}\right| \|(g_t - 1)\|_{C^{l,\beta}(\overline{\Omega})} + \left|\frac{1}{c_t}\frac{\partial c_t}{\partial t}\right| + \|B_t\|_{C^{l+1,\beta}(\overline{\Omega})}\right], \tag{3.21}$$

where C depends on l, β and Ω. Turning now to the estimations of g_t and B_t, we have the following lemma proved in the appendix.

Lemma 8 $\forall\, l \in \mathbb{N}$, $\forall\, \gamma$, $\beta \in [0,1]$, *we have a constant C depending only on Ω, β, γ and l such that* $\forall\, t \ge 0$,

$$\|g_t - 1\|_{l,\gamma} + \|B_t\|_{l,\gamma} \le C\|f - 1\|_{l,\gamma}, \tag{3.22}$$

and

$$\|g_t - 1\|_{l+1,\beta} + \|B_t\|_{l+1,\beta} \le C\left(\frac{1}{t^{1+\beta-\gamma}} + 1\right)\|f - 1\|_{l,\gamma}. \tag{3.23}$$

■

Combining (3.19), (3.21), (3.22) and (3.23), we obtain that, for any $0 < \beta < 1$, $l \in \mathbb{N}, \gamma \in [0,1]$ satisfying $l + \gamma \le k + \alpha$,

$$\|D_t\|_{l,\beta} \le C(l,\beta,\Omega)\frac{\|f\|_0}{a}\|f - 1\|_{l,\beta} \tag{3.24}$$

and

$$\begin{aligned} \|D_t\|_{l+1,\beta} &\le C(l,\beta,\Omega)\left[\frac{\|f-1\|_0}{a}\|g_t - 1\|_{l,\beta} + \left|\frac{1}{c_t}\frac{\partial c_t}{\partial t}\right| + \|B_t\|_{l+1,\beta}\right] \\ &\le C(l,\beta,\gamma,\Omega)\left(\frac{1}{t^{1+\beta-\gamma}} + 1\right)\frac{\|f\|_0}{a}\|f - 1\|_{l,\gamma}. \end{aligned} \tag{3.25}$$

As in § 3.1, in view of solving (1.2), we study the flow (3.7) by paying more attention to the regularity of φ_t. First of all, the existence and uniqueness of φ_t comes from the fact that

$$A_t = \frac{1}{g_t}(D_t - B_t) \in L^1_{loc}(\mathbb{R}_+, W^{1,\infty}(\overline{\Omega})) \cap C^\infty(\mathbb{R}_+^* \times \overline{\Omega}).$$

The condition $A_t \equiv 0$ on $\partial\Omega \times \mathbb{R}_+$ ensures that φ_t is always a transformation from $\overline{\Omega}$ into itself. Moreover, using a classical interpolation result in Hölder spaces (see [6]), one has

$$\|A_t\|_{l+1,\beta} \leq C\left[\|\frac{1}{g_t} - 1\|_{l+1,\beta}\|D_t - B_t\|_0 + \|\frac{1}{g_t}\|_0\|D_t - B_t\|_{l+1,\beta}\right]. \tag{3.26}$$

Since one has also the following estimate in Hölder spaces

$$\|\frac{1}{g_t} - 1\|_{l+1,\beta} \leq \frac{C}{(\inf_\Omega g_t)^{l+3}}\|g_t - 1\|_{l+1,\beta}\|g_t\|_0^{l+1}, \tag{3.27}$$

using (3.21), (3.22), (3.23) (3.24) and (3.25), we deduce that for any $0 < \delta < 1$ and $0 \leq \gamma \leq 1$,

$$\|A_t\|_{l+1,\beta} \leq C(l,\beta,\delta,\gamma,\Omega)\frac{\|f\|_0^{l+2}}{a^{l+4}}\|f\|_{0,\delta}\|f - 1\|_{l,\gamma}\left(\frac{1}{t^{1+\beta-\gamma}} + 1\right), \tag{3.28}$$

and

$$\|A_t\|_{l,\beta} \leq C(l,\beta,\delta,\Omega)\frac{\|f\|_0^{l+1}}{a^{l+3}}\|f\|_{0,\delta}\|f - 1\|_{l,\beta}. \tag{3.29}$$

Remark 7 *We have bounded $\|D_t - B_t\|_0$ in (3.26) by $\|D_t - B_t\|_{0,\delta}$ and finally we can bound it by $C\frac{\|f\|_0}{a}\|f - 1\|_{0,\delta}$ using (29) and (3.24).*

For proving $\varphi_t \in C^{k+1,\alpha}(\overline{\Omega})$ we will use the following lemma, proved in the appendix.

Lemma 9 *Let $l \geq 1, 0 \leq \beta \leq 1$ and $A_t \in L^1_{loc}(\mathbb{R}_+, C^{l,\beta}(\overline{\Omega})) \cap C^\infty(\mathbb{R}_+^* \times \overline{\Omega})$. Considering the solution φ_t of the following flow*

$$\begin{cases} \dfrac{\partial\varphi_t}{\partial t} = A_t(\varphi_t), & \text{in } \overline{\Omega} \times \mathbb{R}_+, \\ \varphi_0 = Id, & \text{in } \overline{\Omega}, \\ A_t \equiv 0, & \text{on } \partial\Omega \times \mathbb{R}_+, \end{cases} \tag{3.30}$$

one has the estimates

$$\|\varphi_t\|_{l,\beta} \leq c\left(\int_0^t \|A_s\|_{l,\beta}\, ds\right) \times \exp\left(c\int_0^t \|\nabla A_s\|_0\, ds\right), \tag{3.31}$$

and

$$\|\varphi_t - Id\|_{l,\beta} \leq c\left(\int_0^t \|A_s\|_{l,\beta}\, ds\right) \times \left[\exp\left(c\int_0^t \|\nabla A_s\|_0 ds\right) - 1\right], \tag{3.32}$$

where the constant c only depends on l, β and Ω. ■

By the previous lemma, for $l = k$, $\beta = \alpha$ and by (3.29), we already establish that $\varphi_t \in C^{k,\alpha}(\overline{\Omega})$. In view of proving $\varphi_t \in C^{k+1,\alpha}(\overline{\Omega})$, we cannot use directly the previous lemma for $l = k+1$ and $\beta = \alpha$. Indeed, by (3.28), we just have $\|A_t\|_{k+1,\alpha} \leq C\|f-1\|_{k,\alpha}/t$ for $t \leq T$, and this cannot imply that $\|A_t\|_{k+1,\alpha} \in L^1_{loc}(\mathbb{R}_+)$. But, on the other hand, since we have the existence and uniqueness of φ_t in $C^{k,\alpha}$, we simply have to establish some a priori estimate for the $C^{k+1,\alpha}(\overline{\Omega})$ norm of φ_t in the case where $A_t \in C^\infty(\mathbb{R}_+ \times \overline{\Omega})$ (i.e. $f \in C^\infty$).

Using a classical result proved in [6], on the $C^{l,\beta}$ estimation of the composition of applications, we have $\forall\, l \in \mathbb{N}$, $\forall\, \beta \in [0,1]$,

$$\|A_t(\varphi_t)\|_{l,\beta} \leq C(l,\beta,\Omega)\Big[\; \|A_t\|_{l,\beta}\|\nabla\varphi_t\|_0^{l+\beta} + \|\nabla A_t\|_0\|\varphi_t\|_{l,\beta} + \|A_t\|_0 \;\Big]. \quad (3.33)$$

This leads to the following inequality: There exists C depending on l, β and Ω such that $\forall\, l \in \mathbb{N}$, $\forall\, \beta \in [0,1]$,

$$\|A_t(\varphi_t)\|_{l,\beta} \leq C\Big[\; \|A_t\|_{l,\beta}\left(1 + \|\varphi_t - Id\|_{0,1}^{l+\beta}\right) + \|\nabla A_t\|_0\|\varphi_t - Id\|_{l,\beta} \;\Big]. \quad (3.34)$$

We will also use the following lemma proved in the appendix. For x, y in $\overline{\Omega}$, $x \neq y$, ψ an application from $\overline{\Omega}$ into $\mathbb{R}^n$ and $l \in \mathbb{N}^*$, we denote by $T^l_\psi(x,y)$ the following quantity

$$T^l_\psi(x,y) = \sum_{|h|=l} |\partial^h\psi(x) - \partial^h\psi(y)|\,. \quad (3.35)$$

Lemma 10 *Let $A \in C^{l,\beta}(\overline{\Omega})$ and $\varphi \in C^{l,\beta}(\overline{\Omega},\overline{\Omega})$. Then there exists a constant $C(l,\beta,\Omega)$ such that $\forall\, x,y \in \overline{\Omega}$,*

$$\begin{aligned} T^l_{A(\varphi)}(x,y) \leq\; & C\,|x-y|^\beta \times \Big[\; + T^l_A(\varphi(x),\varphi(y)) + \|\nabla A\|_0\|\varphi - Id\|_{l,\beta} \\ & + \|A\|_{l,\beta}\|\nabla(\varphi - Id)\|_0\left(1 + \|\varphi - Id\|_1^{l+\beta-1}\right)\;\Big]\,. \end{aligned} \quad (3.36)$$

■

Deriving $(k+1)$ times the flow equation (3.7), we obtain that

$$\forall\, x,y \in \overline{\Omega}, \qquad \frac{\partial}{\partial t}T^{k+1}_{\varphi_t}(x,y) \leq T^{k+1}_{A_t(\varphi_t)}(x,y). \quad (3.37)$$

Applying Lemma 10, with $l = k+1$ and $\beta = \alpha$, we have $\forall\, x,y \in \overline{\Omega}$,

$$\begin{aligned} & \frac{\partial}{\partial t}T^{k+1}_{\varphi_t}(x,y) \\ \leq\; & C\Big\{\; T^{k+1}_{A_t}(\varphi_t(x),\varphi_t(y)) + |x-y|^\alpha \times \Big[\; \|\nabla A_t\|_0\|\varphi_t - Id\|_{k+1,\alpha} \\ & + \|A_t\|_{k+1,\alpha}\|\nabla(\varphi_t - Id)\|_0\left(1 + \|\nabla(\varphi_t - Id)\|_0^{k+\alpha}\right)\;\Big]\;\Big\}\,. \end{aligned} \quad (3.38)$$

First, using (3.28), we have

$$\|\nabla A_t\|_0 \leq C(\delta,\Omega)\frac{\|f\|_0^2}{a^4}\|f\|_{0,\delta}^2\left(\frac{1}{t^{1-\delta}}+1\right), \tag{3.39}$$

and

$$\|A_t\|_{k+1,\alpha} \leq C(\delta,\alpha,k,\Omega)\frac{\|f\|_0^{k+3}}{a^{k+5}}\|f\|_{0,\delta}\|f-1\|_{k,\alpha}\left(\frac{1}{t}+1\right). \tag{3.40}$$

Moreover, by Lemma 9, we have

$$\|\nabla(\varphi_t - Id)\|_0 \leq c\left(\int_0^t \|\nabla A_s\|_0\, ds\right) \times \left[exp\left(c\int_0^t \|\nabla A_s\|_0\, ds\right) - 1\right] . \tag{3.41}$$

By (3.39) this leads to (for $t \leq T$)

$$\|\nabla(\varphi_t - Id)\|_0 \leq ct^\delta \left[exp\left(ct^\delta\right) - 1\right], \tag{3.42}$$

where c only depends on a, δ, T, $\|f\|_{0,\delta}$ and Ω.

Now, we estimate $\int_0^t T_{A_s}^{k+1}\left(\varphi_s(x),\varphi_s(y)\right)\, ds$ for any $x \neq y$ in $\overline{\Omega}$.

$$\begin{aligned}\int_0^t T_{A_s}^{k+1}\left(\varphi_s(x),\varphi_s(y)\right)\, ds = & \int_0^{|x-y|} T_{A_s}^{k+1}\left(\varphi_s(x),\varphi_s(y)\right)\, ds \\ & + \int_{|x-y|}^t T_{A_s}^{k+1}\left(\varphi_s(x),\varphi_s(y)\right)\, ds\ ,\end{aligned} \tag{3.43}$$

the first integral in the right-hand side of (3.43) can be bounded by

$$\begin{aligned}\int_0^{|x-y|} 2\|A_s\|_{k+1}\, ds & \leq & C\int_0^{|x-y|} \|f-1\|_{k,\alpha}\left(\frac{1}{s^{1-\alpha}}+1\right)\, ds \\ & \leq & C\|f-1\|_{k,\alpha}|x-y|^\alpha\ ,\end{aligned} \tag{3.44}$$

where we have used (3.28). The second integral can be bounded in the following way

$$\int_{|x-y|}^t T_{A_s}^{k+1}(\varphi_s(x),\varphi_s(y))\, ds \leq |x-y|^{\alpha'} \times \left(\int_{|x-y|}^t \|A_s\|_{k+1,\alpha'}\|\nabla\varphi_s\|_0^{\alpha'}\, ds\right), \tag{3.45}$$

where we have chosen $\alpha' = \dfrac{1+\alpha}{2} > \alpha$. By (3.28) and (3.31), (3.45) leads to (for $t \leq T$),

$$\begin{aligned}
&\int_{|x-y|}^{t} T_{A_s}^{k+1}(\varphi_s(x), \varphi_s(y))\, ds \\
\leq\ & |x-y|^{\alpha'} \times C \left[\int_{|x-y|}^{t} \left(\frac{1}{s^{1+\alpha'-\alpha}} + 1 \right) \|f-1\|_{k,\alpha}\, ds \right] \\
\leq\ & |x-y|^{\alpha'} \times \left(\int_{|x-y|}^{\infty} \frac{C'}{s^{1+\alpha'-\alpha}} \|f-1\|_{k,\alpha}\, ds \right) \\
\leq\ & C'' \|f-1\|_{k,\alpha} |x-y|^{\alpha} ,
\end{aligned} \tag{3.46}$$

where C'' only depends on a, δ, k, α, T, $\|f\|_{0,\delta}$ and Ω. Combining (3.43), (3.44) and (3.46), we obtain that for any x, $y \in \overline{\Omega}$ and for any $t \leq T$,

$$\int_0^t T_{A_s}^{k+1}(\varphi_s(x), \varphi_s(y))\, ds \leq C \|f-1\|_{k,\alpha} |x-y|^{\alpha}. \tag{3.47}$$

Integrating (3.38) by using (3.39), (3.40), (3.42) and (3.47), we have (for $t \leq T$)

$$T_{\varphi_t}^{k+1}(x,y) \quad \leq \quad C \left(\int_0^t \frac{1}{s^{1-\delta}} \|\varphi_s - Id\|_{k+1,\alpha}\, ds + \|f-1\|_{k,\alpha} \right) \times |x-y|^{\alpha}, \tag{3.48}$$

$\forall\, x, y \in \overline{\Omega}$, where C only depends on a, δ, k, α, T, $\|f\|_{0,\delta}$ and Ω. Taking the sup on x, y, in $\overline{\Omega}$ in (3.48), we obtain

$$\|\varphi_t - Id\|_{k+1,\alpha} \leq C \left[\int_0^t \frac{\|\varphi_s - Id\|_{k+1,\alpha}}{s^{1-\delta}}\, ds + \|f-1\|_{k,\alpha} \right] .$$

Since $\displaystyle\int_0^t \frac{ds}{s^{1-\delta}} < +\infty$, we get the desired result by Gronwall's Lemma. ■

3.2.2 Proof of Theorem 8

Take f in $C^{k,1}(\overline{\Omega})$ for $k \geq 0$ and verifying (H). We consider A_t, f_t and φ_t constructed as in the proof of Theorem 1. By (3.28), for any $0 < \alpha < 1$, we have

$$\|A_t\|_{k+1,\alpha} \leq C \frac{\|f-1\|_{k,1}}{t^{\alpha}}. \tag{3.49}$$

Thus $\|A_t\|_{k+1,\alpha} \in L^1_{loc}(\mathbb{R}_+)$ and by Lemma 9, we get the desired regularity result and the estimate follows from (3.32) and (3.39). ■

3.3 The limiting cases

In this last section, we treat the limiting cases considered in § 2.2 ($f \in C^0, L^\infty$ and BMO) by the path- regularizing flow method. Let U be a neighborhood of Ω. We are looking for u solution of

$$\begin{cases} \det(\nabla u) = f, & \text{on } \Omega \\ u(\overline{\Omega}) \subset U. \end{cases} \tag{3.50}$$

This method leads to exactly to the same gain of regularity of u, established in § 2, but without keeping the boundary fixed except under an additional regularity hypothesis for f closed to $\partial\Omega$.

3.3.1 The C^0 and Dini-continuous cases

Taking $f \neq 1$ in $C^0(\overline{\Omega})$ or in $W^{1,n} \cap L^\infty(\Omega)$ verifying (H), we extend f into $L_0(f)$ on all of $\mathbb{R}^n$ as indicated in Lemma 7. Clearly, in this case, $\inf_{\mathbb{R}^n} L_0(f) = \inf_\Omega f$ and $\sup_{\mathbb{R}^n} L_0(f) = \sup_\Omega f$.

Let U be a neighborhood of $\overline{\Omega}$ included in Ω_1. We can always ensure that (see Lemma 7) $supp(L_0(f) - 1) \subset U$. We also need to have an extension $\tilde{f}$ of f such that

$$supp(\tilde{f} - 1) \subset U, \ \inf_{\Omega_1} \tilde{f} = \inf_\Omega f, \ \sup_{\Omega_1} \tilde{f} = \sup_\Omega f \quad \text{and} \quad \int_{\Omega_1} \tilde{f}\, dx = |\Omega_1|. \tag{3.51}$$

Consider $\tilde{\xi}$ a cut-off function $0 \leq \tilde{\xi} \leq 1$ such that $supp(\tilde{\xi}) \subset U \backslash \overline{\Omega}$. The function

$$\tilde{g} = (1 - \tilde{\xi}) L_0(f) + \tilde{\xi} \inf_\Omega f \tag{3.52}$$

verifies a) and

$$\int_{\Omega_1} \tilde{g}\, dx = \int_{\Omega_1} (1 - \tilde{\xi}) L_0(f)\, dx + \inf_\Omega f \int_{\Omega_1} \tilde{\xi}\, dx. \tag{3.53}$$

Since $\inf_\Omega f < 1$, we may choose $\tilde{\xi}$ such that $\int_{\Omega_1} \tilde{g}\, dx < |\Omega_1|$. (In fact, $\tilde{\xi}$ can be taken as close as we want from the characteristic function of $U \backslash \overline{\Omega}$). Similarly, by using $\sup_\Omega f$ instead of $\inf_\Omega f$, we can construct a function $\tilde{h}$ verifying a) and $\int_{\Omega_1} \tilde{h}\, dx > |\Omega_1|$. Finally, a convex combination of $\tilde{g}$ and $\tilde{h}$ may ensure (3.51).

Let η, ψ, η_t and ψ_t be as in § 3.1. Denote by g_t and B_t the following applications

$$g_t = \eta_t * f \quad \text{and} \quad B_t = \psi_t * f. \tag{3.54}$$

Clearly $\inf_{t\geq 0} g_t \geq \inf_{\Omega} f$ and

$$\|B_t\|_0 \leq C\|f-1\|_0. \tag{3.55}$$

We need the following lemma (a refinement of Lemma 8).

Lemma 11 *Let g_t and B_t be as above. Then, for any $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that $\forall\, t \leq T_\varepsilon$,*

$$\|\nabla g_t\|_0 + \|\nabla B_t\|_0 \leq \frac{\varepsilon}{t}\,. \tag{3.56}$$

■

Proof of Lemma 11. Suppose first $f \in C^0(\overline{\Omega})$. We have

$$\nabla g_t(x) = \int_{\mathbb{R}^n} \nabla\eta_t(x-y) f(y)dy = \int_{\mathbb{R}^n} \frac{1}{t}\nabla\eta\left(\frac{x-y}{t}\right)(f(y)-f(x))\frac{1}{t^n}dy \tag{3.57}$$

where we have used the fact $\int_{\mathbb{R}^n} \nabla\eta\, dx = 0$. This leads to

$$\nabla g_t(x) = \int_{supp(\eta)} \frac{1}{t}\nabla\eta(z)\Big(f(x+tz)-f(x)\Big)\,dz.$$

Since f is uniformly continuous on $\mathbb{R}^n$, we get $\|\nabla g_t(x)\|_0 \leq \varepsilon/t$ for t sufficiently small. We can estimate in the same way $\|\nabla B_t\|_0$, and this proves the lemma for $f \in C^0(\overline{\Omega})$. ■

Take $A_t = B_t/g_t$ and consider the flow (3.7) on all of $\mathbb{R}^n$. Combining (3.54), (3.55) and (3.56), clearly, we deduce that if $\|A_t\|_0 \leq C\frac{\|f-1\|_0}{a}$ and $\forall\, \varepsilon > 0$, there exists $T_\varepsilon > 0$ such that $\forall\, t \leq T_\varepsilon$,

$$\|\nabla A_t\|_0 \leq \frac{\varepsilon}{t}. \tag{3.58}$$

For any $x \in \mathbb{R}^n$, the existence of $\varphi_t(x)$ for all $t \geq 0$ is given by classical arguments. We prove now the uniqueness and some regularity result on φ_t (a crucial point of our proof!).

Lemma 12 $\forall\, \mu \in [\,0,1)$, *there exists $\tilde{T}_\mu > 0$ such that for any two continuous solutions φ_t^1 and φ_t^2 of (3.7), for any $t \leq \tilde{T}_\mu$, $x \neq y \in \mathbb{R}^n$*

$$|\varphi_t^1(x) - \varphi_t^2(y)| \leq C(|x-y|^\mu + |x-y|) \tag{3.59}$$

where C only depends on $\|f-1\|_0$ and a. ■

Proof of Lemma 12. Let $x \neq y$ be two points of $\mathbb{R}^n$. Let $\varphi_t^1(x), \varphi_t^2(y)$ be two continuous solutions of (3.7). Defining $\tau = |x-y|$ we have

$$\varphi_t^1(x) - \varphi_t^2(y) = \int_0^t A_s(\varphi_s^1(x)) - A_s(\varphi_s^2(y))\, ds + (x-y)\,. \tag{3.60}$$

For $t \leq \tau$, since $\|A_s\|_0 \leq C\dfrac{\|f-1\|_0}{a}$,

$$|\varphi_t^1(x) - \varphi_t^2(y)| \leq \tau C\frac{\|f-1\|_0}{a} + |x-y| \leq C'|x-y|. \tag{3.61}$$

And for $t \geq \tau$, (3.60) yields

$$\begin{aligned} |\varphi_t^1(x) - \varphi_t^2(y)| &\leq \int_0^\tau \|A_s\|_0\, ds + \int_\tau^t \left|A_s(\varphi_s^1(x)) - A_s(\varphi_s^2(y))\right|\, ds \\ &\quad + |x-y| \\ &\leq C|x-y| + \int_\tau^t \left|A_s(\varphi_s^1(x)) - A_s(\varphi_s^2(y))\right|\, ds. \end{aligned} \tag{3.62}$$

Denote by $\tilde{\varphi}$ the solution of the following flow

$$\begin{cases} \dfrac{\partial \tilde{\varphi}_t}{\partial t} = A_{t+\tau}(\tilde{\varphi}_t), & \text{in } \Omega \\ \tilde{\varphi}_0 = Id, & \text{in } \Omega. \end{cases}$$

Clearly, since $A_{t+\tau} \in C^\infty(\mathbb{R}^n \times \mathbb{R}_+)$, $\tilde{\varphi}$ exists and is unique, then we have $\varphi_t^1(x) = \tilde{\varphi}_{t-\tau}(\varphi_\tau^1(x))$ and $\varphi_t^2(y) = \tilde{\varphi}_{t-\tau}(\varphi_\tau^2(y))$ for any $t \geq \tau$. This implies

$$\begin{aligned} &\int_\tau^t \left|A_s(\varphi_s^1(x)) - A_s(\varphi_s^2(y))\right|\, ds \\ \leq\ &\int_\tau^t \|\nabla A_s\|_0 \times \|\nabla \tilde{\varphi}_{s-\tau}\|_0 \times |\varphi_\tau^1(x) - \varphi_\tau^2(y)|\, ds \\ \leq\ &C|x-y| \int_\tau^t \|\nabla A_s\|_0 \times \|\nabla \tilde{\varphi}_{s-\tau}\|_0\, ds. \end{aligned} \tag{3.63}$$

Let $\varepsilon > 0$, if $t \leq T_\varepsilon$, by (3.58), $\|\nabla A_s\|_0 \leq \dfrac{\varepsilon}{s}$ for s in $[\tau, t]$. So we get, for any $s \leq t - \tau$,

$$\frac{\partial}{\partial s}\|\nabla \tilde{\varphi}_s\|_0 \leq \frac{\varepsilon}{s+\tau}\|\nabla \tilde{\varphi}_s\|_0$$

and this yields

$$\forall\, s \leq t-\tau, \qquad \|\nabla \tilde{\varphi}_s\|_0 \leq \left(\frac{s+\tau}{\tau}\right)^\varepsilon. \tag{3.64}$$

Combining (3.62) (3.63) and (3.64), we get for $t \leq \min(1, T_\varepsilon)$

$$\begin{aligned} |\varphi_t^1(x) - \varphi_t^2(y)| &\leq C|x-y| + C\tau \int_\tau^t \frac{\varepsilon}{s}\left(\frac{s}{\tau}\right)^\varepsilon ds \\ &\leq C|x-y| + C\tau^{1-\varepsilon}(t^\varepsilon - \tau^\varepsilon) \\ &\leq C\left[\, |x-y|^{1-\varepsilon} + |x-y| \,\right]. \end{aligned}$$

By taking $\tilde{T}_\mu = \min(1, T_{1-\mu})$, this proves Lemma 12. ∎

Lemma 12 implies that φ_t is unique for $t \leq T_\mu$. Moreover, since $A_t \in C^\infty(\mathbb{R}^n \times \mathbb{R}_+^*)$, if there exists some $1 > \alpha > 0$ and some $t > 0$ such that φ_t lies in $C^{0,\alpha}(\mathbb{R}^n)$, then $\varphi_s \in C^{0,\alpha}(\mathbb{R}^n)$ for any $s \geq t$. Combining this remark with Lemma 12, we deduce that φ_t, solution of (3.7), is unique and belongs to $\cap_{\alpha<1} C^{0,\alpha}(\mathbb{R}^n)$.

Let $T > 0$. Now we are going to prove that φ_T is an homeomorphism from $\overline{\Omega}$ into $\varphi_T(\overline{\Omega})$, which is a solution of (1.3). Let $\overline{\varphi}_t$ be the solution for $t \leq T$ of the following flow

$$\begin{cases} \dfrac{\partial \overline{\varphi}_t}{\partial t} = -A_{T-t}(\overline{\varphi}_t), & \text{in } \mathbb{R}^n \\ \overline{\varphi}_0 = Id, & \text{in } \mathbb{R}^n. \end{cases} \tag{3.65}$$

Since $A_{T-t} \in C^\infty([0,T) \times \mathbb{R}^n)$ and $\cup_{t \leq T} supp(A_{T-t})$ is included in some compact set, $\overline{\varphi}_t$ is well defined in $C^\infty([0,T) \times \mathbb{R}^n)$. Furthermore, since $\|A_{T-t}\|_0$ is uniformly bounded on $[0,T]$, $\overline{\varphi}_t$ converges in $C^0(\mathbb{R}^n)$ when $t \longrightarrow T$, so $\overline{\varphi}_T$ is continuous on $\mathbb{R}^n$.

Moreover, by a simple computation similar to the proof of Lemma 5, we have

$$\forall\, t < T, \qquad \frac{\partial}{\partial t}\left[f_{T-t}(\overline{\varphi}_t) \det(\nabla \overline{\varphi}_t) \right] \equiv 0,$$

and this implies

$$\forall\, t < T, \qquad \det(\nabla \overline{\varphi}_t) = \frac{f_T}{f_{T-t}(\overline{\varphi})}. \tag{3.66}$$

This can be written in a weaker form

$$\forall\, E \text{ open subset of } \mathbb{R}^n, \qquad \int_E f_T \, dx = \int_{\overline{\varphi}_t(E)} f_{T-t} \, dx. \tag{3.67}$$

By the uniqueness of the flows on $[0,T)$ we have

$$\forall\, s \in (0,T], \qquad \overline{\varphi}_{T-s} \circ \varphi_T = \varphi_s,$$

implying, in particular, that $\overline{\varphi}_T \circ \varphi_T = Id$ on $\mathbb{R}^n$. Using the same argument, we also have $\varphi_T \circ \overline{\varphi}_T = Id$. Thus φ_T and $\overline{\varphi}_T$ are two homeomorphisms of $\mathbb{R}^n$ and (3.67) implies

$$\forall\, E \text{ open subset of } \mathbb{R}^n, \qquad \int_{\varphi_T(E)} dx = \int_E f \, dx. \tag{3.68}$$

Finally, since $supp(f-1) \subset U$, we may choose T sufficiently small such that $supp(f_t - 1) \subset U$, so $\int_U f_t\, dx = |U|$ for $t \leq T$. Moreover, $f_T \in C^\infty(\mathbb{R}^n)$ and verifies (H), thus there exists some $v \in C^\infty(\overline{U})$ solution of

$$\begin{cases} \det(\nabla v) = f_T, & \text{on } \overline{U} \\ v(x) = x, & \text{in } \partial U, \end{cases}$$

and for $u = v \circ \varphi_T$, we have

$$\forall\, E \text{ open subset of } \Omega, \qquad |u(E)| = \int_E f\, dx \quad \text{and} \quad u(\overline{\Omega}) \subset U.$$

This proves the case where $f \in C^0(\overline{\Omega})$. Suppose now f to be verifying (1.4), then there exists an increasing function $d : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\int_0^1 \frac{d(s)}{s}\, ds < +\infty \quad \text{and} \quad \forall\, x, y \in \overline{\Omega},\ |f(x) - f(y)| \leq d(|x-y|).$$

This property is preserved by the extension construction in Lemma 7. Let A_t, f_t, φ_t and $\overline{\varphi}_t$ be chosen as above. One verifies easily that $\|\nabla A_t\|_0 \leq C\dfrac{d(tR)}{t}$, thus $A_t \in L^1_{loc}(\mathbb{R}_+, W^{1,\infty}(\mathbb{R}^n))$, implying that φ_T and $\overline{\varphi}_T$ are Lipschitz. Furthermore, since

$$\overline{\varphi}_{T-t} \in C^\infty, \text{ for } T > t > 0, \text{ and } \|\overline{\varphi}_{T-t} - \overline{\varphi}_T\|_{0,1} \longrightarrow 0 \text{ as } t \to 0,$$

we get $\overline{\varphi}_T \in C^1$. Then by the fact that $\det(\nabla\overline{\varphi}_T) \geq a > 0$, we easily obtain that $\varphi_T \in C^1$ and thus $v \circ \varphi_T \in C^1$. This proves then the following result:

Theorem 9 *Let Ω be a regular domain, and f be verifying (H) and (1.4). Then we have $u \in C^1$ solution of $det(\nabla u) = f$.* ■

3.3.2 The L^∞ and BMO cases

a). The C^0 case: Let Ω_0 be a regular bounded domain of $\mathbb{R}^n$ and η, ψ, η_t and ψ_t be as in § 3.1. The proof comes from the following lemma.

Lemma 13 *There exists $c > 0$ depending only on Ω_0, η and ψ, such that for any $0 < \alpha < 1$ and $g \in L^\infty(\Omega_0)$ verifying (H), if there exists a positive constant c_g such that $\dfrac{\|g - c_g\|_0}{\inf_{\Omega_0} g} \leq \dfrac{\alpha}{c}$. Then there exists a homeomorphism v in $C^{0,1-\alpha}(\overline{\Omega}_0)$ solution of (3.50).*

Moreover, if $supp(g - c_g) \subset \Omega_0$, v is an homeomorphism from Ω_0 into itself verifying $v(x) = x$ on $\partial\Omega_0$. ■

Proof of Lemma 13. Suppose $supp(g - c_g) \subset \Omega_0$. We extend g by c_g out of Ω_0. Let g_t and B_t be defined as in § 3.1. Define $A_t = \frac{B_t}{g_t}$. Clearly, for $t > 0$, we have

$$\|\nabla A_t\|_0 \leq C \frac{\|g - c_g\|_0}{\inf\limits_{\Omega_0} g} \frac{1}{t}, \tag{3.69}$$

where C only depends on Ω_0, η and ψ. Let $0 < \alpha < 1$, if $C\,\|g - c_g\|_\infty < \alpha \inf\limits_{\Omega_0} g$. Since $\|A_t\|_0$ is bounded independently of t, following the arguments of the proof of Lemma 4, we get that φ_t, solution of the flow (3.7), is in $C^{0,1-\alpha}(\overline{\Omega}_0)$.

Furthermore for t sufficiently small (depending only on $\text{dist}(\partial\Omega_0, supp(g - c_g))$ we have $B_t \equiv 0$ on $\partial\Omega_0$ and φ_t is a homeomorphism from Ω_0 into itself and $\varphi_t(x) = x$ on $\partial\Omega_0$. Solve now the following equation

$$\begin{cases} \det(\nabla w(x)) = g_t(x), & \text{in } \Omega_0 \\ w(x) = x, & \text{on } \partial\Omega_0 . \end{cases}$$

This can be done as in § 3.1, since $g_t \in C^\infty(\overline{\Omega}_0)$ and $\int_{\Omega_0} g_t\, dx \equiv \int_{\Omega_0} g\, dx = |\, \Omega_0 \,|$ for t chosen as above. Now $v = w \circ \varphi_t$ is a solution of Lemma 13.
In the general case, g can be extended on all of $\mathbb{R}^n$ as in § 3.3.1, but by using 1 instead of c_g, and the rest of the proof is similar as in § 3.3.1. This concludes the proof of this lemma. ∎

Consider f verifying the hypothesis of Theorem 2. We proceed as in § 2.2.2, and by using Lemma 13, we prove that the existence of the sequence $w_k \in C^{0,1-\alpha}$ with

$$p \geq \max\left\{ \frac{-\ln a}{\ln\left(1 + \frac{\alpha}{c}\right)}, \frac{\ln b}{\ln\left(1 + \frac{\alpha}{c} a\right)} \right\}. \tag{3.70}$$

where $a = \inf\limits_{\Omega} f$ and $b = \sup\limits_{\Omega} f$. Then $w = w_p \circ \ldots \circ w_1$ belongs to $C^{0,\beta}$, with $\beta = (1 - \alpha)^p$. Taking α tending to ∞ and p as in (3.70), we get the desired result.

b). The *BMO* case: The proof is based on the following lemma.

Lemma 14 *Let Ω_0 be a regular bounded domain of $\mathbb{R}^n$. Then there exists $c > 0$ depending only on Ω_0, η and ψ (defined in § 3.1), such that for any $0 < \alpha < 1$ and $g \in BMO(\Omega_0)$ verifying (H), if $\frac{\|g\|_{BMO}}{\inf\limits_{\Omega_0} g} \leq \frac{\alpha}{c}$, then there exists a homeomorphism v in $C^{0,1-\alpha}(\overline{\Omega}_0)$ solution of (3.50).*

Moreover, if there exists c_g such that $supp(g - c_g) \subset \Omega_0, v$ is a homeomorphism from Ω_0 into itself verifying $v(x) = x$ on $\partial\Omega_0$. ∎

Proof of Lemma 14. Suppose that $supp(g - c_g) \subset \Omega_0$, we extend g by c_g out of Ω_0. Let g_t and B_t be defined as in § 3.1.2 and define $A_t = B_t/g_t$. We have then

$$\begin{aligned} | A_t(x) | &= \frac{\displaystyle\int_{\mathbb{R}^n} \frac{|x-y|}{t} \eta_t(x-y) g(y)\, dy}{\displaystyle\int_{\mathbb{R}^n} \eta_t(x-y) g(y)\, dy} = \frac{\displaystyle\int_{supp(\eta)} \eta(y) g(x+ty) | \, y \, | \, dy}{\displaystyle\int_{supp(\eta)} \eta(y) g(x+ty)\, dy} \\ &\leq \operatorname{diam}(supp(\eta)). \end{aligned}$$

On the other hand,

$$\begin{aligned} \|\nabla A_t\|_0 &= \left\| \frac{\displaystyle\int_{\mathbb{R}^n} \nabla\psi_t(x-y) g(y)\, dy}{\displaystyle\int_{\mathbb{R}^n} \eta_t(x-y) g(y)\, dy} - A_t \frac{\displaystyle\int_{\mathbb{R}^n} \nabla\eta_t(x-y) g(y)\, dy}{\displaystyle\int_{\mathbb{R}^n} \eta_t(x-y) g(y)\, dy} \right\|_0 \\ &\leq \frac{\displaystyle\frac{C}{t} \int_{B_{x,tR}} |g(y) - g_{x,tR}|\, dy}{t^n \inf_{\Omega_0} g}, \end{aligned} \tag{3.71}$$

where $R = \operatorname{diam}(supp(\eta))$, (we have used the fact that $\displaystyle\int_{\mathbb{R}^n} \nabla\psi_t\, dx = \int_{\mathbb{R}^n} \nabla\eta_t\, dx = 0$). The constant C in (3.71) only depends on η and ψ. Thus we have established the following estimate

$$\|\nabla A_t\|_0 \leq C \frac{\|g\|_{BMO}}{\inf_{\Omega_0} g} \frac{1}{t}. \tag{3.72}$$

We finish the proof of Lemma 14 by following the arguments of Lemma 13. ∎

We extend f to all of $\mathbb{R}^n$ as in § 3.3.1 such that $supp(f - 1) \subset U$. Let $M > \inf_{\Omega} f$, we use the same notation as in § 2.2.2, we decompose f as $c_h h \times T_M(f)$ where $h = f/T_M(f)$, we use Lemma 14 to resolve $\det(\nabla v) = c_h h$ on U, and then we resolve $\det(\nabla w) = T_M(f) \circ v^{-1}$, $u = w \circ v$ is the desired solution. ∎

A Proof of Lemma 1

Suppose first α to be close to 1/2. We will construct Φ as the composition of two Lipschitz homeomorphisms Ψ and φ ($\phi = \Psi \circ \varphi$) where φ realizes a homeomorphism from A into A (resp. B into B) such that $\varphi(x) = x$ on ∂A (resp. ∂B). On the other hand, Ψ realizes a homeomorphism from A into $\tilde{A}$ (resp. B into $\tilde{B}$) such that $\tilde{A}$ and $\tilde{B}$ is a partition of D with $\mid \tilde{A} \mid = \alpha$, $\mid \tilde{B} \mid = \beta$ and $\Psi(x) = x$ on ∂D.

More precisely, we are looking for Ψ of the following form, (for α close to 1/2)

$$\begin{cases} \Psi(x) = \left(x', 2x_n \left[\alpha + \left(\frac{1}{2} - \alpha\right) h(x', x_n)\right]\right) & \text{in } A \\ \Psi(x) = \left(x', 1 - 2(1 - x_n) \left[\beta + \left(\frac{1}{2} - \beta\right) \tilde{h}(x', x_n)\right]\right) & \text{in } B \end{cases} \tag{A.1}$$

where $x = (x', x_n)$ (i.e. $x' = (x_1, \dots, x_{n-1})$) and where h and $\tilde{h}$ will be determined below with the conditions

$$h, \tilde{h} \equiv 1 \qquad \text{on} \qquad \partial D \cap \partial([0,1]^{n-1} \times \mathbb{R}) \tag{A.2}$$

and

$$h(x', 1/2) = \tilde{h}(x', 1/2) \qquad \text{for } x' \in [0,1]^{n-1}. \tag{A.3}$$

Thus we have

$$\nabla\Psi(x) = \begin{pmatrix} & & 0 \\ & I_{n-1} & \vdots \\ & & 0 \\ (1-2\alpha)x_n \nabla_{x'} h, & & 2\alpha + (1-2\alpha)\frac{\partial}{\partial x_n}(x_n h) \end{pmatrix} \qquad \text{in } A. \tag{A.4}$$

Then we get

$$\det \nabla\Psi(x) = 2\alpha + (1-2\alpha)\frac{\partial}{\partial x_n}(x_n h) \qquad \text{in } A, \tag{A.5}$$

and

$$\det \nabla\Psi(x) = 2\beta + (2\beta - 1)\frac{\partial}{\partial x_n}\left((1-x_n)\tilde{h}\right) \qquad \text{in } B. \tag{A.6}$$

Since we would like to ensure

$$\det \nabla\Phi(x) = \det \nabla\Psi(\varphi(x)) \times \det \nabla\varphi(x) = \begin{cases} 2\alpha & \text{in } A \\ 2\beta & \text{in } B, \end{cases} \tag{A.7}$$

it suffices to construct φ, homeomorphism of A (resp. B) verifying

$$\begin{cases} \det\nabla\varphi^{-1}(x',x_n) &= 1+\left(\dfrac{1-2\alpha}{2\alpha}\right)\dfrac{\partial}{\partial x_n}(x_n h) \quad \text{in } A \\ \varphi^{-1}(x) &= x \quad \text{on } \partial A, \end{cases} \tag{A.8}$$

and

$$\begin{cases} \det\nabla\varphi^{-1}(x',x_n) &= 1+\left(\dfrac{2\beta-1}{2\beta}\right)\dfrac{\partial}{\partial x_n}\left((1-x_n)\tilde{h}\right) \quad \text{in } B \\ \varphi^{-1}(x) &= x \quad \text{on } \partial B, \end{cases} \tag{A.9}$$

(A.8) and (A.9) will be solved by following the flow-method of Dacorogna and Moser [4]. The difficulty here comes from the fact that we are working on non smooth domains which are only Lipschitz and we want to obtain the Lipschitz estimation (2.4)(iii) (the same regularity). This is the reason why we should introduce some particular functions for h and $\tilde{h}$ so as to solve the first step of the method (that is to solve $\operatorname{div}(v) = f - 1$) with the desired regularity.

We are looking for h and w defined in A such that

$$\begin{cases} \operatorname{div}(w) &= \dfrac{\partial}{\partial x_n}(x_n h) \quad \text{in } A \\ w &= 0 \quad \text{on } \partial A. \end{cases} \tag{A.10}$$

Indeed, by taking $v = \dfrac{1-2\alpha}{2\alpha} w$ we have solved

$$\operatorname{div}(v) = f - 1 \tag{A.11}$$

where $f(x',x_n) = 1 + \dfrac{1-2\alpha}{2\alpha}\dfrac{\partial}{\partial x_n}(x_n h)$. We remark that, if (A.10) is true, we have

$$\begin{cases} h(x',x_n) &= \displaystyle\int_0^1 \operatorname{div}(w)(x',sx_n)ds \quad \text{in } A \\ w &= 0 \quad \text{on } \partial A\,. \end{cases} \tag{A.12}$$

This is equivalent to (A.10). With view to reaching a Lipschitz solution h of (A.12) verifying (A.2), we would construct w such that

$$\begin{cases} w &\in W^{1,\infty}(A) \\ \operatorname{div}(w) &\in W^{1,\infty}(A) \\ w &= 0, \quad \text{in } \partial A \\ \operatorname{div}(w) &= 1, \quad \text{on } \partial A. \end{cases} \tag{A.13}$$

Remark 8 *The combination of the two last constraints on w in (A.13) is an obstruction to a better regularity for w. But we will see that w is C^∞ by parts.* ■

Define

$$\overline{w}(x) = \Big((1-2x_i)d(x,\partial D)\Big)_{1\le i\le n} \qquad \text{in } D. \tag{A.14}$$

We have $\overline{w} \in W^{1,\infty}$ and $\operatorname{div}(\overline{w}) \in W^{1,\infty}$, but this is not the case for $\nabla\overline{w}$. In fact, we have

$$\operatorname{div}(\overline{w}) = 1 - 2(n+1)d(x,\partial D). \tag{A.15}$$

Take now

$$w(x',x_n) = \Big(\overline{w}_1(x',2x_n),\ldots,\overline{w}_{n-1}(x',2x_n),\frac{1}{2}\overline{w}_n(x',2x_n)\Big) \qquad \text{in } A, \tag{A.16}$$

we have $\operatorname{div}(w)(x) = \operatorname{div}(\overline{w})(x',2x_n) \in W^{1,\infty}(A)$. Thus w clearly verifies (A.13). Then take h verifying (A.12) and $\tilde{h}$ such that

$$\tilde{h}(x',x_n) = h(x',1-x_n) \qquad \text{in } B. \tag{A.17}$$

We are looking for $\tilde{w}(x)$ in B satisfying

$$\begin{cases} \operatorname{div}(\tilde{w}) &= \dfrac{\partial}{\partial x_n}\Big((1-x_n)\tilde{h}\Big) \quad \text{in } B \\ \tilde{w} &= 0 \qquad\qquad\qquad\qquad \text{on } \partial B. \end{cases} \tag{A.18}$$

From (A.10), we have

$$\begin{aligned} \operatorname{div}(w)(x',1-x_n) &= -\frac{\partial}{\partial x_n}\Big((1-x_n)h(x',1-x_n)\Big) \\ &= -\frac{\partial}{\partial x_n}\Big((1-x_n)\tilde{h}\Big) \quad \text{in } B. \end{aligned} \tag{A.19}$$

Using (A.17) and (A.19), we consider

$$\tilde{w}(x'x_n) = \Big(-w_1(x',1-x_n),\ldots,-w_{n-1}(x',x_n),w_n(x',1-x_n)\Big) \quad \text{in } B, \tag{A.20}$$

and verify that (A.18) holds for $\tilde{w}$ defined above.

Then we will construct φ , verifying (A.8) and (A.9) with some estimates for $\|\nabla(\varphi - Id)\|_\infty$, by the flow method in [4]. With this aim, we solve the following equation

$$\begin{cases} \dfrac{d\varphi_t^{-1}}{dt} &= \dfrac{v(\varphi_t^{-1})}{t+(1-t)f(\varphi_t^{-1})} \quad \text{in } A \\ \varphi_0^{-1}(x) &= x \qquad\qquad\qquad\qquad \text{in } A, \end{cases} \tag{A.21}$$

where v and f are defined by (A.11). By [4], φ_1^{-1} is a solution of (A.8). Now we establish a bound for $\|\nabla(\varphi - Id)\|_\infty$. Denote $H(t,x) = |\nabla(\varphi_t^{-1} - Id)|^2(x)$. We deduce from (A.21) the following differential inequality

$$\begin{aligned}\frac{d}{dt}H(t,x) \leq \;& C\left|\frac{1-2\alpha}{2\alpha}\right| \times \left(H(t,x) + \sqrt{H(t,x)}\right)\\ &\times\left[\frac{\|\nabla w\|_\infty}{f_t(\varphi_t^{-1})} + (1-t)\left|\frac{1-2\alpha}{2\alpha}\right|\frac{\|\mathrm{div}(w)\|_{1,\infty}\|w\|_\infty}{f_t^2(\varphi_t^{-1})}\right]\end{aligned} \tag{A.22}$$

where $f_t = t + (1-t)f$.

If α is close to $1/2$, we have a positive lower bound for f_t depending only on $\|\mathrm{div}(w)\|_\infty$ and not on t and x. Thus, in this case

$$\frac{d}{dt}H(t,x) \leq C\left|\frac{1-2\alpha}{2\alpha}\right| \times \left(H(t,x) + \sqrt{H(t,x)}\right), \tag{A.23}$$

where C is a constant. From (A.23), we deduce

$$\|\nabla(\varphi - Id)\|_{L^\infty(A)} = \|\sqrt{H(1,x)}\|_\infty \leq C\left|\frac{1-2\alpha}{2\alpha}\right| ; \tag{A.24}$$

then, for α close to $1/2$, we get

$$\|\nabla(\varphi - Id)\|_{L^\infty(A)} \leq C|\,1-2\alpha\,|. \tag{A.25}$$

Similarly we get

$$\|\nabla(\varphi - Id)\|_{L^\infty(B)} \leq C|\,1-2\alpha\,|. \tag{A.26}$$

For α close to $1/2$, (2.4)(iii) can be deduced from (A.25), (A.26) and from the explicit expression of Ψ given by (A.1).

More generally, let $0 < \eta < 1/2$, for $\alpha \in [\eta, 1-\eta]$, we construct Φ_α solution of (2.4)(ii) such that

$$\|\nabla(\Phi_\alpha - Id)\|_{L^\infty(D)} \leq C(\eta), \tag{A.27}$$

where $C(\eta)$ only depends on η. Let Ψ_η be the following homeomorphism of D such that $|\,\Psi_\eta(A)\,| = 1-\eta$,

$$\begin{cases} \Psi_\eta(x', x_n) \;=\; \left(x', [1+g_\eta(x')]x_n\right) & \text{in } A\\ \Psi_\eta(x', x_n) \;=\; \left(x', 1-[1-g_\eta(x')](1-x_n)\right) & \text{in } B\end{cases} \tag{A.28}$$

where g_η is defined in the following way: $g_\eta \in C_0^\infty([0,1]^{n-1}, [0,1))$ in $[0,1]^{n-1}$ with

$$\int_{[0,1]^{n-1}} g_\eta(x')dx' = 1-2\eta. \tag{A.29}$$

This is possible since $(1-2\eta)<1$. Let $\lambda = \dfrac{2\alpha-1}{1-2\eta}$, define Ψ_α as follows

$$\begin{cases} \Psi_\alpha(x',x_n) = \Big(x', [1+\lambda g_\eta(x')]x_n\Big) & \text{in } A \\ \Psi_\alpha(x',x_n) = \Big(x', 1-[1-\lambda g_\eta(x')](1-x_n)\Big) & \text{in } B. \end{cases} \tag{A.30}$$

We have $|\ \Psi_\alpha(A)\ | = \alpha$. As in the previous case, we will find Φ_α in the form $\Psi_\alpha \circ \varphi_\alpha$ where φ_α realizes a homeomorphism from A into A and from B into B, satisfying

$$\begin{cases} \det(\nabla\varphi_\alpha^{-1}) = \dfrac{\det(\nabla\Psi_\alpha)}{2\alpha} = \dfrac{1+\lambda g_\eta(x')}{2\alpha} & \text{in } A \\ \varphi_\alpha^{-1}(x) = x & \text{on } \partial A, \end{cases} \tag{A.31}$$

and

$$\begin{cases} \det(\nabla\varphi_\alpha^{-1}) = \dfrac{\det(\nabla\Psi_\alpha)}{2\alpha} = \dfrac{1-\lambda g_\eta(x')}{2\alpha} & \text{in } B \\ \varphi_\alpha^{-1}(x) = x & \text{on } \partial B. \end{cases} \tag{A.32}$$

For solving (A.31) and (A.32), using a Lipschitz homeomorphism T from A into the ball of radius $r_0 = \left(\dfrac{1}{2\omega_n}\right)^{1/n}$ in $\mathbb{R}^n$ (where ω_n is the volume of B_1) such that $\det(\nabla T) \equiv 1$ in A, we are left with solving the equivalent problem on the ball $B_{r_0}(0)$, i.e. consider

$$\begin{cases} \det(\nabla\xi) = \dfrac{\det(\nabla\Psi_\alpha)\circ T^{-1}}{2\alpha}(x) = \dfrac{1-\lambda g_\eta(T^{-1}(x'))}{2\alpha} & \text{in } B_{r_0}(0) \\ \xi(x) = x & \text{on } \partial B_{r_0}(0). \end{cases} \tag{A.33}$$

Since $g_\eta \in C^\infty$ and $T \in W^{1,\infty}$, we deduce the existence of $\xi \in W^{1,\infty}$ solution of (A.33), using the flow as (A.21). Take

$$\varphi_\alpha = T^{-1}\circ\xi^{-1}\circ T \qquad \text{in } A, \tag{A.34}$$

it verifies (A.31). Similarly we can construct a solution of (A.32). (A.27) can be obtained by the resolution of (A.33) and by an estimation of the $W^{1,\infty}$ norm of Ψ_α given by (A.30).

Finally we get (2.4)(iii) by combining (A.25), (A.26) and (A.27). ■

Remark 9 *Since the best regularity that we can expect for T and T^{-1} is $W^{1,\infty}$, we cannot preserve the estimation of type (2.4)(iii) after an operation as in (A.34). Thus we cannot get the desired estimation globally following the second construction.* ■

B Proof of Lemma 7

Let $m \in \mathbb{N}, 0 \le \beta \le 1$. Let U be a tubular neighborhood of $\partial\Omega$, included in Ω_1 and such that the orthogonal projection π from U onto $\partial\Omega$ is well defined and is regular. For x on $\partial\Omega$, we denote by $\nu(x)$ the unit exterior normal vector of $\partial\Omega$ at x. We define $\overline{f}$ on $U \cup \Omega$ in the following way :

$$\overline{f}(x) = \begin{cases} f(x), & \text{for } x \in \overline{\Omega} \\ \sum_{l=1}^{m+1} a_l f\Big(\pi(x) - \lambda l \nu(\pi(x))\Big), & \text{for } x \in U \backslash \overline{\Omega}, \end{cases} \tag{B.1}$$

where λ is a constant chosen small enough such that $\forall\, x \in U\backslash\overline{\Omega}$, $\pi(x) - \lambda l \nu(\pi(x)) \in U$ and a_l verify:

$$\text{for } 0 \le i \le m, \qquad \sum_{l=1}^{m+1} (-\lambda l)^i a_l = 1. \tag{B.2}$$

Clearly $(a_l)_{1 \le l \le m+1}$ is uniquely determined. One verifies that for $m' \in \mathbb{N}$, $m' \le m$, for $0 \le \alpha \le 1$ and for $f \in C^{m',\alpha}$, one has $\overline{f}$ lies in $C^{m,\alpha}(U \cup \overline{\Omega})$.

Let ξ be a positive function in $C_0^\infty(\mathbb{R}^n)$ whose support is included in $U \cup \overline{\Omega}$ and such that $\xi \equiv 1$ in $\overline{\Omega}$. Take $L_m(f) = \xi\overline{f} + (1 - \xi)$ in $\mathbb{R}^n$. (i) - (iv) hold true obviously. ∎

C Proof of Lemma 8

Since $g_t = \eta_t * f$, we have for $\tilde{g}_t = g_t - 1$

$$\|\tilde{g}_t\|_{l,\gamma} \le \int_{\mathbb{R}^n} \eta_t \times \|f - 1\|_{l,\gamma}\, dx = \|f - 1\|_{l,\gamma}.$$

The same holds for B_t, proving (3.22). To prove (3.23), we consider first $\beta = 0$ and $\beta = 1$.

$$\forall\, x \in \mathbb{R}^n, \qquad \begin{aligned} |\nabla^{l+1} g_t| &= |\nabla\eta_t * \nabla^l f| \\ &= \left| \int_{\mathbb{R}^n} \frac{1}{t^{n+1}} \nabla\eta\left(\frac{x-z}{t}\right) \nabla^l f(z)\, dz \right|. \end{aligned} \tag{C.1}$$

Since $\int_{\mathbb{R}^n} \nabla\eta \, dx = 0$, (C.1) leads to $\forall\, x \in \mathbb{R}^n$,

$$\begin{aligned} |\nabla^{l+1}\tilde{g}_t(x)| &\leq \int_{\mathbb{R}^n} \frac{1}{t^{n+1}} \left|\nabla\eta(\frac{x-z}{t})\right| \times \left|\nabla^l f(z) - \nabla^l f(x)\right| \, dz \\ &\leq \left(\int_{\mathbb{R}^n} \frac{1}{t^{n+1}} \left|\nabla\eta(\frac{x-z}{t})\right| \times |x-z|^\gamma \, dz\right) \times \|f-1\|_{l,\gamma} \\ &= \frac{\|f-1\|_{l,\gamma}}{t^\gamma} \int_{supp(\xi)} C|\, y\,|^\gamma \, dy \\ &\leq \frac{C}{t^{1-\gamma}} \|f-1\|_{l,\gamma}. \end{aligned} \tag{C.2}$$

In the same way, using one more time $\int_{\mathbb{R}^n} \nabla\eta \, dx = 0$, $\forall\, x, y \in \mathbb{R}^n$

$$\begin{aligned} &|\nabla^{l+1}\tilde{g}_t(x) - \nabla^{l+1}\tilde{g}_t(y)| = |\nabla\eta_t * \nabla^l f(x) - \nabla\eta_t * \nabla^l f(y)| \\ \leq\ & \int_{\mathbb{R}^n} \frac{1}{t^{n+1}} \left|\nabla\eta\left(\frac{x-z}{t}\right) - \nabla\eta\left(\frac{y-z}{t}\right)\right| \times |\nabla^l f(z) - \nabla^l f(x)| \, dz. \end{aligned} \tag{C.3}$$

If $|x-y| \leq t$, recalling that $supp(\eta) \subset B_R(0)$ and noting $R' = R+1$, (C.3) yields

$$\begin{aligned} &|\nabla^{l+1}\tilde{g}_t(x) - \nabla^{l+1}\tilde{g}_t(y)| \\ \leq\ & \int_{B_{x,tR'}} \frac{1}{t^{n+1}} \|\nabla^2\eta\|_0 \left|\frac{x-y}{t}\right| \times |\nabla^l f(z) - \nabla^l f(x)| \, dz \\ \leq\ & C\frac{|\, x-y\,|}{t^{n+2}} \int_{B_{x,tR'}} |x-z|^\gamma \, dz \times \|f-1\|_{l,\gamma} \\ \leq\ & C\frac{|x-y|}{t^{2-\gamma}} \|f-1\|_{l,\gamma}. \end{aligned} \tag{C.4}$$

If $|x-y| \geq t$,

$$|\nabla^{l+1}\tilde{g}_t(x) - \nabla^{l+1}\tilde{g}_t(y)| \leq 2\, \|\nabla^{l+1}\tilde{g}_t\|_0 \leq \frac{C}{t^{1-\gamma}}\|f-1\|_{l,\gamma} \leq C\frac{|x-y|}{t^{2-\gamma}}\|f-1\|_{l,\gamma}.$$

Thus we get

$$\|\tilde{g}_t\|_{l+1,1} \leq C\frac{\|f-1\|_{l,\gamma}}{t^{2-\gamma}}. \tag{C.5}$$

Let $0 < \beta < 1$, we have the classical interpolation estimate (see [6])

$$\|\tilde{g}_t\|_{l+1,\beta} \leq C(l,\beta)\|\tilde{g}_t\|_{l+1,0}^{1-\beta} \times \|\tilde{g}_t\|_{l+1,1}^{\beta}. \tag{C.6}$$

Combining (C.2), (C.5) and (C.6), we establish (3.22) for $\tilde{g}_t$. The estimations for B_t can be obtained exactly in the same way, this proves Lemma 8. ■

D Proof of Lemma 9

By classical arguments, φ_t solution of (3.30) is unique $C^1(\mathbb{R}_+, W^{1,\infty}(\overline{\Omega}))$. Thus we just need to establish (3.31) for the case $A_t \in C^\infty(\mathbb{R}_+ \times \overline{\Omega})$. We will use the following elementary lemma

Lemma 15 *Let $h \in C^\infty([0,T] \times \overline{\Omega})$, let $l \in \mathbb{N}$ and $0 \le \beta \le 1$, one has*

$$|\partial_t \|h\|_{l,\beta}| \le \|\partial_t h\|_{l,\beta}, \qquad \forall\, t > 0. \tag{D.1}$$

■

Proof of Lemma 15. Let $w_1, w_2 \in L^\infty(E)$ where $E \subset \mathbb{R}^n$, it is clear that

$$\left| \sup_{z\in E} w_1(z) - \sup_{z\in E} w_2(z) \right| \le \sup_{z\in E} |w_1(z) - w_2(z)|. \tag{D.2}$$

Let $i \le l$, $j = 1, 2$ and $t_j \in \mathbb{R}_+$. Consider

$$w_j^i(x,y) = \left| \frac{\nabla^i h_{t_j}(x) - \nabla^i h_{t_j}(y)}{|x-y|^\beta} \right|,$$

where $E = \overline{\Omega} \times \overline{\Omega} \backslash \triangle$ with $\triangle = \{(x,x), x \in \overline{\Omega}\}$ (resp. $w_j^i(x) = |\nabla^i h_{t_j}(x)|$ with $E = \overline{\Omega}$), one has

$$\sup_{z\in E} w_1(z) - \sup_{z\in E} w_2(z) = |h_{t_1}|_{i,\beta} - |h_{t_2}|_{i,\beta}, \qquad (\textit{resp. } |h_{t_1}|_i - |h_{t_2}|_i)$$

and

$$\begin{aligned}
\sup_{z\in E} |w_1(z) - w_2(z)| &= \sup_{x\neq y\in\overline{\Omega}} |\ w_1^i(x,y) - w_2^i(x,y)\ | \\
&\le \sup_{x\neq y\in\overline{\Omega}} \left| \frac{\nabla^i h_{t_1}(x) - \nabla^i h_{t_1}(y)}{|x-y|^\beta} - \frac{\nabla^i h_{t_2}(x) - \nabla^i h_{t_2}(y)}{|x-y|^\beta} \right| \\
&\le \sup_{x\neq y\in\Omega} \int_{t_1}^{t_2} \left| \frac{\partial}{\partial t} \left[\frac{\nabla^i h_t(x) - \nabla^i h_t(y)}{|x-y|^\beta} \right] \right| dt \\
&\le \int_{t_1}^{t_2} |\partial_t h|_{i,\beta}\, dt,
\end{aligned}$$

$\left(\text{resp. } \le \int_{t_1}^{t_2} |\partial_t h|_i\, dt\right)$. This implies for $t_2 > t_1$

$$\Big| \|h_{t_1}\|_{i,\beta} - \|h_{t_2}\|_{i,\beta} \Big| \le \int_{t_1}^{t_2} \|\partial_t h\|_{i,\beta}\, dt,$$

which implies (D.1) by multiplying $\dfrac{1}{t_2 - t_1}$ and making t_2 tending to t_1. ■

Turning now to the proof of (3.31), (3.30) implies

$$\frac{\partial}{\partial t}\nabla\varphi_t = \nabla A_t(\varphi_t).\nabla\varphi_t \tag{D.3}$$

by Lemma 15, we get

$$\frac{\partial}{\partial t}\|\nabla\varphi_t\|_0 \leq \|\nabla A_t\|_0\|\nabla\varphi_t\|_0. \tag{D.4}$$

Integrating this inequality, we obtain

$$\|\nabla\varphi_t\|_0 \leq exp\left(\int_0^t \|\nabla A_s\|_0\, ds\right). \tag{D.5}$$

Applying Lemma 15 to $h = \varphi_t$, we get

$$|\partial_t\|\varphi_t\|_{l,\beta}| \leq \|A_t(\varphi_t)\|_{l,\beta}. \tag{D.6}$$

We consider the following classical estimate, proved in [6],

$$\|f\circ g\|_{l,\beta} \leq C(l,\beta,\Omega)\Big[\ \|f\|_{l,\beta}\|g\|_1^{l+\beta} + \|f\|_1\|g\|_{l,\beta} + \|f\|_0\ \Big], \tag{D.7}$$

and we have then

$$\begin{aligned} \frac{\partial}{\partial t}\|\varphi_t\|_{l,\beta} &\leq C(l,\beta,\Omega)\Big[\ \|A_t\|_{l,\beta}\|\nabla\varphi_t\|_0^{l+\beta} + \|\nabla A_t\|_0\|\varphi_t\|_{l,\beta}\ \Big] \\ \leq\ & C(l,\beta,\Omega)\Big[\ \|\nabla A_t\|_0\|\varphi_t\|_{l,\beta} + exp\left((l+\beta)\int_0^t \|\nabla A_s\|_0\, ds\right)\|A_t\|_{l,\beta}\ \Big]. \end{aligned} \tag{D.8}$$

Integrating this inequality we get (3.31). (3.32) could be proved in the same way, by using (3.34) instead of (D.7). This proves the lemma. ■

E Proof of Lemma 10

We prove Lemma 10 by induction on l. For $l = 1$ we have

$$\begin{aligned} T^1_{A\circ\varphi}(x,y) &= \sum_{i=1}^n \Big|\partial_i(A\circ\varphi)(x) - \partial_i(A\circ\varphi)(y)\Big| \\ &\leq \sum_{i,k=1}^n \Big|\partial_k A(\varphi(x))\partial_i\varphi^k(x) - \partial_k A(\varphi(y))\partial_i\varphi^k(y)\Big| \\ &\leq C\Big[\ \|\nabla A\|_0 T^1_\varphi(x,y) + \|\nabla\varphi\|_0 T^1_A(\varphi(x),\varphi(y))\ \Big] \\ &\leq C\Big[\ \|\nabla A\|_0 T^1_{\varphi - Id}(x,y) + \Big(|\nabla\varphi - Id\|_0 + 1\Big)T^1_A(\varphi(x),\varphi(y))\ \Big] \\ &\leq C\Big[\ \|\nabla A\|_0\|\varphi - Id\|_{1,\beta}|\ x-y\ |^\beta \\ &\quad + \|A\|_{1,\beta}\|\nabla(\varphi - id)\|_0^\beta|\ x-y\ |^\beta + T^1_A(\varphi(x),\varphi(y))\ \Big]. \end{aligned} \tag{E.1}$$

Suppose the proposition is true for $l \leq m$, we prove it for $l = m+1$.

$$\begin{aligned} T^{m+1}_{A\circ\varphi}(x,y) \leq\ & T^m_{\nabla(A\circ\varphi)}(x,y) = T^m_{\nabla A(\varphi)\nabla\varphi}(x,y) \\ \leq\ & T^m_{\nabla A(\varphi)(\nabla\varphi - Id)}(x,y) + T^m_{\nabla A(\varphi)}(x,y) \\ \leq\ & C\Big\{|x-y|^\beta \times \Big[\ \|\nabla A(\varphi)\nabla(\varphi - Id)\|_{m,\beta} \\ & + \|A\|_{m+1,\beta}\|\nabla(\varphi - Id)\|_0\Big(1 + \|\varphi - Id\|_1^{m+\beta-1}\Big) \\ & + \|\nabla A\|_1\|\varphi - Id\|_{m,\beta}\Big] + T^{m+1}_A(\varphi(x),\varphi(y))\Big\}. \end{aligned} \tag{E.2}$$

By a classical estimate, we have (see [5])

$$\begin{aligned} \|\nabla A(\varphi)\nabla(\varphi - Id)\|_{m,\beta} \leq\ & C\Big[\ \|\nabla A(\varphi)\|_{m,\beta}\|\nabla(\varphi - Id)\|_0 \\ & + \|\nabla A\|_0\|\nabla(\varphi - Id)\|_{m,\beta}\Big], \end{aligned} \tag{E.3}$$

and using (3.34) for estimating $\|\nabla A(\varphi)\|_{m,\beta}$, we get

$$\|\nabla A(\varphi)\|_{m,\beta} \leq C\Big[\ \|A\|_{m+1,\beta}\Big(1 + \|\varphi - Id\|_1^{m+\beta}\Big) + \|\nabla A\|_1\|\varphi - Id\|_{m,\beta}\Big]. \tag{E.4}$$

Moreover, using interpolation inequality for the Hölder spaces (see [6]), we obtain

$$\|\nabla A\|_1\|\varphi - Id\|_{m,\beta} \leq C\Big(\|\nabla A\|_0\|\varphi - Id\|_{m+1,\beta} + \|A\|_{m+1,\beta}\|\varphi - Id\|_1\Big). \tag{E.5}$$

Combining (E.3) to (E.5), the worst term obtained is: $\|\nabla A\|_1\|\varphi - Id\|_{m,\beta}\|\varphi - Id\|_1$. Using one more time the following general interpolation result (see [6]): $\forall\, a, b \in \mathbb{R}^+$ and $\lambda \in [0,1]$, there exists $C(\Omega, a, b, \lambda) > 0$ such that

$$\forall\, g \in C^{\max(a,b)}(\overline{\Omega}), \qquad \|g\|_{\lambda a + (1-\lambda)b} \leq C(\Omega, a, b, \lambda)\|g\|_a^\lambda \|g\|_b^{1-\lambda},$$

where $\|g\|_p = \|g\|_{[p],p-[p]}$. We obtain then

$$\begin{aligned} \|\nabla A\|_1\|\varphi - Id\|_{m,\beta}\|\varphi - Id\|_1 \leq\ & C\Big(\|\nabla A\|_0\|\varphi - Id\|_{m+1,\beta}\Big)^{1-\frac{1}{m+\beta}} \\ & \times\Big(\|A\|_{m+1,\beta}\|\varphi - Id\|_1\Big)^{\frac{1}{m+\beta}}\|\varphi - Id\|_1. \end{aligned} \tag{E.6}$$

Applying Young's inequality in (E.4), we get

$$\begin{aligned} \|\nabla A\|_1\|\varphi - Id\|_{m,\beta}\|\varphi - Id\|_1 \leq\ & C\Big(\|\nabla A\|_0\|\varphi - Id\|_{m+1,\beta} \\ & + \|A\|_{m+1,\beta}\|\varphi - Id\|_1^{m+\beta+1}\Big). \end{aligned} \tag{E.7}$$

Combining (E.2)–(E.7) we prove the lemma. ■

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